L 33
Diagonalization

Diagonalization

- Diagonalization problem:

  For a square matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?

- Diagonalizable matrix:

  A square matrix A is called diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

- Notes: (P diagonalizes A)

  (1) If there exists an invertible matrix P such that $B = P^{-1}AP$, then two square matrices A and B are called similar.

  (2) The eigenvalue problem is related closely to the diagonalization problem.

- Thm : (Similar matrices have the same eigenvalues)

  If A and B are similar $n \times n$ matrices, then they have the same eigenvalues.

Pf:

$$A \text{ and } B \text{ are similar} \Rightarrow B = P^{-1}AP$$

$$\begin{aligned}
|\lambda I - B| &= |\lambda I - P^{-1}AP| = |P^{-1}\lambda I P - P^{-1}AP| = |P^{-1}(\lambda I - A)P| \\
&= |P^{-1}||\lambda I - A||P| = |P^{-1}||P||\lambda I - A| = |P^{-1}P||\lambda I - A| \\
&= |\lambda I - A|
\end{aligned}$$

Thus A and B have the same eigenvalues.

- Ex 1: (A diagonalizable matrix) check the following matrix is diagonal or not.

$$A=\begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}$$

Sol: Characteristic equation:

$$\left|\lambda \mathrm{I}-A\right|=\begin{vmatrix}\lambda-1 & -3 & 0\\ -3 & \lambda-1 & 0\\ 0 & 0 & \lambda+2\end{vmatrix}=(\lambda-4)(\lambda+2)^2=0$$

$$\text{The eigenvalues: } \lambda_1=4,\ \lambda_2=-2,\ \lambda_3=-2$$

$$(1)\ \lambda=4 \Rightarrow \text{the eigenvector } p_1=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

$$(2)\lambda = -2 \Rightarrow \text{the eigenvector } p_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = [p_1 \quad p_2 \quad p_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\text{such that } P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Note: If $P = [p_2 \quad p_1 \quad p_3]$

$$= \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Thm : (Condition for diagonalization)

  An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

- Ex 4: (A matrix that is not diagonalizable)

Show that the following matrix is not diagonalizable.

$$A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I-A|=\begin{vmatrix}\lambda-1 & -2\\ 0 & \lambda-1\end{vmatrix}=(\lambda-1)^2=0$$

The eigenvalue : $\lambda_1=1$

$$\boldsymbol{\lambda I-A=I-A=\begin{bmatrix}0 & -2\\ 0 & 0\end{bmatrix}\sim\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\Rightarrow \text{eigenvector } p_1=\begin{bmatrix}1\\ 0\end{bmatrix}}$$

A does not have two linearly independent eigenvectors,

so A is not diagonalizable.

- Steps for diagonalizing *an n×n* square matrix:

Step 1: Find *n* linearly independent eigenvectors

$\boldsymbol{p}_1, \boldsymbol{p}_2, \cdots \boldsymbol{p}_n$ for A with corresponding eigenvalues.

Step 2: Let $\boldsymbol{P} = [\boldsymbol{p}_1 \mid \boldsymbol{p}_2 \mid \cdots \mid \boldsymbol{p}_n]$

Step 3:

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

where, $\boldsymbol{A}\boldsymbol{p}_i = \lambda_i \boldsymbol{p}_i, \quad \boldsymbol{i} = 1, 2, \ldots, \boldsymbol{n}$

- Thm 5.6: (Sufficient conditions for diagonalization)

If an $n \times n$ matrix A has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent and A is diagonalizable.

- Ex 5: (Determining whether a matrix is diagonalizable)

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

Sol: Because A is a triangular matrix, its eigenvalues are

$$\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -3$$

These three values are distinct, so A is diagonalizable.

- Ex 6: (Diagonalizing a matrix)

$$A=\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Find a matrix P such that $P^{-1}AP$ is diagonal.

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & 1 & 1 \\ -1 & \lambda-3 & -1 \\ 3 & -1 & \lambda+1 \end{vmatrix} = (\lambda-2)(\lambda+2)(\lambda-3)=0$$

The eigenvalues: $\lambda_1=2, \lambda_2=-2, \lambda_3=3$

$$\lambda_1 = 2$$

$$\Rightarrow \lambda_1 \mathrm{I} - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} \Rightarrow \text{ eigenvector } p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -2$$

$$\Rightarrow \lambda_2 \mathrm{I} - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t \\ -\frac{1}{4}t \\ t \end{bmatrix} \Rightarrow \text{ eigenvector } p_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\lambda_3 = 3$$

$$\Rightarrow \lambda_3 \mathrm{I} - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} \Rightarrow \text{ eigenvector } p_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$P = [p_1 \quad p_2 \quad p_3] = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}, \therefore P^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ 1/5 & 0 & 1/5 \\ 1/5 & 1 & 1/5 \end{bmatrix}$$

$$\text{s.t.} P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$