Chapter 4 Motion in 2-D plane

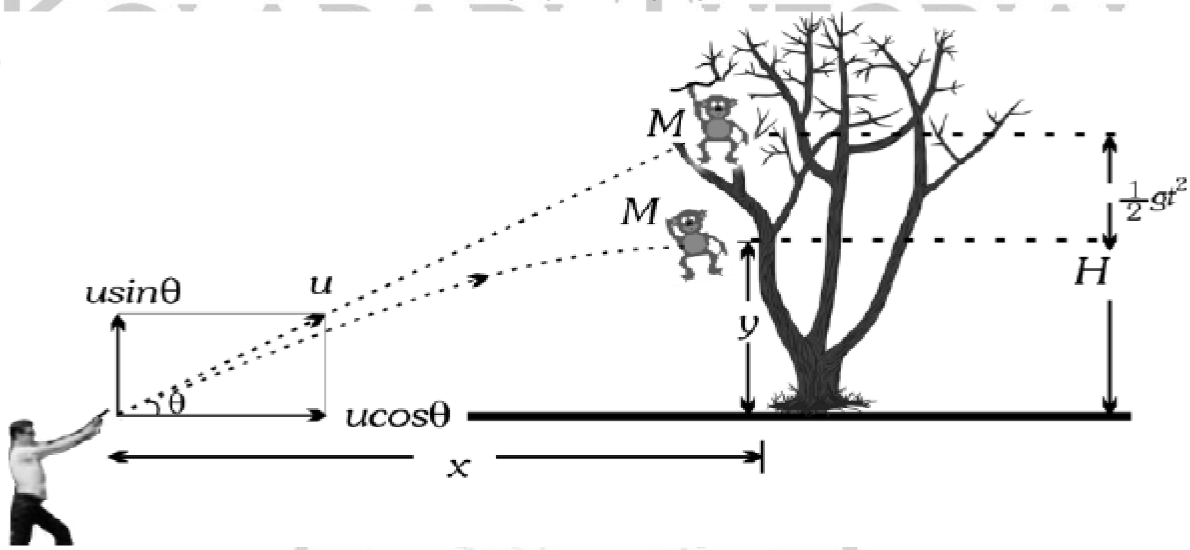
Introduction:

The motion of an object is called **two dimensional**, if two of the three co-ordinates are required to specify the position of the object in space changes w.r.t time.

In such a motion, the object moves in a plane. For example, a billiard ball moving over the billiard table, an insect crawling over the floor of a room, earth revolving around the sun etc.

- Two special cases of motion in two dimension are 1. Projectile motion 2. Circular motion

PROECTILE MOTION



A hunter aims his gun and fires a bullet directly towards a monkey sitting on a distant tree. If the monkey remains in his position, he will be safe but at the instant the bullet leaves the barrel of gun, if the monkey drops from the tree, the bullet will hit the monkey because the bullet will not follow the linear path.

If the force acting on a particle is oblique with initial velocity then the motion of particle is called **projectile motion**.

A body thrown with some initial velocity and then allowed to move under the action of gravity alone (not being propelled by any fuel), is known as a projectile. If we observe the path of the projectile, we find that the projectile moves in a path, which can be considered as a part of parabola. Such a motion is known as projectile motion. A few examples of projectiles are (i) a bomb thrown from an aeroplane (ii) a javelin or a shot-put thrown by an athlete (iii) motion of a ball hit by a cricket bat etc.

A body can be projected in two ways: (i) It can be thrown from the ground in a direction inclined to it. (ii) It can be projected horizontally from a certain height. The different types of projectiles are shown in Fig.

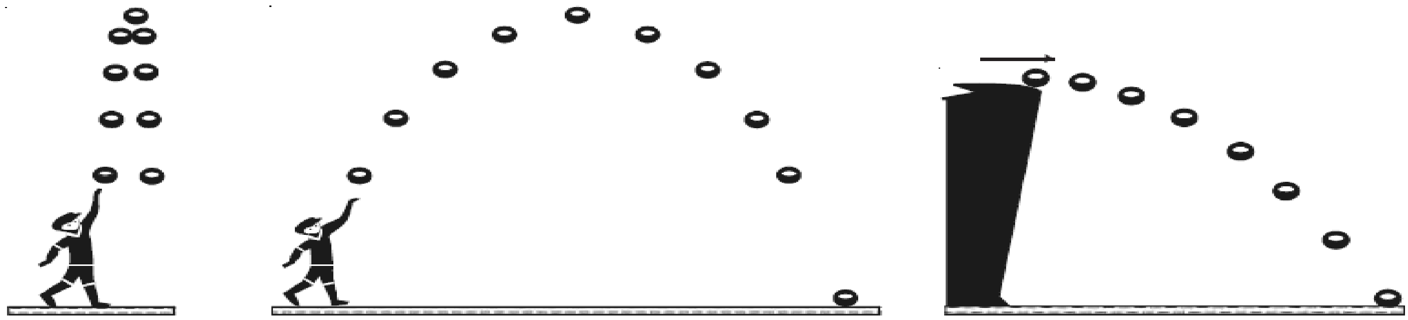


Fig Different types of projectiles

The projectiles undergo a vertical motion as well as horizontal motion. The two components of the projectile motion are (i) vertical component and (ii) horizontal component. These two perpendicular components of motion are independent of each other.

A body projected with an initial velocity making an angle with the horizontal direction possess uniform horizontal velocity and variable vertical velocity, due to force of gravity. The object therefore has horizontal and vertical motions simultaneously. The resultant motion would be the vector sum of these two motions and the path following would be curvilinear.

This discussion can be summarised as **Principles of Physical Independence of Motions**:

(1) The motion of a projectile is a two-dimensional motion. So, it can be discussed in two parts. Horizontal motion and vertical motion. These two motions take place independent of each other. This is called the principle of physical independence of motions.

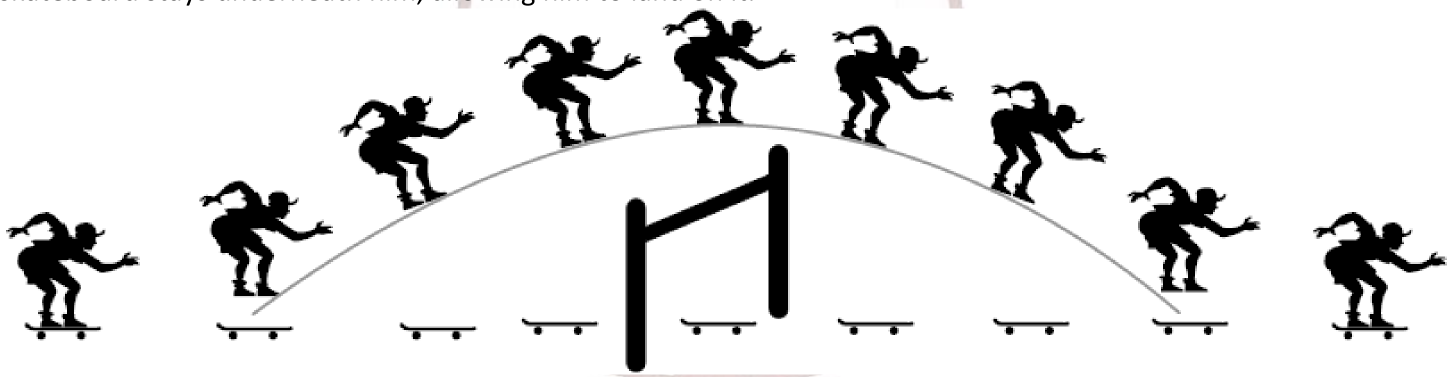
(2) The velocity of the particle can be resolved into two mutually perpendicular components. Horizontal component and vertical component.

(3) The horizontal component remains unchanged throughout the flight. The force of gravity continuously affects the vertical component.

(4) The horizontal motion is a uniform motion and the vertical motion is a uniformly accelerated retarded motion.

Two independent motions of a projectile			
Motion	**Forces**	**Velocity**	**Acceleration**
Horizontal	No force acts	Constant	Zero
Vertical	The force of gravity acts downwards	Changes ($\sim$10 m s^{-1})	Downwards ($\sim$10 m s^{-2})

<u>Ex</u>: When a man jumps over the hurdle leaving behind its skateboard then vertical component of his velocity is changing, but not the horizontal component, which matches with the skateboard velocity. As a result, the skateboard stays underneath him, allowing him to land on it.

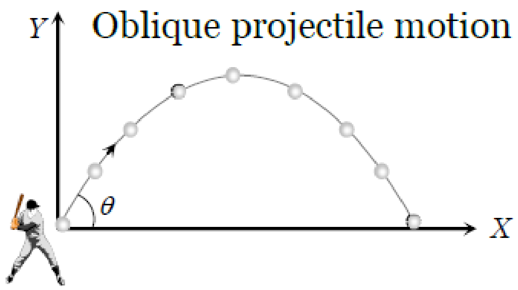


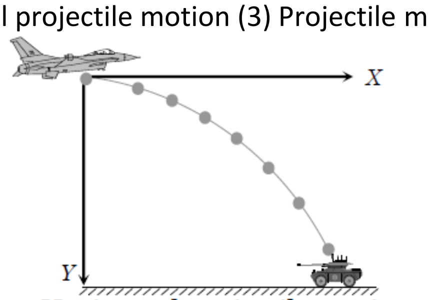
- **Angle of projection:** The angle between the initial direction of projection and the horizontal direction through the point of projection is called the angle of projection.
- **Velocity of projection**: The velocity with which the body is projected is known as velocity of projection.
- **Range:** Range of a projectile is the horizontal distance between the point of projection and the point where the projectile hits the ground.
- **Trajectory:** The path described by the projectile is called the trajectory.
- **Time of flight:** Time of flight is the total time taken by the projectile from the instant of projection till it strikes the ground.
- **Assumptions in the study of Projectile Motion**

(1) There is no resistance due to air.
(2) The effect due to curvature of earth is negligible.
(3) The effect due to rotation of earth is negligible.
(4) For all points of the trajectory, the acceleration due to gravity 'g' is constant in magnitude and direction.

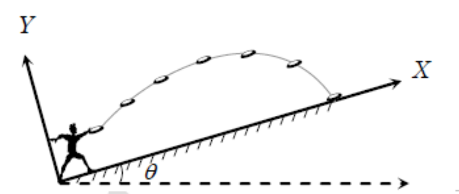
- **Types of Projectile Motion**

(1) Oblique projectile motion (2) Horizontal projectile motion (3) Projectile motion on an inclined plane





Horizontal projectile motion



Projectile motion on an inclined

<u>Motion of a projectile projected at an angle with the horizontal (oblique projection)</u>

Consider a body projected from a point O on the surface of the Earth with an initial velocity u at an angle θ with the horizontal as shown in Fig. The velocity u can be resolved into two components

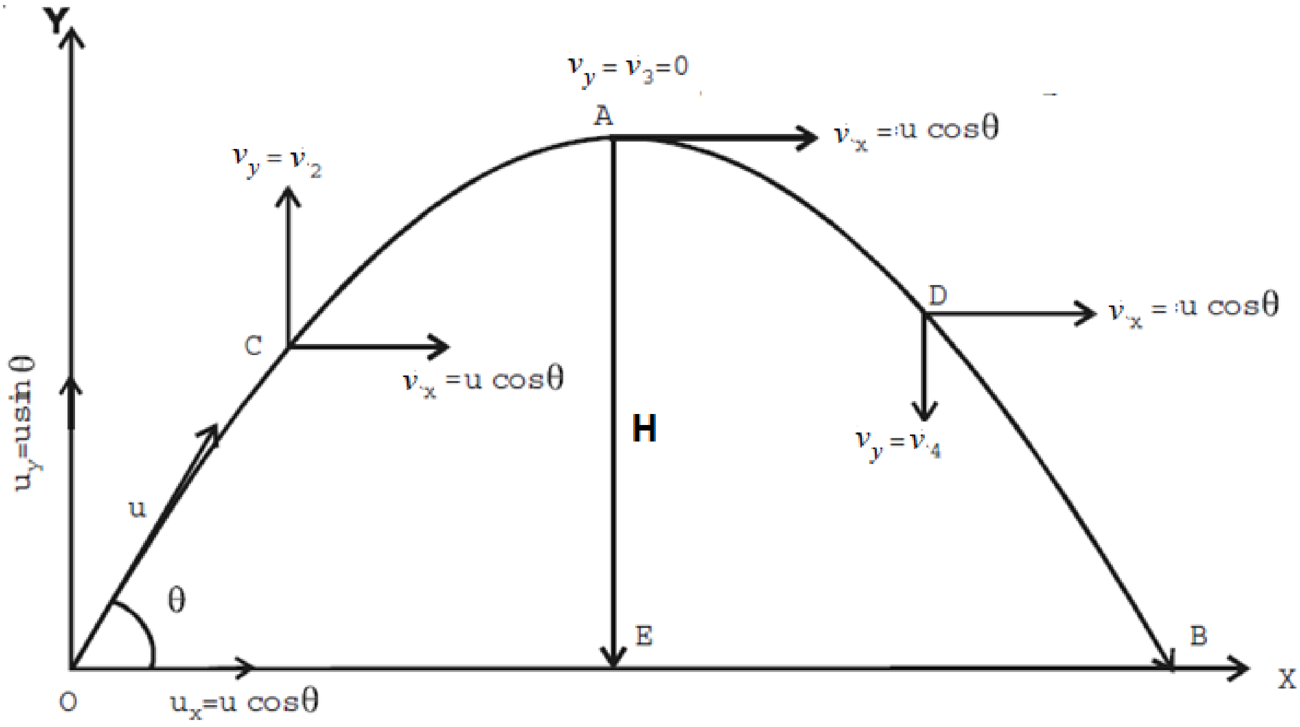


Fig Motion of a projectile projected at an angle with horizontal

(i) $u_x = u \cos\theta$, along the horizontal direction OX and

(ii) $u_y = u \sin\theta$, along the vertical direction OY

In projectile motion, horizontal component of velocity of the object v_x (=u cos θ), shall remain constant as no acceleration is acting in the horizontal direction. But the vertical component v_y of the object continuously decreases due to the effect of the gravity and it becomes zero when the body is at the highest point of its path. After this, the vertical component v_y is directed downwards and increases with time till the body strikes the ground at B.

The acceleration (g) and mechanical energy remains constant while, speed, velocity, vertical component of velocity (u sin θ), momentum kinetic energy and potential energy all changes. Velocity, and KE are maximum at the point of projection while minimum (but not zero) at highest point.

- **Path of the projectile (Equation of trajectory)**

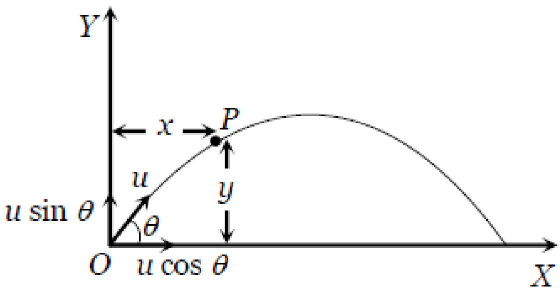
A projectile thrown with velocity u at an angle θ with the horizontal. The velocity u can be resolved into two rectangular components u cos θ component along X-axis and u sin θ component along Y-axis.

Let t_1 be the time taken by the projectile to reach the point C from the instant of projection.

Horizontal distance travelled by the projectile in time t is,

x = horizontal velocity $\times$ time

$x = u\cos\theta \times t$ (or) $t = \frac{x}{u\cos\theta}$ ...(1)



Let the vertical distance travelled by the projectile in time t = s = y

At O, initial vertical velocity $u_y = u \sin\theta$

From the equation of motion $s = u_y t - \frac{1}{2} g t^2$

Substituting the known values,

$y = (u\sin\theta)\, t - \frac{1}{2} g t^2$ ...(2)

Substituting equation (1) in equation (2),

$y = (u\sin\theta)\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2} g \left(\frac{x}{u\cos\theta}\right)^2$

$\mathbf{y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}}$ ...(3)

The above equation is of the form $y = Ax + Bx^2$ and represents a parabola. Thus the path of a projectile is a parabola.

NOTE: Equation of oblique projectile also can be written as $\mathbf{y = x\tan\theta\left(1 - \frac{x}{R}\right)}$ (where R = horizonal range)

- **Resultant velocity of the projectile at any instant t_1(instantaneous velocity)**

In projectile motion, vertical component of velocity changes but horizontal component of velocity remains always constant.
Let v be the instantaneous velocity of projectile at time t direction of this velocity is along the tangent to the trajectory at point P.

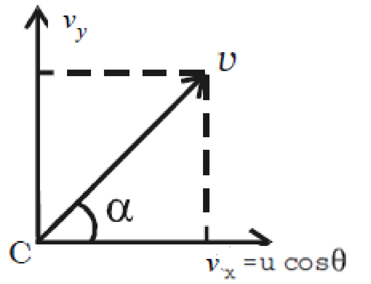


Fig Resultant velocity of the projectile at any instant

$\vec{v} = v_x\hat{i} + v_y\hat{j}$ => v = $\sqrt{v_x^2 + v_y^2}$

At C, the velocity along the horizontal direction is $v_x = u_x$ = u cos θ and the velocity along the vertical direction is
$v_y = u_y - gt$ (From the equation of motion,)
v_y = u sin θ – gt

∴ The resultant velocity at C is v = $\sqrt{v_x^2 + v_y^2} = \sqrt{(u\cos\theta)^2 + (u\sin\theta - gt)^2}$

v = $\sqrt{\mathbf{u^2 + g^2t^2 - 2gtu\sin\theta}}$

The direction of instantaneous velocity v is given by
$\tan\alpha = \frac{v_y}{v_x} = \frac{u\sin\theta - gt}{u\cos\theta}$ (or) $\alpha = \tan^{-1}\left(\frac{u\sin\theta - gt}{u\cos\theta}\right)$
where α is the angle made by v with the horizontal line.

- **Maximum height reached by the projectile**

The maximum vertical displacement produced by the projectile is known as the maximum height reached by the projectile. It is the maximum height from the point of projection, a projectile can reach. In Fig, EA is the maximum height attained by the projectile. It is represented as H.
At O, the initial vertical velocity (u) = u sin θ
At A, the final vertical velocity (v_3) = 0
The vertical distance travelled by the object = s_y = H
From equation of motion, $v_3^2 = u^2 - 2gs_y$
Substituting the known values, $(0)^2 = (u\sin\theta)^2 - 2gH$
$2gH = u^2\sin^2\theta$ (or) **H =** $\frac{u^2\sin^2\theta}{2g}$ ...(4)
$H_{max} = \frac{u^2}{2g}$ (when $\sin^2\theta$ = max = 1 i.e., θ = 90°)
i.e., for maximum height body should be projected vertically upward.

- **Time taken to attain maximum height**

Let t be the time taken by the projectile to attain its maximum height.
From equation of motion v_3 = u – g t
Substituting the known values 0 = u sin θ – g t
g t = u sin θ
t = $\frac{u\sin\theta}{g}$ ...(5)

- **Time of flight**

The total time taken by the projectile to go up and come down to the same level from which it was projected is called time of flight.
Let t_f be the time of flight (i.e) the time taken by the projectile to reach B from O through A. When the body returns to the ground, the net vertical displacement made by the projectile
s_y = H – H = 0
From the equation of motion $s_y = u\,t_f - \frac{1}{2}g\,t_f^2$
Substituting the known values $0 = (u\sin\theta)\,t_f - \frac{1}{2}g\,t_f^2$
$\frac{1}{2}g\,t_f^2 = (u\sin\theta)\,t_f$ (or) $\mathbf{t_f} = \frac{2u\sin\theta}{g}$ ...(6)
From equations (5) and (6) $\mathbf{t_f = 2t}$ ...(7)
(i.e) the time of flight is twice the time taken to attain the maximum height.

- **Horizontal range**

It is the horizontal distance travelled by a body during the time of flight. The horizontal distance OB is called the range of the projectile.

Horizontal range = horizontal velocity × time of flight

(i.e) R = u cos θ × t_f

Substituting the value of t_f, R = u cosθ $\times \frac{2u \sin \theta}{g}$

R = $\frac{u^2 \, 2\cos\theta \sin \theta}{g}$

∴ **R =** $\frac{\mathbf{u^2 \sin 2\theta}}{\mathbf{g}}$ ...(8)

If angle of projection is changed from θ to θ' = (90 − θ) then range remains unchanged. These angles are called complementary angle of projection.

- **Maximum Range**

From (8), it is seen that for the given velocity of projection, the horizontal range depends on the angle of projection only. The range is maximum only if the value of sin 2θ is maximum.

For maximum range R_{max}, sin 2θ = 1

(i.e) θ = 45°

Therefore the range is maximum when the angle of projection is 45°.

$R_{max} = \frac{u^2 \times 1}{g}$

⇒ $\mathbf{R_{max}} = \frac{\mathbf{u^2}}{\mathbf{g}}$ ...(9)

When the range is maximum, the height H reached by the projectile H = $\frac{u^2\sin^2\theta}{2g} = \frac{u^2\sin^2 45^0}{2g} = \frac{u^2}{4g} = \frac{R_{max}}{4}$

⇒ **H =** $\frac{\mathbf{R_{max}}}{\mathbf{4}}$

- **Relation between horizontal range and maximum height**

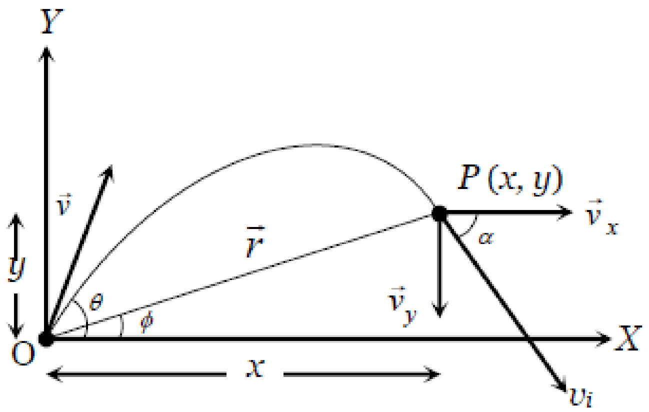
R = 4H cot θ

If θ = 45°, then R = 4H

- **Displacement of projectile**

Let the particle acquires a position P having the coordinates (x, y) just after time t from the instant of projection.

The corresponding position vector of the particle at time t is shown in the figure.



$\vec{r} = x\,\hat{i} + y\,\hat{j}$ ...(i)

The horizontal distance covered during time t given as

x = v_x t ⇒ x = u cos θ t ...(ii)

The vertical velocity of the particle at time t is given as

y = u sin θ t − $\frac{1}{2}$ gt^2 ...(iii)

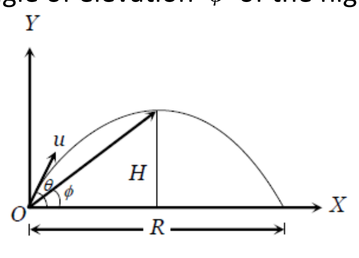
and ϕ = $\tan^{-1}$ (y/x)

Note : $\vec{\mathbf{r}}$ **= u cos θ t $\hat{\mathbf{i}}$ +** $\left(\mathbf{u \sin\theta\, t - \frac{1}{2}\, gt^2}\right)\hat{\mathbf{j}}$

r = $\sqrt{(u\cos\theta\ t)^2 + \left(u\sin\theta\ t - \frac{1}{2}\ gt^2\right)^2}$ **= ut** $\sqrt{\mathbf{1 + \left(\frac{gt}{2u}\right)^2 - \frac{gt\sin\theta}{u}}}$

$\boldsymbol{\phi}$ = $\tan^{-1}\left(\frac{u\sin\theta\ t - \frac{1}{2}gt^2}{u\cos\theta\ t}\right)$ = **tan**$^{-1}\left(\frac{\mathbf{2u\sin\theta - gt}}{\mathbf{2u\cos\theta}}\right)$

The angle of elevation ϕ of the highest point of the projectile and the angle of projection θ are related to each other as



tan ϕ = $\frac{1}{2}$ tan θ .

Motion of a projectile thrown horizontally:

Let us consider an object thrown horizontally with a velocity u from a point A, which is at a height h from the horizontal plane OX (Fig).

The object acquires the following motions simultaneously:

(i) Uniform velocity with which it is projected in the horizontal direction OX

(ii) Vertical velocity, which is non-uniform due to acceleration due to gravity.

The two velocities are independent of each other. The horizontal velocity of the object shall remain constant as no acceleration is acting in the horizontal direction. The velocity in the vertical direction shall go on changing because of acceleration due to gravity.

- **Path of a projectile**

Let the time taken by the object to reach C from A = t

Vertical distance travelled by the object in time t = s = y

From equation of motion, $s = ut + \frac{1}{2}at^2$ ...(1)

Substituting the known values in equation (1),

$y = (0)\,t + \frac{1}{2}gt^2 = \frac{1}{2}gt^2$ ...(2)

At A, the initial velocity in the horizontal direction is u.

Horizontal distance travelled by the object in time t is x.

$\therefore$ x = horizontal velocity $\times$ time = u t (or) $t = \frac{x}{u}$ ...(3)

Substituting t in equation (2), $\mathbf{y = \frac{1}{2}\,g\left(\frac{x}{u}\right)^2 = \frac{1}{2}\,g\,\frac{x^2}{u^2}}$ ...(4)

(or) $y = kx^2$

where $k = \frac{g}{2u^2}$ is a constant.

The above equation is the equation of a parabola. Thus the path taken by the projectile is a parabola.

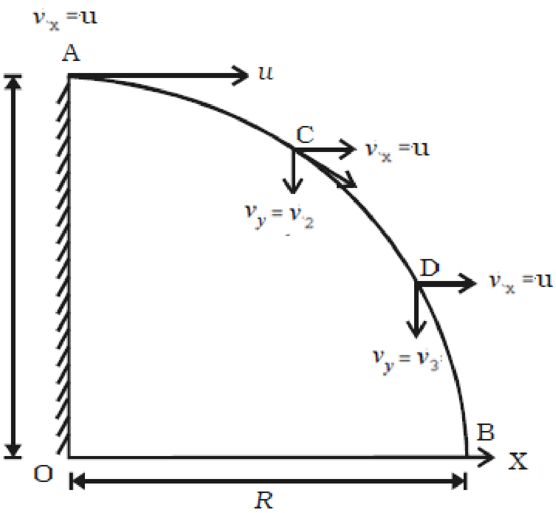


Fig Projectile projected horizontally from the top of a tower

- **Resultant velocity at C**

At an instant of time t, let the body be at C.

At A, initial vertical velocity (u_y) = 0

At C, the horizontal velocity (v_x) = u

At C, the vertical velocity = $v_y = v_2$

From equation of motion, $v_2 = u_y + g\,t$

Substituting all the known values, $v_2 = 0 + g\,t$ ...(5)

The resultant velocity at C is $\mathbf{v = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + g^2t^2}}$ ...(6)

The direction of v is given by $\mathbf{\tan\theta = \frac{v_y}{v_x} = \frac{gt}{u}}$

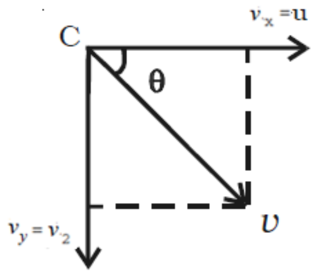


Fig Resultant velocity at any point

- **Time of flight and range**

The distance OB = R, is called as range of the projectile.

Range = horizontal velocity × time taken to reach the ground

$R = u\,t_f$ ...(8)

where t_f is the time of flight (time taken by the body to reach the ground)

At A, initial vertical component of the velocity (u_y) = 0

The vertical distance travelled by the object in time $t_f = s_y = h$

From the equations of motion $S_y = u_y t_f + \frac{1}{2} g\,t_f^2$ ...(9)

Substituting the known values in equation (9),

$h = (0)\,t_f + \frac{1}{2}g\,t_f^2$ (or) $\mathbf{t_f = \sqrt{\frac{2h}{g}}}$ = ...(10)

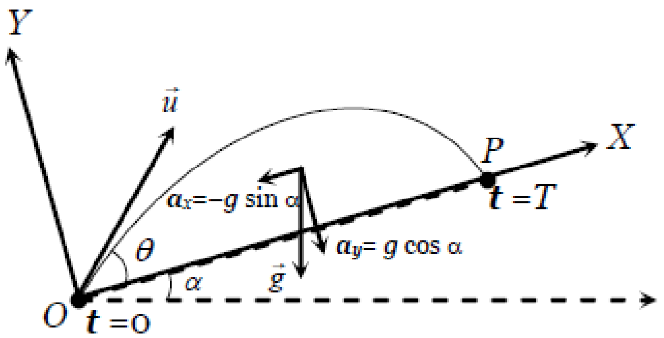
Substituting t_f in equation (8), Range $\mathbf{R = u\sqrt{\frac{2h}{g}}}$ ...(11)

- **Point to note**

! If projectiles A and B are projected horizontally with different initial velocity from same height and third particle C is dropped from same point then

(i) All three particles will take equal time to reach the ground.
(ii) Their net velocity would be different but all three particle possess same vertical component of velocity.
(iii) The trajectory of projectiles A and B will be straight line w.r.t. particle C.
! If various particles thrown with same initial velocity but in different direction then
(i) They strike the ground with same speed at different times irrespective of their initial direction of velocities.
(ii) Time would be least for particles which was thrown vertically downward.
(iii) Time would be maximum for particle A which was thrown vertically upward.

Projectile Motion on an Inclined Plane:



Let a particle be projected up with a speed u from an inclined plane which makes an angle α with the horizontal velocity of projection makes an angle θ with the inclined plane.
We have taken reference x-axis in the direction of plane.
Hence the component of initial velocity parallel and perpendicular to the plane are equal to u cosθ and u sinθ respectively i.e.
$u_{||}$ = u cosθ and $u_\perp$ = u sinθ .
The component of g along the plane is g sinα and perpendicular to the plane is g cosα as shown in the figure i.e.
$a_{||}$ = - g sinα and $a_\perp$ = g cosα
Therefore the particle decelerates at a rate of g sinα as it moves from O to P.

- **Time of flight**

We know for oblique projectile motion $T = \frac{2u\sin\theta}{g}$

or we can say $T = \frac{2u_\perp}{a_\perp}$

∴ Time of flight on an inclined plane $\mathbf{T = \frac{2u\sin\theta}{g\cos\alpha}}$

- **Maximum height**

We know for oblique projectile motion $H = \frac{u^2\sin^2\theta}{2g}$

or we can say $H = \frac{{u_\perp}^2}{2a_\perp}$

∴ Maximum height on an inclined plane $\mathbf{H = \frac{u^2\sin^2\theta}{2g\cos\alpha}}$

- **Horizontal range**

For one dimensional motion $s = ut + \frac{1}{2}at^2$

Horizontal range on an inclined plane $s = u_{||}T + \frac{1}{2}a_{||}T^2$

$R = u\cos\theta\, T - \frac{1}{2}g\sin\alpha\, T^2$

$R = u\cos\theta\left(\frac{2u\sin\theta}{g\cos\alpha}\right) - \frac{1}{2}g\sin\alpha\left(\frac{2u\sin\theta}{g\cos\alpha}\right)^2$

$\mathbf{R = \frac{2u^2}{g}\frac{\sin\theta\cos(\theta+\alpha)}{\cos^2\alpha}}$

(i) Maximum range occurs when $\theta = \frac{\pi}{4} - \frac{\alpha}{2}$

(ii) The maximum range along the inclined plane when the projectile is thrown upwards is given by

$R_{max} = \frac{u^2}{g(1+\sin\alpha)}$

(iii) The maximum range along the inclined plane when the projectile is thrown downwards is given by

$R_{min} = \frac{u^2}{g(1-\sin\alpha)}$

CIRCULAR MOTION

Circular motion is another example of motion in two dimensions. To create circular motion in a body it must be given some initial velocity and a force must then act on the body which is always directed at right angles to instantaneous velocity.

Since this force is always at right angles to the displacement due to the initial velocity therefore no work is done by the force on the particle. Hence, its kinetic energy and thus speed is unaffected. But due to simultaneous action of the force and the velocity the particle follows resultant path, which in this case is a circle.

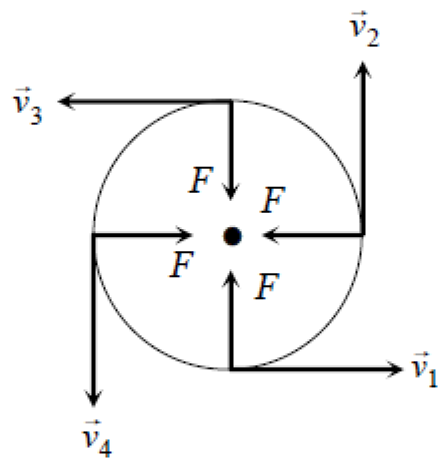
Circular motion can be classified into two types-Uniform circular motion and non-uniform circular motion.

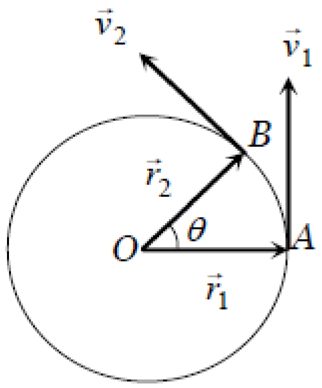
Variables of Circular Motion

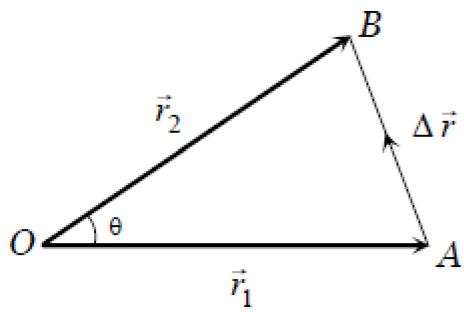
- **Displacement and distance :**

When particle moves in a circular path describing an angle θ during time t (as shown in the figure) from the position A to the position B, we see that the magnitude of the position vector $\vec{r}$ (that is equal to the radius of the circle) remains constant, i.e., $|\vec{r_1}| = |\vec{r_2}| = r$ and the direction of the position vector changes from time to time.

(i) Displacement : The change of position vector or the displacement $\Delta\vec{r}$ of the particle from position A to position B is given by referring the figure.







$\Delta\vec{r} = \vec{r_2} - \vec{r_1}$

$\Delta r = |\Delta\vec{r}| = |\vec{r_2} - \vec{r_1}| = \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos\theta}$

Putting $r_1 = r_2 = r$ we obtain

$\Delta r = \sqrt{r^2 + r^2 + 2rr\cos\theta}$

=> $\Delta r = \sqrt{2r^2(1-\cos\theta)}$

$\mathbf{\Delta r = \sqrt{2r^2\left(2\sin^2\frac{\theta}{2}\right)} = 2r\sin\frac{\theta}{2}}$

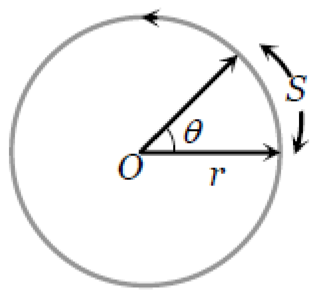
(ii) Distance: The distance covered by the particle during the time t is given as d = length of the arc AB = rθ

- **Angular displacement (θ) :**

The angle turned by a body moving on a circle from some reference line is called angular displacement.

(i) Dimension = $[M^0L^0T^0]$ (as θ = arc/radius).

(ii) Units = Radian or Degree. It is some times also specified in terms of fraction or multiple of revolution.



(iii) 2π rad = 360° = 1 Revolution

(iv) Angular displacement is a axial vector quantity. Its direction depends upon the sense of rotation of the object can be given by Right Hand Rule; which states that if the curvature of the fingers of right hand represents the sense of rotation of the object, then the thumb, held perpendicular to the curvature of the fingers, represents the direction of angular displacement vector.

(v) Relation between linear displacement and angular displacement

$\vec{s} = \vec{\theta} \times \vec{r}$ or $s = r\theta$.

- **Angular velocity (ω) :**

Angular velocity of an object in circular motion is defined as the time rate of change of its angular displacement.

(i) Angular velocity $\omega = \frac{\text{angle traced}}{\text{time taken}} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$

$\omega = \mathbf{\frac{d\theta}{dt}}$

(ii) Dimension : $[M^0 L^0 T^{-1}]$

(iii) Units : Radians per second (rad. s^{-1}) or Degree per second.

(iv) Angular velocity is an axial vector.

Its direction is the same as that of $\Delta\theta$. For anticlockwise rotation of the point object on the circular path, the direction of ω, according to Right hand rule is along the axis of circular path directed upwards. For clockwise rotation of the point object on the circular path, the direction of ω is along the axis of circular path directed downwards.

! Note : It is important to note that nothing actually moves in the direction of the angular velocity vector $\vec{\omega}$. The direction of $\vec{\omega}$ simply represents that the rotational motion is taking place in a plane perpendicular to it.

(v) Relation between angular velocity and linear velocity.

$\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$

(vi) For uniform circular motion co remains constant where as for non-uniform motion ω varies with respect to time.

- **Change in velocity :**

We want to know the magnitude and direction of the change in velocity of the particle which is performing uniform circular motion as it moves from A to B during time t as shown in figure. The change in velocity vector is given as

$\Delta\vec{v} = \vec{v_2} - \vec{v_1}$

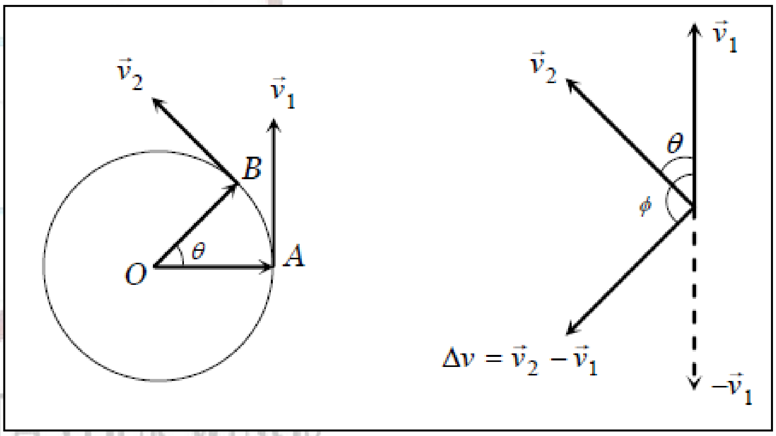
or $\Delta v = |\Delta\vec{v}| = |\vec{v} - \vec{v}| = \sqrt{{v_1}^2 + {v_2}^2 + 2v_1v_2\cos\theta}$

For uniform circular motion $v_1 = v_2 = v$

So, $\Delta v = \sqrt{v^2 + v^2 + 2vv\cos\theta}$

=> $\Delta v = \sqrt{2v^2(1 - \cos\theta)}$

$\mathbf{\Delta v} = \sqrt{2v^2\left(2\sin^2\frac{\theta}{2}\right)} = \mathbf{2v \sin\frac{\theta}{2}}$



The direction of is shown in figure that can be given as

$\phi = \frac{180° - \theta}{2} = (90^0 - \frac{\theta}{2})$

! Relation between linear velocity and angular velocity. In vector form $\vec{v} = \vec{\omega} \times \vec{r}$

- **Time period (T) :**

In circular motion, the time period is defined as the time taken by the object to complete one revolution on its circular path. (i) Units: second. (ii) Dimension: $[M^0L^0T]$

- **Frequency (n) :**

In circular motion, the frequency is defined as the number of revolutions completed by the object on its circular path in a unit time.

(i) Units: s^{-1} or hertz (Hz).

(ii) Dimension: $[M^0L^0T^{-1}]$

Note :

! Relation between time period and frequency: If n is the frequency of revolution of an object in circular motion, then the object completes n revolutions in 1 second. Therefore, the object will complete one revolution in 1/n second.

$\therefore$ T = 1/n

! Relation between angular velocity, frequency and time period : Consider a point object describing a uniform circular motion with frequency n and time period T. When the object completes one revolution, the angle traced at its axis of circular motion is 2 π radians. It means,

when time t = T, θ = 2π radians. Hence, angular velocity

$\omega = \frac{\theta}{t} = \frac{2\pi}{T} = \mathbf{2\pi n}$ ($\because$T = 1/n)

- **Angular acceleration (α) :**

Angular acceleration of an object in circular motion is defined as the time rate of change of its angular velocity.

(i) $\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt}$

(ii) Units : rad s^{-2}

(iii) Dimension : $[M^0 L^0 T^{-2}]$

(iv) Relation between linear acceleration and angular acceleration $\vec{a} = \vec{\alpha} \times \vec{r}$

(v) For uniform circular motion since θ is constant so $\alpha = \frac{d\omega}{dt} = 0$.

(vi) For non-uniform circular motion $\alpha \neq 0$.

Note :

! Relation between linear (tangential) acceleration and angular acceleration $\vec{a} = \vec{\alpha} \times \vec{r}$

! For uniform circular motion angular acceleration is zero, so tangential acceleration also is equal to zero.

! For non-uniform circular motion a $\neq$ 0 (because $\alpha \neq 0$).

Centripetal Acceleration

(1) Acceleration acting on the object undergoing uniform circular motion is called centripetal acceleration.

(2) It always acts on the object along the radius towards the centre of the circular path.

(3) Magnitude of centripetal acceleration $\alpha = \frac{v^2}{r} = \omega^2 r = 4\pi n^2 r = \frac{4\pi^2}{T^2} r$.

(4) Direction of centripetal acceleration: It is always the same as that of $\Delta\vec{v}$. When Δt decreases, $\Delta\theta$ also decreases. Due to which $\Delta\vec{v}$ becomes more and more perpendicular to $\vec{v}$. When $\Delta t \to 0$, $\Delta\vec{v}$ becomes perpendicular to the velocity vector. As the velocity vector of the particle at an instant acts along the tangent to the circular path, therefore $\Delta\vec{v}$ and hence the centripetal acceleration vector acts along the radius of the circular path at that point and is directed towards the centre of the circular path.

Equation of circular motion:

(i) $\omega = \omega_0 + \alpha t$ (ii) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ (iii) $\omega^2 = \omega_0^2 + 2\alpha\theta$

RELATIVE MOTION

! Position of B relative to A is represented as $\vec{r_{BA}} = \vec{r_B} - \vec{r_A}$

Thus $\vec{r_{BC}} - \vec{r_{AC}} = (\vec{r_B} - \vec{r_C}) - (\vec{r_A} - \vec{r_C}) = \vec{r_B} - \vec{r_A} = \vec{r_{BA}}$

So, $\vec{r_{BA}} = \vec{r_{BC}} - \vec{r_{AC}}$

! Position of A relative to B is represented as $\vec{r_{AB}} = \vec{r_A} - \vec{r_B}$

So, $\vec{r_{AB}} = -\vec{r_{BA}}$

! Velocity of B relative to A is represented as $\vec{V_{BA}} = \vec{V_B} - \vec{V_A}$ or $\vec{V_{BC}} - \vec{V_{AC}}$

! Velocity of A relative to B is represented as $\vec{V_{AB}} = \vec{V_A} - \vec{V_B} = -\vec{V_{BA}}$

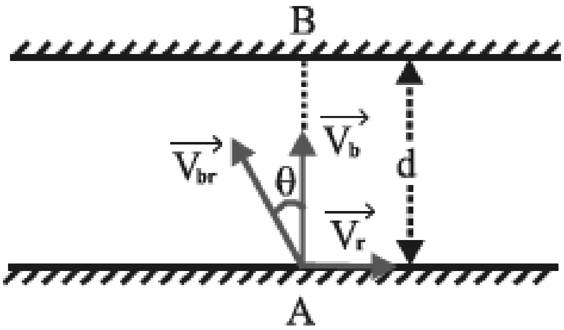
RIVER-BOAT PROBLEMS

A boatman starts from point A on one bank of a river with velocity $\vec{V_{br}}$ in the direction shown in figure. River is flowing along positive X-direction with velocity $\vec{V_r}$. Width of the river is d.

$\vec{V_r}$ = absolute velocity of river/ velocity of river relative to ground ($\vec{V_{rg}}$),

$\vec{V_{br}}$ = velocity of boat with respect to river or the velocity of boatman with which he steers,

$\vec{V_b}$ = absolute or actual velocity of boat / velocity of boat relative to ground ($\vec{V_{bg}}$).



We know that $\vec{V_{bg}} = \vec{V_{br}} - \vec{V_{gr}}$

$\quad\quad => \overrightarrow{V_{bg}} = \overrightarrow{V_{br}} + \overrightarrow{V_{rg}}$

Thus, $\quad \overrightarrow{V_b} = \overrightarrow{V_{br}} + \overrightarrow{V_r}$

Therefore, $V_{b_x} = V_{br_x} + V_{r_x} = -V_{br}\sin\theta + V_r = V_r - V_{br}\sin\theta$

and $V_{b_y} = V_{br_y} + V_{r_y} = V_{br}\cos\theta + 0 = V_{br}\cos\theta$

So,

Time taken by the boat to cross the river is: $t = \frac{d}{V_{b_y}} = \frac{d}{V_{br}\cos\theta}$

or, $\mathbf{t = \frac{d}{V_{br}\cos\theta}}$ .....(i)

Displacement along X-axis when boat reaches on the bank (also called drift) is

$X = V_{b_x} t = (V_r - V_{br}\sin\theta)\left(\frac{d}{V_{br}\cos\theta}\right)$

or, $\mathbf{X = (V_r - V_{br}\sin\theta)\left(\frac{d}{V_{br}\cos\theta}\right)}$ .....(ii)

! Condition for crossing river in **shortest interval**:

From equation (i) we can see that time (t) will be minimum when **θ = 0°** i.e. the boatman shoud steer his boat perpendicular to the river current.

! Condition for reach point B, a point just opposite from starting point A (**shortest distance**):

From equation (i) we can see that shift (t) will be minimum (i.e. X = 0) when

$V_r - V_{br}\sin\theta = 0 \;=>\; \sin\theta = \left(\frac{V_r}{V_{br}}\right)$ or, $\theta = \sin^{-1}\left(\frac{V_r}{V_{br}}\right)$

Hence to reach point B the boatman should row at an angle $\boldsymbol{\theta = \sin^{-1}\left(\frac{V_r}{V_{br}}\right)}$ upstream from AB

<u>RELATIVE VELOCITY OF RAIN W.R.T. THE MOVING MAN</u>

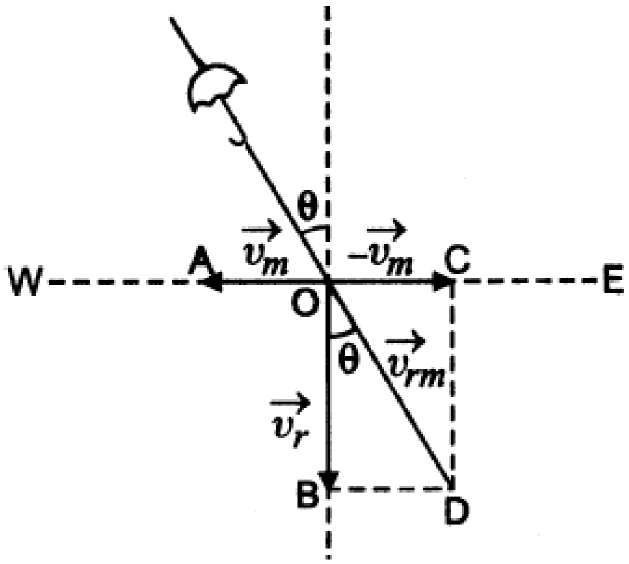
Let us consider a man walking wets with velocity, $\overrightarrow{V_m}$ represented by $\overline{OA}$. Let the rain be falling vertically downwards with velocity $\overrightarrow{V_r}$, represented by $\overline{OB}$.

The relative velocity of rain with respect to man (i.e. $\overrightarrow{V_{rm}}$)

$\overrightarrow{V_{rm}} = \overrightarrow{V_r} - \overrightarrow{V_m}$

Now the relative velocity of rain with respect to man will be the resultant velocity of $\overrightarrow{V_r}$ ($=\overrightarrow{OB}$) and $-\overrightarrow{V_m}$ ($=\overrightarrow{OC}$), which will be represented by diagonal of rectangle OBDC.

$\therefore V_{rm} = \sqrt{{V_r}^2 + {V_m}^2 + 2V_rV_m\cos 90°} = \sqrt{{V_r}^2 + {V_m}^2}$



If θ is the angle which $\overrightarrow{V_{rm}}$ makes with the vertical direction then

$\tan\theta = \frac{BD}{OB} = \left(\frac{V_m}{V_r}\right)$ or, $\boldsymbol{\theta = \tan^{-1}\left(\frac{V_m}{V_r}\right)}$

Here, angle θ is from vertical towards west and is written as θ, west of vertical.

<u>Note:</u> In the above problem if the man wants to protect himself from the rain, he should hold his umbrella in the direction of relative velocity of rain with respect to man i.e. the umbrella should be held making an angle θ (= $\tan^{-1}V_m/V_r$) west of vertical.