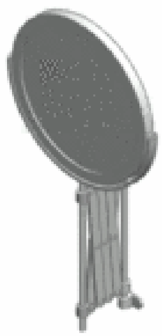
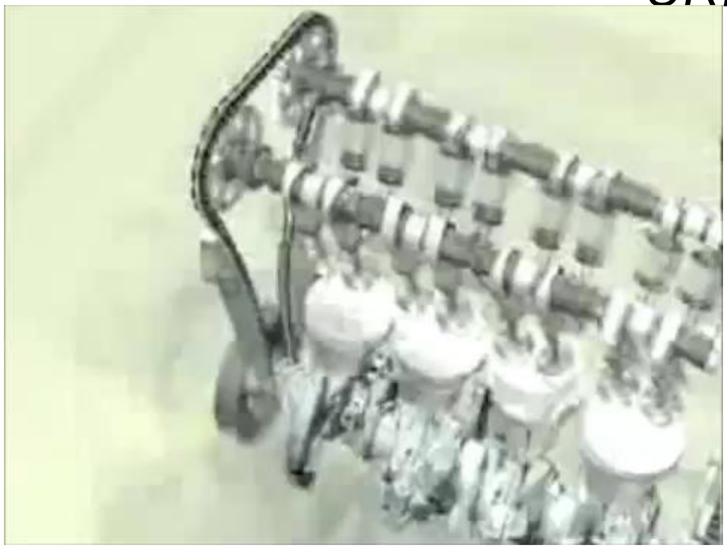
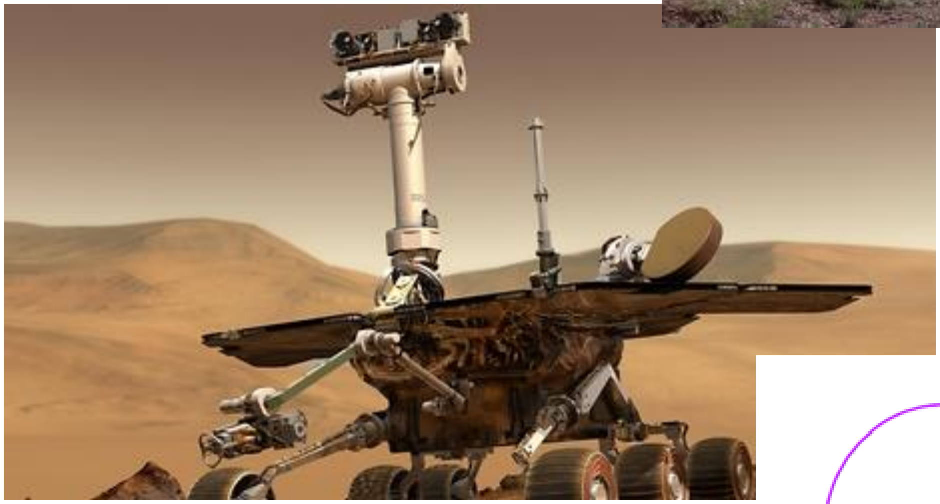
Kinematics Of Rigid Bodies

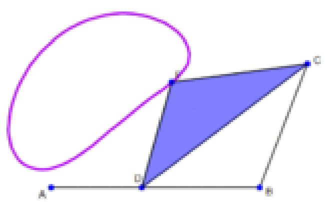
Prof. Nicholas Zabaras
Warwick Centre for Predictive Modelling
University of Warwick
Coventry CV4 7AL
United Kingdom

Email: nzabaras@gmail.com
URL: http://www.zabaras.com/









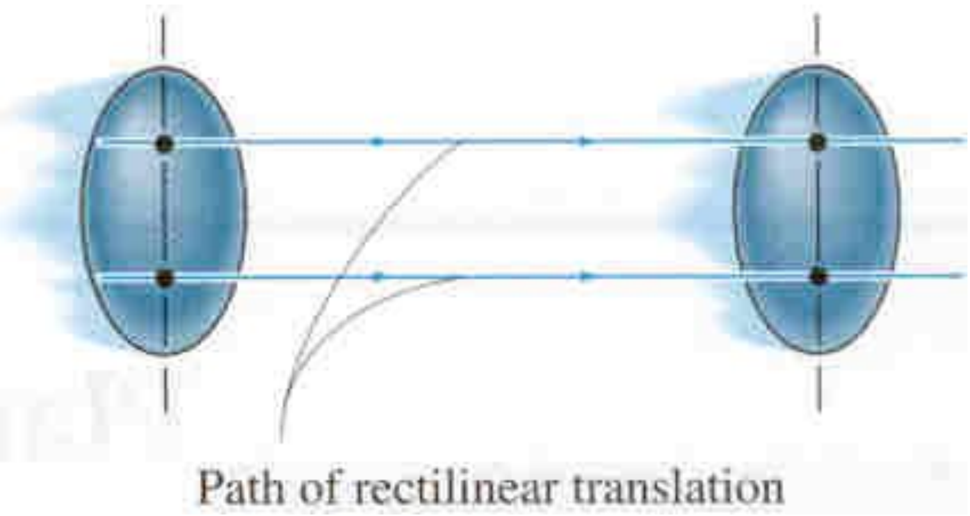
April 10, 2016

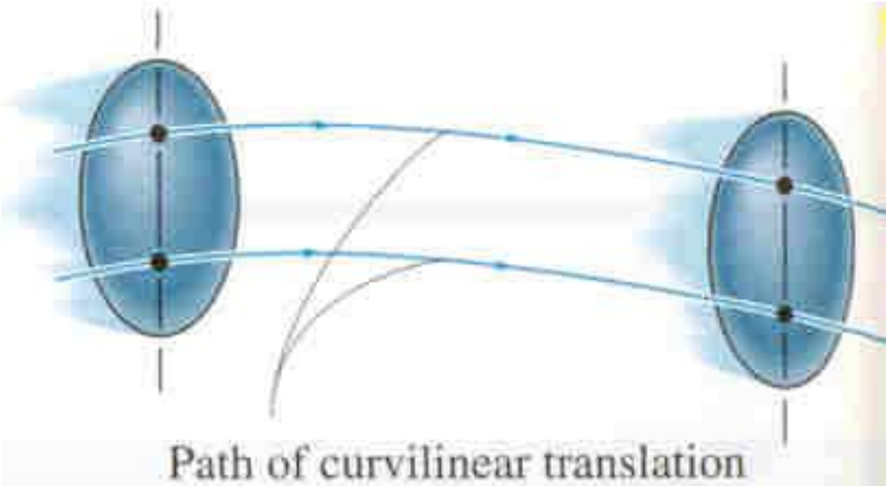
Rigid Bodies

- On a rigid body all points remain the same distance from each other
- E.g. the length *AB* is fixed regardless of motion

- General Planar Motion = Translation + Fixed-axis Rotation

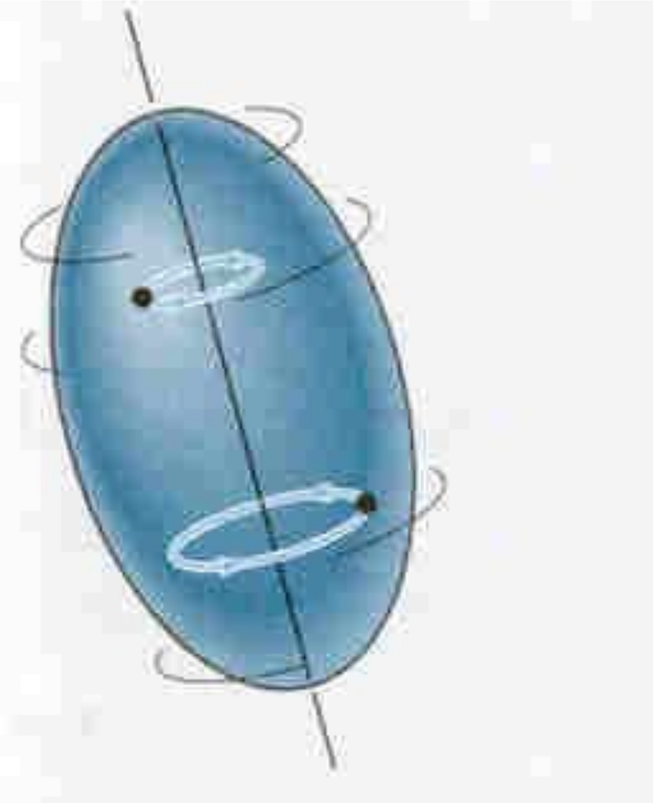
Translation





- Every line segment on the body remains parallel to its original direction during the motion
- In both cases all points on the object move in the same direction, with the same velocity and acc'n

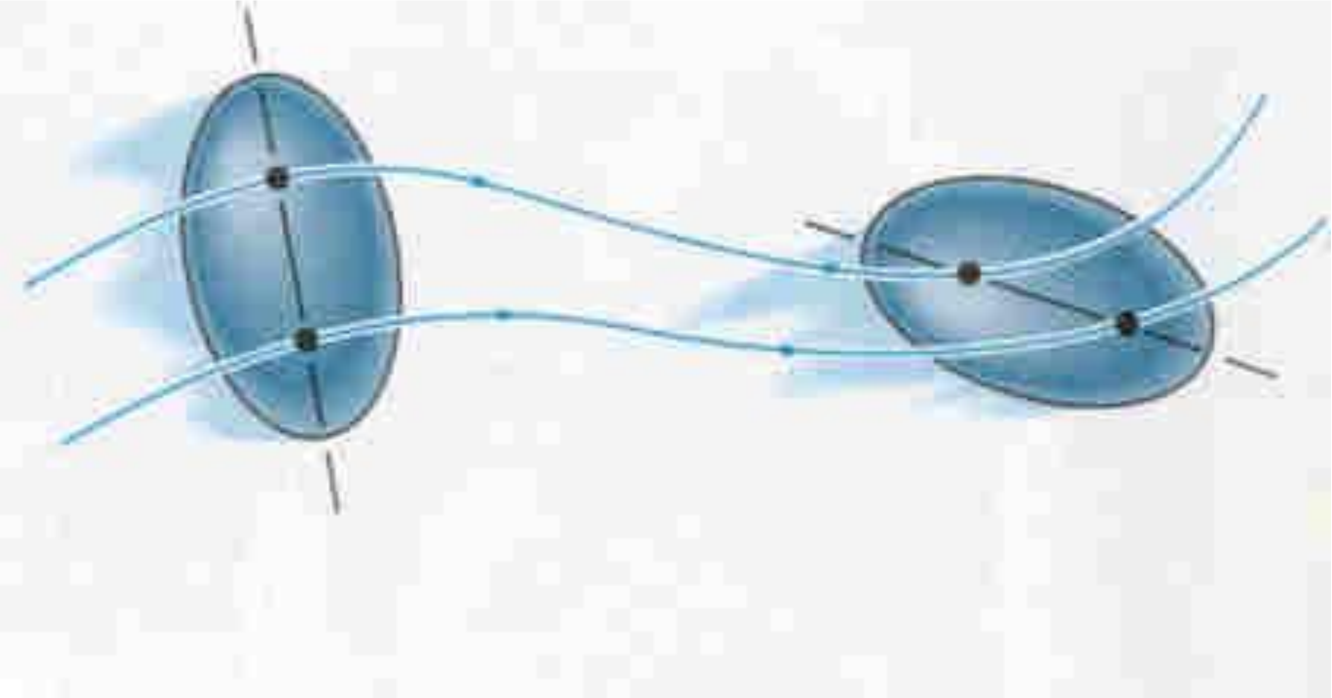
Rotation about fixed axis



Rotation about a fixed axis

All particles of the body move along circular paths except those which lie on the axis of rotation

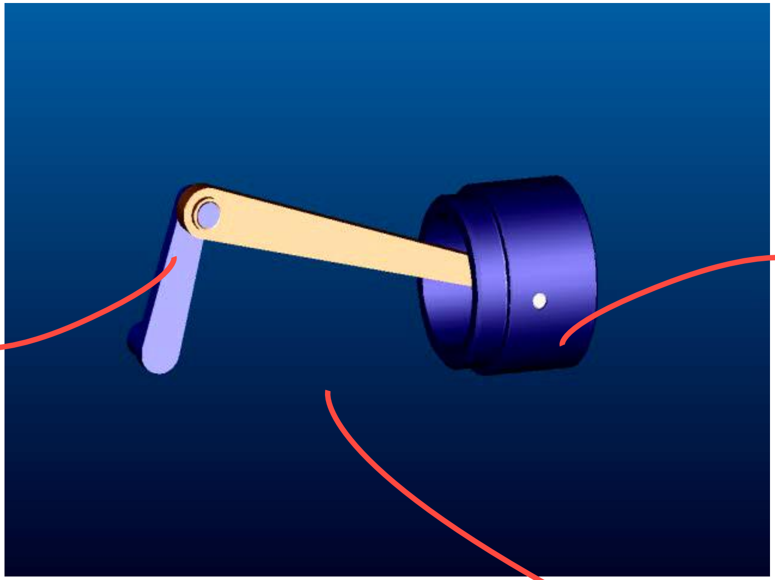
General plane motion



General plane motion

Combination of translation and rotation

General Planar Motion



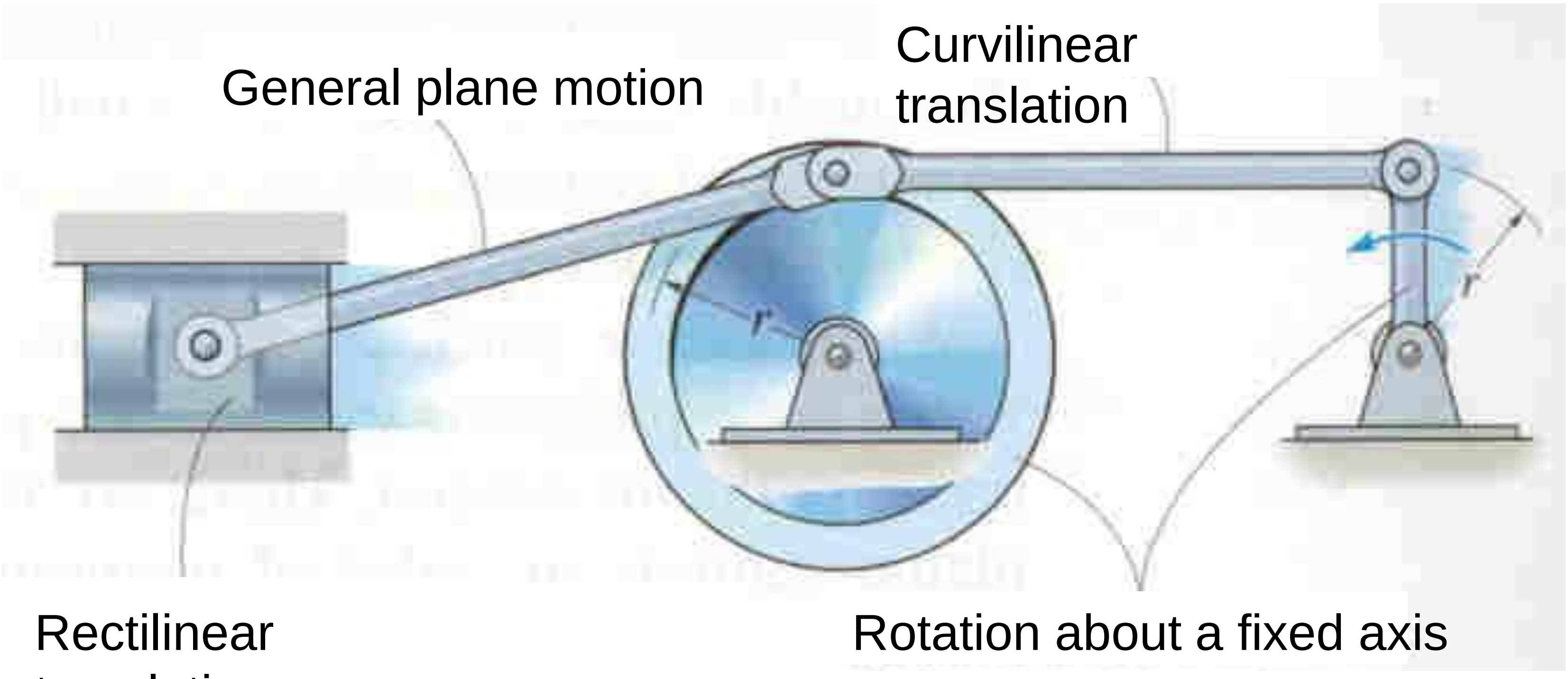
fixed-axis rotation

rectilinear translation

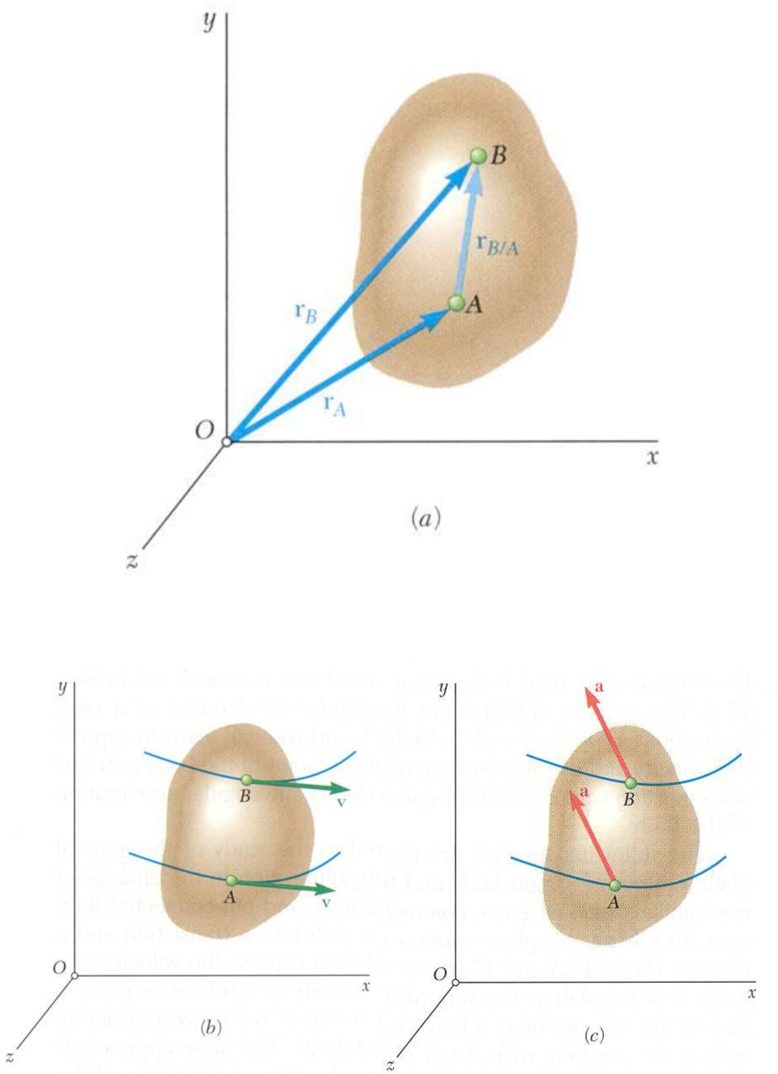
general planar motion

General Planar Motion = Translation + Fixed-Axis Rotation

Example


General plane motion
Curvilinear
translation
Rectilinear
translation
Rotation about a fixed axis

Translation



- Consider rigid body in translation:
  - direction of any straight line inside the body is constant,
  - all particles forming the body move in parallel lines.

- For any two particles in the body,

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

- Differentiating with respect to time,

$$\dot{\vec{r}}_B = \dot{\vec{r}}_A + \dot{\vec{r}}_{B/A} = \dot{\vec{r}}_A$$

$$\vec{v}_B = \vec{v}_A$$

All particles have the same velocity.

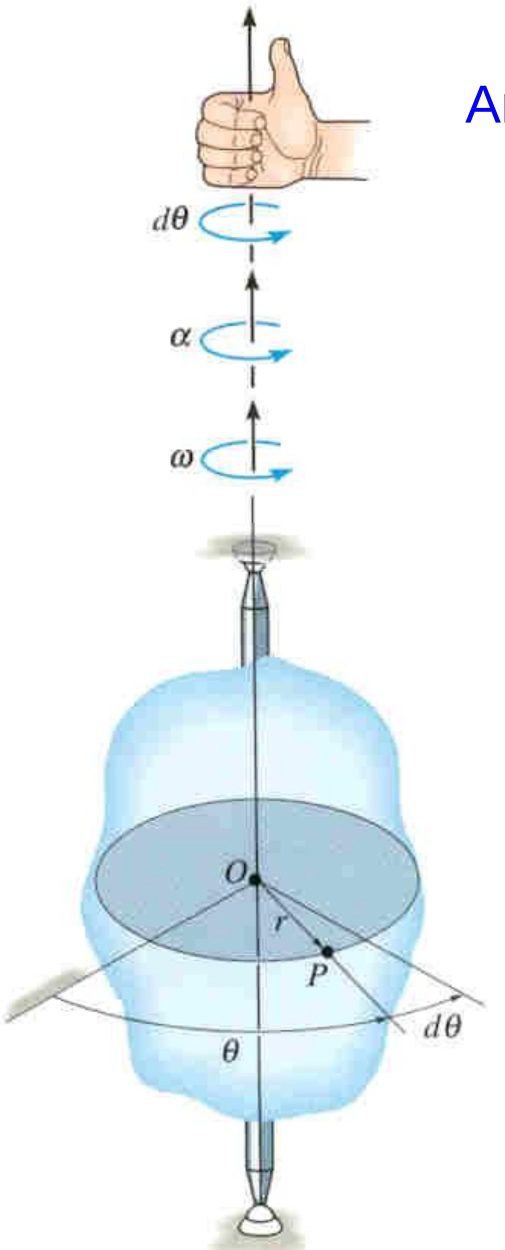
- Differentiating with respect to time again,

$$\ddot{\vec{r}}_B = \ddot{\vec{r}}_A + \ddot{\vec{r}}_{B/A} = \ddot{\vec{r}}_A$$

$$\vec{a}_B = \vec{a}_A$$

All particles have the same acceleration.

Rotation About a Fixed Axis



Angular velocity (ω)

the time rate of change in the angular position

$$\omega = \frac{d\theta}{dt} = \dot{\theta}$$

Angular acceleration

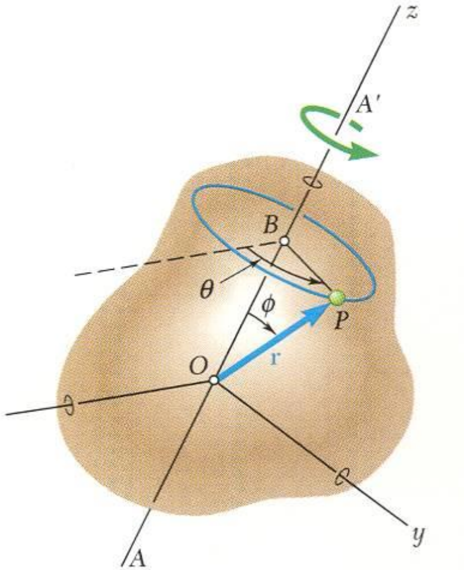
the time rate of change of the angular velocity

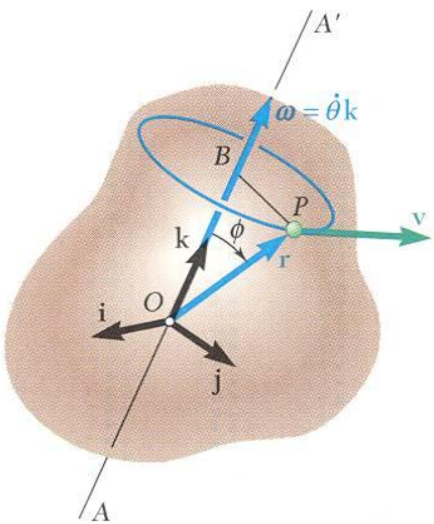
$$\alpha = \frac{d\omega}{dt} = \ddot{\theta} \qquad \alpha = \frac{d^2\theta}{dt^2} = \ddot{\theta}$$

$\alpha = f(\theta)$

$$\alpha \, d\theta = \omega \, d\omega$$

Rotation About a Fixed Axis. Velocity





- Consider rotation of rigid body about a fixed axis *AA'*

- Velocity vector $\vec{v} = d\vec{r}/dt$ of the particle *P* is tangent to the path with magnitude $v = ds/dt$

$$\Delta s = (BP)\Delta\theta = (r\sin\phi)\Delta\theta$$

$$v = \frac{ds}{dt} = \lim_{\Delta t \to 0} (r\sin\phi)\frac{\Delta\theta}{\Delta t} = r\dot{\theta}\sin\phi$$

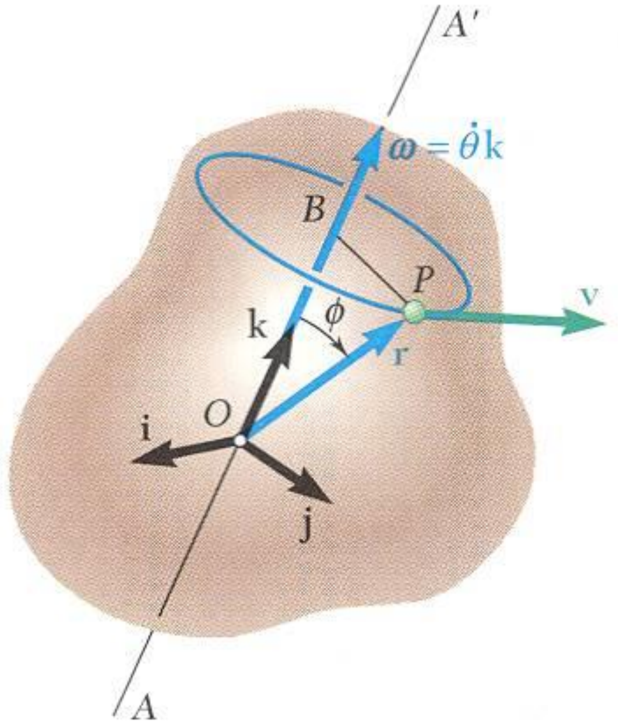
- The same result is obtained from

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r}$$

$$\vec{\omega} = \omega\vec{k} = \dot{\theta}\vec{k} = angular\ velocity$$

$$\vec{k} = unit\ vector\ in\ the\ z\ direction$$

Rotation About a Fixed Axis. Acceleration



- Differentiating to determine the acceleration,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r})$$
$$= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$
$$= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \vec{v}$$

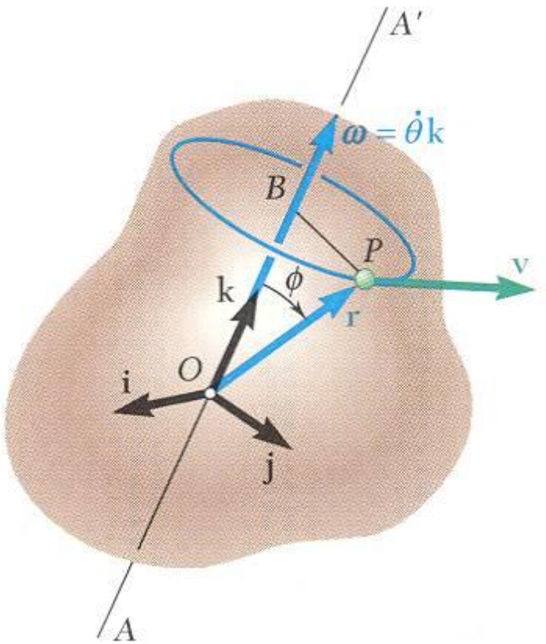
- $\frac{d\vec{\omega}}{dt} = \vec{\alpha} = angular\ acceleration$

$$= \alpha \vec{k} = \dot{\omega}\vec{k} = \ddot{\theta}\vec{k}$$

- Acceleration of *P* is combination of two vectors,

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{\omega} \times \vec{r}$$
$$\vec{\alpha} \times \vec{r} = \text{tangential acceleration component}$$
$$\vec{\omega} \times \vec{\omega} \times \vec{r} = \text{radial acceleration component}$$

Equations Defining the Rotation of a Rigid Body About a Fixed Axis



- Motion of a rigid body rotating around a fixed axis is often specified by the type of angular acceleration.
- Recall $\omega = \dfrac{d\theta}{dt}$ or $dt = \dfrac{d\theta}{\omega}$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \omega \frac{d\omega}{d\theta}$$

- *Uniform Rotation,* $\alpha = 0$:

$$\theta = \theta_0 + \omega t$$

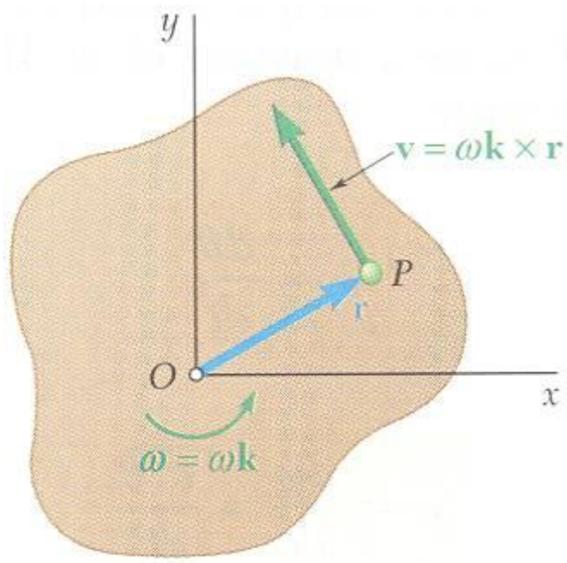
- *Uniformly Accelerated Rotation,* $\alpha =$ constant:

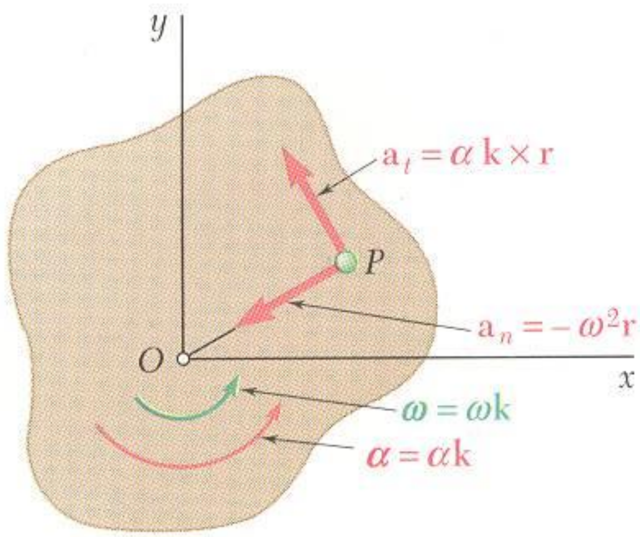
$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

Rotation About a Fixed Axis. Representative Slab





- Consider the motion of a representative slab in a plane perpendicular to the axis of rotation. Note that in the following slides $\vec{r}$ is defined in the plane xy as shown.
- Velocity of any point *P* of the slab,

$$\vec{v} = \vec{\omega} \times \vec{r} = \omega \vec{k} \times \vec{r}$$

$$v = r\omega$$

- Acceleration of any point *P* of the slab,

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{\omega} \times \vec{r}$$

$$= \alpha \vec{k} \times \vec{r} - \omega^2 \vec{r}$$

Recall the vector triple product identify

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{\omega}(\underbrace{\vec{\omega} \cdot \vec{r}}_{0}) - \vec{r}(\vec{\omega} \cdot \vec{\omega}) = -\omega^2 \vec{r}$$

- Resolving the acceleration into tangential and normal components,

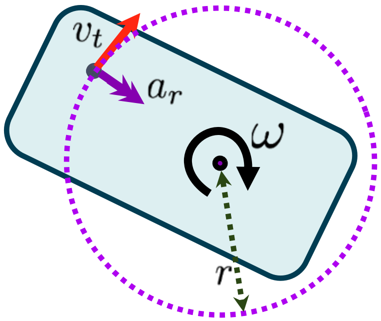
$$\vec{a}_t = \alpha \vec{k} \times \vec{r} \qquad a_t = r\alpha$$

$$\vec{a}_n = -\omega^2 \vec{r} \qquad a_n = r\omega^2$$

Fixed-Axis Rotation: Acceleration

- Suppose the space station is 200m in diameter.

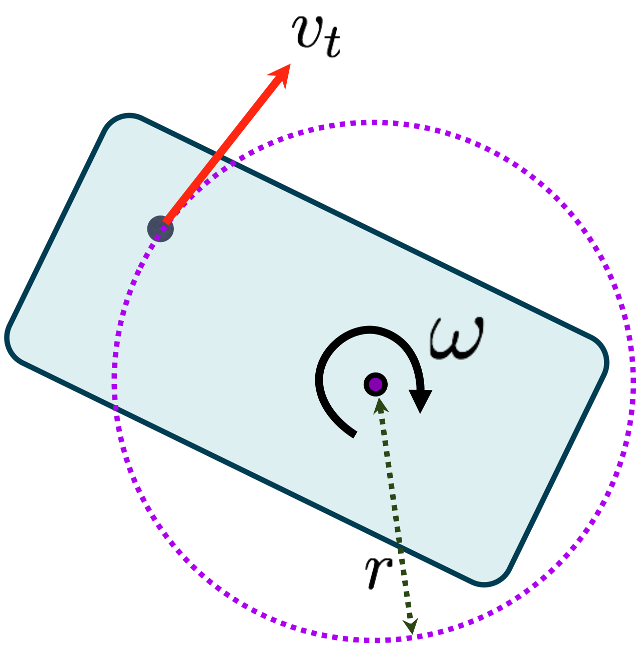
$$a_r = r\omega^2$$



- What must be the angular velocity to simulate gravity on earth? ω = 0.316 rad/s ≈ 3 rpm

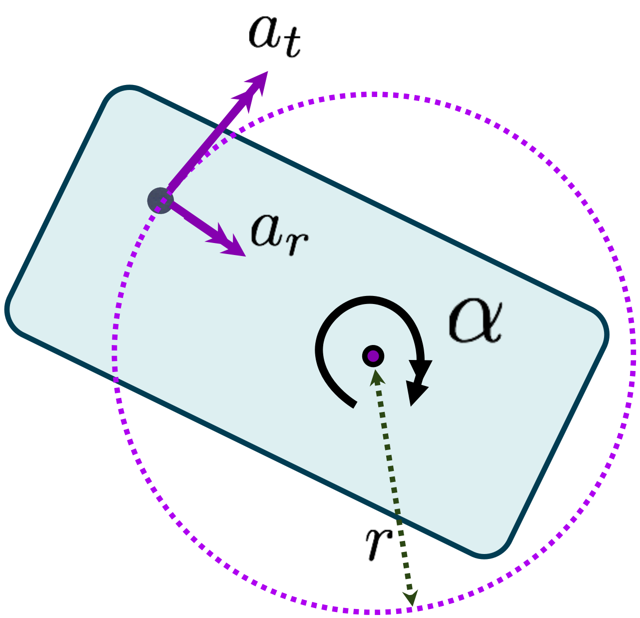
Fixed-Axis Rotation: Summary

- one component of velocity: $v_t = r\omega$



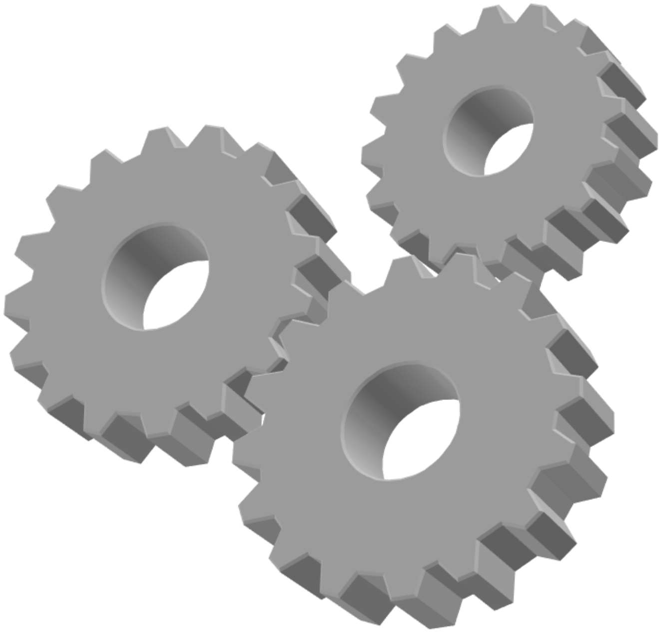
- two components of acceleration: $a_t = r\alpha$

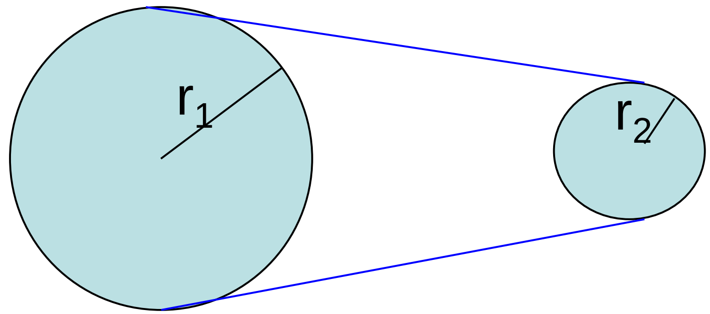
$$a_r = r\omega^2$$

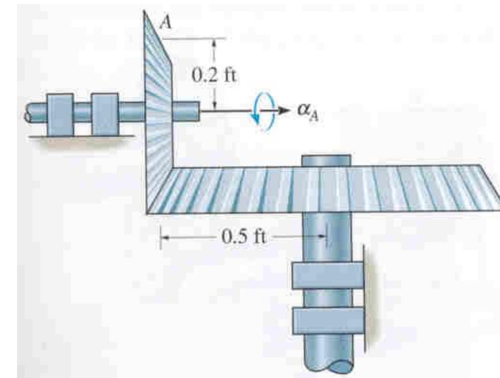


$$|a| = \sqrt{a_t^2 + a_r^2}$$

Velocity & Tangential Acceleration at Points of Contact



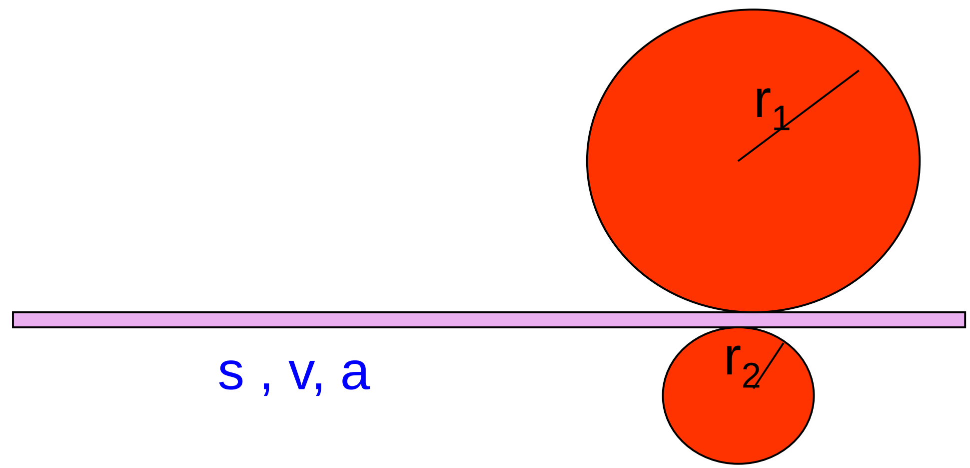




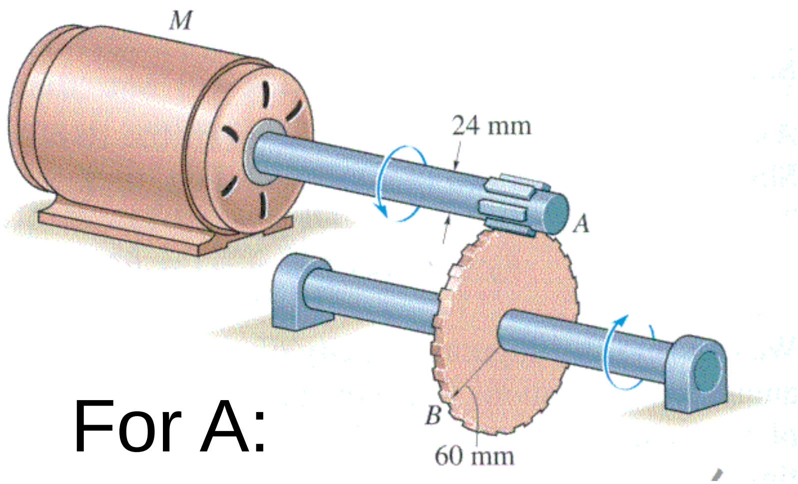
$$S = \theta_1 r_1 = \theta_2 r_2$$

$$\upsilon = \omega_1 r_1 = \omega_2 r_2$$

$$a = \alpha_1 r_1 = \alpha_2 r_2$$



Example



$\alpha = 0.06\theta^2 \ rad / s^2$

$\omega_0 = 50 \ rad / s$

$Compute \ \omega_B = ? \ when \ \Delta\theta_A = 10 \ rev.$

For A:

$$\alpha \, d\theta = \omega \, d\omega$$

$$\int_{50}^{\omega} \omega \, d\omega = \int_{0}^{2\pi(10)} 0.06\,\theta^2 d\theta$$

$$\frac{1}{2}\omega^2 \Big|_{50}^{\omega} = 0.02\theta^3 \Big|_{0}^{2\pi 10}$$

$$0.5\omega^2 - 1250 = 4961$$

$$\omega_A = 111.45 \ \ rad / s$$

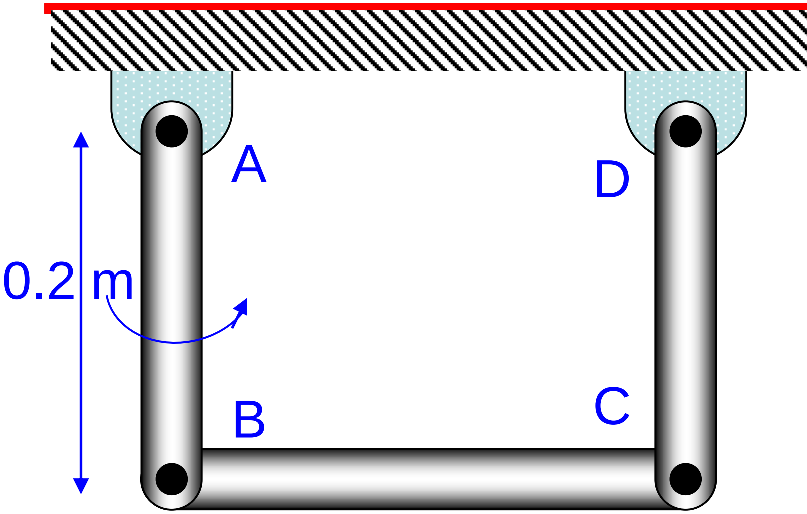
To compute ω_B

$$\omega_A r_A = \omega_B r_B$$

$$(111.45)(12) = \omega_B (60)$$

$$\omega_B = 22.3 \, rad / s$$

Example

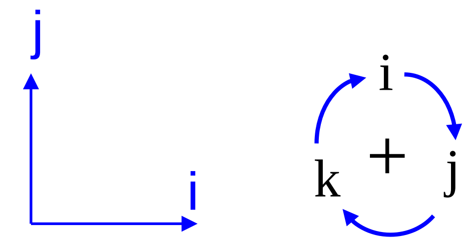


$$\omega_{AB} = 10\, rad/s$$

$$\upsilon_B = ?$$

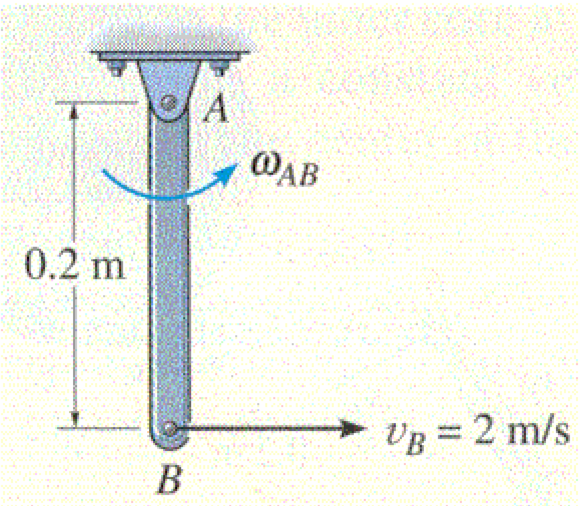
$$\upsilon_C = ?$$

$$\omega_{DC} = ?$$



$$\vec{i} \times \vec{j} = \vec{k}, \vec{j} \times \vec{k} = \vec{i}, \vec{k} \times \vec{i} = \vec{j}$$

$$\vec{j} \times \vec{i} = -\vec{k}, \vec{k} \times \vec{j} = -\vec{i}, \vec{i} \times \vec{k} = -\vec{j}$$



$$\mathrm{v}_B = \omega_{AB} \times r_B$$

$$\upsilon_B \vec{i} = \omega_{AB} \vec{k} \times (-0.2\vec{j})$$

$$\upsilon_B \vec{i} = 10\vec{k} \times (-0.2\vec{j})$$

$$\upsilon_B \vec{i} = 2\vec{i}$$

$$\upsilon_B = 2 \text{ m/s} \longrightarrow$$

$$\mathrm{v}_C = \mathrm{v}_B$$

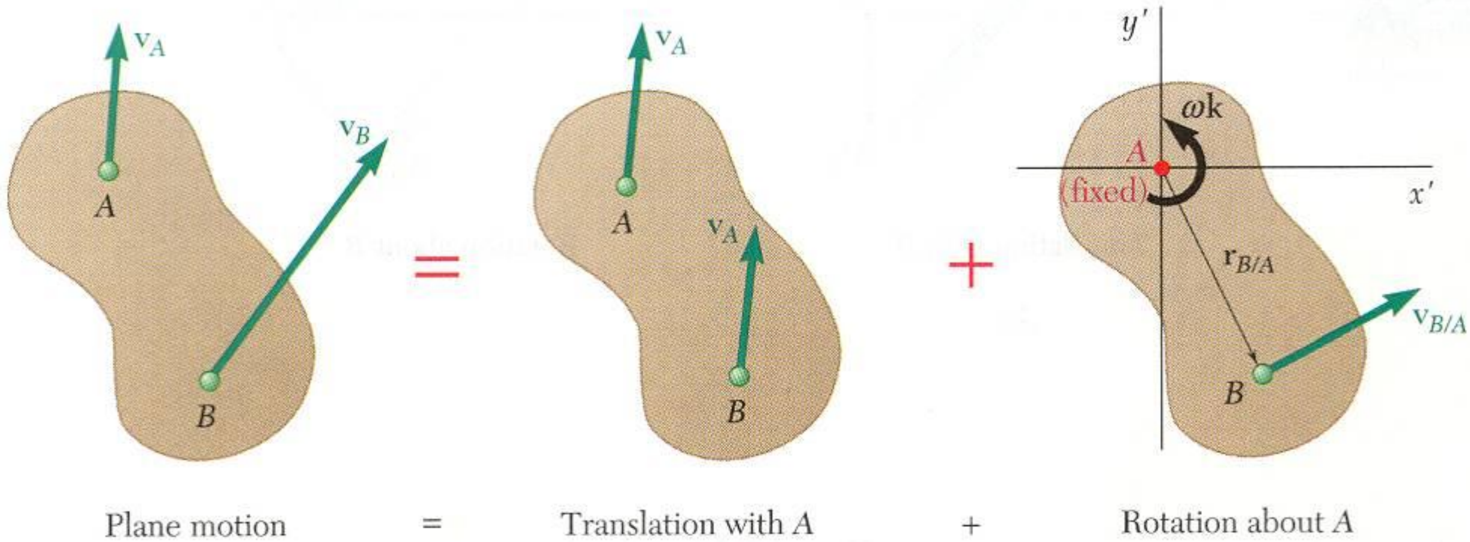
$$\mathrm{v}_C = \omega_{DC} \times r_C$$

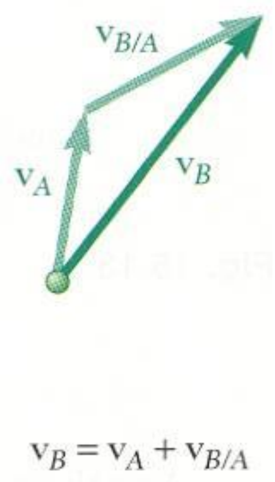
$$2\vec{i} = \omega_{DC} \vec{k} \times (-0.2\vec{j})$$

$$2 = 0.2\,\omega_{DC}$$

$$\omega_{DC} = 10\, rad/s$$

Absolute and Relative Velocity in Plane Motion





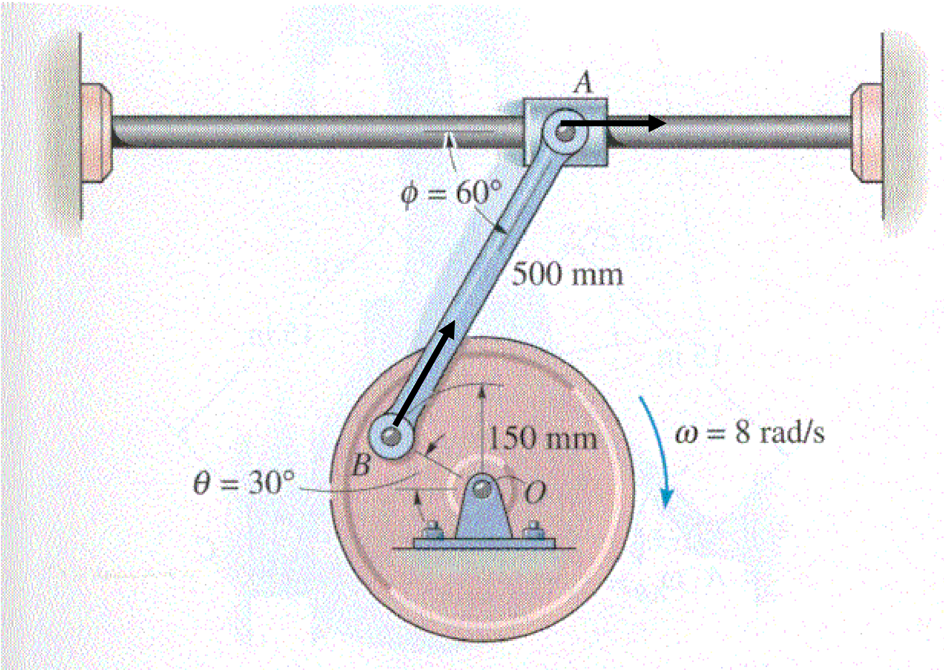
- *Any plane motion can be replaced by a translation of an arbitrary reference point A and a simultaneous rotation about A.*

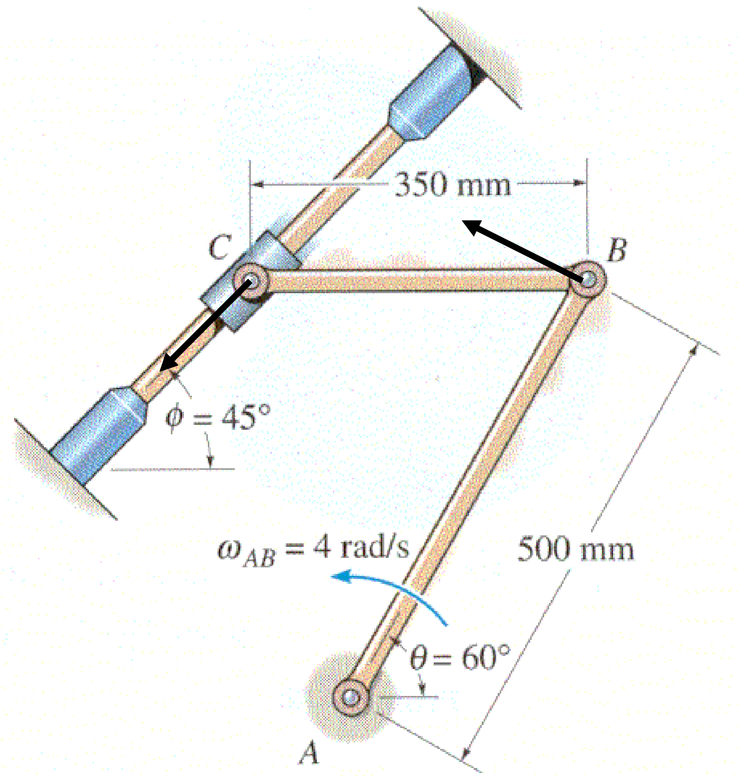
$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$

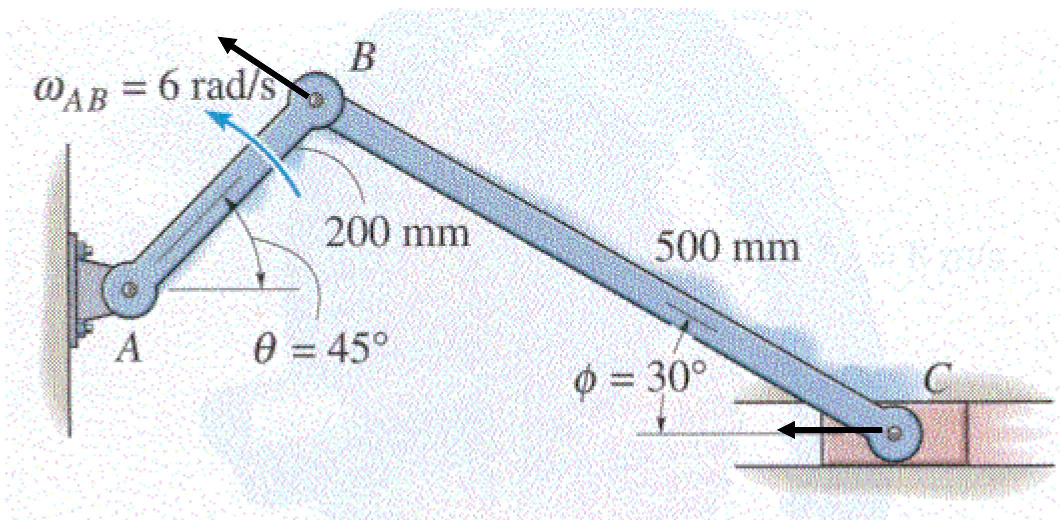
$$\vec{v}_{B/A} = \omega\vec{k} \times \vec{r}_{B/A} \qquad v_{B/A} = r\omega$$

$$\vec{v}_B = \vec{v}_A + \omega\vec{k} \times \vec{r}_{B/A}$$

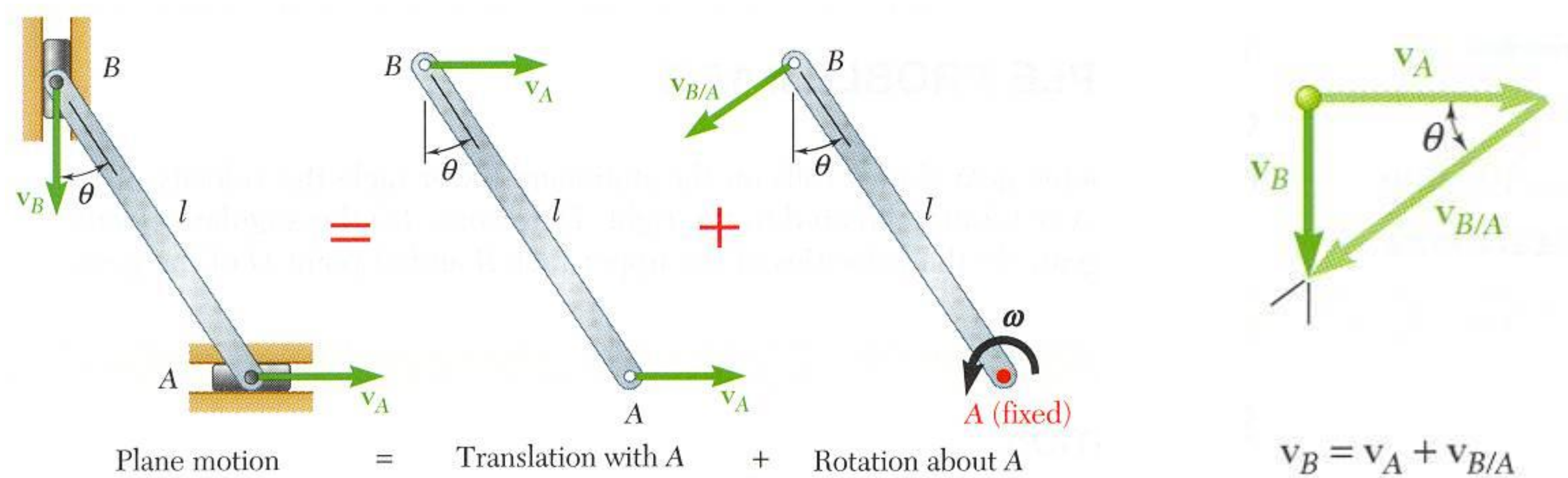
Indicate the direction of velocities of A, B, C







Absolute and Relative Velocity in Plane Motion



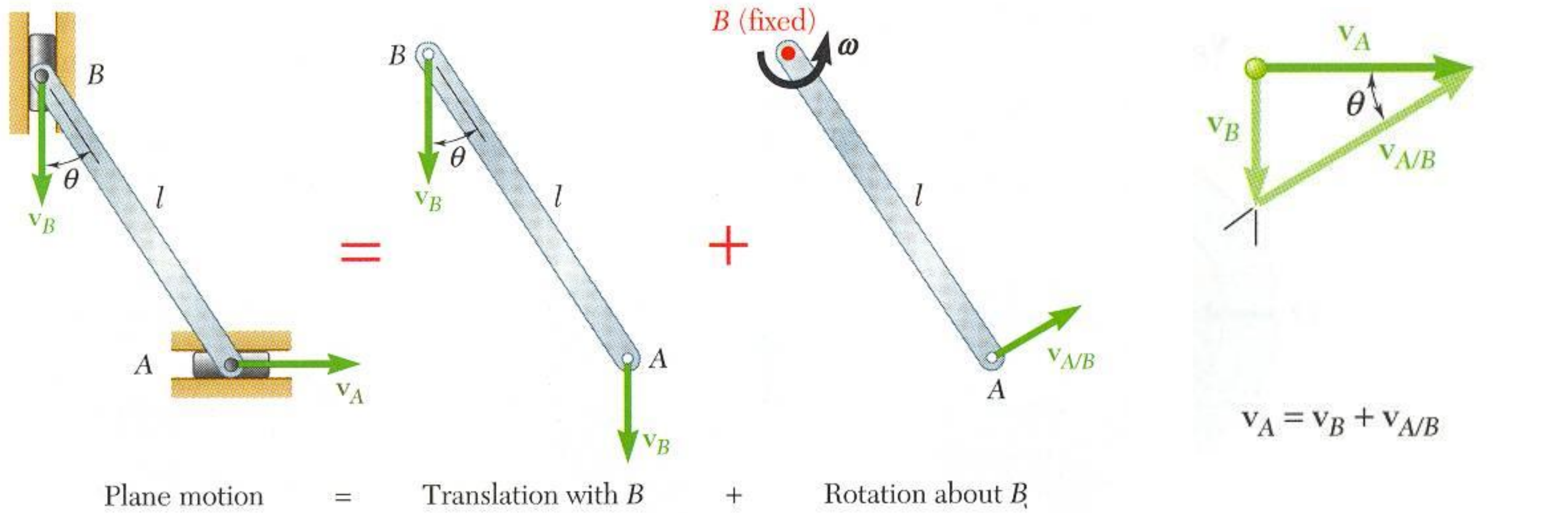
- Assuming that the velocity v_A of end A is known, wish to determine the velocity v_B of end B and the angular velocity ω in terms of v_A, l, and θ.

- The directions of v_B and $v_{B/A}$ are known. Complete the velocity diagram.

$$\frac{v_B}{v_A} = \tan\theta \qquad \cos\theta = \frac{v_A}{v_{B/A}} = \frac{v_A}{l\omega}$$

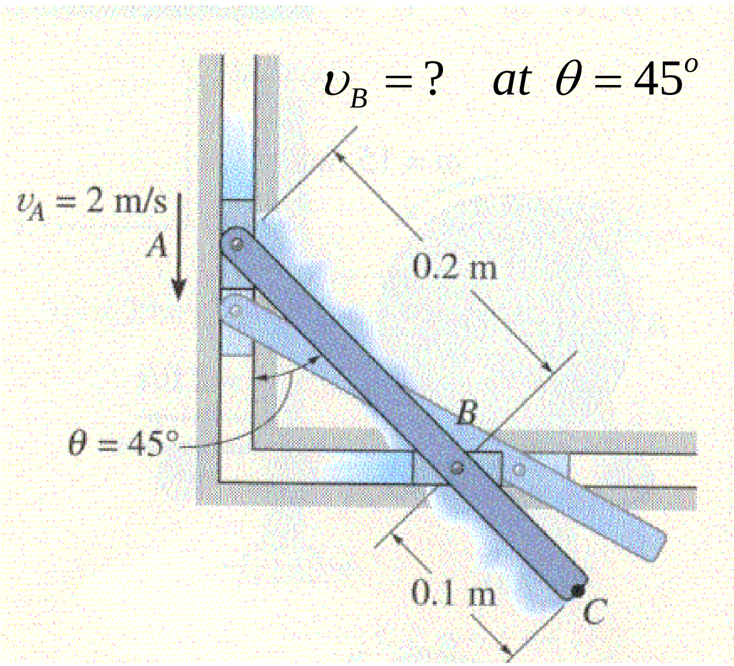
$$v_B = v_A \tan\theta \qquad \omega = \frac{v_A}{l\cos\theta}$$

Absolute and Relative Velocity in Plane Motion



- Selecting point *B* as the reference point and solving for the velocity v_A of end *A* and the angular velocity ω leads to an equivalent velocity triangle.
- $v_{A/B}$ has the same magnitude but opposite sense of $v_{B/A}$. The sense of the relative velocity is dependent on the choice of reference point.
- Angular velocity ω of the rod in its rotation about *B* is the same as its rotation about *A*. *Angular velocity is not dependent on the choice of reference point.*

Example

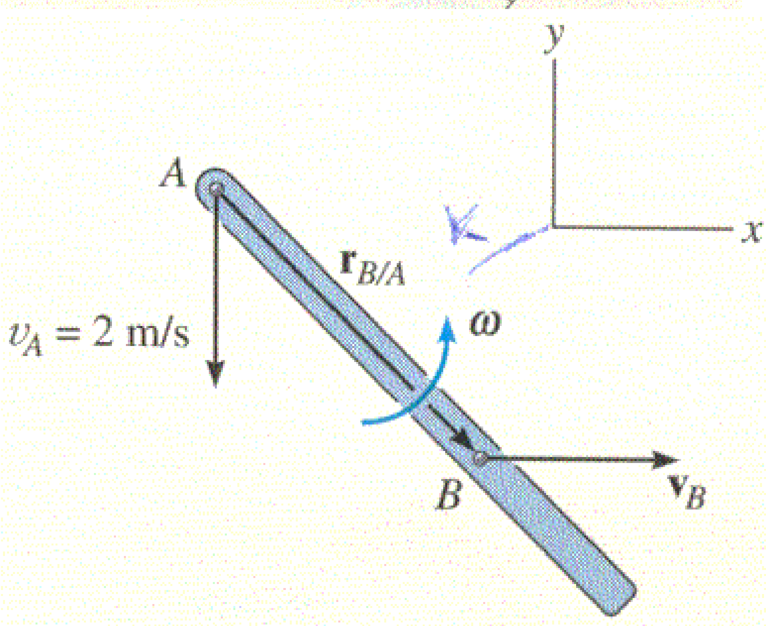


The link shown is guided by two blocks at *A* and *B*, which move in the fixed slots. If the velocity of *A* is 2 m/s downward, determine the velocity of *B at* the instant *θ=45* °.

$$\mathrm{v}_B = \mathrm{v}_A + \omega \times \mathrm{r}_{B/A}$$

$$\upsilon_B \vec{i} = -2\vec{j} + [\omega \vec{k} \times (0.2\sin 45^o \vec{i} - 0.2\cos 45^o \vec{j})]$$

$$\upsilon_B \vec{i} = -2\vec{j} + 0.2\omega \sin 45^o \vec{j} + 0.2\omega \cos 45^o \vec{i}$$

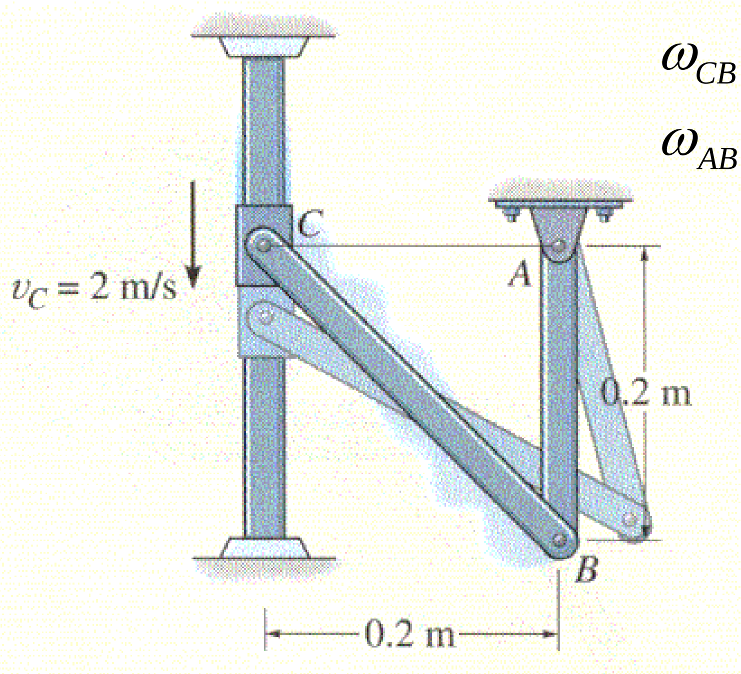


$$\upsilon_B = 0.2\omega \cos 45^o \qquad 0 = -2 + 0.2\,\omega \sin 45^o$$

$$\omega = 14.1 \quad rad/s$$

$$\upsilon_B = 2 \quad m/s$$

Example



$\omega_{CB} = ?$

$\omega_{AB} = ?$

The collar C is moving downward with a velocity of 2 m/s. Determine the angular velocity of *CB and AB* at this instant.

$$\mathbf{v}_B = \mathbf{v}_C + \omega_{CB} \times \mathbf{r}_{B/C}$$

$$\upsilon_B \vec{i} = -2\vec{j} + \omega_{CB}\vec{k} \times (0.2\vec{i} - 0.2\vec{j})$$

$$\upsilon_B \vec{i} = -2\vec{j} + 0.2\omega_{CB}\vec{j} + 0.2\omega_{CB}\vec{i}$$

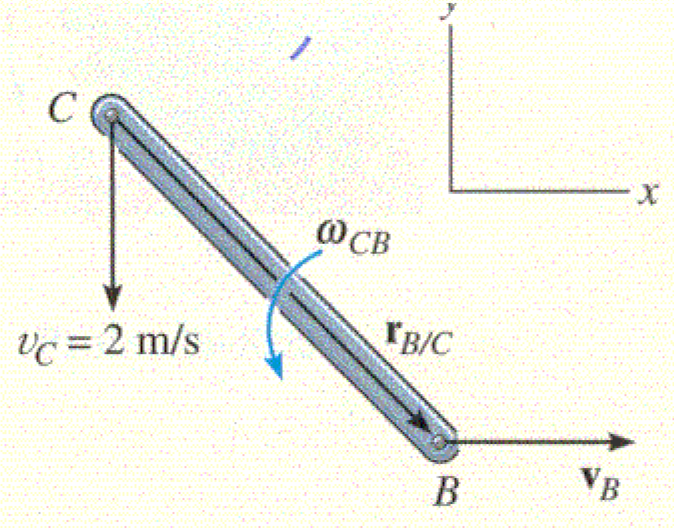
$$\upsilon_B = 0.2\omega_{CB} \qquad 0 = -2 + 0.2\omega_{CB}$$

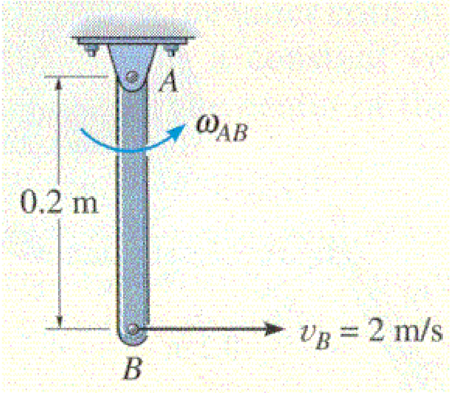
$$\omega_{CB} = 10 \quad rad/s$$

$$\upsilon_B = 2\ m/s$$

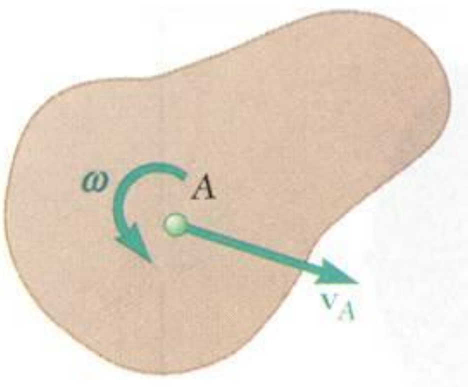
$$\upsilon_B = \omega_{AB} r$$

$$\omega_{AB} = \frac{2}{0.2} = 10\ rad/s$$



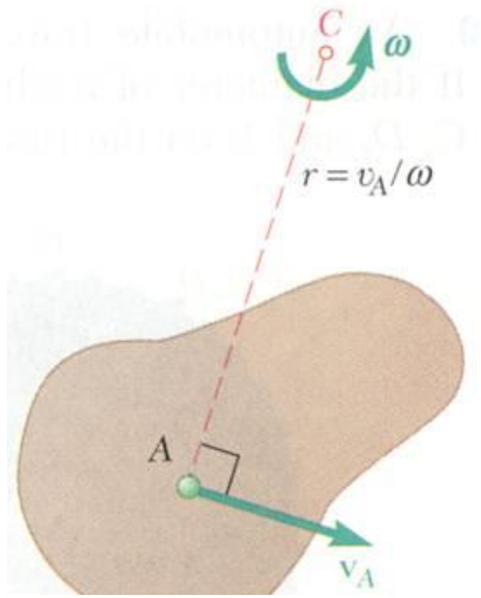


Instantaneous Center of Rotation in Plane Motion



- Plane motion of all particles in a slab can always be replaced by the translation of an arbitrary point *A* and a rotation about *A* with an angular velocity that is independent of the choice of *A.*

- The same translational and rotational velocities at *A* are obtained by allowing the slab to rotate with the same angular velocity about the point *C* on a perpendicular to the velocity at *A.*



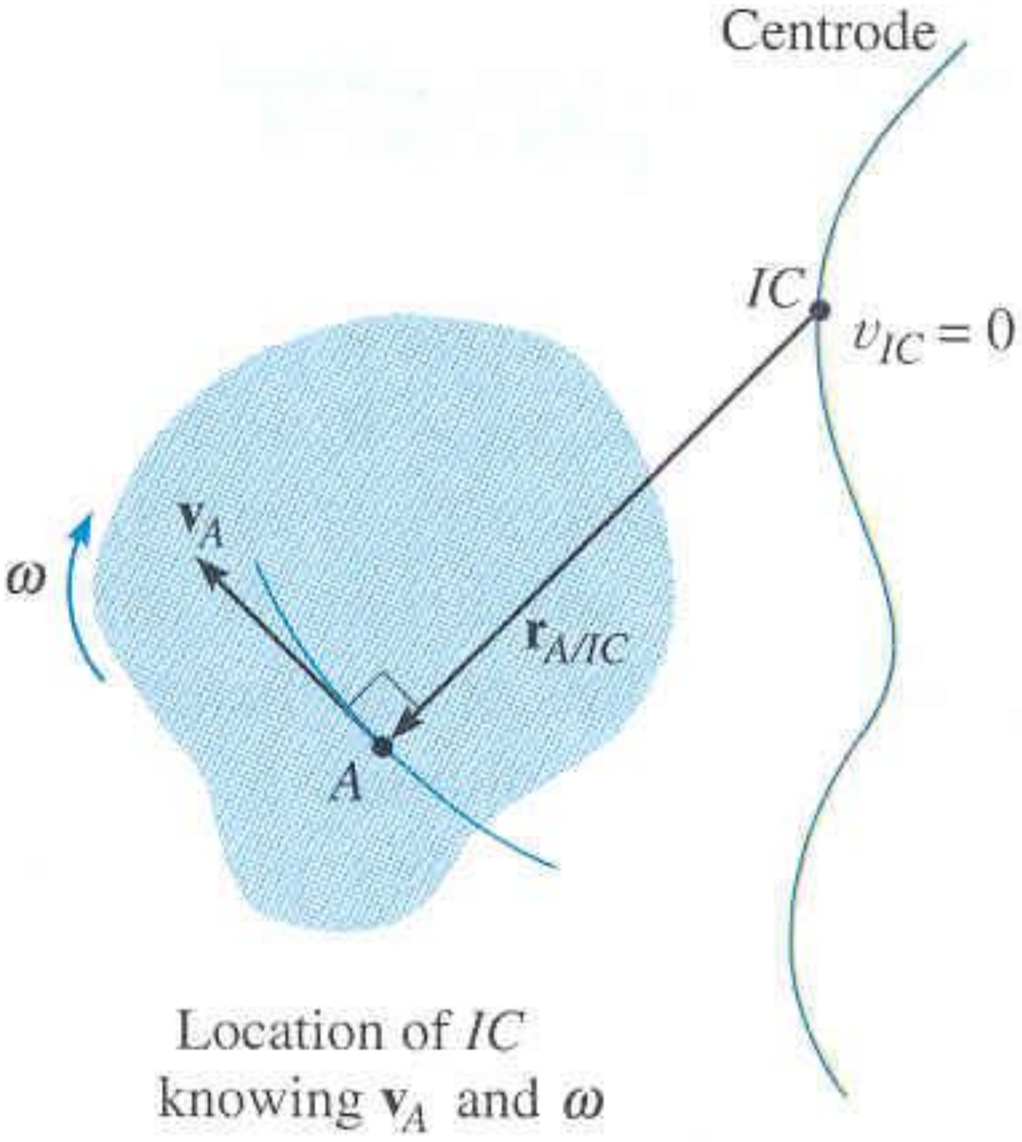
- The velocity of all other particles in the slab are the same as originally defined since the angular velocity and translational velocity at *A* are equivalent.

- As far as the velocities are concerned, the slab seems to rotate about the *instantaneous center of rotation C*.

$$\mathbf{v}_A = \underbrace{\mathbf{v}_{IC}}_{0} + \omega \times \mathbf{r}_{A/IC}$$

$$= \omega \times \mathbf{r}_{A/IC}$$

Instantaneous Center of Rotation



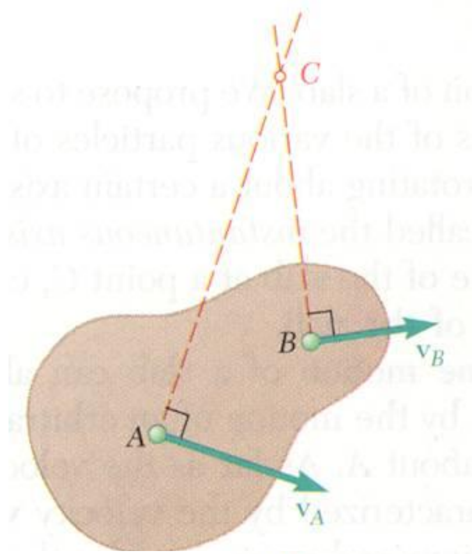
$$\mathrm{v}_A = \mathrm{v}_{IC} + \omega \times \mathrm{r}_{A/IC}$$

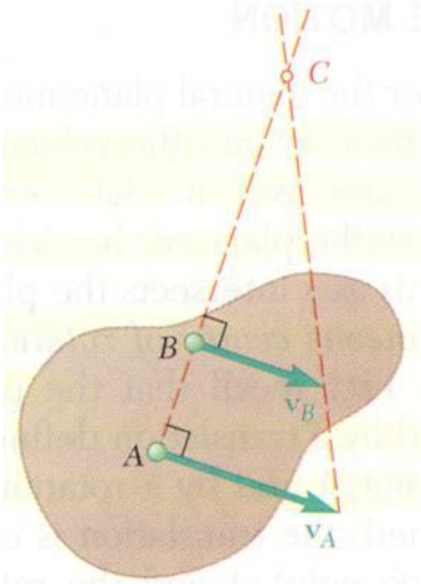
0

$$= \omega \times \mathrm{r}_{A/IC}$$

$$r_{A/IC} = \frac{\upsilon_A}{\omega}$$

Instantaneous Center of Rotation in Plane Motion





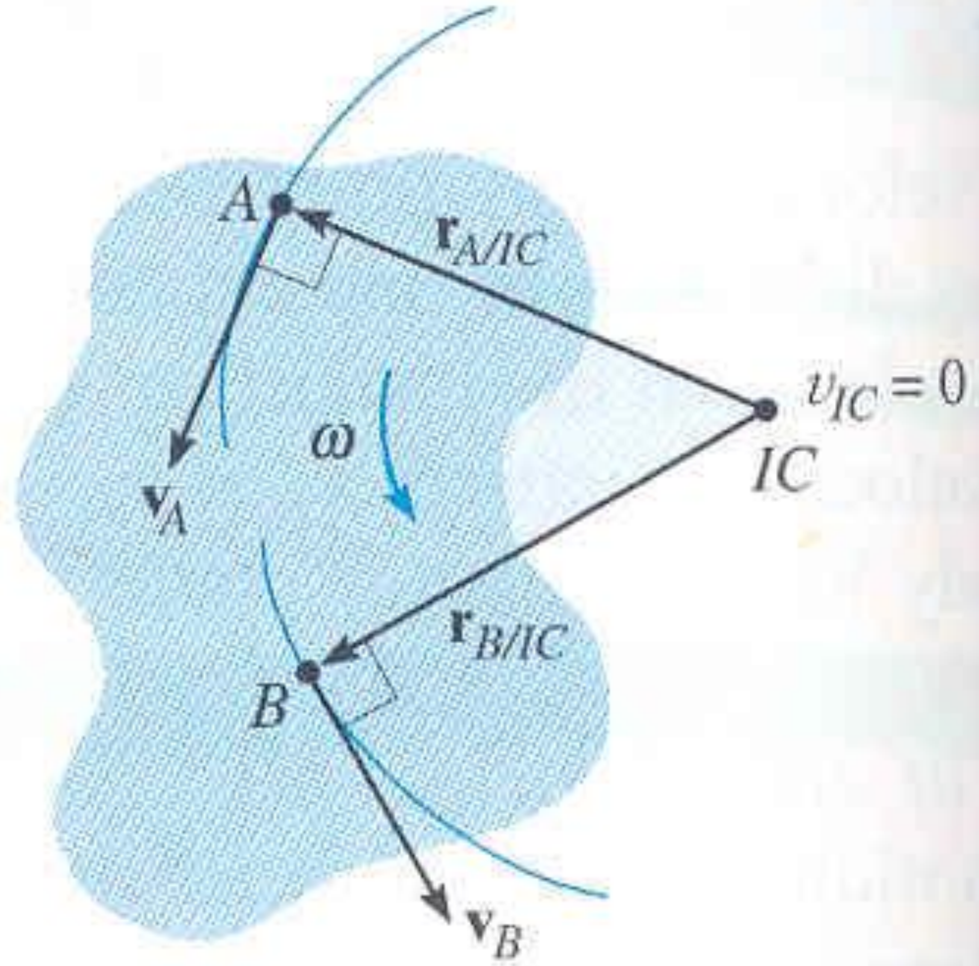
- If the velocity at two points *A* and *B* are known, the instantaneous center of rotation lies at the intersection of the perpendiculars to the velocity vectors through *A* and *B* .

- If the velocity vectors are parallel, the instantaneous center of rotation is at infinity and the angular velocity is zero.

- If the velocity vectors at *A* and *B* are perpendicular to the line *AB*, the instantaneous center of rotation lies at the intersection of the line *AB* with the line joining the extremities of the velocity vectors at *A* and *B*.

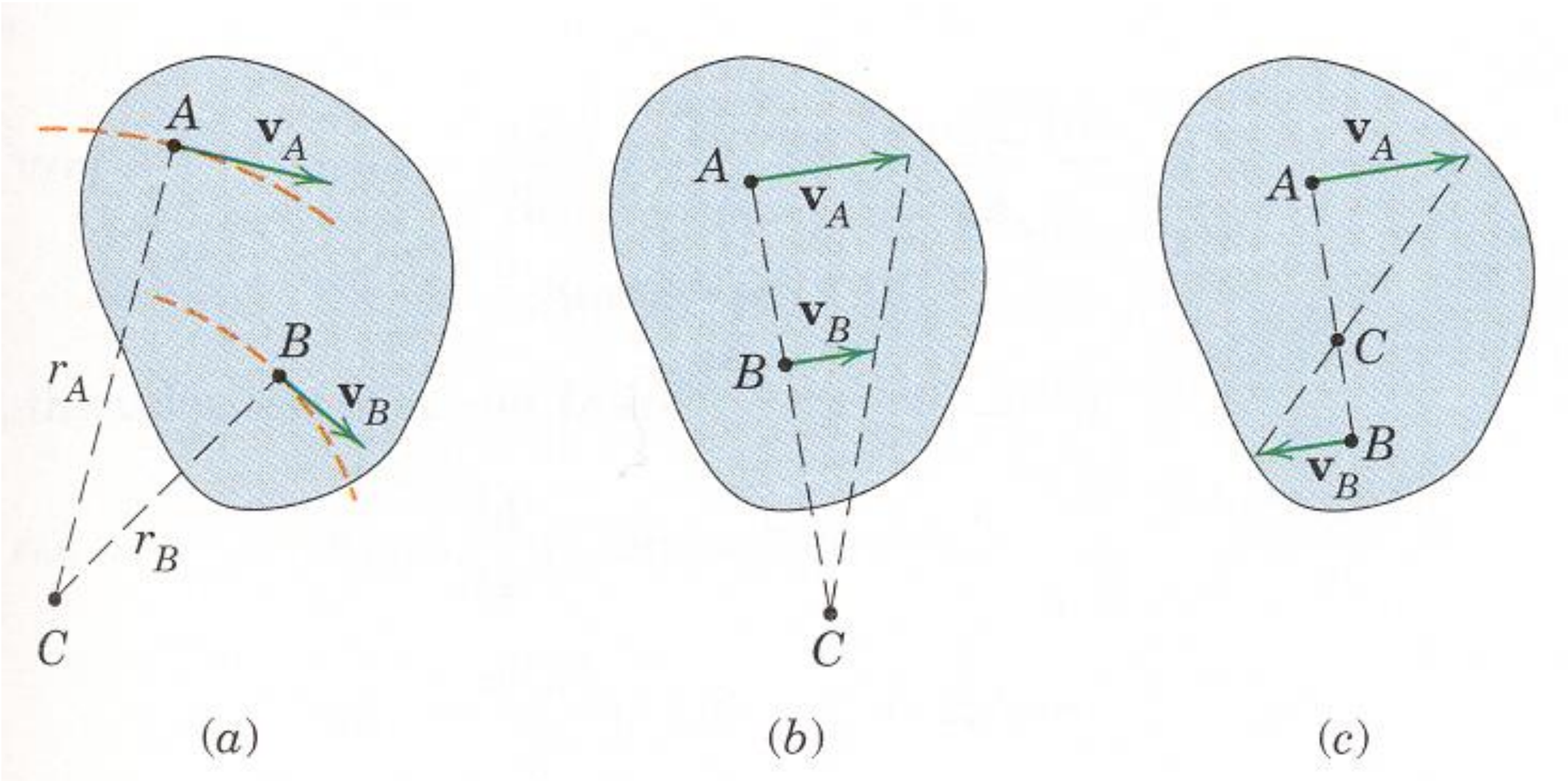
- If the velocity magnitudes are equal, the instantaneous center of rotation is at infinity and the angular velocity is zero.

Location of IC knowing the line of action of v_A and v_B

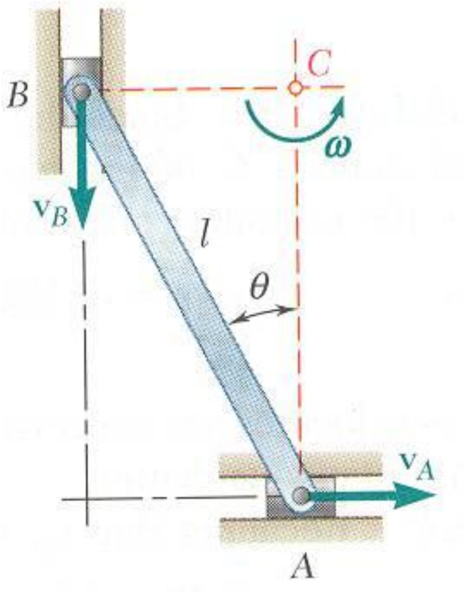


Location of *IC*
knowing the lines of action of $\mathbf{v}_A$ and $\mathbf{v}_B$

Location of IC knowing the line of action of v_A and v_B



Instantaneous Center of Rotation in Plane Motion

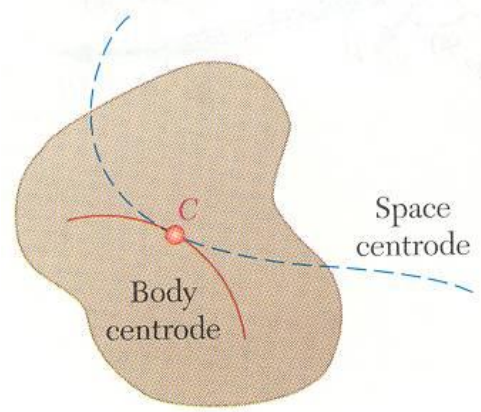


- The instantaneous center of rotation lies at the intersection of the perpendiculars to the velocity vectors through *A* and *B* .

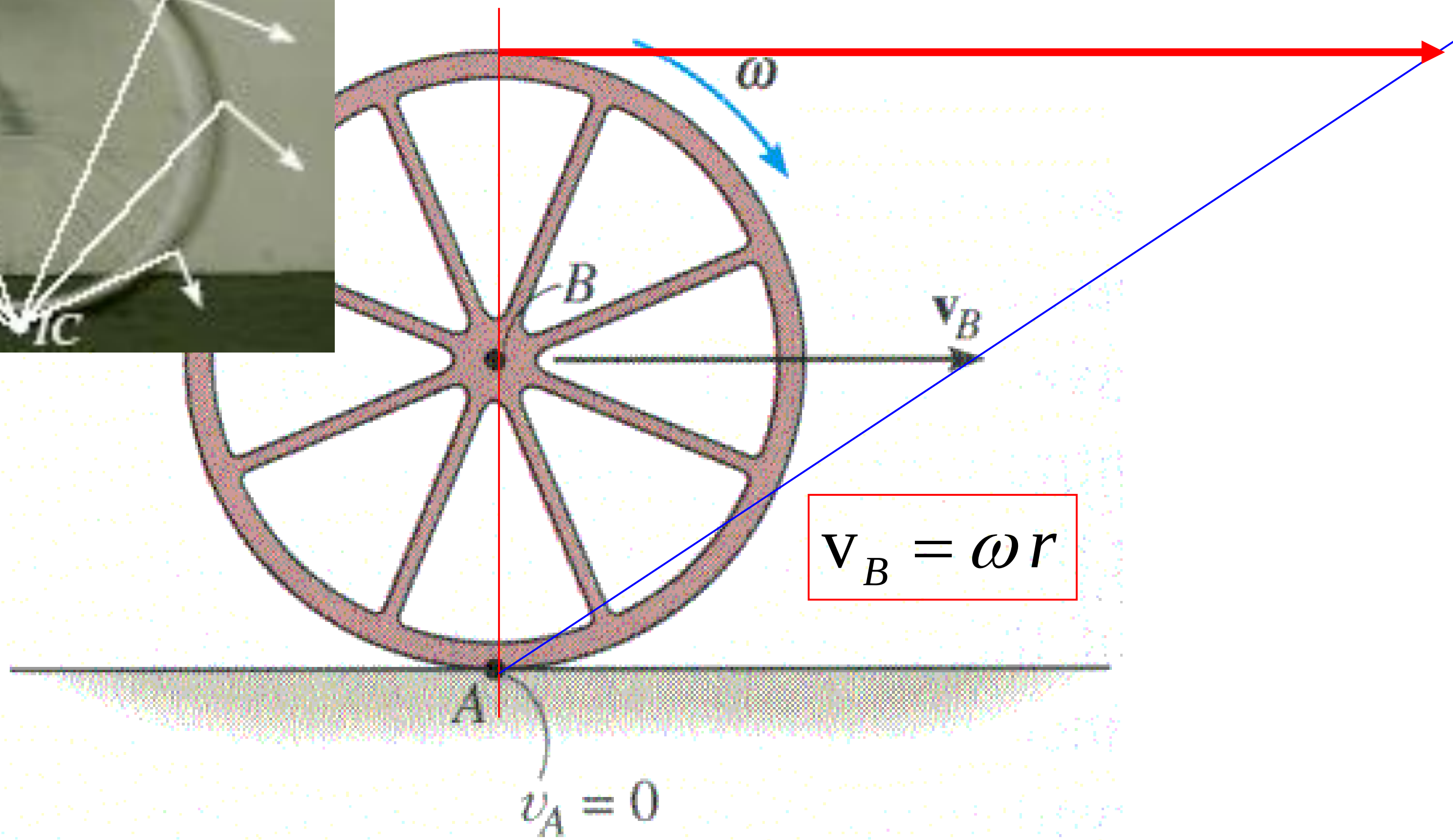
$$\omega = \frac{v_A}{AC} = \frac{v_A}{l\cos\theta}$$

$$v_B = (BC)\omega = (l\sin\theta)\frac{v_A}{l\cos\theta} = v_A\tan\theta$$

- The velocities of all particles on the rod are as if they were rotated about *C.*
- The particle at the center of rotation has zero velocity.
- The particle coinciding with the center of rotation changes with time and the acceleration of the particle at the instantaneous center of rotation is not zero.
- *The acceleration of the particles in the slab cannot be determined as if the slab were simply rotating about C.*
- The trace of the locus of the center of rotation on the body is the body centrode and in space is the space centrode.



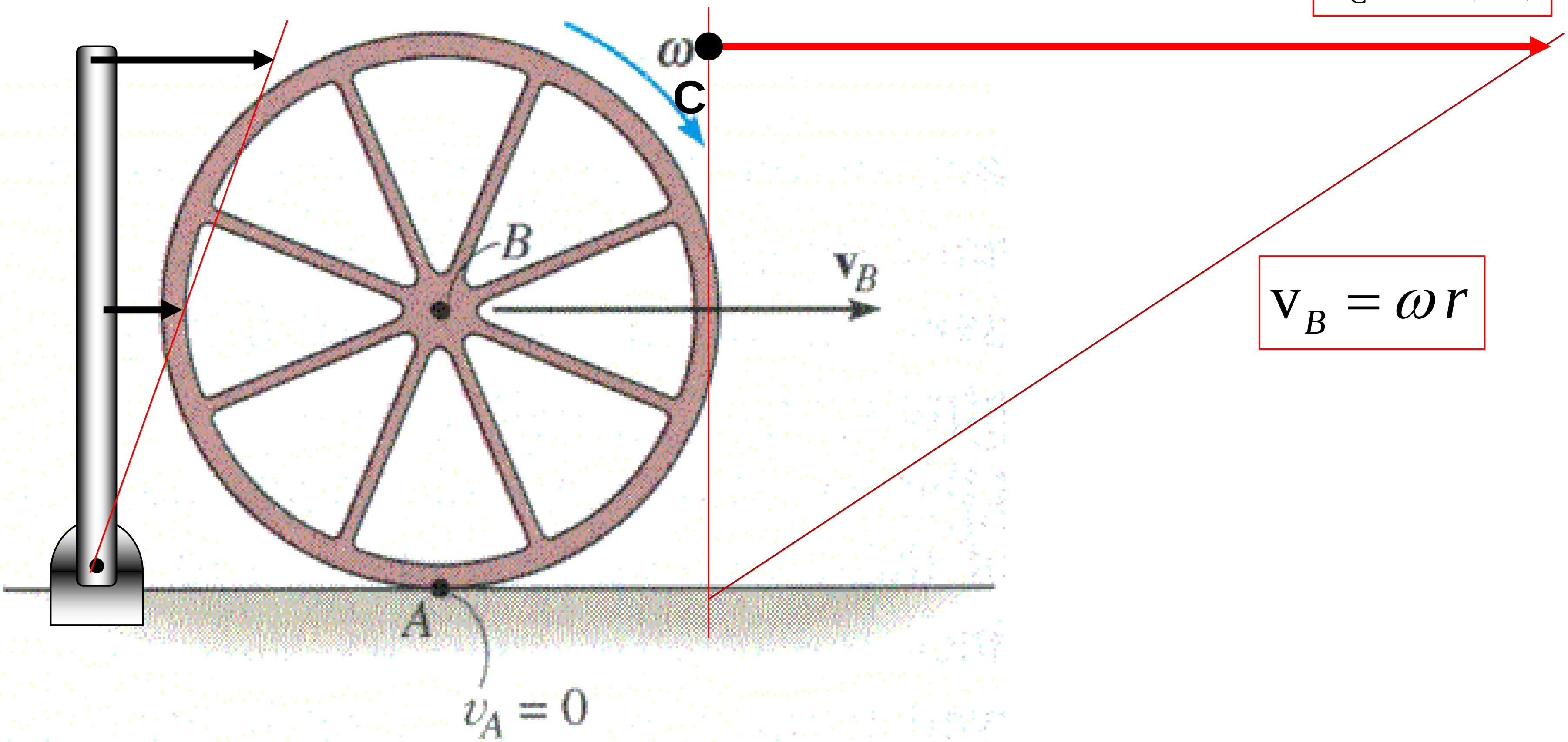
Rolling wheels



$$\mathrm{v}_B = \omega r$$

Rolls without slipping

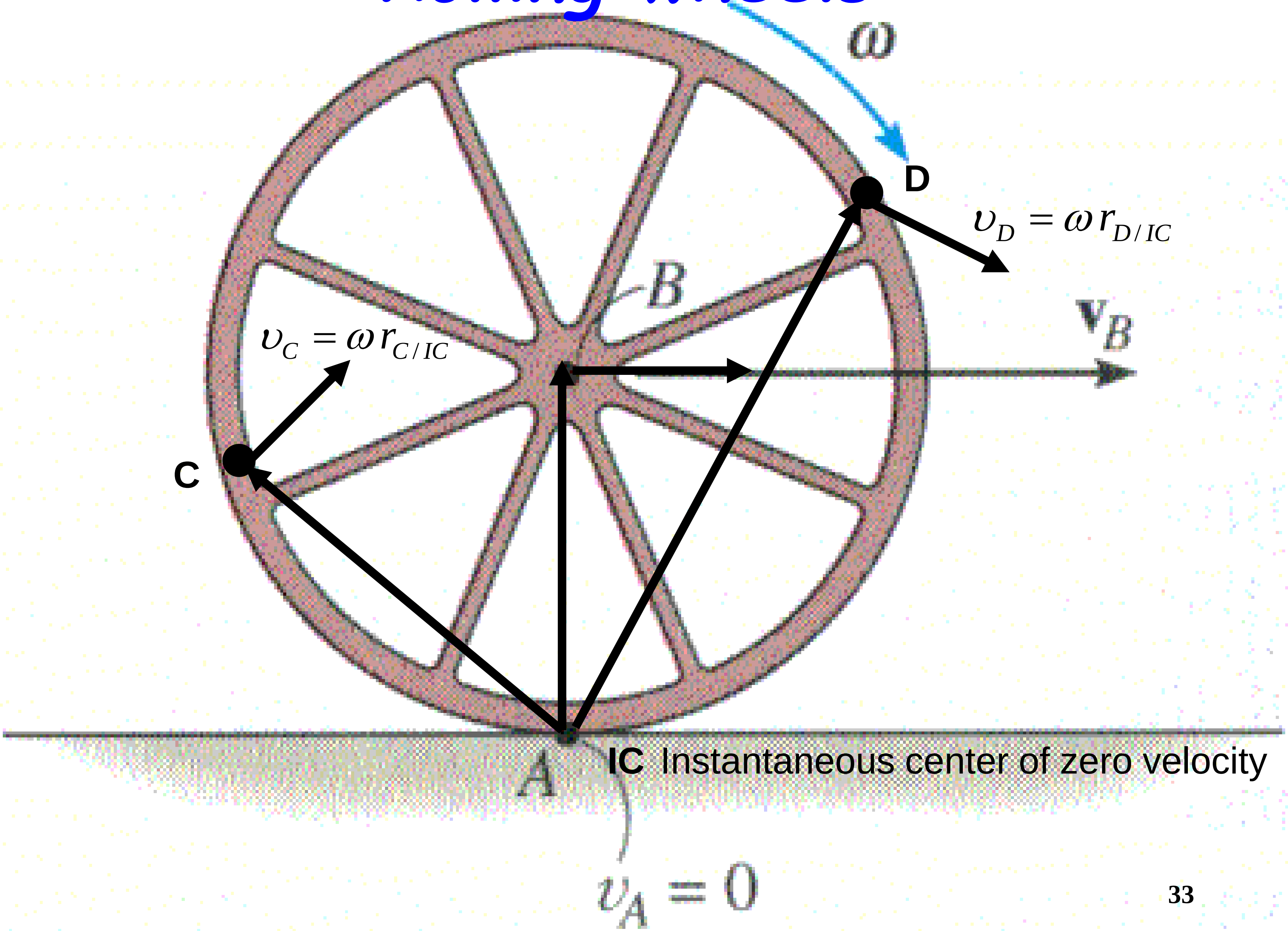
Rolling wheels



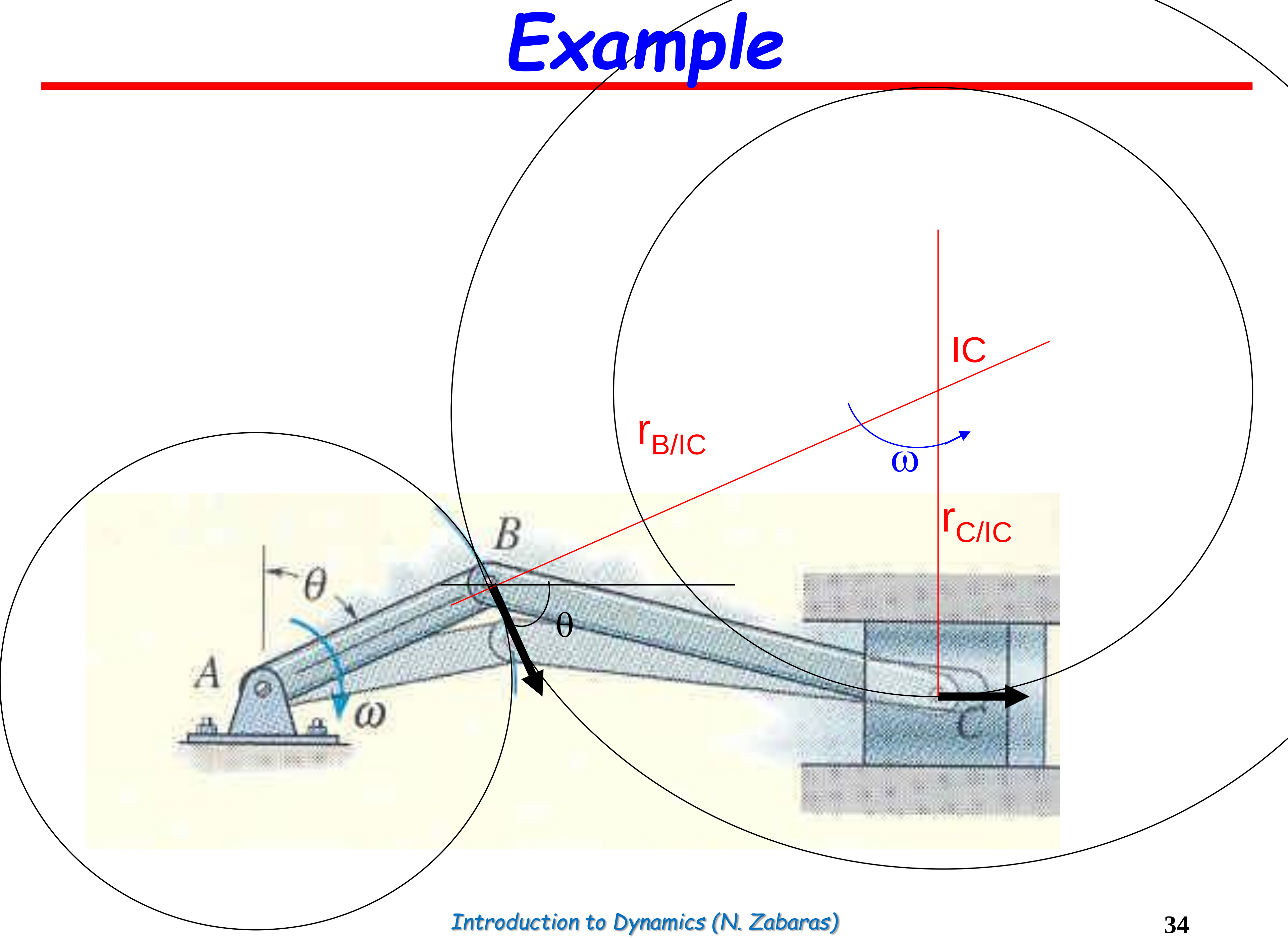
$$v_C = \omega(2r)$$

$$v_B = \omega r$$

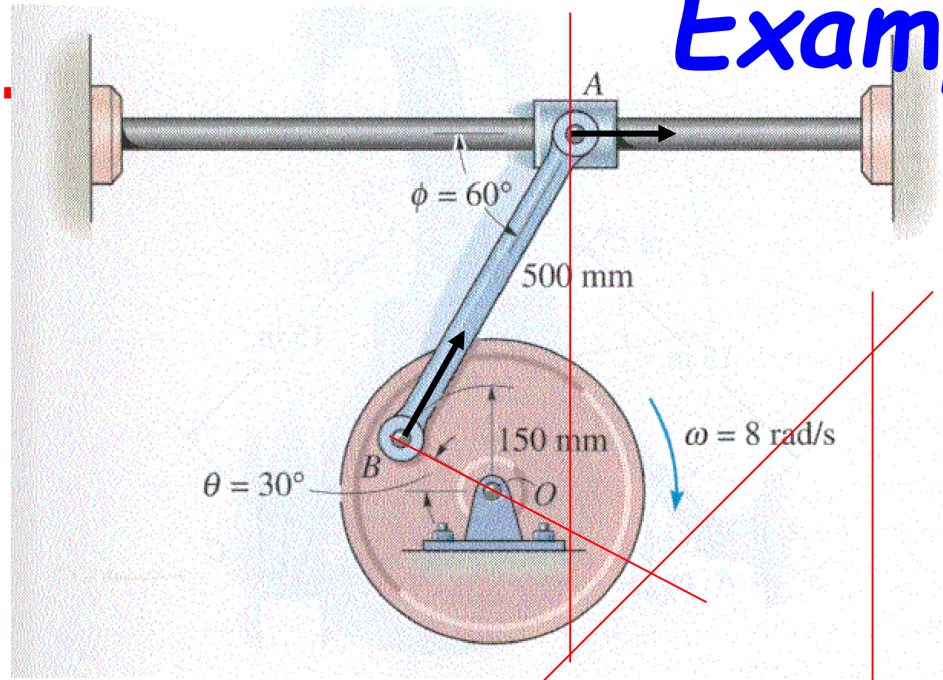
Rolling wheels



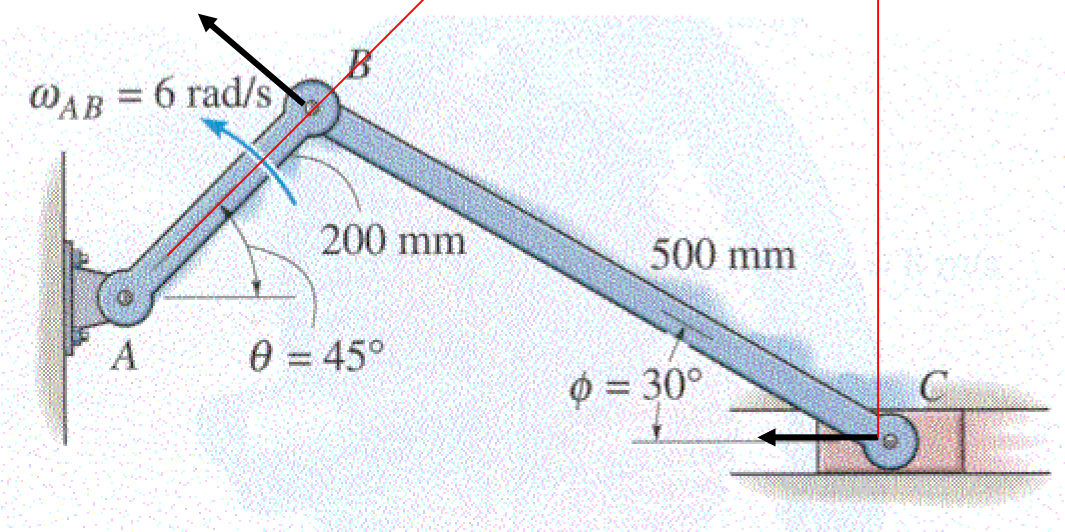
Example



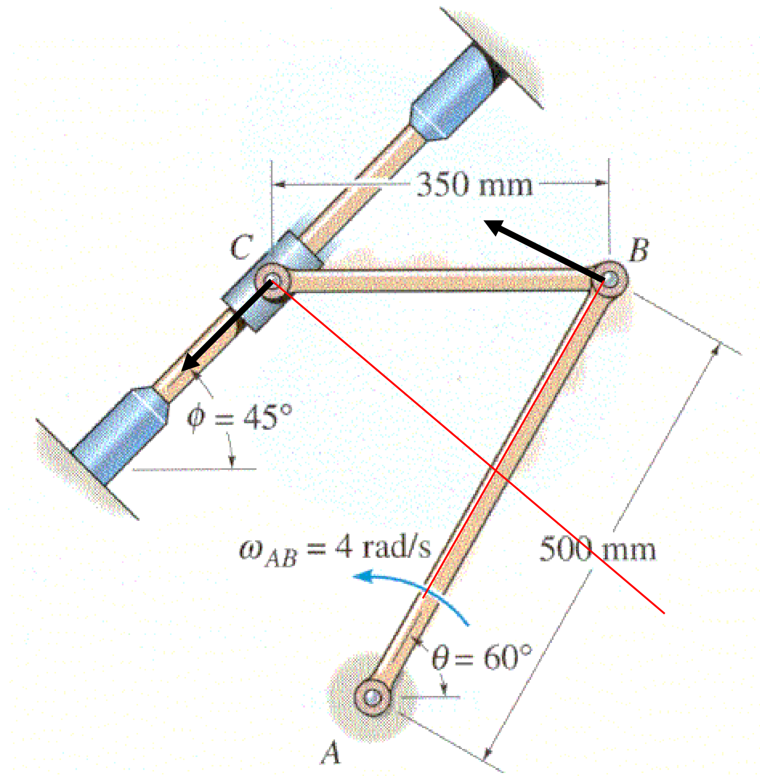
Examples



A
φ = 60°
500 mm
150 mm
ω = 8 rad/s
θ = 30°
B
O

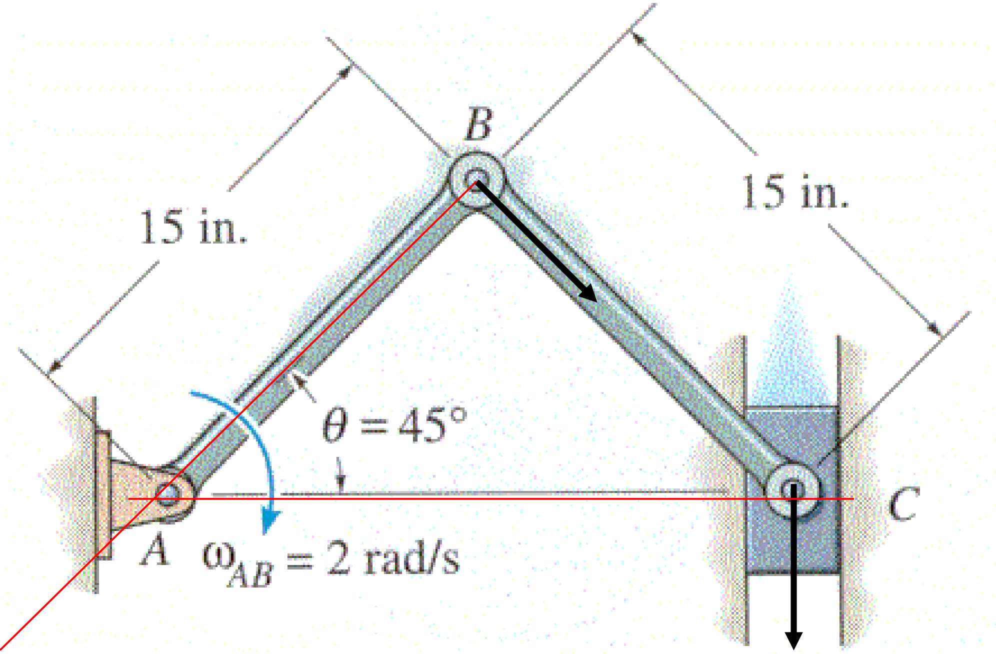


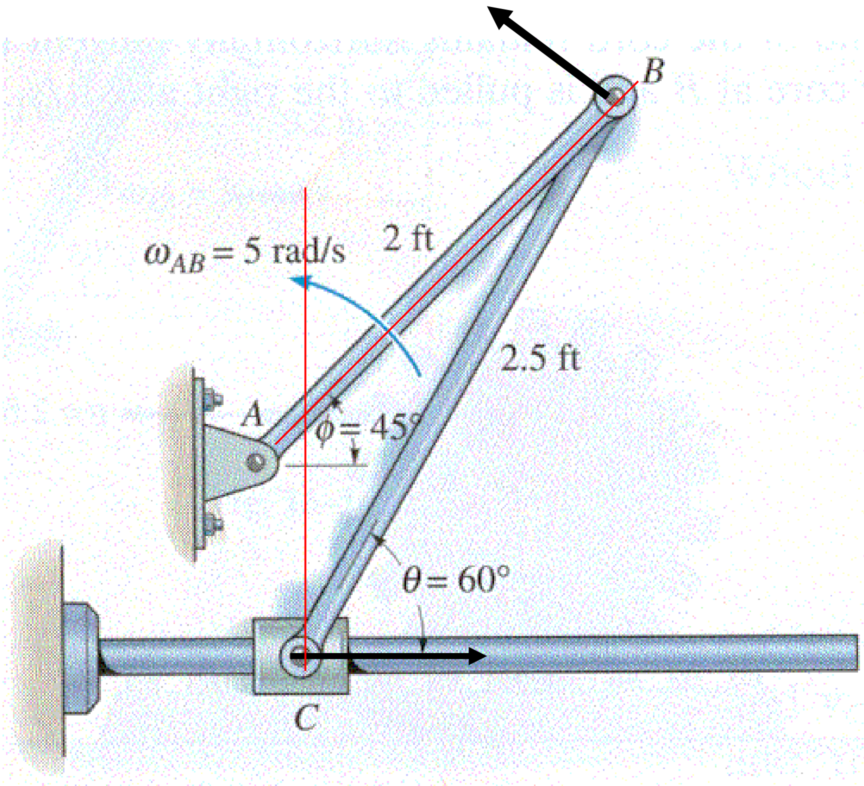
ω_AB = 6 rad/s
B
200 mm
500 mm
A
θ = 45°
φ = 30°
C



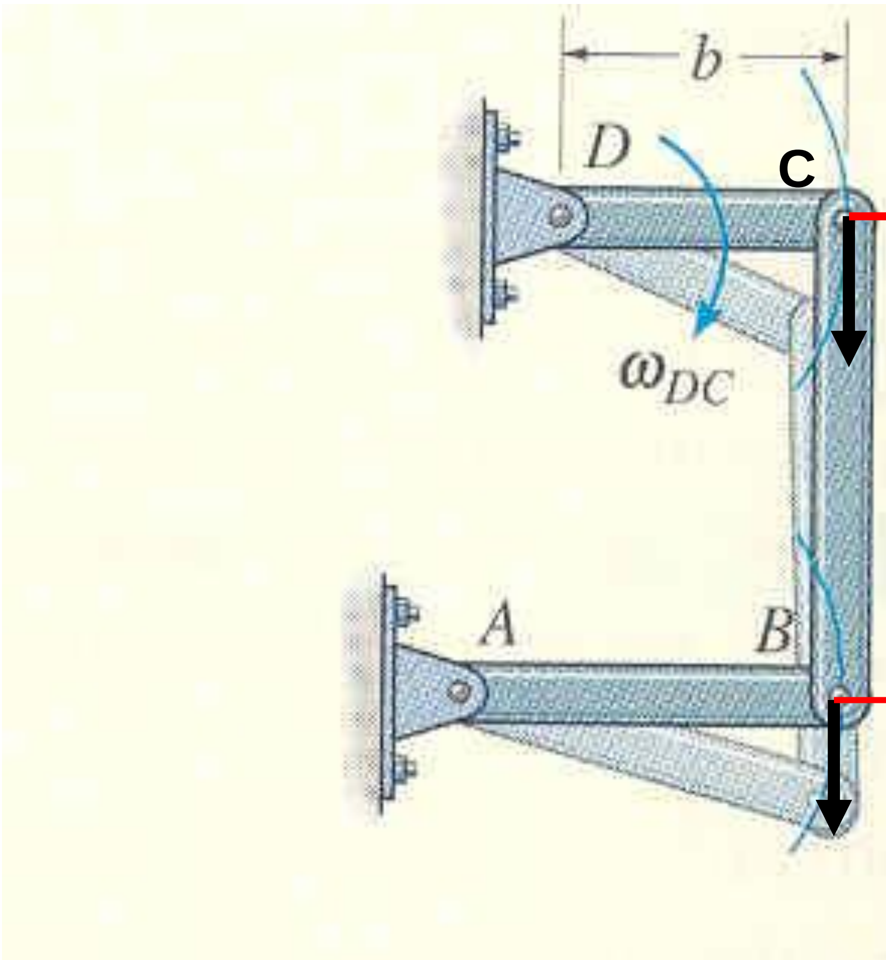
350 mm
C
B
φ = 45°
ω_AB = 4 rad/s
500 mm
θ = 60°
A

Examples





Example



$$r_{C/IC} = \infty$$

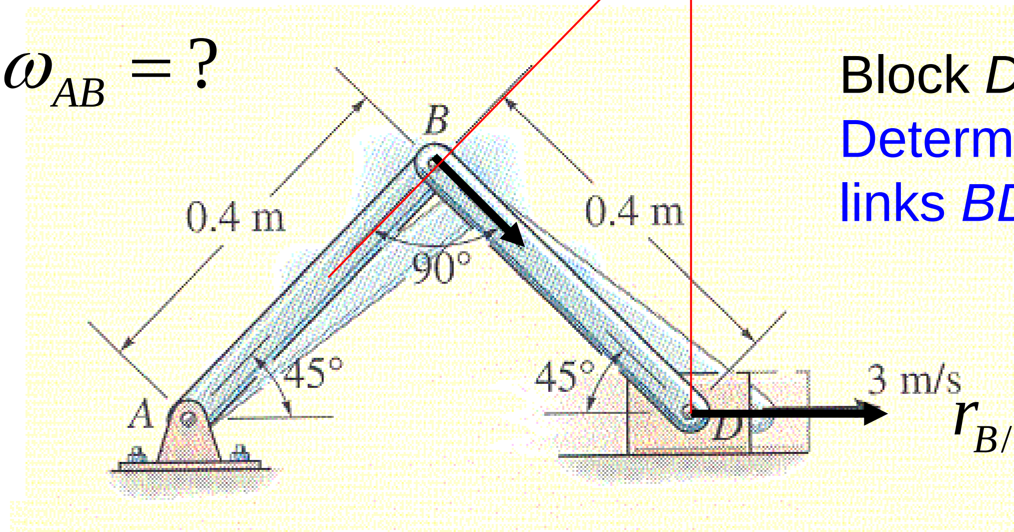
$$\omega_{CB} = \frac{\upsilon_C}{r_{C/IC}} = 0$$

$$r_{B/IC} = \infty$$

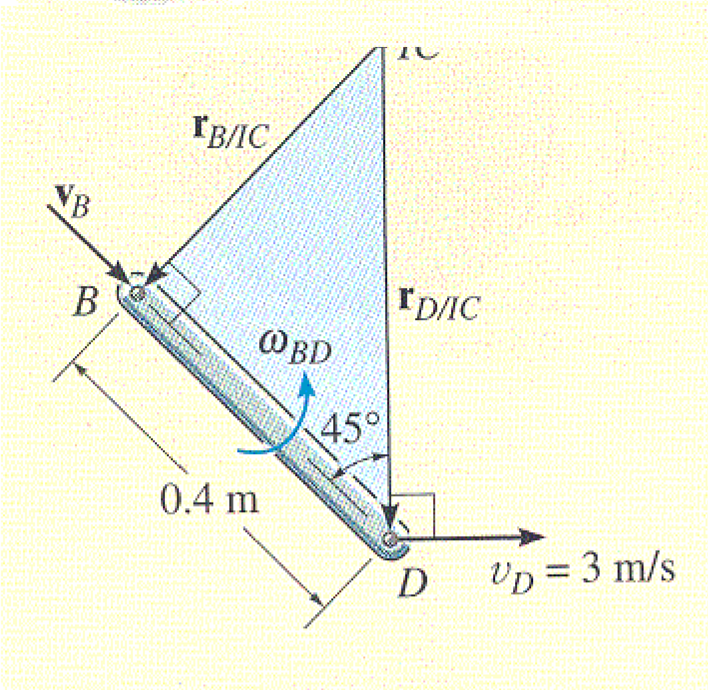
$$\mathrm{v}_B = \mathrm{v}_C$$

Example

$\omega_{AB} = ?$



Block *D* moves with a speed of 3 m/ s. Determine the angular velocities of links *BD* and *AB*, at the instant shown.



$$r_{B/IC} = 0.4 \tan 45^o m = 0.4\, m$$

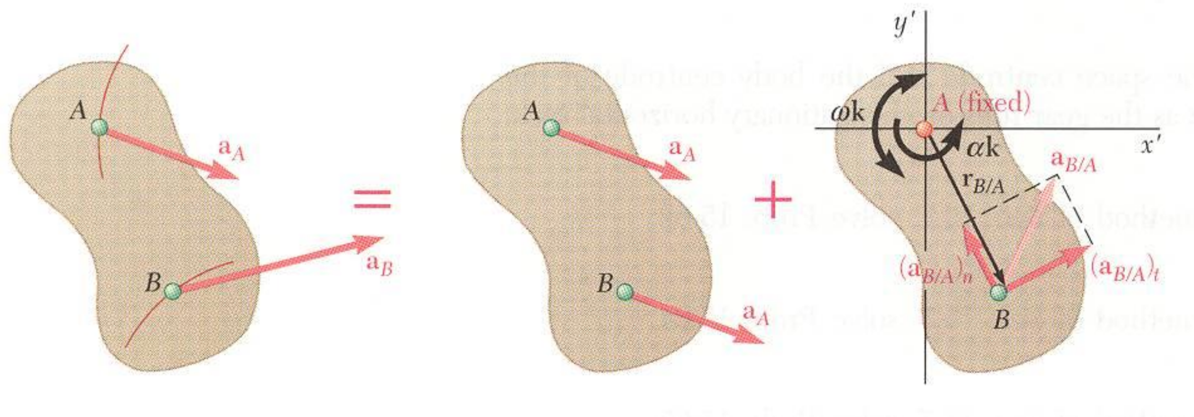
$$r_{D/IC} = \frac{0.4\, m}{\cos 45^o} = 0.566\, m$$

$$\omega_{BD} = \frac{\upsilon_D}{r_{D/IC}} = \frac{3\, m/s}{0.566\, m} = 5.30\, rad/s$$

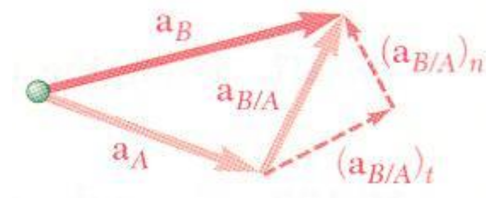
$$\upsilon_B = \omega_{BD}(r_{B/IC}) = 5.3(0.4) = 2.12\, m/s$$

$$\omega_{AB} = \frac{\upsilon_B}{r_{B/A}} = \frac{2.12\, m/s}{0.4\, m} = 5.3\, rad/s$$

Absolute and Relative Acceleration in Plane Motion



Plane motion = Translation with A + Rotation about A



- Absolute acceleration of a particle of the slab,

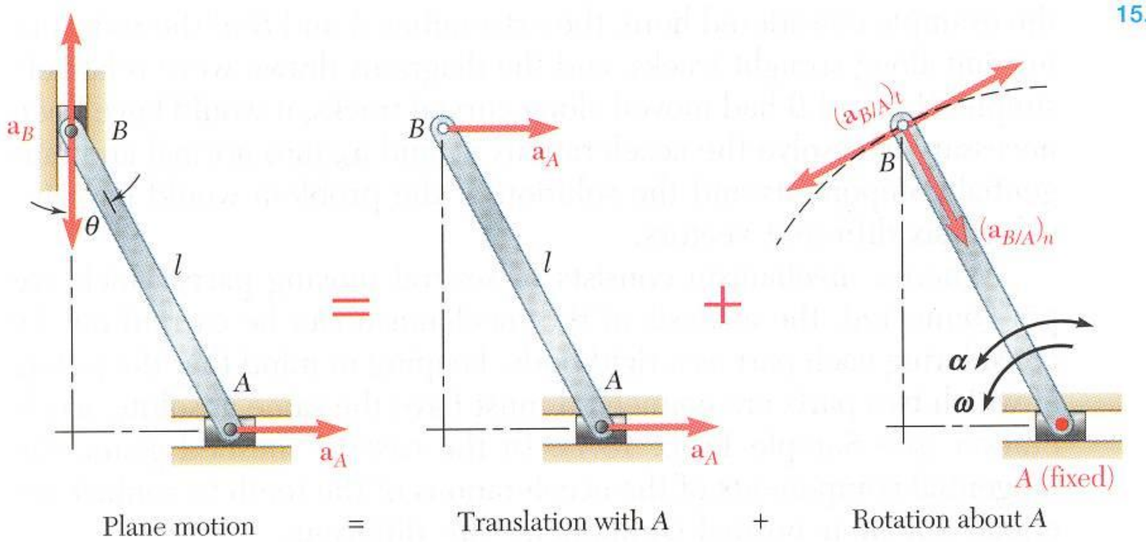
$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

- Relative acceleration $\vec{a}_{B/A}$ associated with rotation about *A* includes tangential and normal components,

$$\left(\vec{a}_{B/A}\right)_t = \alpha\vec{k} \times \vec{r}_{B/A} \qquad \left(a_{B/A}\right)_t = r\alpha$$

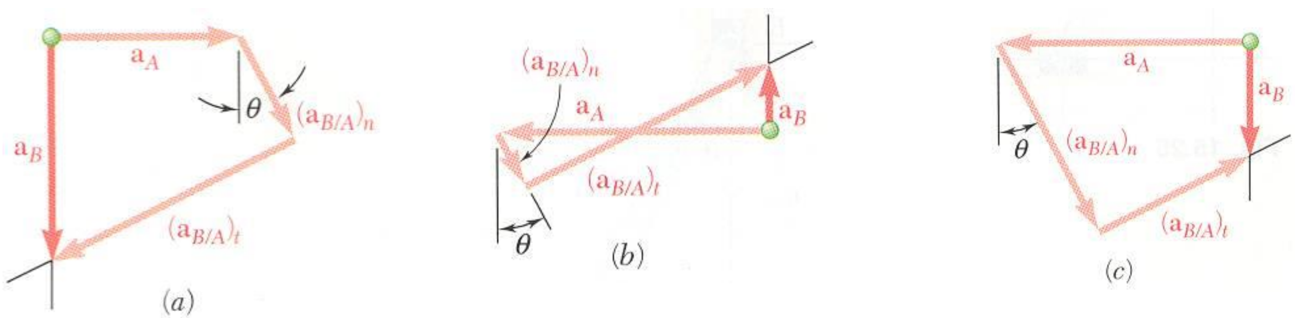
$$\left(\vec{a}_{B/A}\right)_n = -\omega^2 \vec{r}_{B/A} \qquad \left(a_{B/A}\right)_n = r\omega^2$$

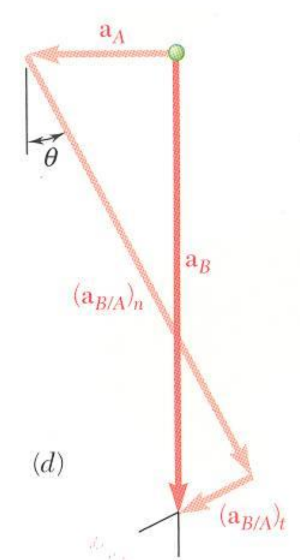
Absolute and Relative Acceleration in Plane Motion



- Given $\vec{a}_A$ and $\vec{v}_A$, determine $\vec{a}_B$ and $\vec{\alpha}$.

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$
$$= \vec{a}_A + \left(\vec{a}_{B/A}\right)_n + \left(\vec{a}_{B/A}\right)_t$$



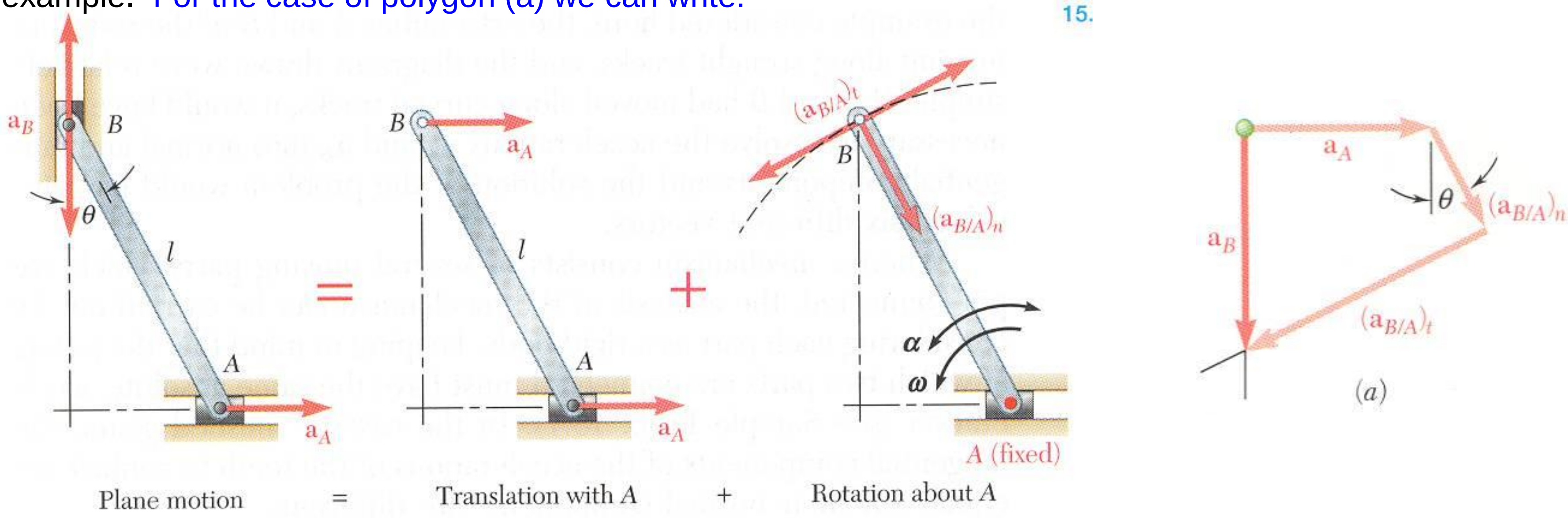


- Vector result depends on sense of $\vec{a}_A$ and the relative magnitudes of a_A and $\left(a_{B/A}\right)_n$

- Must also know angular velocity ω.

Absolute and Relative Acceleration in Plane Motion

As shown in the earlier slide, four different vector polygons can be obtained, depending upon the sense of $\mathbf{a}_A$ and the relative magnitude of a_A and $(a_{B/A})_t$. Only one of those four cases will be applicable for a given example. For the case of polygon (a) we can write:



- Write $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$ in terms of the two component equations,

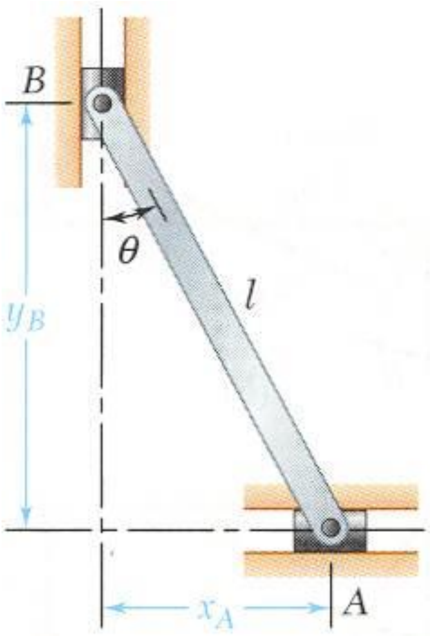
$$\overset{+}{\rightarrow} \quad x \text{ components:} \qquad 0 = a_A + l\omega^2 \sin\theta - l\alpha\cos\theta$$

$$+\uparrow \quad y \text{ components:} \quad -a_B = -l\omega^2 \cos\theta - l\alpha\sin\theta$$

a_A and a_B are the magnitudes of the vectors $\mathbf{a}_A$ and $\mathbf{a}_B$ that have directions as shown in polygon (a)

- Solve for a_B and α.

Analysis of Plane Motion in Terms of a Parameter



- In some cases, it is advantageous to determine the absolute velocity and acceleration of a mechanism directly.

$$x_A = l\sin\theta \qquad y_B = l\cos\theta$$

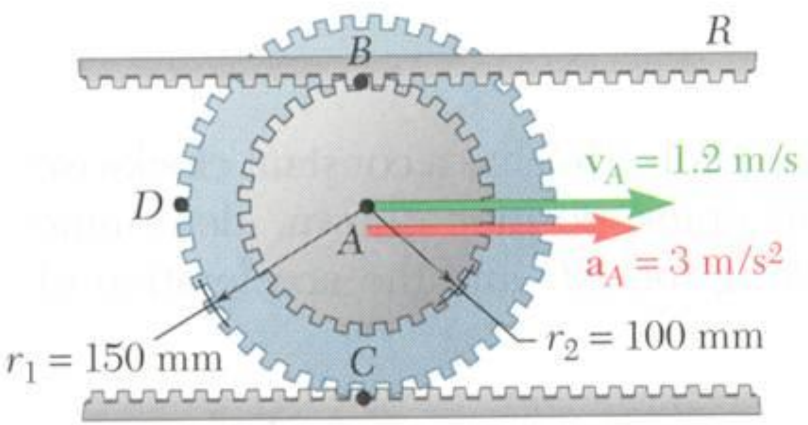
$$v_A = \dot{x}_A = l\dot{\theta}\cos\theta = l\omega\cos\theta \qquad v_B = \dot{y}_B = -l\dot{\theta}\sin\theta = -l\omega\sin\theta$$

$$a_A = \ddot{x}_A = -l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta = -l\omega^2\sin\theta + l\alpha\cos\theta$$

$$a_B = \ddot{y}_B = -l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta = -l\omega^2\cos\theta - l\alpha\sin\theta$$

We note that a positive sign for $\mathbf{v}_A$ or $\mathbf{a}_A$ indicates that the velocity $\mathbf{v}_A$ or the acceleration $\mathbf{a}_A$ is directed to the right; a positive sign for $\mathbf{v}_B$ or $\mathbf{a}_B$ indicates that $\mathbf{v}_B$ or $\mathbf{a}_B$ is directed upward. Note the positive direction of $\mathbf{a}_B$ is different here from that in the earlier slide (still the corresponding Eqs are the same).

Sample Problem



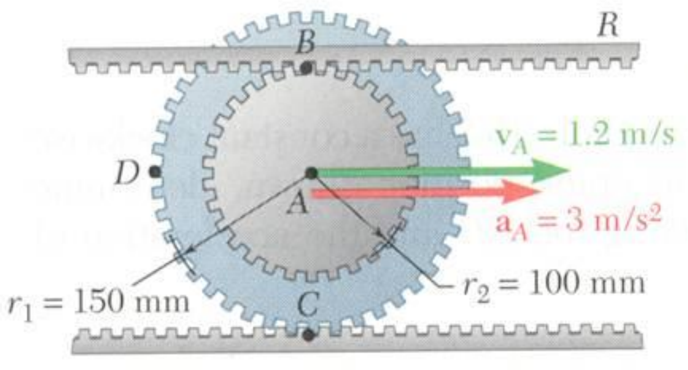
The center of the double gear has a velocity and acceleration to the right of 1.2 m/s and 3 m/s^2, respectively. The lower rack is stationary.

Determine *(a)* the angular acceleration of the gear, and *(b)* the acceleration of points *B*, *C*, and *D*.

SOLUTION:

- The expression of the gear position as a function of θ is differentiated twice to define the relationship between the translational and angular accelerations.
- The acceleration of each point on the gear is obtained by adding the acceleration of the gear center and the relative accelerations with respect to the center. The latter includes normal and tangential acceleration components.

Sample Problem



SOLUTION:

- The expression of the gear position as a function of θ is differentiated twice to define the relationship between the translational and angular accelerations.

$$x_A = -r_1\theta$$

$$v_A = -r_1\dot{\theta} = -r_1\omega$$

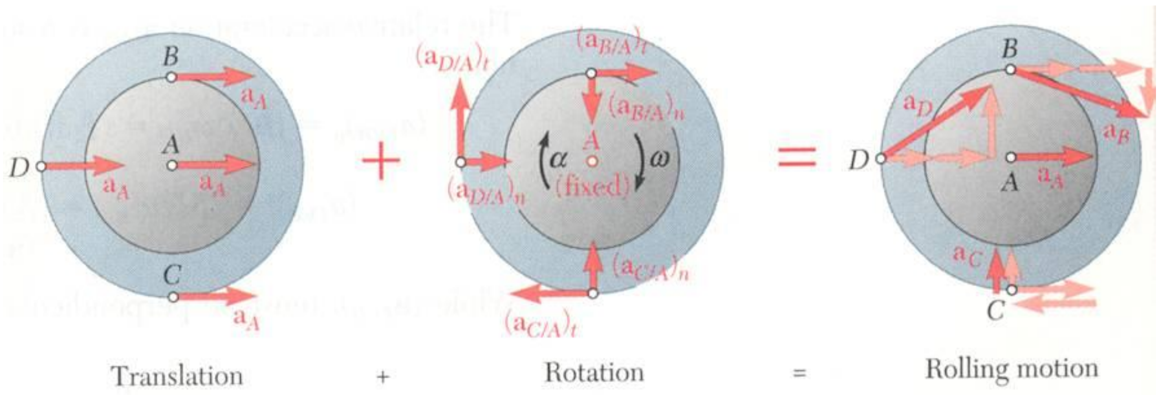
$$\omega = -\frac{v_A}{r_1} = -\frac{1.2\,\mathrm{m/s}}{0.150\,\mathrm{m}} = -8\,\mathrm{rad/s}$$

$$a_A = -r_1\ddot{\theta} = -r_1\alpha$$

$$\alpha = -\frac{a_A}{r_1} = -\frac{3\,\mathrm{m/s^2}}{0.150\,\mathrm{m}}$$

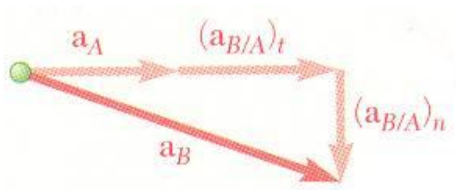
$$\vec{\alpha} = \alpha\vec{k} = -\left(20\,\mathrm{rad/s^2}\right)\vec{k}$$

Sample Problem



- The acceleration of each point is obtained by adding the acceleration of the gear center and the relative accelerations with respect to the center.

  The latter includes normal and tangential acceleration components.



$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A} = \vec{a}_A + \left(\vec{a}_{B/A}\right)_t + \left(\vec{a}_{B/A}\right)_n$$

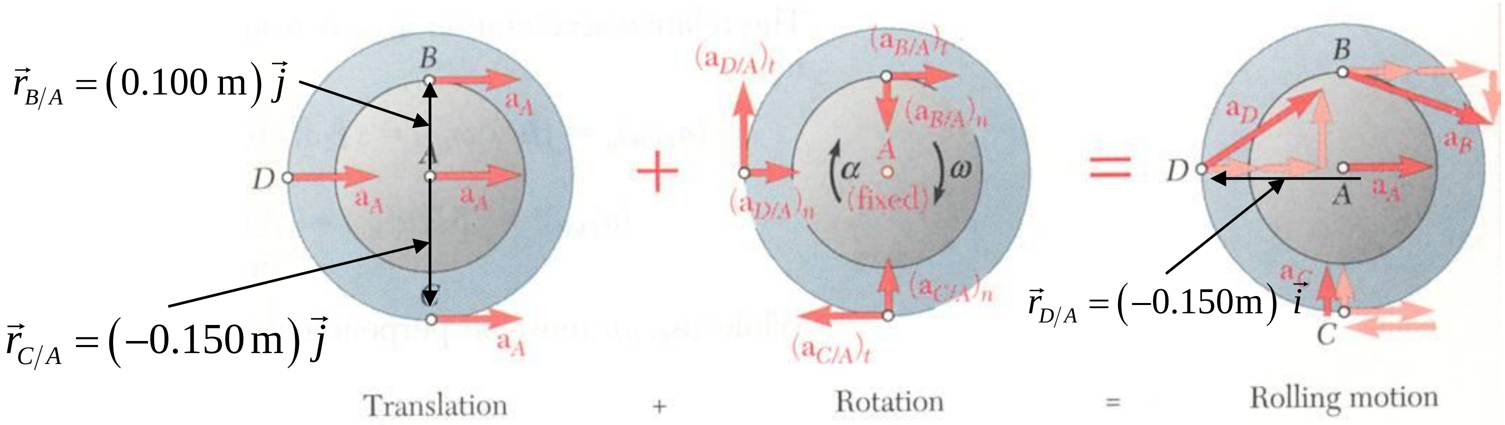
$$= \vec{a}_A + \alpha\vec{k} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A}$$

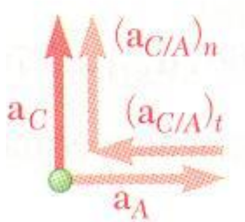
$$= \left(3\,\mathrm{m/s^2}\right)\vec{i} - \left(20\,\mathrm{rad/s^2}\right)\vec{k} \times \left(0.100\,\mathrm{m}\right)\vec{j} - \left(8\,\mathrm{rad/s}\right)^2\left(0.100\,\mathrm{m}\right)\vec{j}$$

$$= \left(3\,\mathrm{m/s^2}\right)\vec{i} + \left(2\,\mathrm{m/s^2}\right)\vec{i} - \left(6.40\,\mathrm{m/s^2}\right)\vec{j}$$

$$\vec{a}_B = \left(5\,\mathrm{m/s^2}\right)\vec{i} - \left(6.40\,\mathrm{m/s^2}\right)\vec{j} \qquad a_B = 8.12\,\mathrm{m/s^2}$$

Sample Problem



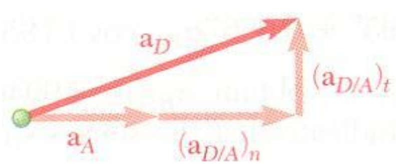


$$\vec{a}_C = \vec{a}_A + \vec{a}_{C/A} = \vec{a}_A + \alpha\vec{k}\times\vec{r}_{C/A} - \omega^2\vec{r}_{C/A}$$

$$= (3\,\mathrm{m/s^2})\,\vec{i} - (20\,\mathrm{rad/s^2})\,\vec{k}\times(-0.150\,\mathrm{m})\,\vec{j} - (8\,\mathrm{rad/s})^2\,(-0.150\,\mathrm{m})\,\vec{j}$$

$$= (3\,\mathrm{m/s^2})\,\vec{i} - (3\,\mathrm{m/s^2})\,\vec{i} + (9.60\,\mathrm{m/s^2})\,\vec{j}$$

$$\boxed{\vec{a}_c = (9.60\,\mathrm{m/s^2})\vec{j}}$$



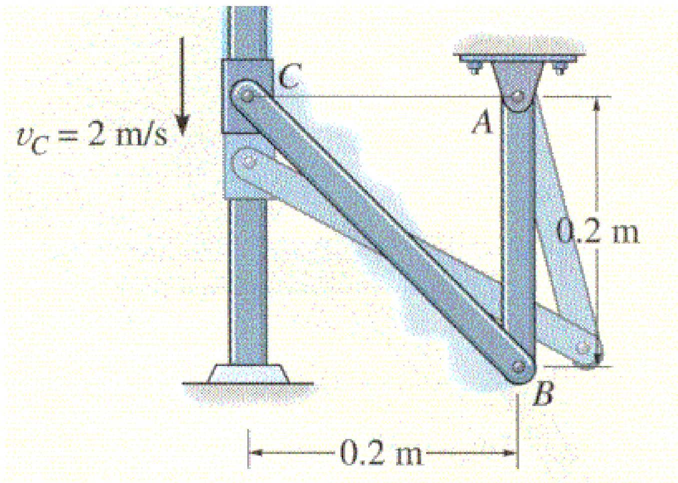
$$\vec{a}_D = \vec{a}_A + \vec{a}_{D/A} = \vec{a}_A + \alpha\vec{k}\times\vec{r}_{D/A} - \omega^2\vec{r}_{D/A}$$

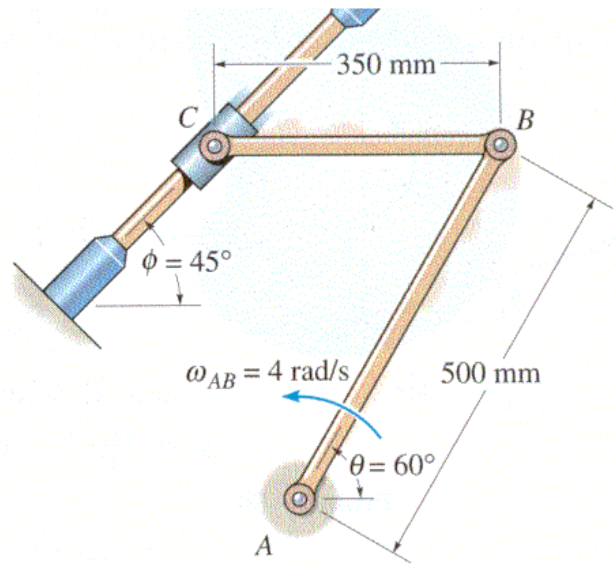
$$= (3\,\mathrm{m/s^2})\,\vec{i} - (20\,\mathrm{rad/s^2})\,\vec{k}\times(-0.150\,\mathrm{m})\,\vec{i} - (8\,\mathrm{rad/s})^2\,(-0.150\mathrm{m})\,\vec{i}$$

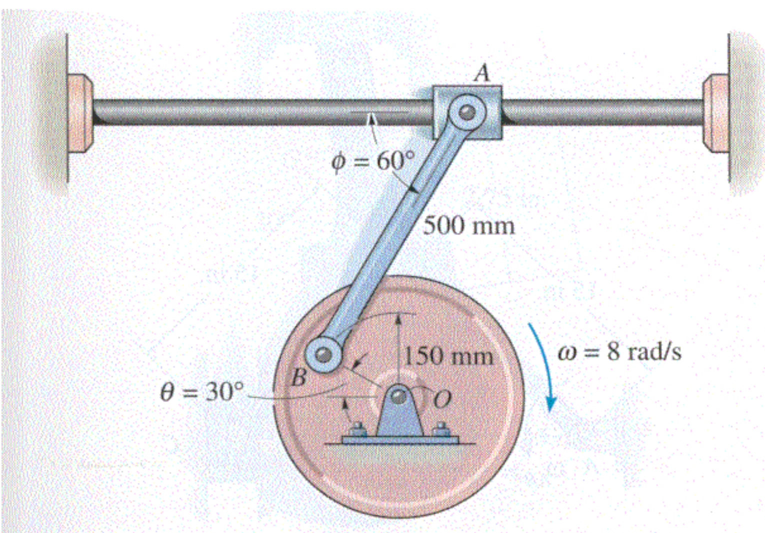
$$= (3\,\mathrm{m/s^2})\,\vec{i} + (3\,\mathrm{m/s^2})\,\vec{j} + (9.60\,\mathrm{m/s^2})\,\vec{i}$$

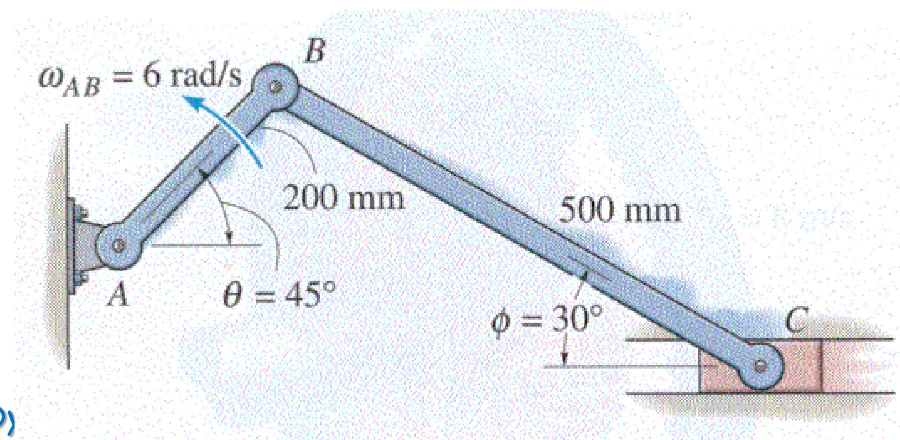
$$\boxed{\vec{a}_D = (12.6\,\mathrm{m}/s^2)\,\vec{i} + (3\,\mathrm{m/s^2})\,\vec{j} \qquad a_D = 12.95\,\mathrm{m/s^2}}$$

Relative motions "motion of one part lead to the motion of other parts" (rigid bodies and pin-connected rigid bodies)









Relative-Motion Analysis

Relative Velocity

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

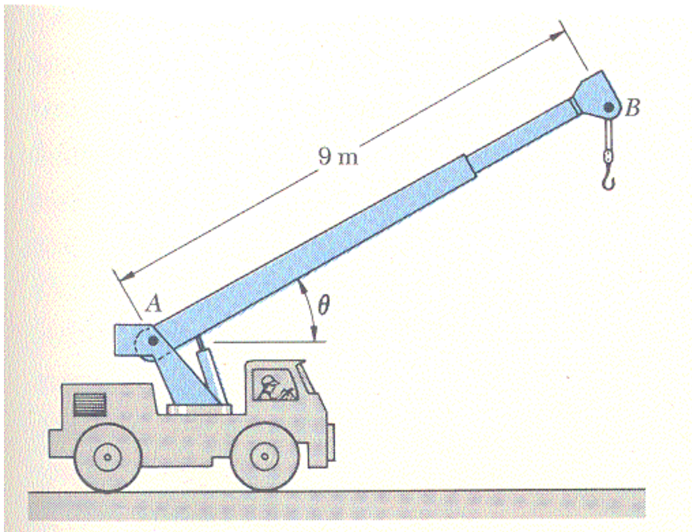
Instantaneous Center of Zero Velocity

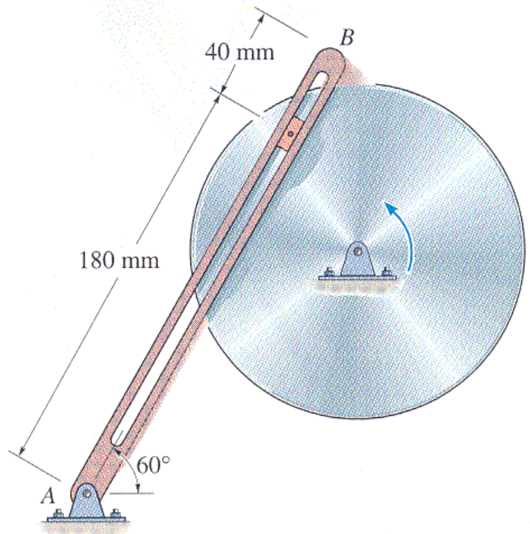
$$\upsilon_B = \omega \, r_{B/IC}$$

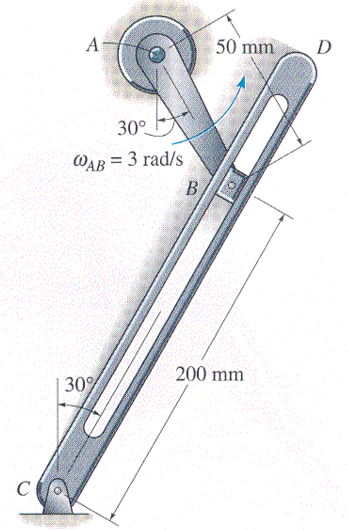
Relative Acceleration

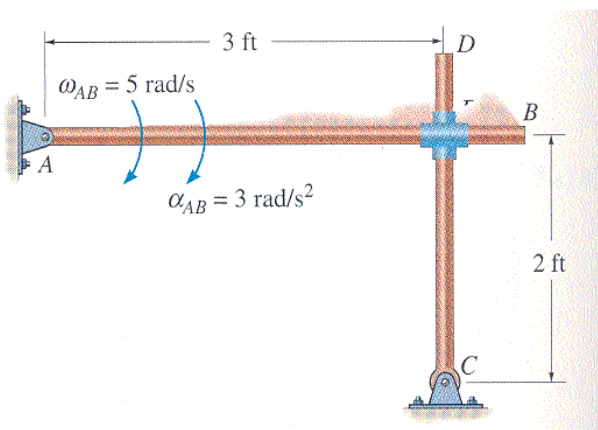
$$\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2 \, \vec{r}_{B/A}$$

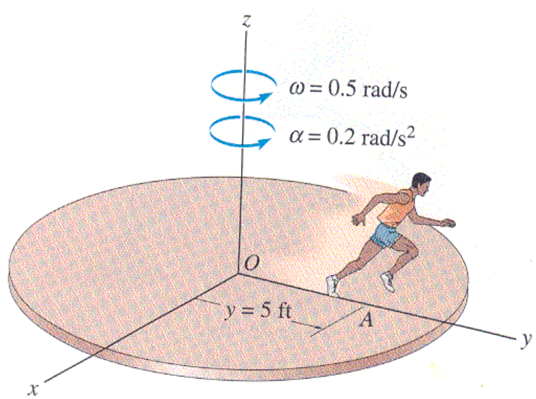
Examples of Non-Relative-Motions (sliding connections – coordinate system that translates and rotates)

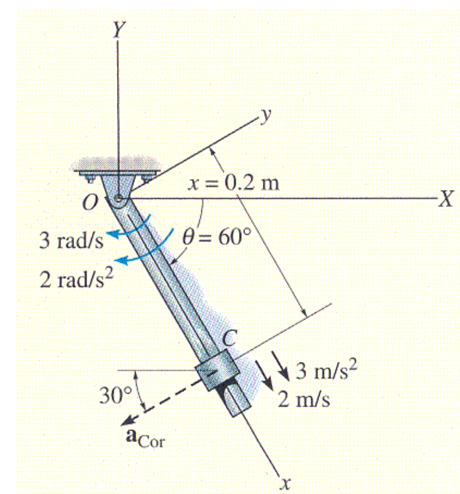










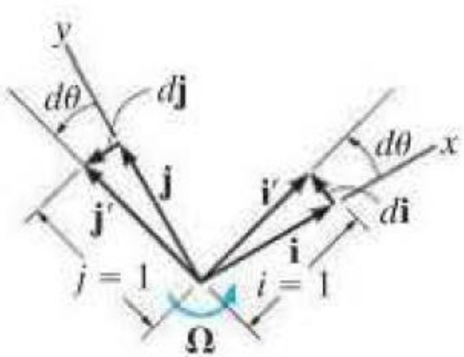


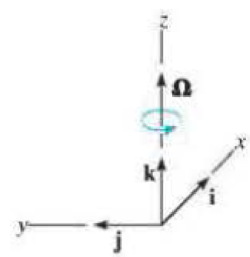
Position & Velocity in Rotating Frames

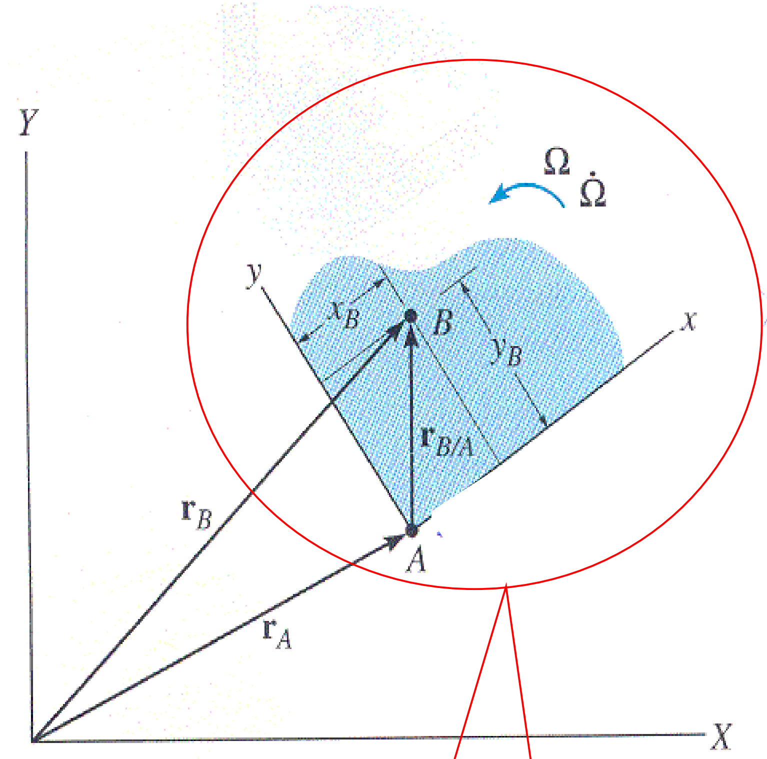
$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

$$\Omega = \omega = \text{angular velocity} = \dot{\theta}$$

$$\dot{\Omega} = \alpha = \text{angular acceleration} = \ddot{\theta}$$







Rotating axes

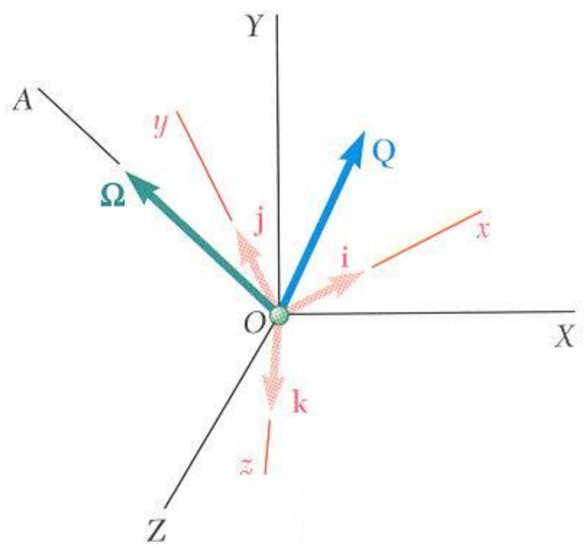
$$\dot{\vec{\imath}} = \Omega\,\vec{\jmath} = \vec{\Omega} \times \vec{\imath}, \dot{\vec{\jmath}} = -\Omega\,\vec{\imath} = \vec{\Omega} \times \vec{\jmath}, where\ \vec{\Omega} = \Omega\vec{k}$$

$$\vec{r}_{B/A} = x_{B/A}\vec{\imath} + y_{B/A}\vec{\jmath}$$

$$\dot{\vec{r}}_{B/A} = \dot{x}_{B/A}\vec{\imath} + \dot{y}_{B/A}\vec{\jmath} + x_{B/A}\dot{\vec{\imath}} + y_{B/A}\dot{\vec{\jmath}} = \dot{\vec{r}}_{B/A} + x_{B/A}\vec{\Omega} \times \vec{\imath} + y_{B/A}\vec{\Omega} \times \vec{\jmath} = \left(\vec{v}_{B/A}\right)_{xyz} + \vec{\Omega} \times \vec{r}_{B/A}$$

$$\vec{v}_B = \vec{v}_A + \vec{\Omega} \times \vec{r}_{B/A} + \left(\vec{v}_{B/A}\right)_{xyz}$$

Rate of Change of a Vector With Respect to a Rotating Frame



- Frame *OXYZ* is fixed.
- Frame *Oxyz* rotates about fixed axis *OA* with angular velocity $\vec{\Omega}$
- Vector function $\vec{Q}(t)$ varies in direction and magnitude. Particular case is when it is taken as a position vector.

- With respect to the rotating *Oxyz* frame,

$$\vec{Q} = Q_x \vec{i} + Q_y \vec{j} + Q_z \vec{k}$$

$$\left(\dot{\vec{Q}}\right)_{Oxyz} = \dot{Q}_x \vec{i} + \dot{Q}_y \vec{j} + \dot{Q}_z \vec{k}$$

- With respect to the fixed *OXYZ* frame,

$$\left(\dot{\vec{Q}}\right)_{OXYZ} = \dot{Q}_x \vec{i} + \dot{Q}_y \vec{j} + \dot{Q}_z \vec{k} + Q_x \dot{\vec{i}} + Q_y \dot{\vec{j}} + Q_z \dot{\vec{k}}$$

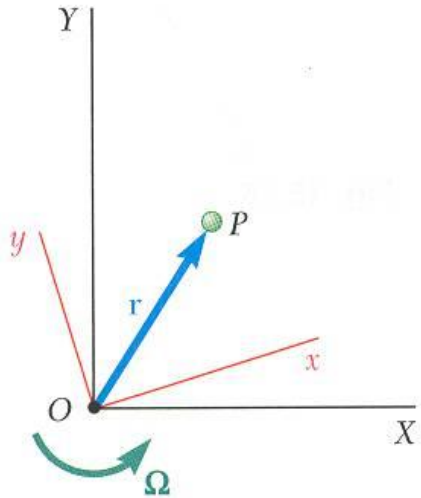
- $\dot{Q}_x \vec{i} + \dot{Q}_y \vec{j} + \dot{Q}_z \vec{k} = \left(\dot{\vec{Q}}\right)_{Oxyz}$ = rate of change with respect to rotating frame.

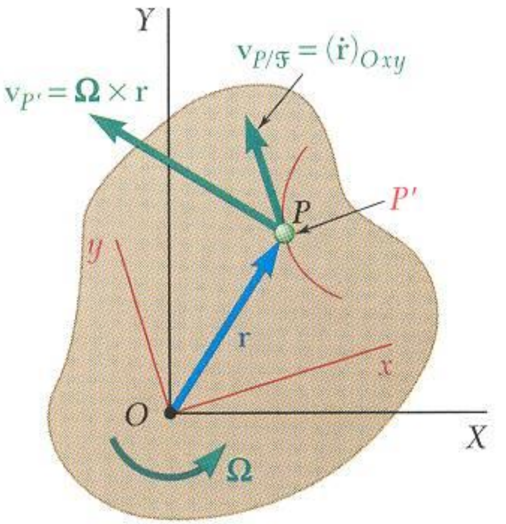
- If $\vec{Q}$ were fixed within *Oxyz* then $\left(\dot{\vec{Q}}\right)_{OXYZ}$ is equivalent to velocity of a point in a rigid body attached to *Oxyz* and $Q_x \dot{\vec{i}} + Q_y \dot{\vec{j}} + Q_z \dot{\vec{k}} = \vec{\Omega} \times \vec{Q}$

  - With respect to the fixed *OXYZ* frame,

$$\left(\dot{\vec{Q}}\right)_{OXYZ} = \left(\dot{\vec{Q}}\right)_{Oxyz} + \vec{\Omega} \times \vec{Q}$$

Velocity for Rotating Frames: Big Picture





- Frame *OXY* is fixed and frame *Oxy* rotates with angular velocity $\vec{\Omega}$.
- Position vector $\vec{r}_P$ for the particle *P* is the same in both frames but the rate of change depends on the choice of frame.
- The absolute velocity of the particle *P* is

$$\vec{v}_P = \left(\dot{\vec{r}}\right)_{OXY} = \vec{\Omega} \times \vec{r} + \left(\dot{r}\right)_{Oxy}$$

- Imagine a rigid slab attached to the rotating frame *Oxy* or $\mathfrak{F}$ for short. Let *P'* be a point on the slab which corresponds instantaneously to position of particle *P*.

$\vec{v}_{P/\mathfrak{F}} = \left(\dot{\vec{r}}\right)_{Oxy} =$ velocity of *P* along its path on the slab

$\vec{v}_{P'} =$ absolute velocity of point *P'* on the slab

- Absolute velocity for the particle P may be written as

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/\mathfrak{F}}$$

Acceleration for Rotating Frames

$$\vec{v}_B = \vec{v}_A + \vec{\Omega} \times \vec{r}_{B/A} + \left(\vec{v}_{B/A}\right)_{xyz}$$

$$\frac{d\vec{v}_B}{dt} = \frac{d\vec{v}_A}{dt} + \frac{d\vec{\Omega}}{dt} \times \vec{r}_{B/A} + \vec{\Omega} \times \frac{d\vec{r}_{B/A}}{dt} + \frac{d\left(\vec{v}_{B/A}\right)_{xyz}}{dt}$$

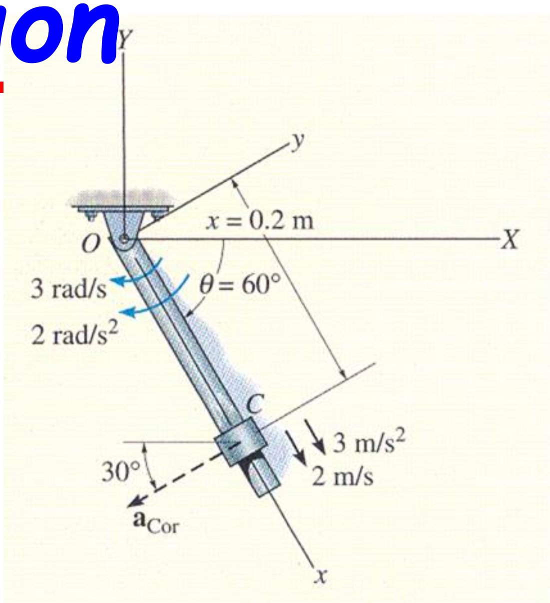
$$\vec{a}_B = \vec{a}_A + \dot{\vec{\Omega}} \times \vec{r}_{B/A} + \vec{\Omega} \times \frac{d\vec{r}_{B/A}}{dt} + \frac{d\left(\vec{v}_{B/A}\right)_{xyz}}{dt}$$

$$\dot{\vec{r}}_{B/A} = \left(\vec{v}_{B/A}\right)_{xyz} + \vec{\Omega} \times \vec{r}_{B/A} \qquad \dot{\vec{v}}_{B/A} = \left(\vec{a}_{B/A}\right)_{xyz} + \vec{\Omega} \times \left(\vec{v}_{B/A}\right)_{xyz}$$

Here we used the general expression for rotating systems:

$$\left(\dot{\vec{Q}}\right)_{OXYZ} = \left(\dot{\vec{Q}}\right)_{Oxyz} + \vec{\Omega} \times \vec{Q}$$

Coriolis Acceleration



Acceleration of origin

Acceleration of the object

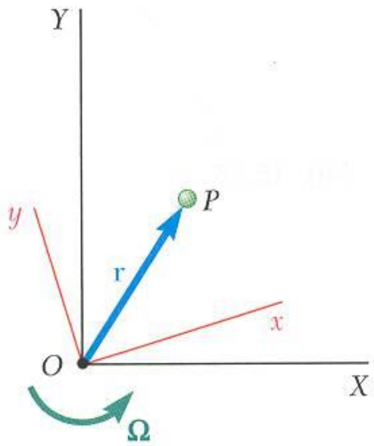
$$\mathbf{a}_B = \mathbf{a}_A + \dot{\Omega}\times \mathbf{r}_{B/A} + \Omega\times(\Omega\times \mathbf{r}_{B/A}) + 2\Omega\times(\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}$$

α r
Tangential acceleration

ω^2 r
Normal acceleration

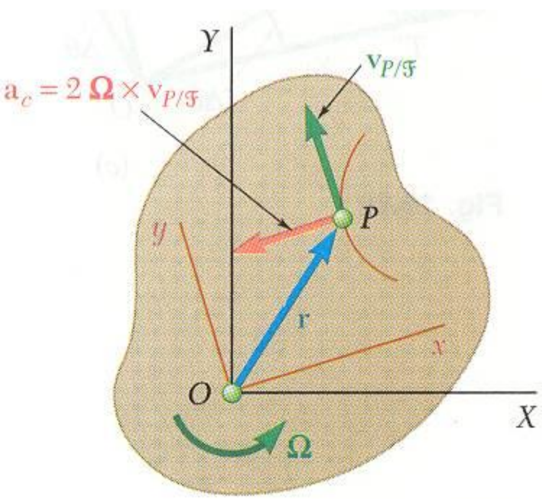
$2\dot{\theta}\,\dot{r}$
Coriolis acceleration

Coriolis Acceleration: Big Picture



$$\vec{v}_P = \vec{\Omega}\times\vec{r} + (\dot{r})_{Oxy}$$

$$= \vec{v}_{P'} + \vec{v}_{P/\mathfrak{F}}$$



- Absolute acceleration for the particle *P* is

$$\vec{a}_P = \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times(\dot{\vec{r}})_{OXY} + \frac{d}{dt}\left[(\dot{\vec{r}})_{Oxy}\right]$$

but, $(\dot{\vec{r}})_{OXY} = \vec{\Omega}\times\vec{r} + (\dot{\vec{r}})_{Oxy}$

$$\frac{d}{dt}\left[(\dot{\vec{r}})_{Oxy}\right] = (\ddot{\vec{r}})_{Oxy} + \vec{\Omega}\times(\dot{\vec{r}})_{Oxy}$$

$$\vec{a}_P = \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) + 2\vec{\Omega}\times(\dot{\vec{r}})_{Oxy} + (\ddot{\vec{r}})_{Oxy}$$

- Utilizing the conceptual point *P'* on the slab,

$$\vec{a}_{P'} = \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$\vec{a}_{P/\mathfrak{F}} = (\ddot{\vec{r}})_{Oxy}$$

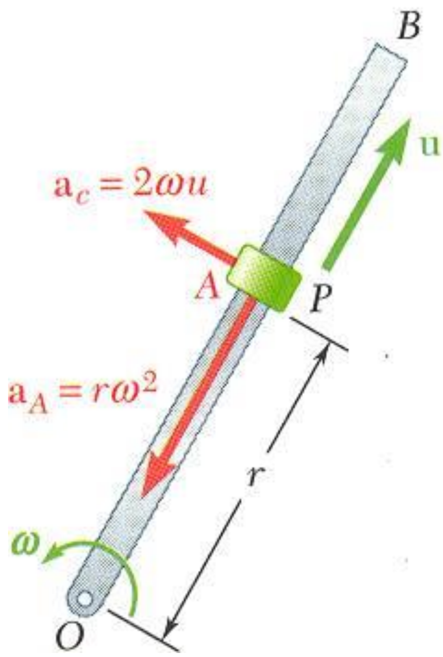
- Absolute acceleration for the particle *P* becomes

$$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/\mathfrak{F}} + 2\vec{\Omega}\times(\dot{\vec{r}})_{Oxy}$$

$$= \vec{a}_{P'} + \vec{a}_{P/\mathfrak{F}} + \vec{a}_c$$

$$\vec{a}_c = 2\vec{\Omega}\times(\dot{\vec{r}})_{Oxy} = 2\vec{\Omega}\times\vec{v}_{P/\mathfrak{F}} = \text{Coriolis acceleration}$$

Coriolis Acceleration



- Consider a collar *P* which is made to slide at constant relative velocity *u* along rod *OB.* The rod is rotating at a constant angular velocity ω. The point *A* on the rod corresponds to the instantaneous position of *P*.
- Absolute acceleration of the collar is

$$\vec{a}_P = \vec{a}_A + \vec{a}_{P/\mathfrak{F}} + \vec{a}_c$$

where

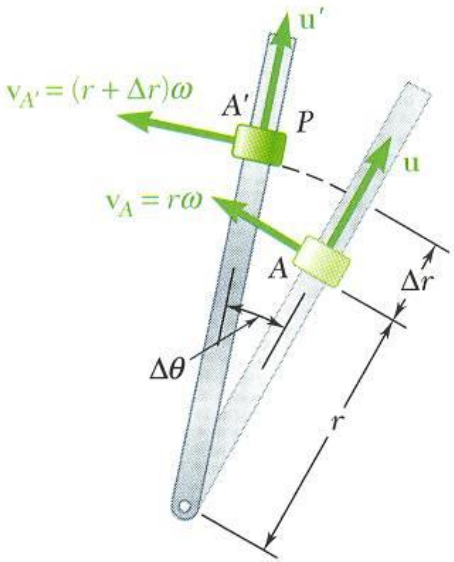
$$\vec{a}_A = \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) \qquad a_A = r\omega^2$$

$$\vec{a}_{P/\mathfrak{F}} = \left(\ddot{\vec{r}}\right)_{Oxy} = 0$$

$$\vec{a}_c = 2\vec{\Omega} \times \vec{v}_{P/\mathfrak{F}} \qquad a_c = 2\omega u$$

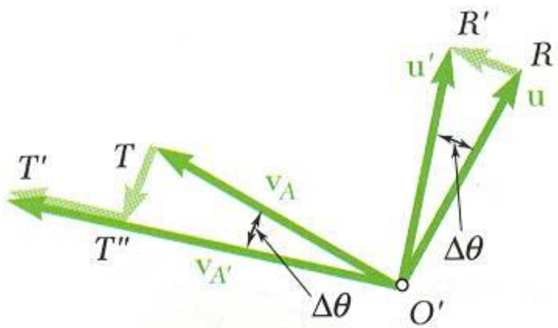
- The absolute acceleration consists of the radial and tangential vectors shown

Coriolis Acceleration



at t, $\quad \vec{v} = \vec{v}_A + \vec{u}$

at $t + \Delta t$, $\quad \vec{v}\,' = \vec{v}_{A'} + \vec{u}\,'$



- Change in velocity over Δt is represented by the sum of three vectors

$$\Delta\vec{v} = \overrightarrow{RR'} + \overrightarrow{TT''} + \overrightarrow{T''T'}$$

- $\overrightarrow{TT''}$ is due to change in direction of the velocity of point *A* on the rod,

$$\lim_{\Delta t \to 0} \frac{\overrightarrow{TT''}}{\Delta t} = \lim_{\Delta t \to 0} v_A \frac{\Delta\theta}{\Delta t} = r\omega\omega = r\omega^2 = a_A$$

$$\text{recall, } \vec{a}_A = \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) \qquad a_A = r\omega^2$$

- $\overrightarrow{RR'}$ and $\overrightarrow{T''T'}$ result from combined effects of relative motion of *P* and rotation of the rod

$$\lim_{\Delta t \to 0}\left(\frac{\overrightarrow{RR'}}{\Delta t} + \frac{\overrightarrow{T''T'}}{\Delta t}\right) = \lim_{\Delta t \to 0}\left(u\frac{\Delta\theta}{\Delta t} + \omega\frac{\Delta r}{\Delta t}\right)$$

$$= u\omega + \omega u = 2\omega u$$

$$\text{recall, } \vec{a}_c = 2\vec{\Omega} \times \vec{v}_{P/\mathcal{F}} \qquad a_c = 2\omega u$$

Example

$$a_C = a_O + \dot{\Omega} \times r_{C/O} + \Omega \times (\Omega \times r_{C/O}) + 2\Omega \times (v_{C/O})_{xyz} + (a_{C/O})_{xyz}$$

$$a_O = 0, v_0 = 0, \Omega = -3\vec{k}\ rad/s, \dot{\Omega} = -2\vec{k}\ rad/s^2$$

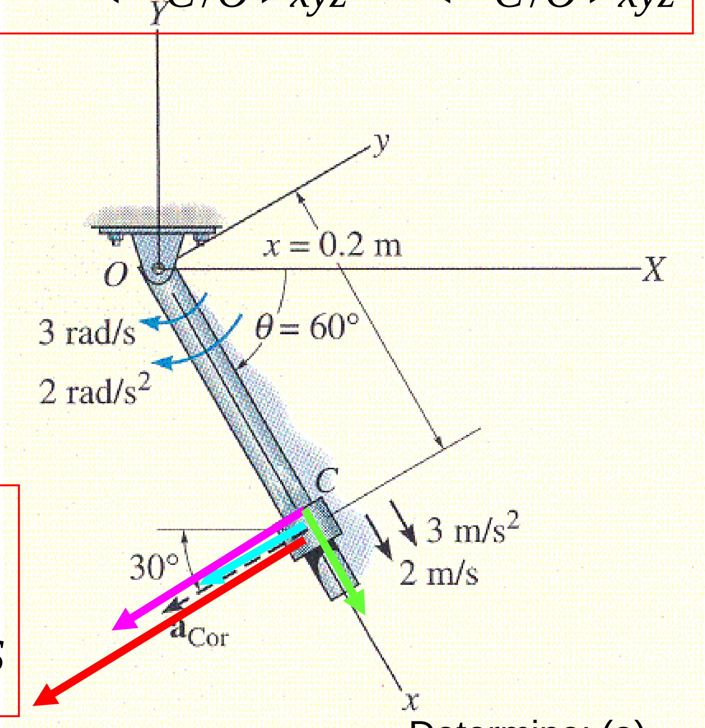
$$(\mathbf{r}_{C/O})_{xyz} = 0.2\vec{i}\ m, (\mathbf{v}_{C/O})_{xyz} = 2\vec{i}\ m/s, (\mathbf{a}_{C/O})_{xyz} = 3\vec{i}\ m/s^2$$

$$a_{Cor} = 2\Omega \times (v_{C/O})_{xyz}$$

$$= 2(-3\vec{k}) \times \left(2\vec{i}\right) = -12\,\vec{j}\ m/s^2$$

$$\mathbf{v}_C = \mathbf{v}_O + \mathbf{\Omega} \times \mathbf{r}_{C/O} + (\mathbf{v}_{C/O})_{xyz}$$

$$= 0 + (-3\vec{k}) \times (0.2\vec{i}) + 2\vec{i} = \left(2\vec{i} - 0.6\vec{j}\right) m/s$$



$$\mathbf{a}_C = \mathbf{a}_O + \dot{\mathbf{\Omega}} \times \mathbf{r}_{C/O} + \mathbf{\Omega}\,\mathrm{x}\,(\Omega\,\mathrm{x}\,\mathbf{r}_{C/O}) + 2\Omega \times (\mathbf{v}_{C/O})_{xyz} + (\mathbf{a}_{C/O})_{xyz}$$

$$= 0 + (-2\vec{k}) \times (0.2\vec{i}) + (-3\vec{k}) \times \left[(-3\vec{k}) \times (0.2\vec{i})\right] + 2(-3\vec{k}) \times (2\vec{i}) + 3\vec{i}$$

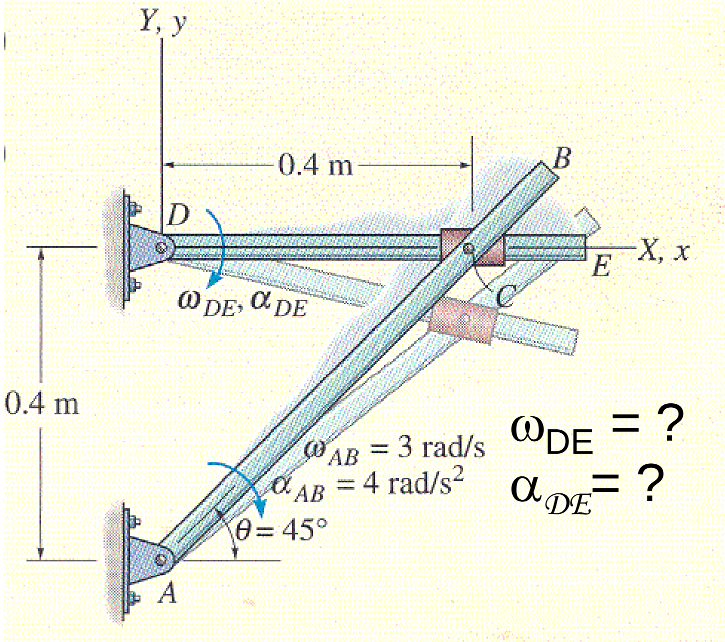
$$= \left(1.2\vec{i} - 12.4\vec{j}\right) m/s^2$$

$$x - axes = 3 - 1.8 = 1.2\ m/s^2$$

$$y - axes = -0.4 - 12 = -12.4\ m/s^2$$

Determine: (a) The Coriolis acceleration and (b) the velocity and acceleration of the collar at the instant shown.

Example



AB rotates clockwise such that it has an ω_{AB} = 3 rad/ s and α_{AB} = 4 rad/s^2 when θ =45°. Determine the angular motion of rod *DE* at this instant. The collar at Cis pin connected to *AB* and slides over rod *DE*.

The origin of both the fixed and moving frames of reference is at D. The xyz reference is attached to and rotates with *DE* so that the relative motion of the collar is easy to follow.

Motion of moving framework: $\boldsymbol{v}_D = \mathbf{0}, \boldsymbol{a}_D = \mathbf{0}, \boldsymbol{\Omega} = -\omega_{DE}\vec{k}, \dot{\boldsymbol{\Omega}} = -\alpha_{DE}\vec{k}$

Motion of C with respect to moving framework: $r_{C/D} = 0.4\boldsymbol{i}\ m$

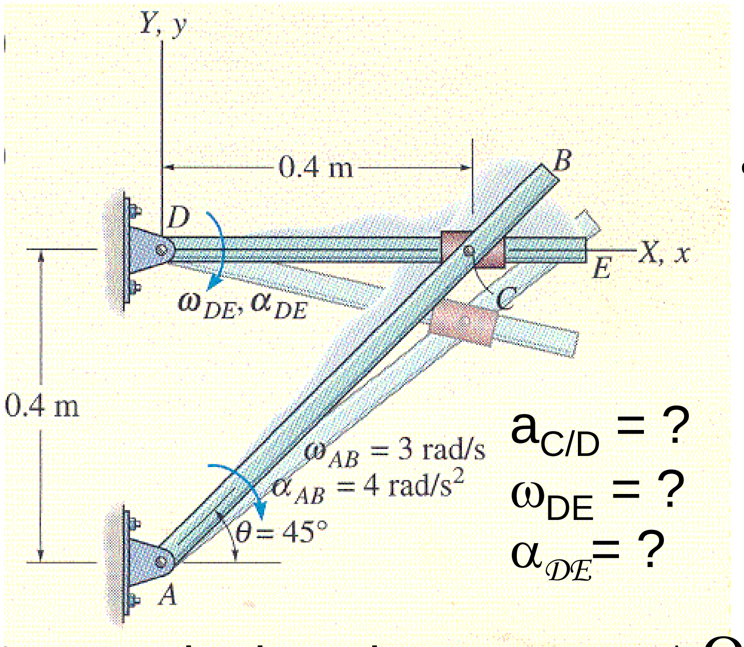
$$\left(\boldsymbol{v}_{C/D}\right)_{xyz} = \left(v_{C/D}\right)_{xyz}\vec{\imath}, \left(\boldsymbol{a}_{C/D}\right)_{xyz} = \left(a_{C/D}\right)_{xyz}\vec{\imath}$$

Since the collar moves on a circular path of radius AC, we can compute:

$$\mathrm{v}_C = \omega_{AB} \times r_{C/A} = (-3\vec{k}) \times (0.4\vec{i} + 0.4\vec{j}) = \{1.2\vec{i} - 1.2\vec{j}\}\,m/s$$

$$\mathrm{a}_C = \alpha_{AB} \times r_{C/A} - \omega_{AB}^{\ 2} r_{C/A} = (-4\vec{k}) \times (0.4\vec{i} + 0.4\vec{j}) - (3)^2(0.4\vec{i} + 0.4\vec{j}) = \{-2\vec{i} - 5.2\vec{j}\}\,m/s^2$$

Example



$a_{C/D} = ?$

$\omega_{DE} = ?$

$\alpha_{DE} = ?$

$$\mathrm{v}_C = \mathrm{v}_D + \Omega \times \mathrm{r}_{C/D} + (\mathrm{v}_{C/D})_{xyz}$$

$$\mathrm{a}_C = \mathrm{a}_D + \dot{\Omega} \times \mathrm{r}_{C/D} + \Omega \times (\Omega \times \mathrm{r}_{C/D}) + 2\Omega \times (\mathrm{v}_{C/D})_{xyz} + (\mathrm{a}_{C/D})_{xyz}$$

$$\mathrm{r}_{C/D} = \{0.4\vec{i}\}\, m \qquad \boldsymbol{v}_D = \mathbf{0}$$

$$(\mathrm{v}_{C/D})_{xyz} = \upsilon_{C/D}\vec{i}\ m/s \qquad \boldsymbol{\Omega} = -\omega_{DE}\vec{k}$$

$$(a_{C/D})_{xyz} = a_{C/D}\vec{i}\ m/s^2$$

$$\mathrm{v}_C = \{1.2\vec{i} - 1.2\vec{j}\}\, m/s \qquad \mathrm{a}_C = \{-2\vec{i} - 5.2\vec{j}\}\, m/s^2$$

We substitute in: $\mathrm{v}_C = \mathrm{v}_D + \Omega \times \mathrm{r}_{C/D} + (\mathrm{v}_{C/D})_{xyz}$

$$1.2\vec{i} - 1.2\vec{j} = 0 + (-\omega_{DE}\vec{k}) \times (0.4\vec{i}) + (\upsilon_{C/D})_{xyz}\vec{i} = -0.4\omega_{DE}\vec{j} + (\upsilon_{C/D})_{xyz}\vec{i} \Rightarrow$$

$$\omega_{DE} = 3\, rad/s \qquad (\upsilon_{C/D})_{xyz} = 1.2\ m/s$$

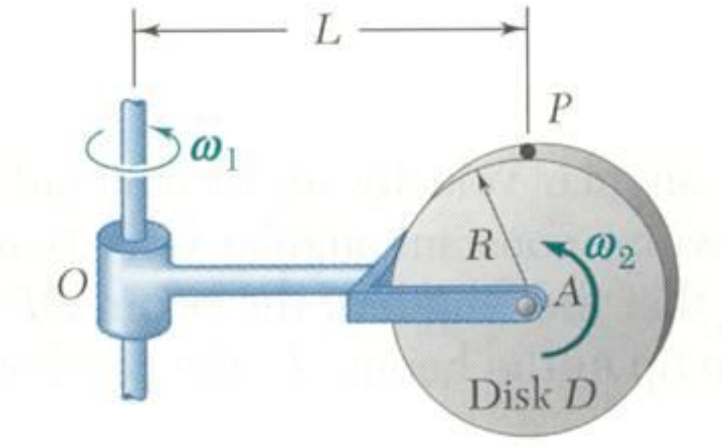
Similarly: $\mathrm{a}_C = \mathrm{a}_D + \dot{\Omega} \times \mathrm{r}_{C/D} + \Omega \times (\Omega \times \mathrm{r}_{C/D}) + 2\Omega \times (\mathrm{v}_{C/D})_{xyz} + (\mathrm{a}_{C/D})_{xyz}$

$$-2\vec{i} - 5.2\vec{j} = 0 + (-\alpha_{DE}\vec{k}) \times (0.4\vec{i}) + (-3\vec{k}) \times [(-3\vec{k}) \times (0.4\vec{i})] + 2(-3\vec{k}) \times (1.2\vec{i}) + a_{C/D}\vec{i}$$

$$-2\vec{i} - 5.2\vec{j} = 0 - 0.4\alpha_{DE}\vec{j} - 3.6\vec{i} - 7.2\vec{j} + a_{C/D}\vec{i}$$

$$a_{C/D} = 1.6\ m/s^2 \qquad \alpha_{DE} = -5\, rad/s^2 = 5\, rad/s^2 \circlearrowright$$

Sample Problem



For the disk mounted on the arm, the indicated angular rotation rates are constant.

Determine:

- the velocity of the point *P*,
- the acceleration of *P*, and
- angular velocity and angular acceleration of the disk.

SOLUTION:

- Define a fixed reference frame *OXYZ* at *O* and a moving reference frame *Axyz* or $\mathfrak{F}$ attached to the arm at *A*.
- With *P'* of the moving reference frame coinciding with *P*, the velocity of the point *P* is found from

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/\mathfrak{F}}$$

- The acceleration of *P* is found from

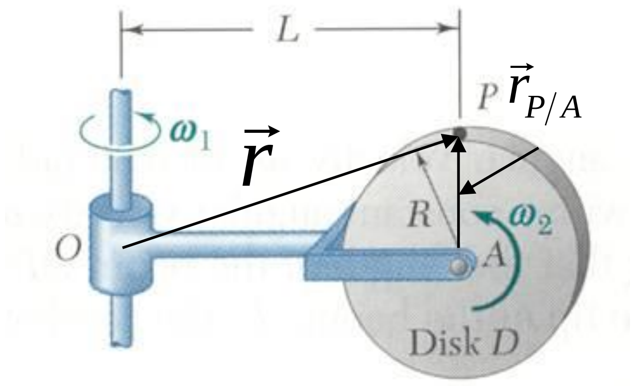
$$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/\mathfrak{F}} + \vec{a}_c$$

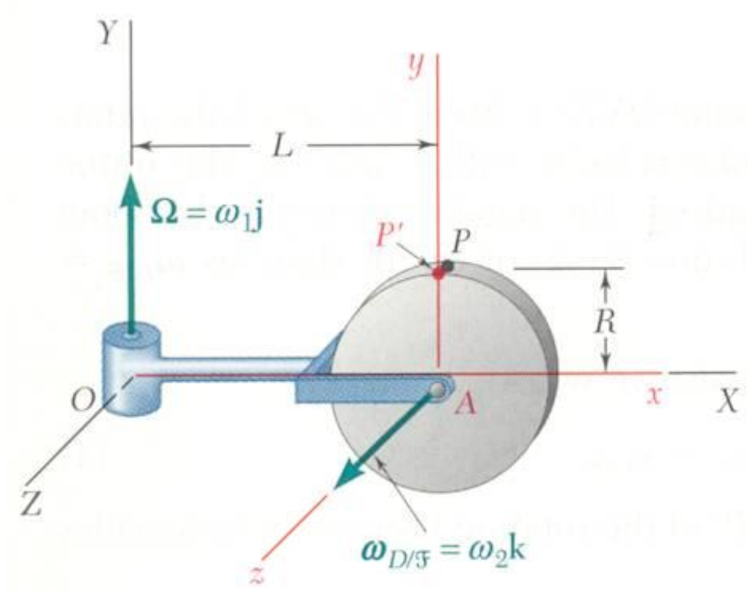
- The angular velocity and angular acceleration of the disk are

$$\vec{\omega} = \vec{\Omega} + \vec{\omega}_{D/\mathfrak{F}}$$

$$\vec{\alpha} = \left(\dot{\vec{\omega}}\right)_{\mathfrak{F}} + \vec{\Omega} \times \vec{\omega}$$ (applying the general Eq. $\left(\dot{\vec{Q}}\right)_{OXYZ} = \left(\dot{\vec{Q}}\right)_{Oxyz} + \vec{\Omega} \times \vec{Q}$)

Sample Problem





SOLUTION:

- Define a fixed reference frame *OXYZ* at *O* and a moving reference frame *Axyz* or $\mathfrak{F}$ attached to the arm at *A*.

$$\vec{r} = L\vec{i} + R\vec{j} \qquad \vec{r}_{P/A} = R\vec{j}$$

$$\vec{\Omega} = \omega_1 \vec{j} \qquad \vec{\omega}_{D/\mathfrak{F}} = \omega_2 \vec{k}$$

- With *P'* of the moving reference frame coinciding with *P*, the velocity of the point *P* is found from

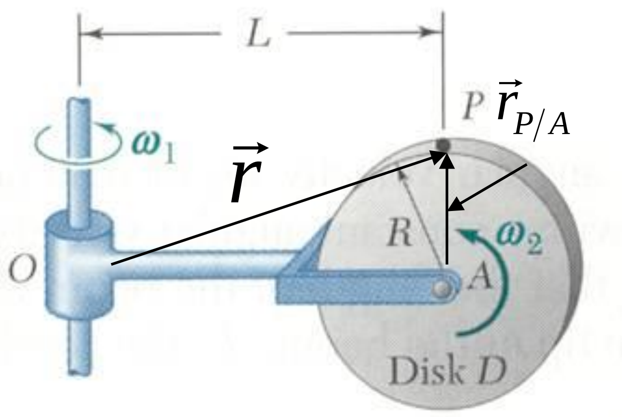
$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/\mathfrak{F}}$$

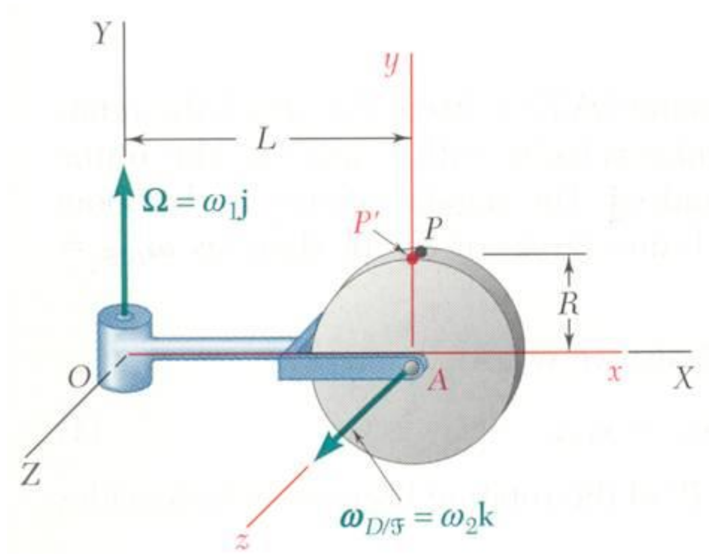
$$\vec{v}_{P'} = \vec{\Omega} \times \vec{r} = \omega_1 \vec{j} \times \left(L\vec{i} + R\vec{j}\right) = -\omega_1 L \vec{k}$$

$$\vec{v}_{P/\mathfrak{F}} = \vec{\omega}_{D/\mathfrak{F}} \times \vec{r}_{P/A} = \omega_2 \vec{k} \times R\vec{j} = -\omega_2 R \vec{i}$$

$$\boxed{\vec{v}_P = -\omega_2 R \vec{i} - \omega_1 L \vec{k}}$$

Sample Problem





- The acceleration of *P* is found from

$$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/\mathcal{F}} + \vec{a}_c$$

$$\vec{a}_{P'} = \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) = \omega_1 \vec{j} \times \left(-\omega_1 L \vec{k}\right) = -\omega_1^2 L \vec{i}$$

$$\vec{a}_{P/\mathcal{F}} = \vec{\omega}_{D/\mathcal{F}} \times \left(\vec{\omega}_{D/\mathcal{F}} \times \vec{r}_{P/A}\right)$$

$$= \omega_2 \vec{k} \times \left(-\omega_2 R \vec{i}\right) = -\omega_2^2 R \vec{j}$$

$$\vec{a}_c = 2\vec{\Omega} \times \vec{v}_{P/\mathcal{F}}$$

$$= 2\omega_1 \vec{j} \times \left(-\omega_2 R \vec{i}\right) = 2\omega_1 \omega_2 R \vec{k}$$

$$\boxed{\vec{a}_P = -\omega_1^2 L \vec{i} - \omega_2^2 R \vec{j} + 2\omega_1 \omega_2 R \vec{k}}$$

- Angular velocity and acceleration of the disk,

$$\vec{\omega} = \vec{\Omega} + \vec{\omega}_{D/\mathcal{F}}$$

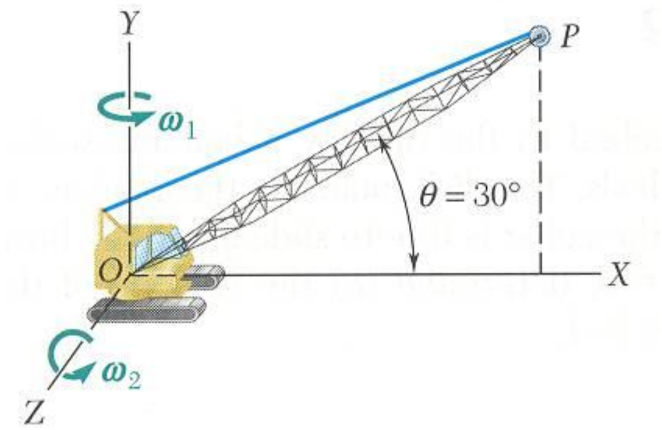
$$\boxed{\vec{\omega} = \omega_1 \vec{j} + \omega_2 \vec{k}}$$

$$\vec{\alpha} = \left(\dot{\vec{\omega}}\right)_{\mathbf{F}} + \vec{\Omega} \times \vec{\omega}$$

$$= 0 + \omega_1 \vec{j} \times \left(\omega_1 \vec{j} + \omega_2 \vec{k}\right)$$

$$\boxed{\vec{\alpha} = \omega_1 \omega_2 \vec{i}}$$

Sample Problem



The crane rotates with a constant angular velocity $\omega_1 = 0.30$ rad/s and the boom is being raised with a constant angular velocity $\omega_2 = 0.50$ rad/s relative to the cab. The length of the boom is *l* = 12 m.

Determine:

- angular velocity of the boom,
- angular acceleration of the boom,
- velocity of the boom tip, and
- acceleration of the boom tip.

SOLUTION

- The frame *OXYZ* is fixed. We attach the rotating frame *Oxyz* to the cab. Its angular velocity with respect to the frame *OXYZ* is therefore

$$\vec{\Omega} = \vec{\omega}_1 = (0.3 rad/s)\vec{j}$$

- The angular velocity of the boom relative to the cab and the rotating frame *Oxyz* (or $\mathfrak{F}$ for short) is

$$\vec{\omega}_{B/\mathfrak{F}} = \vec{\omega}_2 = \left(0.5 rad/s\right)\vec{k}$$

- For the velocity, we write:

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/\mathfrak{F}}$$

$$\vec{v}_{P'} = \vec{\Omega} \times \vec{r} = (0.30 \text{ rad/s})\,\vec{j} \times [(10.39 \text{ m})\,\vec{i} + (6 \text{ m})\,\vec{j}]$$

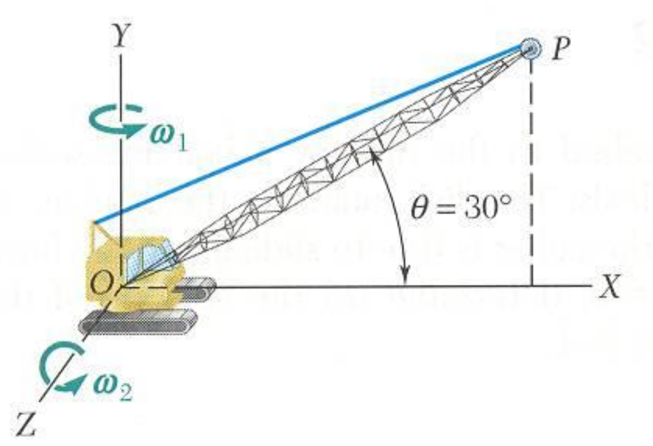
$$= -(3.12 \text{ m/s})\,\vec{k}$$

$$\vec{v}_{P/\mathfrak{F}} = \vec{\omega}_{B/\mathfrak{F}} \times \vec{r}$$

$$= (0.50 rad/s)\,\vec{k} \times [(10.39m)\,\vec{i} + (6m)\,\vec{j}]$$

$$= -(3m/s)\,\vec{i} + (5.20m/s)\,\vec{j}$$

Sample Problem



The crane rotates with a constant angular velocity $\omega_1 = 0.30$ rad/s and the boom is being raised with a constant angular velocity $\omega_2 = 0.50$ rad/s relative to the cab. The length of the boom is l = 12 m.

Determine:

- angular velocity of the boom,
- angular acceleration of the boom,
- velocity of the boom tip, and
- acceleration of the boom tip.

SOLUTION:

- For the acceleration

$$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/\mathfrak{F}} + \vec{a}_c$$

- Note that $\vec{\Omega}, \vec{\omega}_{B/\mathfrak{F}}$ are constant.

$$\vec{a}_{P'} = \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) = (0.30 \text{ rad/s})\,\vec{j} \times (-3.12 \text{ m/s})\,\vec{k} = -(0.94 \text{ m/s}^2)\,\vec{i}$$

$$\vec{a}_{P/\mathfrak{F}} = \vec{\omega}_{B/\mathfrak{F}} \times \left(\vec{\omega}_{B/\mathfrak{F}} \times \vec{r}\right)$$

$$= (0.50 \text{ rad/s})\,\vec{k} \times [-(3 \text{ m/s})\,\vec{i} + (5.20 \text{ m/s})\,\vec{j}]$$

$$= -(1.50 \text{ m/s}^2)\,\vec{j} - (2.60 \text{ m/s}^2)\,\vec{i}$$

$$\vec{a}_c = 2\vec{\Omega} \times \vec{v}_{P/\mathfrak{F}}$$

$$= 2(0.30 \text{ rad/s})\,\vec{j} \times [-(3 \text{ m/s})\,\vec{i} + (5.20 \text{ m/s})\,\vec{j}]$$

$$= (1.80 \text{ m/s}^2)\,\vec{k}$$

- Substituting gives:

$$\vec{a} = -\left(3.54\,\text{m/s}^2\right)\vec{i} - \left(1.50\,\text{m/s}^2\right)\vec{j} + \left(1.80\,\text{m/s}^2\right)\vec{k}$$