Knots, Graphs and Geometry

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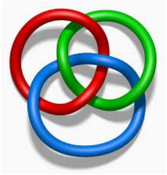
What is a knot ?

A knot is a (smooth) embedding of the circle S^1 in S^3. Similarly, a link of k-components is a (smooth) embedding of a disjoint union of k circles in S^3.

Figure-8 knot

Whitehead link



Borromean rings

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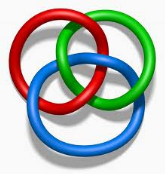

Borromean rings

Two knots are equivalent if there is continuous deformation (ambient isotopy) of S^3 taking one to the other.

Goals: To classify knots upto equivalence.

Knot Diagrams

A common way to describe a knot is using a planar projection of the knot which is a 4-valent planar graph indicating the over and under crossings called a knot diagram.

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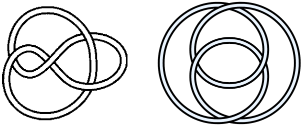

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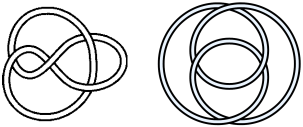
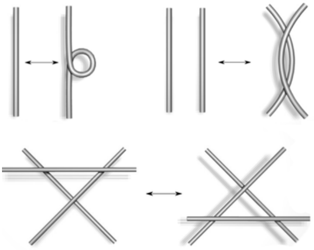
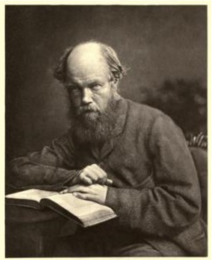


Two knot diagrams represent the same knot if and only if they are related by a sequence of three kinds of moves on the diagram called the Reidemeister moves.



Origins of Knot theory

In 1867, Lord Kelvin conjectured that atoms were knotted tubes of ether and the variety of knots were thought to mirror the variety of chemical elements. This theory inspired the celebrated Scottish physicist Peter Tait to undertake an extensive study and tabulation of knots (in collaboration with C. N. Little).



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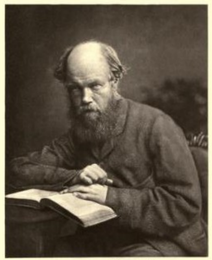
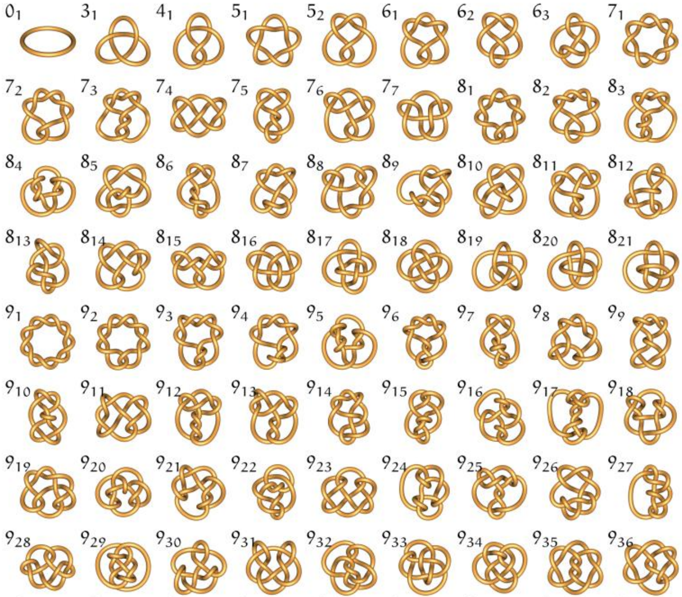


Tait enumerated knots using their diagrammatic complexity called the crossing number of a knot, defined as the minimal number of crossings over all knot diagrams.

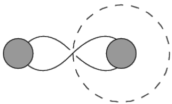
Knots with low crossing number



Tait Conjectures

A knot diagram is alternating if the crossings alternate under, over, under, over, as one travels along each component of the link. A knot is alternating if it has an alternating diagram.

A reduced diagram is a diagram with no reducible crossings.



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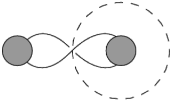

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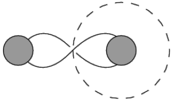

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Tait made more conjectures, about relating alternating diagrams (Flyping Conjecture), and about writhe of alternating knots.

Knot invariants

How to tell knots apart ?

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The crossing number is a knot invariant, however very hard to compute. Tait Conjecture 1 gives a way to compute it for alternating knots.

Knots invariants

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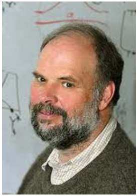

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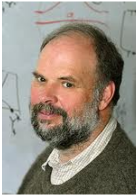

Bill Thruston used hyperbolic geometry to introduce geometric invariants, which results in "hyperbolic knot theory".

Examples of knots invariants

- Topological: Arising from topology of the $S^3 - K$ e.g. Fundamental group, Alexander polynomial (1927), Seifert surfaces (1934), Knot Heegaard Floer homology (Ozsvath-Szabo-Rasmussen, 2003).

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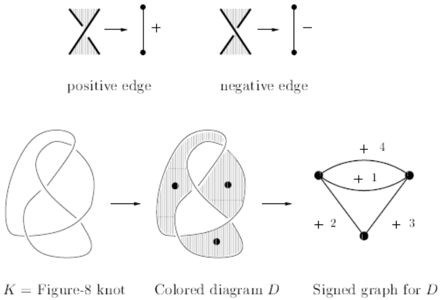
A big problem in knot theory is to relate different kind of invariants.

Knots and Graphs

The Tait graph G_K of a knot diagram K is a plane signed graph arising from a checkboard coloring of K as follows: shaded regions correspond to vertices , crossings corresponding to signed edges.



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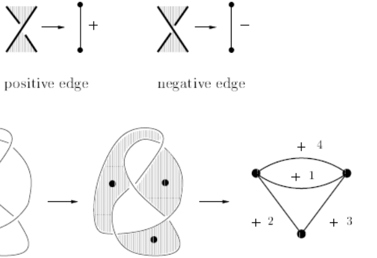

K = Figure-8 knot   Colored diagram D Signed graph for D

The other checkboard coloring gives the planar dual of G_K.

Knots and Graphs

Thistlethwaite (1987)
(1) Jones polynomial of K can be written in terms of spanning trees of G_K: $V_K(t) = \sum_{T \subset G_K} \mu(T)$.
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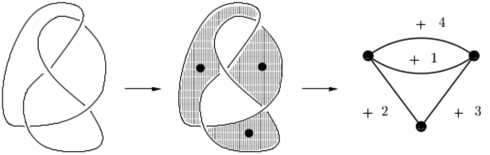
Corollary

(1) Proves Tait Conjecture 1.

(2) If K is connected, reduced alternating diagram, Knot determinant $\det(K) =$ number of spanning trees of G_K.

Knots and Graphs



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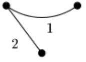
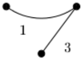
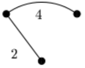
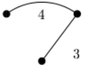
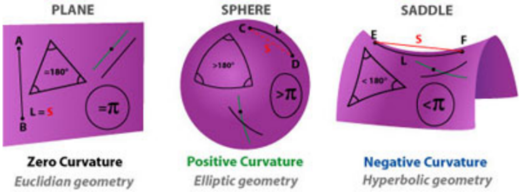
Spanning trees	 T_1	 T_2	 T_3	 T_4	 T_5
Activities	$LLdd$	$LdDd$	ℓDDd	ℓLdD	$\ell\ell DD$
Weights	A^{-8}	$-A^{-4}$	$-A^{4}$	1	A^{8}

$\langle D\rangle = A^{-8} - A^{-4} + 1 - A^{4} + A^{8}$ writhe $w(D) = 0$

$V_K(t) = t^{-2} - t^{-1} + 1 - t + t^{2}$

Knots and Geometry

2-dimensional geometries



Thurston (1980s) Most knot complements i.e. $S^3 - K$ can be modeled on the 3-dimensional negatively curved geometry i.e. hyperbolic geometry.

Basic hyperbolic geometry I

Escher's work using hyperbolic plane

Hyperbolic plane crochet by Daina Taimina

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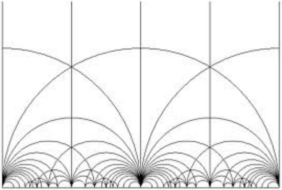
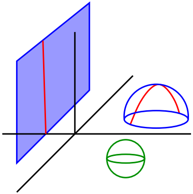


Hyperbolic upper-half plane



Hyperbolic upper-half space

Basic hyperbolic geometry II

- The Upper Half-Space model $\mathbb{H}^3 = \{(x, y, t) | t > 0\}$ with metric $\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}t^2}{t^2}$. Straight lines (geodesics) are lines or half circles orthogonal to the xy-plane.

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- Other models include Poincare ball model, Klein model and the Hyperboloid model.

Hyperbolic building blocks

How to build hyperbolic knots or manifolds ?

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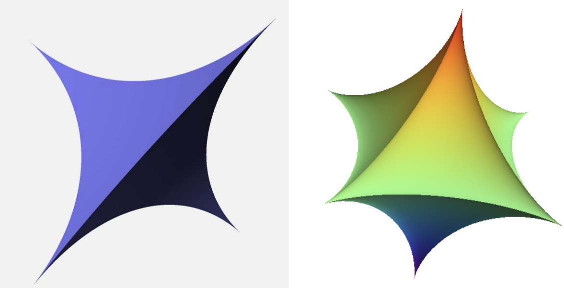
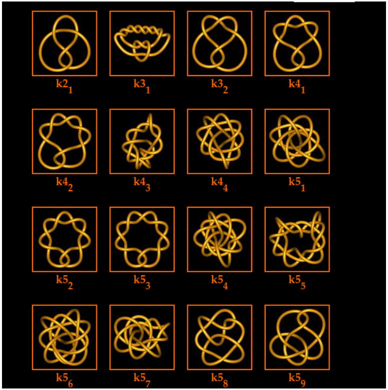
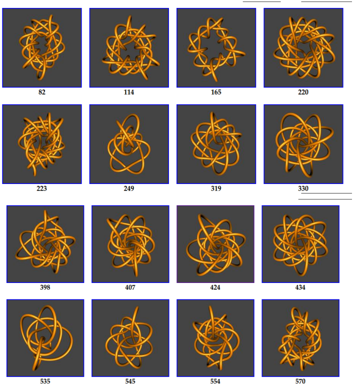
Ideal tetrahedra & polyhedra in hyperbolic 3-space can be glued together to make knot complements. This is a geometric way of describing knots.



The least number of hyperbolic tetrahedra gives a geometric complexity for knots.

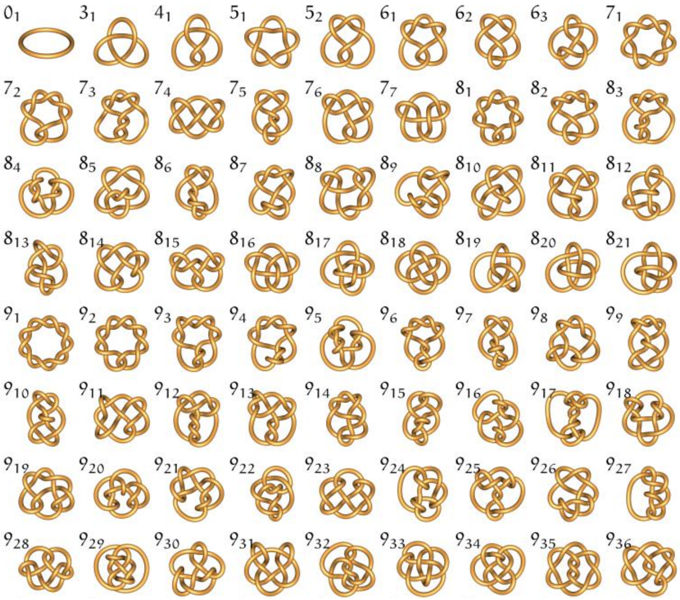
Knots with low tetrahedral number





Hyperbolic knots with geometric complexity up to 6 tetrahedra were found by Callahan-Dean-Weeks (1999), extended to 7 tetrahedra by Champanerkar-Kofman-Paterson (2004), and to 8 tetrahedra by Champanerkar-Kofman-Mullen (2013).

Knots with low crossing number



Computing knot invariants

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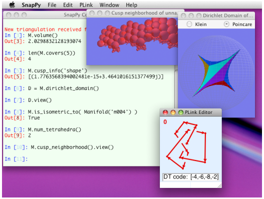
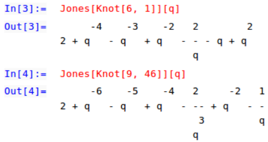
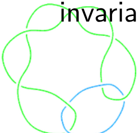
SnapPy by Culler and Dunfield, based on SnapPea by Jeff Weeks computes hyperbolic invariants.

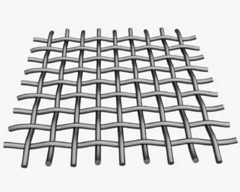




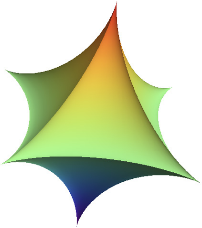
KnotTheory by Bar-Natan, is a Mathematica package which computes diagrammatic invariants.



Asymptotic knot theory



Infinite alternating weave $\mathcal{W}$



Regular ideal octahedron in $\mathbb{H}^3$ with volume $= v_8$

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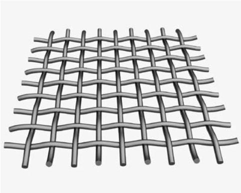

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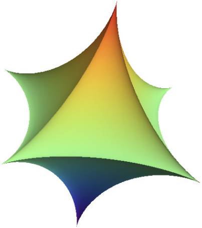

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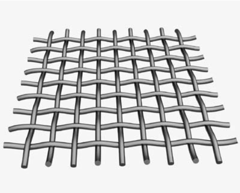

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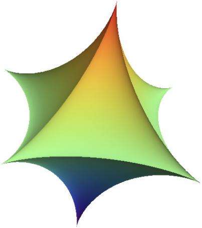

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if K_n is $n \times n$ grid which give an exhaustion of $\mathcal{W}$, then

$$\lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = v_8 = \lim_{n\to\infty} \frac{\mathrm{vol}(K_n)}{c(K_n)}$$

Abhijit's Home page:
http://www.math.csi.cuny.edu/abhijit/

KnotAtlas: http://katlas.math.toronto.edu/wiki/

SnapPy: http://www.math.uic.edu/~t3m/SnapPy/

Knot Invariants: http://www.indiana.edu/~knotinfo/

KnotPlot: http://www.knotplot.com/

Thank you