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## ABSTRACT

In this article we prove that Kontsevich’s category  $\mathrm{NC}_{\mathrm{num}}(k)_F$  of noncommutative numerical motives is equivalent to the one constructed by the authors in [Marcolli and Tabuada, *Noncommutative motives, numerical equivalence, and semisimplicity*, Amer. J. Math., to appear, available at arXiv:1105.2950]. As a consequence, we conclude that  $\mathrm{NC}_{\mathrm{num}}(k)_F$  is abelian semi-simple as conjectured by Kontsevich.

## 1. Introduction and statement of results

Over the past two decades Bondal, Drinfeld, Kaledin, Kapranov, Kontsevich, Van den Bergh, and others have been promoting a broad noncommutative (algebraic) geometry program where ‘geometry’ is performed directly on dg categories; see [BK89, BK90, BV03, Dri02, Dri04, Kal10, Kon98, Kon05, Kon09, Kon10]. Among many developments, Kontsevich introduced a rigid symmetric monoidal category  $\mathrm{NC}_{\mathrm{num}}(k)_F$  of noncommutative numerical motives (over a ground field  $k$  and with coefficients in a field  $F$ ); consult § 4 for details. The key ingredient in his approach is the existence of a well-behaved bilinear form on the Grothendieck group of each smooth and proper dg category.

Recently, the authors introduced in [MT11a] an alternative rigid symmetric monoidal category  $\mathrm{NNum}(k)_F$  of noncommutative numerical motives; see § 5. In contrast to Kontsevich’s approach, the authors used Hochschild homology to formalize the ‘intersection number’ in the noncommutative world.

The main result of this article is the following theorem.

**THEOREM 1.1.** *The categories  $\mathrm{NC}_{\mathrm{num}}(k)_F$  and  $\mathrm{NNum}(k)_F$  are equivalent (as rigid symmetric monoidal categories).*

By combining Theorem 1.1 with [MT11a, Theorem 1.9] and [MT11b, Theorem 4.6], we then obtain the following result.

**THEOREM 1.2.** *Let  $k$  and  $F$  be fields of the same characteristic. Then the category  $\mathrm{NC}_{\mathrm{num}}(k)_F$  is abelian semi-simple.*

In Theorem 1.2,  $k$  and  $F$  can be of characteristic zero or of positive characteristic.

Assuming several (polarization) conjectures, Kontsevich conjectured Theorem 1.2 in the particular case where  $F = \mathbb{Q}$  and  $k$  is of characteristic zero; see [Kon05]. We observe that

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Kontsevich’s beautiful insight not only holds much more generally but, moreover, does not require the assumption of any (polarization) conjecture.

**Notation**

We will work over a (fixed) ground field  $k$ . The field of coefficients will be denoted by  $F$ . Let  $(\mathcal{C}(k), \otimes_k, k)$  be the symmetric monoidal category of complexes of  $k$ -vector spaces. We will use *cohomological* notation, i.e. the differential increases the degree.

**2. Differential graded categories**

A *differential graded (dg) category*  $\mathcal{A}$  (over  $k$ ) is a category enriched over  $\mathcal{C}(k)$ , i.e. the morphism sets  $\mathcal{A}(x, y)$  are complexes of  $k$ -vector spaces and the composition operation satisfies the Leibniz rule  $d(f \circ g) = d(f) \circ g + (-1)^{\deg(f)} f \circ d(g)$ ; consult Keller’s ICM address [Kel06] for further details.

The *opposite* dg category  $\mathcal{A}^{\text{op}}$  has the same objects as  $\mathcal{A}$ , with complexes of morphisms given by  $\mathcal{A}^{\text{op}}(x, y) := \mathcal{A}(y, x)$ . The  $k$ -linear category  $H^0(\mathcal{A})$  has the same objects as  $\mathcal{A}$  and morphisms given by  $H^0(\mathcal{A})(x, y) := H^0\mathcal{A}(x, y)$ , where  $H^0$  denotes 0th cohomology. A *right dg  $\mathcal{A}$ -module*  $M$  (or simply an  $\mathcal{A}$ -module) is a dg functor  $M : \mathcal{A}^{\text{op}} \rightarrow \mathcal{C}_{\text{dg}}(k)$  with values in the dg category  $\mathcal{C}_{\text{dg}}(k)$  of complexes of  $k$ -vector spaces. We will denote by  $\mathcal{C}(\mathcal{A})$  the category of  $\mathcal{A}$ -modules. Recall from [Kel06, §3] that  $\mathcal{C}(\mathcal{A})$  carries a *projective* model structure. Moreover, the differential graded structure of  $\mathcal{C}_{\text{dg}}(k)$  naturally turns  $\mathcal{C}(\mathcal{A})$  into a dg category  $\mathcal{C}_{\text{dg}}(\mathcal{A})$ . The dg category  $\mathcal{C}_{\text{dg}}(\mathcal{A})$  endowed with the projective model structure is a  $\mathcal{C}(k)$ -*model category* in the sense of [Hov99, Definition 4.2.18]. Let  $\mathcal{D}(\mathcal{A})$  be the *derived category* of  $\mathcal{A}$ , i.e. the localization of  $\mathcal{C}(\mathcal{A})$  with respect to the class of weak equivalences. Its full triangulated subcategory of compact objects (i.e. those  $\mathcal{A}$ -modules  $M$  such that the functor  $\text{Hom}_{\mathcal{D}(\mathcal{A})}(M, -)$  preserves arbitrary sums; see [Nee01, Definition 4.2.7]) will be denoted by  $\mathcal{D}_c(\mathcal{A})$ .

*Notation 2.1.* We will denote by  $\widehat{\mathcal{A}}_{\text{pe}}$  the full dg subcategory of  $\mathcal{C}_{\text{dg}}(\mathcal{A})$  consisting of those cofibrant  $\mathcal{A}$ -modules which become compact in  $\mathcal{D}(\mathcal{A})$ . Since all the objects in  $\mathcal{C}(\mathcal{A})$  are fibrant and  $\mathcal{C}_{\text{dg}}(\mathcal{A})$  is a  $\mathcal{C}(k)$ -model category, we have the natural isomorphisms of  $k$ -vector spaces

$$H^i \widehat{\mathcal{A}}_{\text{pe}}(M, N) \simeq \text{Hom}_{\mathcal{D}_c(\mathcal{A})}(M, N[-i]), \quad i \in \mathbb{Z}. \tag{1}$$

As any  $\mathcal{A}$ -module admits a (functorial) cofibrant approximation, we obtain a natural equivalence of triangulated categories  $H^0(\widehat{\mathcal{A}}_{\text{pe}}) \simeq \mathcal{D}_c(\mathcal{A})$ .

The *tensor product*  $\mathcal{A} \otimes_k \mathcal{B}$  of two dg categories is defined as follows: the set of objects is the Cartesian product of the sets of objects, and the complexes of morphisms are given by  $(\mathcal{A} \otimes_k \mathcal{B})((x, x'), (y, y')) := \mathcal{A}(x, y) \otimes_k \mathcal{B}(x', y')$ . Note that the tensor product of any two dg categories is automatically derived since we are working over a ground field  $k$ . Finally, a  $\mathcal{A}$ - $\mathcal{B}$ -*bimodule*  $X$  is a dg functor  $X : \mathcal{A} \otimes_k \mathcal{B}^{\text{op}} \rightarrow \mathcal{C}_{\text{dg}}(k)$  or, in other words, a  $(\mathcal{A}^{\text{op}} \otimes_k \mathcal{B})$ -module.

**DEFINITION 2.2** (Kontsevich [Kon05, Kon98]). A dg category  $\mathcal{A}$  is *smooth* if the  $\mathcal{A}$ - $\mathcal{A}$ -bimodule

$$\mathcal{A}(-, -) : \mathcal{A} \otimes_k \mathcal{A}^{\text{op}} \longrightarrow \mathcal{C}_{\text{dg}}(k), \quad (x, y) \mapsto \mathcal{A}(y, x) \tag{2}$$

belongs to  $\mathcal{D}_c(\mathcal{A}^{\text{op}} \otimes_k \mathcal{A})$ , and it is *proper* if for each ordered pair of objects  $(x, y)$  we have

$$\sum_i \dim H^i \mathcal{A}(x, y) < \infty.$$

**3. Noncommutative Chow motives**

The rigid symmetric monoidal category  $\text{NChow}(k)_F$  of *noncommutative Chow motives*, which we now recall, was originally constructed (over a commutative ground ring) in [Tab11, Tab05]; see [Tab10, § 4] for a survey. Let  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  be dg categories. The derived tensor product of bimodules (see [Dri04, § 14.3]) gives rise to a bi-triangulated functor

$$\mathcal{D}_c(\mathcal{A}^{\text{op}} \otimes_k \mathcal{B}) \times \mathcal{D}_c(\mathcal{B}^{\text{op}} \otimes_k \mathcal{C}) \rightarrow \mathcal{D}_c(\mathcal{A}^{\text{op}} \otimes_k \mathcal{C}), \quad (X, Y) \mapsto X \otimes_{\mathcal{B}}^{\mathbb{L}} Y.$$

By applying it to the  $F$ -linearized Grothendieck group functor  $K_0(-)_F$ , we then obtain a well-defined bilinear pairing

$$K_0(\mathcal{A}^{\text{op}} \otimes_k \mathcal{B})_F \times K_0(\mathcal{B}^{\text{op}} \otimes_k \mathcal{C})_F \rightarrow K_0(\mathcal{A}^{\text{op}} \otimes_k \mathcal{C})_F, \quad ([X], [Y]) \mapsto [X \otimes_{\mathcal{B}}^{\mathbb{L}} Y].$$

The category  $\text{NChow}(k)_F$  is defined as the pseudo-abelian envelope of the category whose objects are the smooth and proper dg categories, whose morphisms from  $\mathcal{A}$  to  $\mathcal{B}$  are given by the  $F$ -linearized Grothendieck group  $K_0(\mathcal{A}^{\text{op}} \otimes_k \mathcal{B})_F$ , and whose composition operation is given by the above pairing.

In analogy with the commutative world, the morphisms of  $\text{NChow}(k)_F$  are called *correspondences*. Finally, the symmetric monoidal structure on  $\text{NChow}(k)_F$  is induced by the tensor product of dg categories.

**4. Kontsevich's approach**

In this section we recall and enhance Kontsevich's construction of the category  $\text{NC}_{\text{num}}(k)_F$  of noncommutative numerical motives; see [Kon05]. Let  $\mathcal{A}$  be a proper dg category. By construction, the dg category  $\widehat{\mathcal{A}}_{\text{pe}}$  is also proper and we have a natural equivalence of triangulated categories  $\text{H}^0(\widehat{\mathcal{A}}_{\text{pe}}) \simeq \mathcal{D}_c(\mathcal{A})$ . Hence, thanks to the natural isomorphisms (1), we can consider the assignment

$$\text{obj } \mathcal{D}_c(\mathcal{A}) \times \text{obj } \mathcal{D}_c(\mathcal{A}) \longrightarrow \mathbb{Z}, \quad (M, N) \mapsto \chi(M, N),$$

where  $\chi(M, N)$  is the integer

$$\sum_i (-1)^i \dim \text{Hom}_{\mathcal{D}_c(\mathcal{A})}(M, N[-i]).$$

Since the Grothendieck group  $K_0(\mathcal{A})$  of  $\mathcal{A}$  is the Grothendieck group of the triangulated category  $\mathcal{D}_c(\mathcal{A})$ , a simple verification shows us that the above assignment gives rise to a well-defined bilinear form  $K_0(\mathcal{A}) \otimes_{\mathbb{Z}} K_0(\mathcal{A}) \rightarrow \mathbb{Z}$ . By tensoring it with  $F$ , we then obtain

$$\chi(-, -) : K_0(\mathcal{A})_F \otimes_F K_0(\mathcal{A})_F \longrightarrow F. \tag{3}$$

The bilinear form (3) is in general *neither* symmetric *nor* anti-symmetric. For example, let  $\mathcal{A}$  be the dg enhancement  $\mathcal{D}_{\text{dg}}^{\text{perf}}(\mathbb{P}^1)$  of the derived category  $\mathcal{D}^{\text{perf}}(\mathbb{P}^1)$  of perfect complexes of  $\mathcal{O}_{\mathbb{P}^1}$ -modules; see [LO10] for the uniqueness of this enhancement. By construction,  $\mathcal{D}_c(\mathcal{D}_{\text{dg}}^{\text{perf}}(\mathbb{P}^1)) \simeq \text{H}^0(\mathcal{D}_{\text{dg}}^{\text{perf}}(\mathbb{P}^1)) \simeq \mathcal{D}^{\text{perf}}(\mathbb{P}^1)$ , and thanks to the work of Beilinson [Bei78] we have

$$\text{Hom}_{\mathcal{D}^{\text{perf}}(\mathbb{P}^1)}(\mathcal{O}, \mathcal{O}(1)[-i]) \simeq \begin{cases} k \oplus k & \text{if } i = 0, \\ 0 & \text{if } i \neq 0 \end{cases}$$

and  $\text{Hom}_{\mathcal{D}^{\text{perf}}(\mathbb{P}^1)}(\mathcal{O}(1), \mathcal{O}[-i]) = 0$  for all  $i \in \mathbb{Z}$ . Hence,  $\chi([\mathcal{O}], [\mathcal{O}(1)]) = 2$  while  $\chi([\mathcal{O}(1)], [\mathcal{O}]) = 0$ . Now, let us denote by

$$\begin{aligned} \text{Ker}_L(\chi) &:= \{\underline{M} \in K_0(\mathcal{A})_F \mid \chi(\underline{M}, \underline{N}) = 0 \text{ for all } \underline{N} \in K_0(\mathcal{A})_F\}, \\ \text{Ker}_R(\chi) &:= \{\underline{N} \in K_0(\mathcal{A})_F \mid \chi(\underline{M}, \underline{N}) = 0 \text{ for all } \underline{M} \in K_0(\mathcal{A})_F\} \end{aligned}$$

the left and right kernels of the above bilinear form (3). Since (3) is in general not symmetric (nor anti-symmetric), it is expected that  $\text{Ker}_L(\chi) \neq \text{Ker}_R(\chi)$  in some cases, although the authors do not know of an example where this nonequality holds. On the other hand, when  $\mathcal{A}$  is in addition smooth, we will prove in Theorem 4.3 that  $\text{Ker}_L(\chi) = \text{Ker}_R(\chi)$ . In order to prove this result, let us start by recalling Bondal and Kapranov’s notion of a Serre functor. Let  $\mathcal{T}$  be a  $k$ -linear Ext-finite triangulated category, i.e.  $\sum_i \dim \text{Hom}_{\mathcal{T}}(M, N[-i]) < \infty$  for any two objects  $M$  and  $N$  in  $\mathcal{T}$ . Following Bondal and Kapranov [BK89, § 3], a Serre functor  $S : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$  is an autoequivalence together with bifunctorial isomorphisms

$$\text{Hom}_{\mathcal{T}}(M, N) \simeq \text{Hom}_{\mathcal{T}}(N, S(M))^*, \tag{4}$$

where  $(-)^*$  stands for the  $k$ -duality functor. Whenever a Serre functor exists, it is unique up to isomorphism.

**THEOREM 4.1.** *Let  $\mathcal{A}$  be a smooth and proper dg category. Then the triangulated category  $\mathcal{D}_c(\mathcal{A})$  admits a Serre functor.*

*Proof.* Note first that the properness of  $\widehat{\mathcal{A}}_{\text{pe}}$ , the equivalence of categories  $\text{H}^0(\widehat{\mathcal{A}}_{\text{pe}}) \simeq \mathcal{D}_c(\mathcal{A})$ , and the natural isomorphisms (1) imply that  $\mathcal{D}_c(\mathcal{A})$  is Ext-finite. By [BK89, Corollary 3.5], it then suffices to show that  $\mathcal{D}_c(\mathcal{A})$  is saturated in the sense of [BK89, Definition 2.5]. By combining [CT12, Proposition 4.10] with [Kel06, Theorem 4.12], we observe that every dg category  $\mathcal{A}$  is dg Morita-equivalent to a dg algebra  $A$ . Hence, without loss of generality, we may replace  $\mathcal{A}$  by  $A$ . The fact that  $\mathcal{D}_c(A)$  is saturated is now the content of [Shk07, Theorem 3.1].  $\square$

**LEMMA 4.2.** *Let  $\mathcal{A}$  be a smooth and proper dg category and let  $M, N \in \mathcal{D}_c(\mathcal{A})$ . Then we have the equalities*

$$\chi(M, N) = \chi(N, S(M)) = \chi(S^{-1}(N), M),$$

where  $S$  is the Serre functor given by Theorem 4.1.

*Proof.* Consider the following sequence of equalities:

$$\begin{aligned} \chi(M, N) &= \sum_i (-1)^i \dim \text{Hom}_{\mathcal{D}_c(\mathcal{A})}(M, N[-i]) \\ &= \sum_i (-1)^i \dim \text{Hom}_{\mathcal{D}_c(\mathcal{A})}(N[-i], S(M)) \end{aligned} \tag{5}$$

$$= \sum_i (-1)^i \dim \text{Hom}_{\mathcal{D}_c(\mathcal{A})}(N, S(M)[i]) \tag{6}$$

$$= \chi(N, S(M)). \tag{7}$$

Equality (5) follows from the bifunctorial isomorphisms (4) and from the fact that a finite-dimensional  $k$ -vector space and its  $k$ -dual have the same dimension. Equality (6) follows from the fact that the suspension functor is an autoequivalence of the triangulated category  $\mathcal{D}_c(\mathcal{A})$ . Finally, equality (7) follows from a reordering of the finite sum which does not alter the sign of each term. This shows the equality  $\chi(M, N) = \chi(N, S(M))$ . The equality  $\chi(M, N) = \chi(S^{-1}(N), M)$

is proven in a similar way: simply use

$$\text{Hom}_{\mathcal{T}}(M, N) \simeq \text{Hom}_{\mathcal{T}}(S^{-1}(N), M)^*$$

instead of the bifunctorial isomorphisms (4). □

**THEOREM 4.3.** *Let  $\mathcal{A}$  be a smooth and proper dg category. Then  $\text{Ker}_L(\chi) = \text{Ker}_R(\chi)$ ; the resulting well-defined subspace of  $K_0(\mathcal{A})_F$  will be denoted by  $\text{Ker}(\chi)$ .*

*Proof.* We start by proving the inclusion  $\text{Ker}_L(\chi) \subseteq \text{Ker}_R(\chi)$ . Let  $\underline{M}$  be an element of  $\text{Ker}_L(\chi)$ . Since  $K_0(\mathcal{A})_F$  is generated by the elements of shape  $[N]$ , with  $N \in \mathcal{D}_c(\mathcal{A})$ , it suffices to show that  $\chi([N], \underline{M}) = 0$  for every such  $N$ . Note that  $\underline{M}$  can be written as  $[a_1M_1 + \cdots + a_nM_n]$  with  $a_1, \dots, a_n \in F$  and  $M_1, \dots, M_n \in \mathcal{D}_c(\mathcal{A})$ . We then have the equalities

$$\begin{aligned} \chi([N], \underline{M}) &= a_1\chi(N, M_1) + \cdots + a_n\chi(N, M_n) \\ &= a_1\chi(M_1, S(N)) + \cdots + a_n\chi(M_n, S(N)) \\ &= \chi(\underline{M}, [S(N)]), \end{aligned} \tag{8}$$

where (8) follows from Lemma 4.2. Finally, since by hypothesis  $\underline{M}$  belongs to  $\text{Ker}_L(\chi)$ , we have  $\chi(\underline{M}, [S(N)]) = 0$  and so conclude that  $\chi([N], \underline{M}) = 0$ . Using the equality  $\chi(M, N) = \chi(S^{-1}(N), M)$  of Lemma 4.2, the proof of the inclusion  $\text{Ker}_R(\chi) \subseteq \text{Ker}_L(\chi)$  is similar. □

Let  $\mathcal{A}$  and  $\mathcal{B}$  be two smooth and proper dg categories. Recall that, by definition,

$$\text{Hom}_{\text{NChow}(k)_F}(\mathcal{A}, \mathcal{B}) = K_0(\mathcal{A}^{\text{op}} \otimes_k \mathcal{B})_F. \tag{9}$$

Since smooth and proper dg categories are stable under tensor product (see [CT12, §4]), the above bilinear form (3) (applied to  $\mathcal{A} = \mathcal{A}^{\text{op}} \otimes_k \mathcal{B}$ ) gives rise to

$$\chi(-, -) : \text{Hom}_{\text{NChow}(k)_F}(\mathcal{A}, \mathcal{B}) \otimes_F \text{Hom}_{\text{NChow}(k)_F}(\mathcal{A}, \mathcal{B}) \longrightarrow F.$$

By Theorem 4.3 we then obtain a well-defined kernel  $\text{Ker}(\chi) \subset \text{Hom}_{\text{NChow}(k)_F}(\mathcal{A}, \mathcal{B})$ . These kernels (one for each ordered pair of smooth and proper dg categories) assemble themselves into a  $\otimes$ -ideal. Moreover, this  $\otimes$ -ideal extends naturally to the pseudo-abelian envelope, giving rise to a well-defined  $\otimes$ -ideal  $\text{Ker}(\chi)$  on the category  $\text{NChow}(k)_F$ .

**DEFINITION 4.4** (Kontsevich [Kon05]). The category  $\text{NC}_{\text{num}}(k)_F$  of *noncommutative numerical motives* (over  $k$  and with coefficients in  $F$ ) is the pseudo-abelian envelope of the quotient category  $\text{NChow}(k)_F / \text{Ker}(\chi)$ .

*Remark 4.5.* The fact that  $\text{Ker}(\chi)$  is a well-defined  $\otimes$ -ideal of  $\text{NChow}(k)_F$  will become clear(er) after the proof of Theorem 1.1.

### 5. Alternative approach

The authors introduced in [MT11a] an alternative category  $\text{NNum}(k)_F$  of noncommutative numerical motives. Let  $\mathcal{A}$  and  $\mathcal{B}$  be two smooth and proper dg categories, and let  $\underline{X} = [\sum_i a_i X_i]$  and  $\underline{Y} = [\sum_j b_j Y_j]$  be two correspondences. Recall that the  $X_i$  are  $\mathcal{A}$ - $\mathcal{B}$ -bimodules, the  $Y_j$  are  $\mathcal{B}$ - $\mathcal{A}$ -bimodules, and the sums are indexed by a finite set. The *intersection number*  $\langle \underline{X} \cdot \underline{Y} \rangle$  of  $\underline{X}$  with  $\underline{Y}$  is given by the formula

$$\sum_{i,j,n} (-1)^n a_i \cdot b_j \cdot \dim HH_n(\mathcal{A}, X_i \otimes_{\mathcal{B}}^{\mathbb{L}} Y_j) \in F,$$

where  $HH_n(\mathcal{A}, X_i \otimes_{\mathbb{L}}^{\mathbb{L}} Y_j)$  denotes the  $n$ th Hochschild homology group of  $\mathcal{A}$  with coefficients in the  $\mathcal{A}$ - $\mathcal{A}$ -bimodule  $X_i \otimes_{\mathbb{L}}^{\mathbb{L}} Y_j$ . This procedure gives rise to a well-defined bilinear pairing

$$\langle - \cdot - \rangle : \text{Hom}_{\text{NChow}(k)_F}(\mathcal{A}, \mathcal{B}) \otimes_F \text{Hom}_{\text{NChow}(k)_F}(\mathcal{B}, \mathcal{A}) \longrightarrow F. \tag{10}$$

As explained in [MT11a, Proposition 4.3], the intersection number  $\langle \underline{X} \cdot \underline{Y} \rangle$  agrees with the categorical trace of the composed correspondence  $\mathcal{A} \xrightarrow{\underline{X}} \mathcal{B} \xrightarrow{\underline{Y}} \mathcal{A}$  in the rigid symmetric monoidal category  $\text{NChow}(k)_F$ . By standard properties of the categorical trace (see [AK02a, (7.2)]), we then have the equality  $\langle \underline{X} \cdot \underline{Y} \rangle = \langle \underline{Y} \cdot \underline{X} \rangle$ , where the latter pairing is similar to (10) with  $\mathcal{A}$  and  $\mathcal{B}$  interchanged.

A correspondence  $\underline{X}$  is said to be *numerically equivalent to zero* if for every correspondence  $\underline{Y}$  the intersection number  $\langle \underline{X} \cdot \underline{Y} \rangle$  is zero. As proved in [MT11a, Theorem 1.5], the correspondences which are numerically equivalent to zero form a  $\otimes$ -ideal  $\mathcal{N}$  of the category  $\text{NChow}(k)_F$ . The *category of noncommutative numerical motives*  $\text{NNum}(k)_F$  is then defined as the pseudo-abelian envelope of the quotient category  $\text{NChow}(k)_F/\mathcal{N}$ .

### 6. Proof of Theorem 1.1

The proof will consist of showing that the  $\otimes$ -ideals  $\text{Ker}(\chi)$  and  $\mathcal{N}$ , described in §§ 4 and 5, are exactly the same. As explained in the proof of Theorem 4.1, working with smooth and proper dg categories is equivalent to working with smooth and proper dg algebras. In what follows, we will use the latter approach.

Let  $A$  be a dg algebra and  $M$  a right dg  $A$ -module. We will denote by  $D(M)$  its *dual*, i.e. the left dg  $A$ -module  $\mathcal{C}_{\text{dg}}(A)(M, A)$ . This procedure is (contravariantly) functorial in  $M$  and thus gives rise to a triangulated functor  $\mathcal{D}(A) \rightarrow \mathcal{D}(A^{\text{op}})^{\text{op}}$  which restricts to an equivalence  $\mathcal{D}_c(A) \xrightarrow{\sim} \mathcal{D}_c(A^{\text{op}})^{\text{op}}$ . Since the Grothendieck group of a triangulated category is canonically isomorphic to the one of the opposite category, we obtain then an induced isomorphism  $K_0(A)_F \xrightarrow{\sim} K_0(A^{\text{op}})_F$ .

**PROPOSITION 6.1.** *Let  $A$  and  $B$  be two smooth and proper dg algebras and let  $X, Y \in \mathcal{D}_c(A^{\text{op}} \otimes_k B)$ . Then  $\chi(X, Y) \in F$  agrees with the categorical trace of the correspondence  $[Y \otimes_{\mathbb{L}}^{\mathbb{L}} D(X)] \in \text{Hom}_{\text{NChow}(k)_F}(A, A)$ .*

*Proof.* The  $A$ - $B$ -bimodules  $X$  and  $Y$  give rise, respectively, to correspondences  $[X] : A \rightarrow B$  and  $[Y] : A \rightarrow B$  in  $\text{NChow}(k)_F$ . On the other hand, the  $B$ - $A$ -bimodule

$$D(X) := (\widehat{A^{\text{op}} \otimes_k B})_{\text{pe}}(X, A^{\text{op}} \otimes_k B) \in \mathcal{D}_c(B^{\text{op}} \otimes_k A)$$

(see Notation 2.1) gives rise to a correspondence  $[D(X)] : B \rightarrow A$ . We can then consider the composition

$$[Y \otimes_{\mathbb{L}}^{\mathbb{L}} D(X)] : A \xrightarrow{[Y]} B \xrightarrow{[D(X)]} A. \tag{11}$$

Recall from [Tab11] that the  $\otimes$ -unit of  $\text{NChow}(k)_F$  is the ground field  $k$  considered as a dg algebra concentrated in degree zero. Recall also that the dual of  $A$  is  $A^{\text{op}}$  and that the evaluation map  $A \otimes_k A^{\text{op}} \xrightarrow{\text{ev}} k$  is given by the class  $[A] \in K_0(A^{\text{op}} \otimes_k A)_F$  of  $A$  considered as an  $A$ - $A$ -bimodule. Hence, the categorical trace of the correspondence (11) is the composition

$$k \xrightarrow{[Y \otimes_{\mathbb{L}}^{\mathbb{L}} D(X)]} A^{\text{op}} \otimes_k A \simeq A \otimes_k A^{\text{op}} \xrightarrow{[A]} k.$$

Since the composition operation in  $\text{NChow}(k)_F$  is induced by the derived tensor product of bimodules, the above composition corresponds to the class in  $K_0(k)_F \simeq F$  of the complex of  $k$ -vector spaces

$$(Y \otimes_B^{\mathbb{L}} D(X)) \otimes_{A^{\text{op}} \otimes_k A}^{\mathbb{L}} A^{\text{op}}. \tag{12}$$

Now, note that the complex of  $k$ -vector spaces  $Y \otimes_k D(X)$  carries two actions of  $A^{\text{op}}$  and two actions of  $B$ : the actions of  $A^{\text{op}}$  are induced by the left action of  $A$  on  $Y$  and by the right action of  $A$  on  $D(X)$ , while the actions of  $B$  are induced by the right action of  $B$  on  $Y$  and by the left action of  $B$  on  $D(X)$ . The coequalizer of the two  $A^{\text{op}}$ -actions is given by  $(Y \otimes_k D(X)) \otimes_{A^{\text{op}} \otimes_k A}^{\mathbb{L}} A^{\text{op}}$ , which is naturally isomorphic to  $Y \otimes_{A^{\text{op}}}^{\mathbb{L}} D(X)$ . Similarly, the coequalizer of the two  $B$ -actions is given by  $(Y \otimes_k D(X)) \otimes_{B^{\text{op}} \otimes_k B}^{\mathbb{L}} B$ , which is naturally isomorphic to  $Y \otimes_B^{\mathbb{L}} D(X)$ . The complex of  $k$ -vector spaces  $(Y \otimes_B^{\mathbb{L}} D(X)) \otimes_{A^{\text{op}} \otimes_k A}^{\mathbb{L}} A^{\text{op}}$  is therefore the coequalizer of the  $A^{\text{op}}$ - and  $B$ -actions, and hence by the above arguments it is naturally isomorphic to  $Y \otimes_{A^{\text{op}} \otimes_k B}^{\mathbb{L}} D(X)$ . By combining this fact with the natural isomorphism

$$Y \otimes_{A^{\text{op}} \otimes_k B}^{\mathbb{L}} D(X) \simeq (\widehat{A^{\text{op}} \otimes_k B})_{\text{pe}}(X, Y),$$

we deduce that (12) is naturally isomorphic to  $(\widehat{A^{\text{op}} \otimes_k B})_{\text{pe}}(X, Y)$ . As a consequence, these two complexes of  $k$ -vector spaces have the same Euler characteristic,

$$\sum_i (-1)^i \dim H^i((Y \otimes_B^{\mathbb{L}} D(X)) \otimes_{A^{\text{op}} \otimes_k A}^{\mathbb{L}} A^{\text{op}}) = \sum_i (-1)^i \dim H^i((\widehat{A^{\text{op}} \otimes_k B})_{\text{pe}}(X, Y)).$$

The natural isomorphisms of  $k$ -vector spaces (1) (applied to  $\mathcal{A} = A^{\text{op}} \otimes_k B$ ,  $M = X$  and  $N = Y$ ) then allow us to conclude that the right-hand side of the above equality agrees with  $\chi(X, Y) \in \mathbb{Z}$ . On the other hand, the left-hand side is simply the class of the complex (12) in the Grothendieck group  $K_0(k) = \mathbb{Z}$ . As a consequence, this equality holds also on the  $F$ -linearized Grothendieck group  $K_0(k)_F \simeq F$ , and so the proof is finished.  $\square$

Now, let  $A$  and  $B$  be two smooth and proper dg algebras. As explained above, the duality functor induces an isomorphism  $K_0(A^{\text{op}} \otimes_k B)_F \simeq K_0(B^{\text{op}} \otimes_k A)_F$  on the  $F$ -linearized Grothendieck groups. Via the description (9) of the Hom-sets of  $\text{NChow}(k)_F$ , we obtain an induced duality isomorphism

$$D(-) : \text{Hom}_{\text{NChow}(k)_F}(A, B) \xrightarrow{\sim} \text{Hom}_{\text{NChow}(k)_F}(B, A). \tag{13}$$

PROPOSITION 6.2. *The square*

$$\begin{array}{ccc} \text{Hom}_{\text{NChow}(k)_F}(A, B) \otimes_F \text{Hom}_{\text{NChow}(k)_F}(A, B) & \xrightarrow{\chi(-, -)} & F \\ \text{(13)} \otimes \text{id} \downarrow \simeq & & \parallel \\ \text{Hom}_{\text{NChow}(k)_F}(B, A) \otimes_F \text{Hom}_{\text{NChow}(k)_F}(A, B) & \xrightarrow{\langle - \cdot - \rangle} & F \end{array}$$

is commutative.

*Proof.* Since the  $F$ -linearized Grothendieck group  $K_0(A^{\text{op}} \otimes_k B)_F$  is generated by the elements of shape  $[X]$ , with  $X \in \mathcal{D}_c(A^{\text{op}} \otimes_k B)$ , and  $\chi(-, -)$  and  $\langle - \cdot - \rangle$  are bilinear, it suffices to show the commutativity of the above square with respect to the correspondences  $\underline{X} = [X]$  and  $\underline{Y} = [Y]$ . By Proposition 6.1,  $\chi(\underline{X}, \underline{Y}) = \chi(X, Y) \in F$  agrees with the categorical trace in  $\text{NChow}(k)_F$  of the correspondence  $[Y \otimes_B^{\mathbb{L}} D(X)] \in \text{Hom}_{\text{NChow}(k)_F}(A, A)$ .

On the other hand, since the bilinear pairing  $\langle - \cdot - \rangle$  is symmetric (as explained in §5), we have the equality  $\langle D(\underline{X}) \cdot \underline{Y} \rangle = \langle \underline{Y} \cdot D(\underline{X}) \rangle$ . By [MT11a, Corollary 4.4], we then conclude that



the intersection number  $\langle \underline{Y} \cdot D(\underline{X}) \rangle$  also agrees with the categorical trace of the correspondence  $[Y \otimes_B^{\mathbb{L}} D(X)]$ . The proof is then achieved.  $\square$

We now have all the ingredients needed to prove Theorem 1.1. We will show that a correspondence  $\underline{X} \in \text{Hom}_{\text{NChow}(k)_F}(A, B)$  belongs to  $\text{Ker}(\chi)$  if and only if it is numerically equivalent to zero. Assume first that  $\underline{X} \in \text{Ker}_R(\chi) = \text{Ker}(\chi)$ . Then, by Proposition 6.2, the intersection number  $\langle D(\underline{Y}) \cdot \underline{X} \rangle$  is trivial for every correspondence  $\underline{Y} \in \text{Hom}_{\text{NChow}(k)_F}(A, B)$ . The symmetry of the bilinear pairing  $\langle - \cdot - \rangle$ , combined with the isomorphism (13), then allows us to conclude that  $\underline{X}$  is numerically equivalent to zero.

Now, assume that  $\underline{X}$  is numerically equivalent to zero. Once again, the symmetry of the bilinear pairing  $\langle - \cdot - \rangle$ , together with the isomorphism (13), implies that  $\chi(\underline{Y}, \underline{X}) = 0$  for every correspondence  $\underline{Y} \in \text{Hom}_{\text{NChow}(k)_F}(A, B)$ . As a consequence,  $\underline{X} \in \text{Ker}_R(\chi) = \text{Ker}(\chi)$ . These results extend naturally to the pseudo-abelian envelope, and so we conclude that the  $\otimes$ -ideals  $\text{Ker}(\chi)$  and  $\mathcal{N}$ , described respectively in §4 and §5, are exactly the same. This concludes the proof of Theorem 1.1.

*Remark 6.3.* Note that the proof of Theorem 1.1 does not make use of the equality  $\text{Ker}_L(\chi) = \text{Ker}_R(\chi)$ . If in the proof we replace  $\text{Ker}_R(\chi)$  by  $\text{Ker}_L(\chi)$ , we would conclude that this latter  $\otimes$ -ideal also agrees with  $\mathcal{N}$ . As a consequence,  $\text{Ker}_L(\chi) = \mathcal{N} = \text{Ker}_R(\chi)$ , and so we obtain an alternative proof of Theorem 4.3.

### 7. An open question

In this final section, following the suggestion of an anonymous referee, we formulate a precise question relating the classical theory of motives with the recent theory of noncommutative motives. Recall from [MT11c, Proposition 3.1] the construction of the commutative diagram

$$\begin{array}{ccccc}
 \text{Chow}(k)_F & \longrightarrow & \text{Chow}(k)_{F/\otimes\mathbb{Q}(1)} & \xrightarrow{R} & \text{NChow}(k)_F \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{Voev}(k)_F & \longrightarrow & \text{Voev}(k)_{F/\otimes\mathbb{Q}(1)} & \xrightarrow{R_{\otimes\text{nil}}} & \text{NVoev}(k)_F \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{Num}(k)_F & \longrightarrow & \text{Num}(k)_{F/\otimes\mathbb{Q}(1)} & \xrightarrow{R_{\mathcal{N}}} & \text{NNum}(k)_F
 \end{array}$$

where  $R$ ,  $R_{\otimes\text{nil}}$  and  $R_{\mathcal{N}}$  are  $F$ -linear, additive, symmetric monoidal, and fully faithful functors. Some explanations are in order:  $\text{Chow}(k)_F$  stands for the category of Chow motives,  $\text{Voev}(k)_F$  stands for the category of Voevodsky’s (pure) motives (i.e. the pseudo-abelian envelope of the quotient of  $\text{Chow}(k)_F$  by the  $\otimes$ -nilpotence ideal),  $\text{Num}(k)_F$  stands for the category of numerical motives, and  $\text{NChow}(k)_F$ ,  $\text{NVoev}(k)_F$  and  $\text{NNum}(k)_F$  stand for their noncommutative analogues. The categories  $\text{Chow}(k)_{F/\otimes\mathbb{Q}(1)}$ ,  $\text{Voev}(k)_{F/\otimes\mathbb{Q}(1)}$  and  $\text{Num}(k)_{F/\otimes\mathbb{Q}(1)}$  are the orbit categories associated to the action of the Tate motive  $\mathbb{Q}(1)$ . Intuitively speaking, the above commutative diagram formalizes the conceptual idea that all the categories of pure motives can be embedded into their noncommutative analogues after factoring out by the action of the Tate motive. It is then natural to ask the following question.

*Question.* Are the functors  $R$ ,  $R_{\otimes\text{nil}}$  and  $R_{\mathcal{N}}$  essential surjective (and hence equivalences) under appropriate conditions on  $k$  and  $F$ ?

It seems unlikely that the answer to this question will be ‘yes’. Another way of approaching the possible existence of ‘truly noncommutative motives’ was discussed in [MT11b] in terms of motivic Galois groups of suitable Tannakian categories  $\text{Num}^\dagger(k)_k$  and  $\text{NNum}^\dagger(k)_k$  of commutative or noncommutative numerical motives. In [MT11b] the ‘truly noncommutative motives’ are identified with the category of representations of the kernel of the surjective homomorphism

$$\text{Gal}(\text{NNum}^\dagger(k)_k) \twoheadrightarrow \text{Ker}(t : \text{Gal}(\text{Num}^\dagger(k)_k) \rightarrow \mathbb{G}_m)$$

between motivic Galois groups. At present, it is not known whether this surjective homomorphism has a nontrivial kernel, but this is likely to be the case.

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