

Physics 101

Lecture 12

Equilibrium and Angular Momentum

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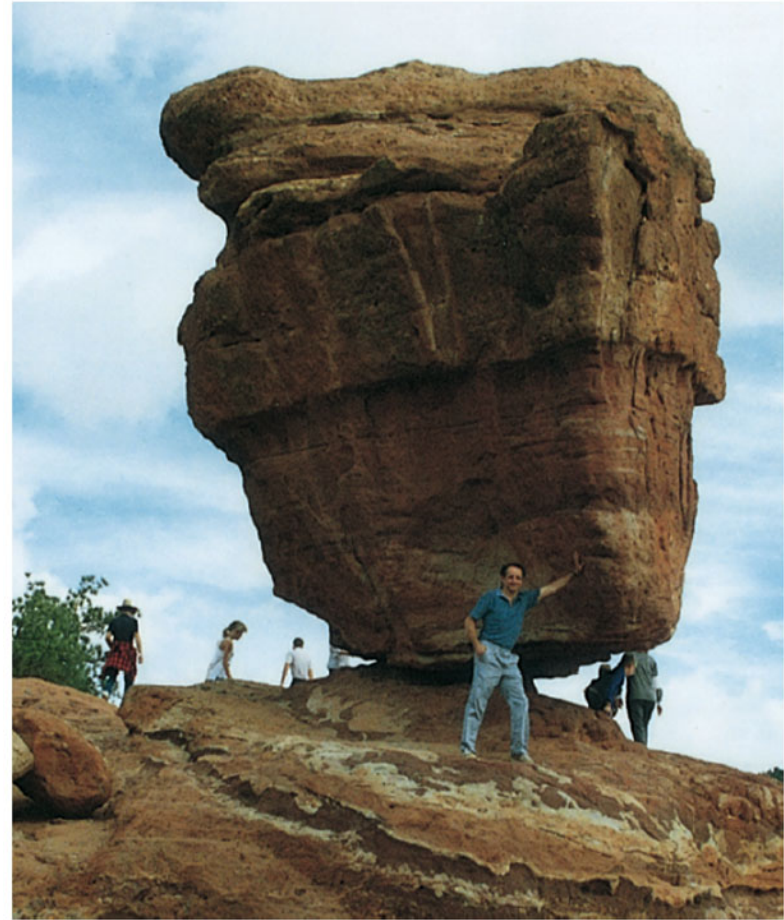
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Static Equilibrium

- ❑ Equilibrium and static equilibrium
- ❑ Static equilibrium conditions
 - Net external force must equal zero
 - Net external torque must equal zero
- ❑ Center of gravity
- ❑ Solving static equilibrium problems



Static and Dynamic Equilibrium

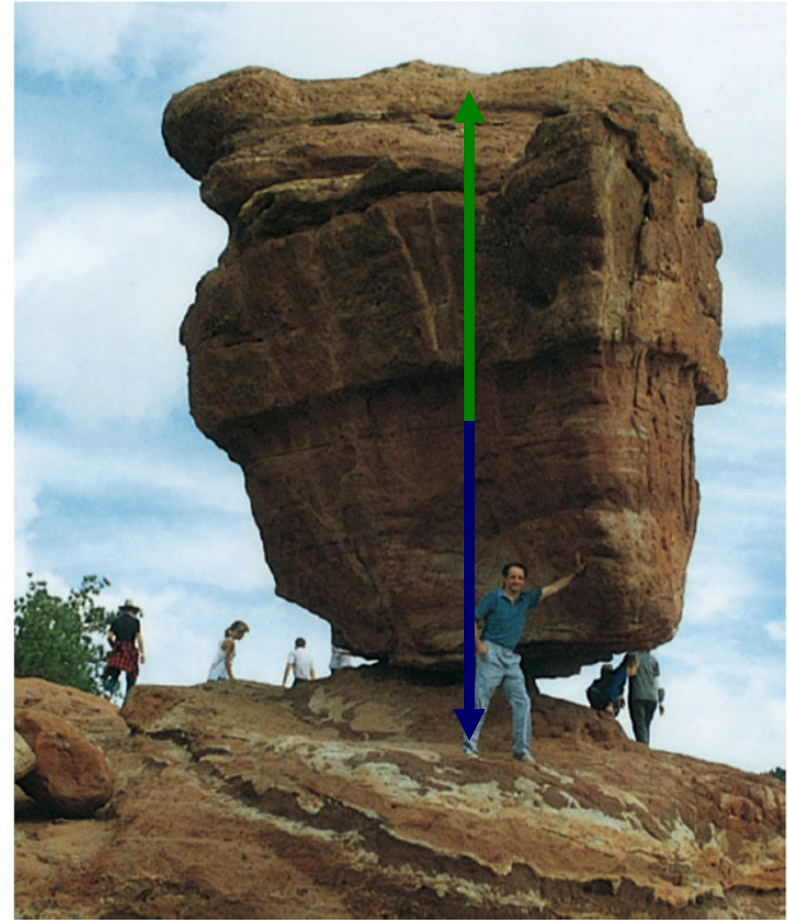
- ❑ Equilibrium implies the object is at rest (static) or its center of mass moves with a constant velocity (dynamic)
- ❑ We will consider only with the case in which linear and angular velocities are equal to zero, called “**static equilibrium**” : $v_{CM} = 0$ and $\omega = 0$
- ❑ Examples
 - *Book on table*
 - *Hanging sign*
 - *Ceiling fan – off*
 - *Ceiling fan – on*
 - *Ladder leaning against wall*

Conditions for Equilibrium

- The first condition of equilibrium is a statement of translational equilibrium
- The net external force on the object must equal zero

$$\vec{F}_{net} = \sum \vec{F}_{ext} = m\vec{a} = 0$$

- It states that the translational acceleration of the object's center of mass must be zero

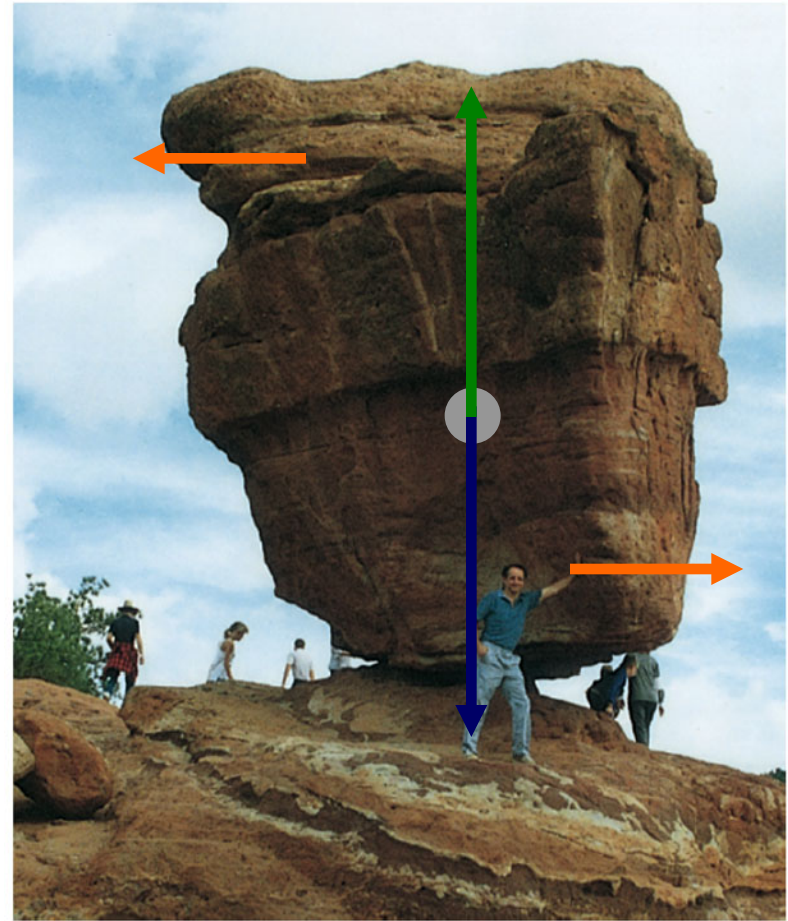


Conditions for Equilibrium

- If the object is modeled as a particle, then this is the only condition that must be satisfied

$$\vec{F}_{net} = \sum \vec{F}_{ext} = 0$$

- For an extended object to be in equilibrium, a second condition must be satisfied
- This second condition involves the rotational motion of the extended object

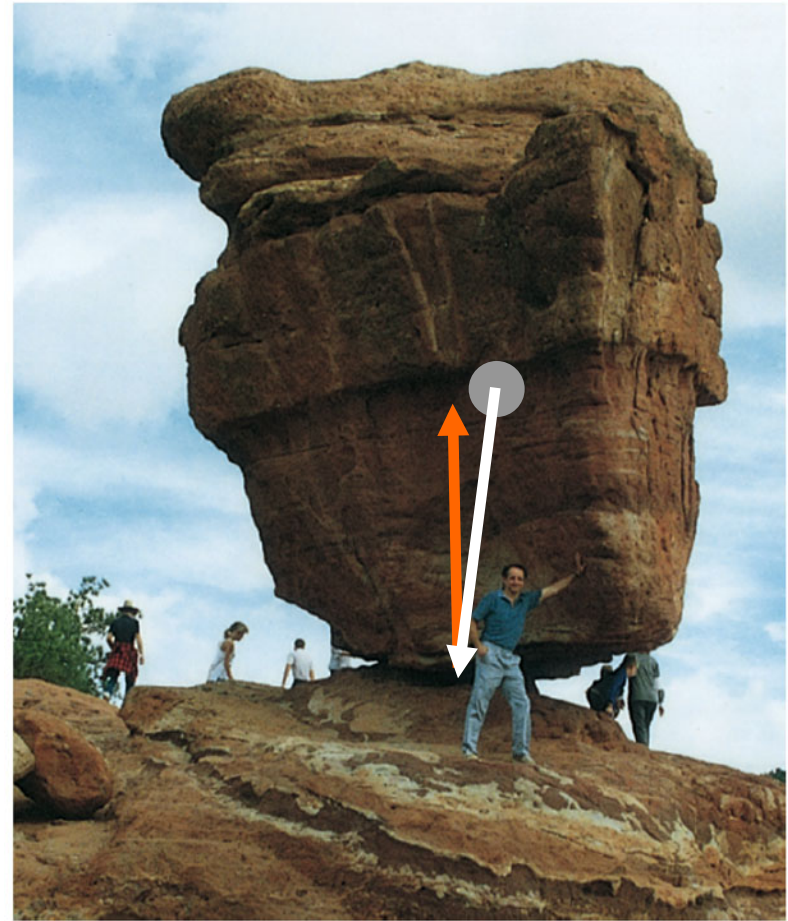


Conditions for Equilibrium

- The second condition of equilibrium is a statement of rotational equilibrium
- The net external torque on the object must equal zero

$$\vec{\tau}_{net} = \sum \vec{\tau}_{ext} = I\vec{\alpha} = 0$$

- It states the angular acceleration of the object to be zero
- This must be true for any axis of rotation

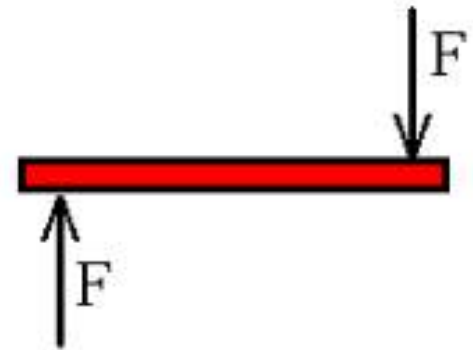


Conditions for Equilibrium

- The net force equals zero $\sum \vec{F} = 0$
 - If the object is modeled as a particle, then this is the only condition that must be satisfied
- The net torque equals zero $\sum \tau^r = 0$
 - This is needed if the object cannot be modeled as a particle
- These conditions describe the rigid objects in the equilibrium analysis model

Static Equilibrium

- Consider a light rod subject to the two forces of equal magnitude as shown in figure. Choose the correct statement with regard to this situation:
- (A) The object is in force equilibrium but not torque equilibrium.
 - (B) The object is in torque equilibrium but not force equilibrium
 - (C) The object is in both force equilibrium and torque equilibrium
 - (D) The object is in neither force equilibrium nor torque equilibrium



Equilibrium Equations

- For simplicity, We will restrict the applications to situations in which all the forces lie in the xy plane.

- Equation 1: $\vec{F}_{net} = \sum \vec{F}_{ext} = 0: F_{net,x} = 0 \quad F_{net,y} = 0 \quad \cancel{F_{net,z} = 0}$

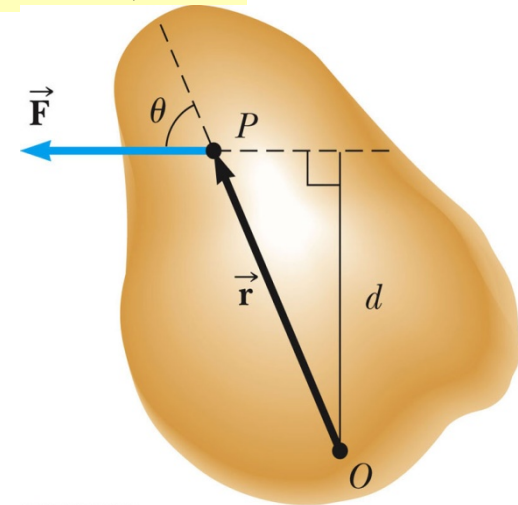
- Equation 2: $\vec{\tau}_{net} = \sum \vec{\tau}_{ext} = 0: \cancel{\tau_{net,x} = 0} \quad \cancel{\tau_{net,y} = 0} \quad \tau_{net,z} = 0$

- There are three resulting equations

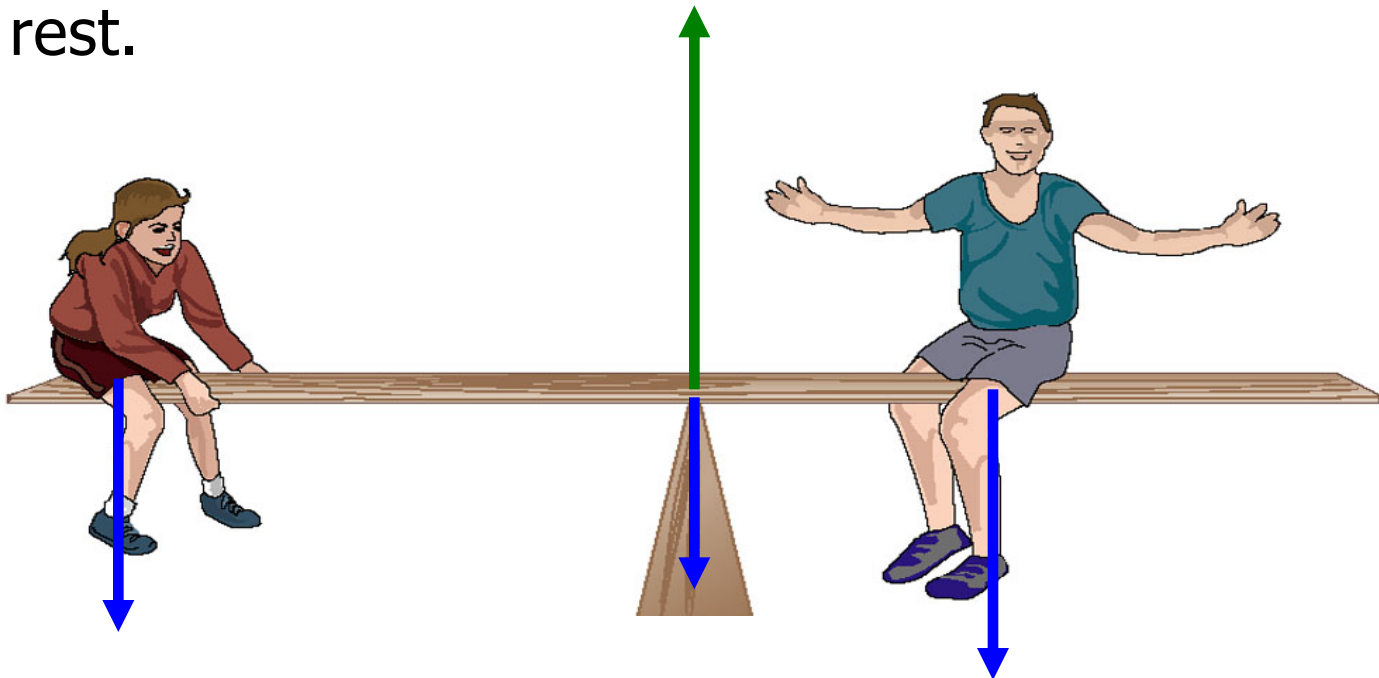
$$F_{net,x} = \sum F_{ext,x} = 0$$

$$F_{net,y} = \sum F_{ext,y} = 0$$

$$\tau_{net,z} = \sum \tau_{ext,z} = 0$$



- ❑ EX: A seesaw consisting of a uniform board of mass m_{pl} and length L supports at rest a father and daughter with masses M and m , respectively. The support is under the center of gravity of the board, the father is a distance d from the center, and the daughter is a distance 2.00 m from the center.
- ❑ A) Find the magnitude of the upward force \mathbf{n} exerted by the support on the board.
- ❑ B) Find where the father should sit to balance the system at rest.



A) Find the magnitude of the upward force \mathbf{n} exerted by the support on the board.

B) Find where the father should sit to balance the system at rest.

$$F_{net,y} = n - mg - Mg - m_{pl}g = 0$$

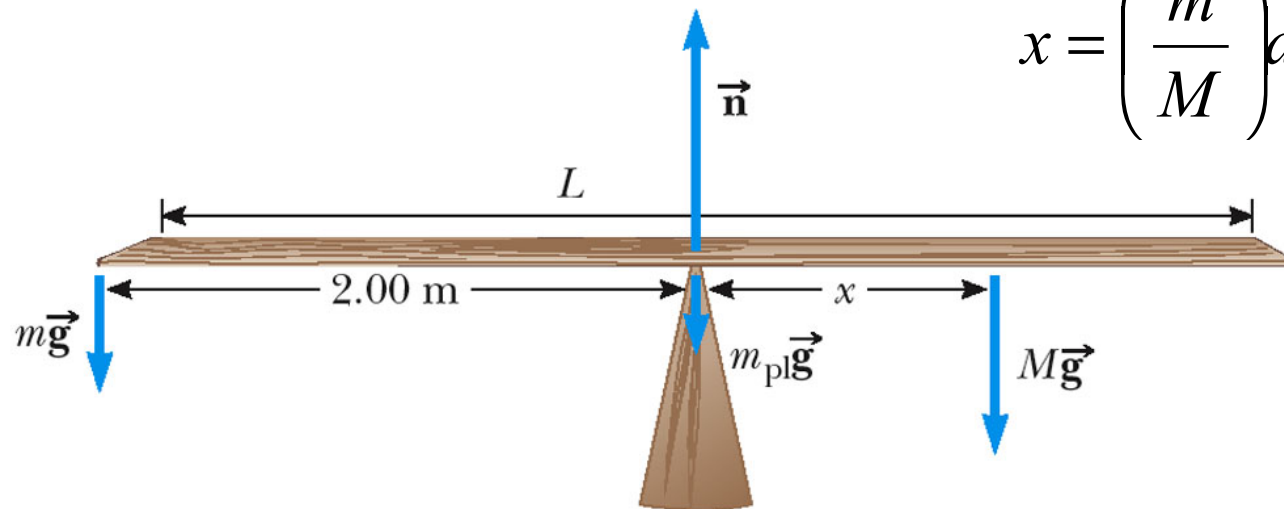
$$n = mg + Mg + m_{pl}g$$

$$\tau_{net,z} = \tau_d + \tau_f + \tau_{pl} + \tau_n$$

$$= mgd - Mgx + 0 + 0 = 0$$

$$mgd = Mgx$$

$$x = \left(\frac{m}{M} \right) d = \frac{2m}{M} < 2.00 \text{ m}$$



$$F_{net,x} = \sum F_{ext,x} = 0$$

$$F_{net,y} = \sum F_{ext,y} = 0$$

$$\tau_{net,z} = \sum \tau_{ext,z} = 0$$

Axis of Rotation

- ❑ The net torque is about an axis through any point in the xy plane
- ❑ Does it matter which axis you choose for calculating torques?
- ❑ NO. The choice of an axis is arbitrary
- ❑ If an object is in translational equilibrium and the net torque is zero about one axis, then the net torque must be zero about any other axis
- ❑ We should be smart to choose a rotation axis to simplify problems

B) Find where the father should sit to balance the system at rest.

Rotation axis O

$$\tau_{net,z} = \tau_d + \tau_f + \tau_{pl} + \tau_n$$

$$= mgd - Mg x + 0 + 0 = 0$$

$$mgd = Mg x$$

$$x = \left(\frac{m}{M} \right) d = \frac{2m}{M}$$

Rotation axis P

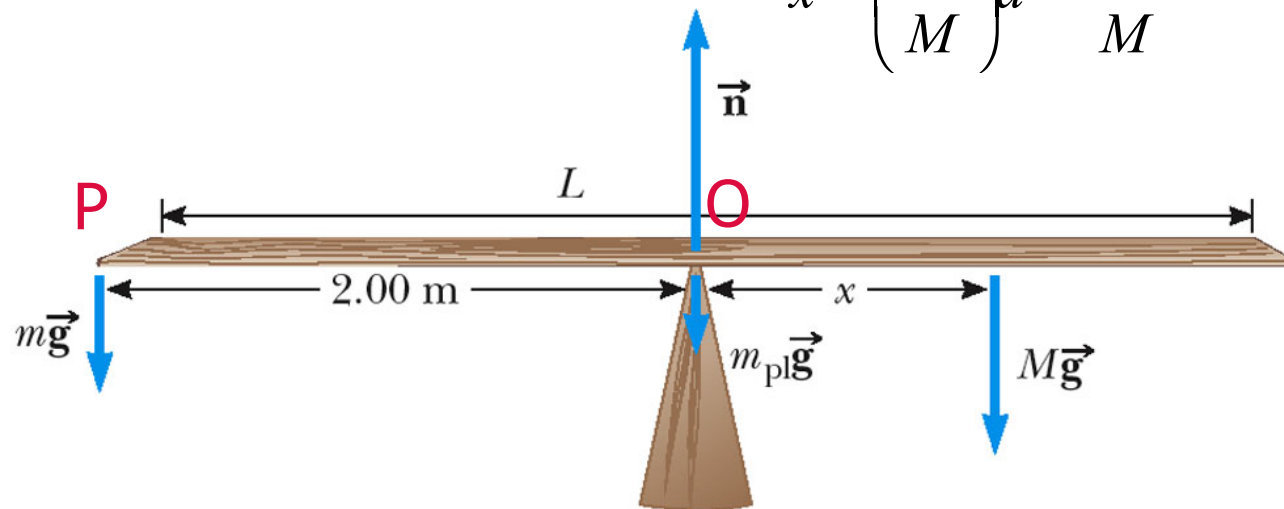
$$\tau_{net,z} = \tau_d + \tau_f + \tau_{pl} + \tau_n$$

$$= 0 - Mg(d + x) - m_{pl}gd + nd = 0$$

$$-Mgd - Mg x - m_{pl}gd + (Mg + mg + m_{pl}g)d = 0$$

$$mgd = Mg x$$

$$x = \left(\frac{m}{M} \right) d = \frac{2m}{M}$$



$$F_{net,x} = \sum F_{ext,x} = 0$$

$$F_{net,y} = \sum F_{ext,y} = 0$$

$$\tau_{net,z} = \sum \tau_{ext,z} = 0$$

Center of Gravity

- ❑ The torque due to the gravitational force on an object of mass M is the force Mg acting at the center of gravity of the object
- ❑ If g is uniform over the object, then the center of gravity of the object coincides with its center of mass
- ❑ If the object is homogeneous and symmetrical, the center of gravity coincides with its geometric center

Where is the Center of Mass?

- Assume $m_1 = 1 \text{ kg}$, $m_2 = 3 \text{ kg}$, and $x_1 = 1 \text{ m}$, $x_2 = 5 \text{ m}$, where is the center of mass of these two objects?

A) $x_{\text{CM}} = 1 \text{ m}$

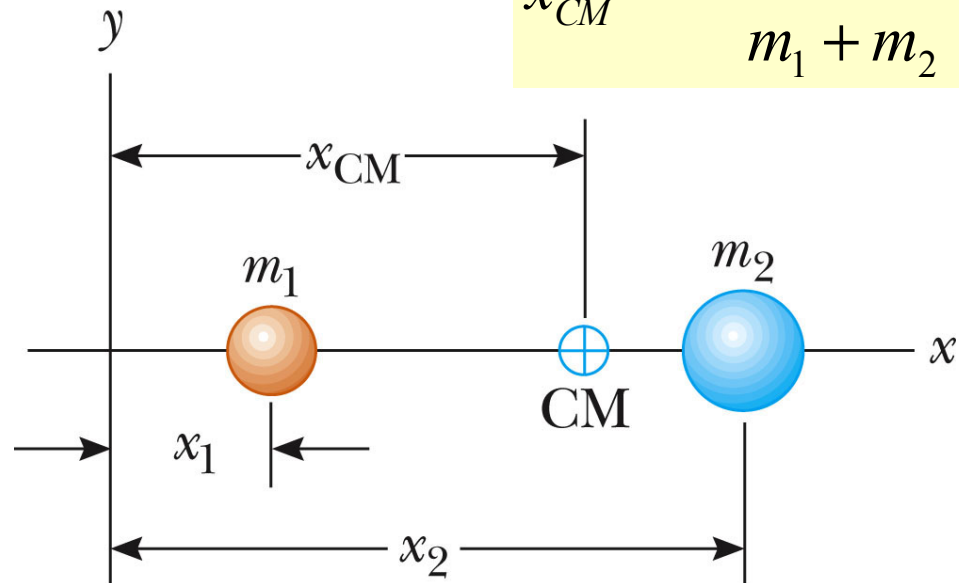
B) $x_{\text{CM}} = 2 \text{ m}$

C) $x_{\text{CM}} = 3 \text{ m}$

D) $x_{\text{CM}} = 4 \text{ m}$

E) $x_{\text{CM}} = 5 \text{ m}$

$$x_{\text{CM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

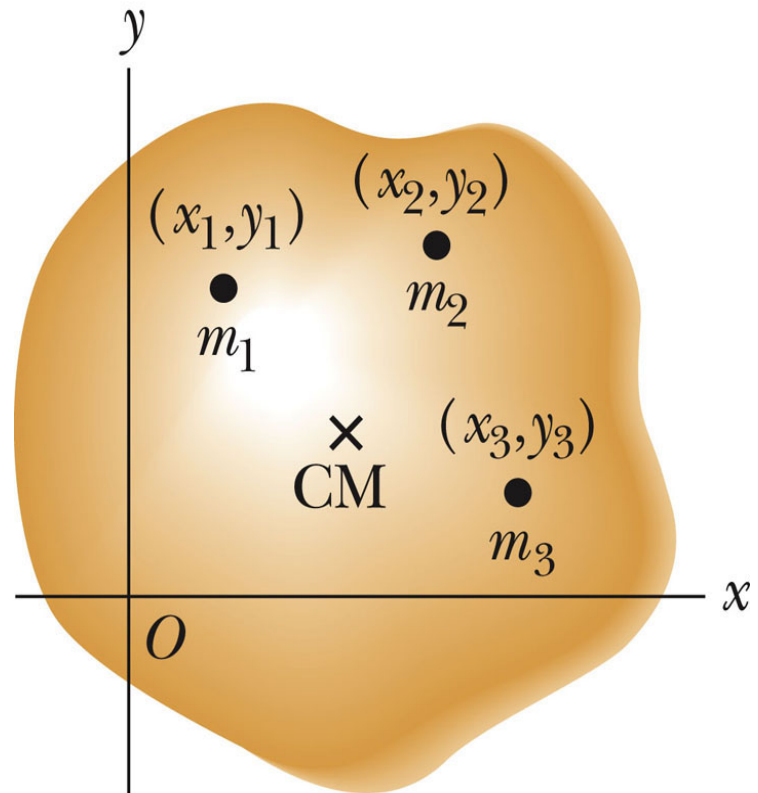


Center of Mass (CM)

- An object can be divided into many small particles
 - Each particle will have a specific mass and specific coordinates
- The x coordinate of the center of mass will be

$$x_{CM} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

- Similar expressions can be found for the y coordinates



Center of Gravity (CG)

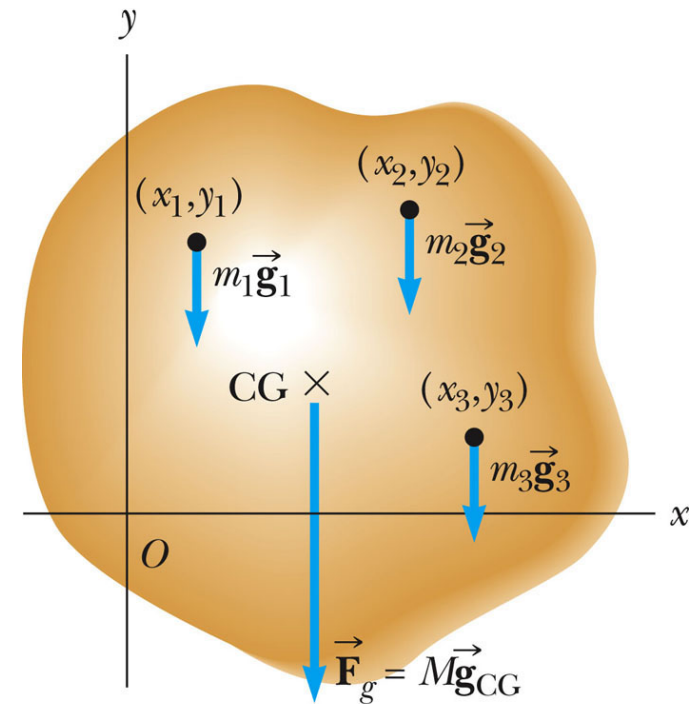
- All the various gravitational forces acting on all the various mass elements are equivalent to a single gravitational force acting through a single point called the center of gravity (CG)

$$\begin{aligned} Mg_{CG}x_{CG} &= (m_1 + m_2 + m_3 + \dots)g_{CG}x_{CG} \\ &= m_1g_1x_1 + m_2g_2x_2 + m_3g_3x_3 + \dots \end{aligned}$$

- If $g_1 = g_2 = g_3 = \dots$

- then

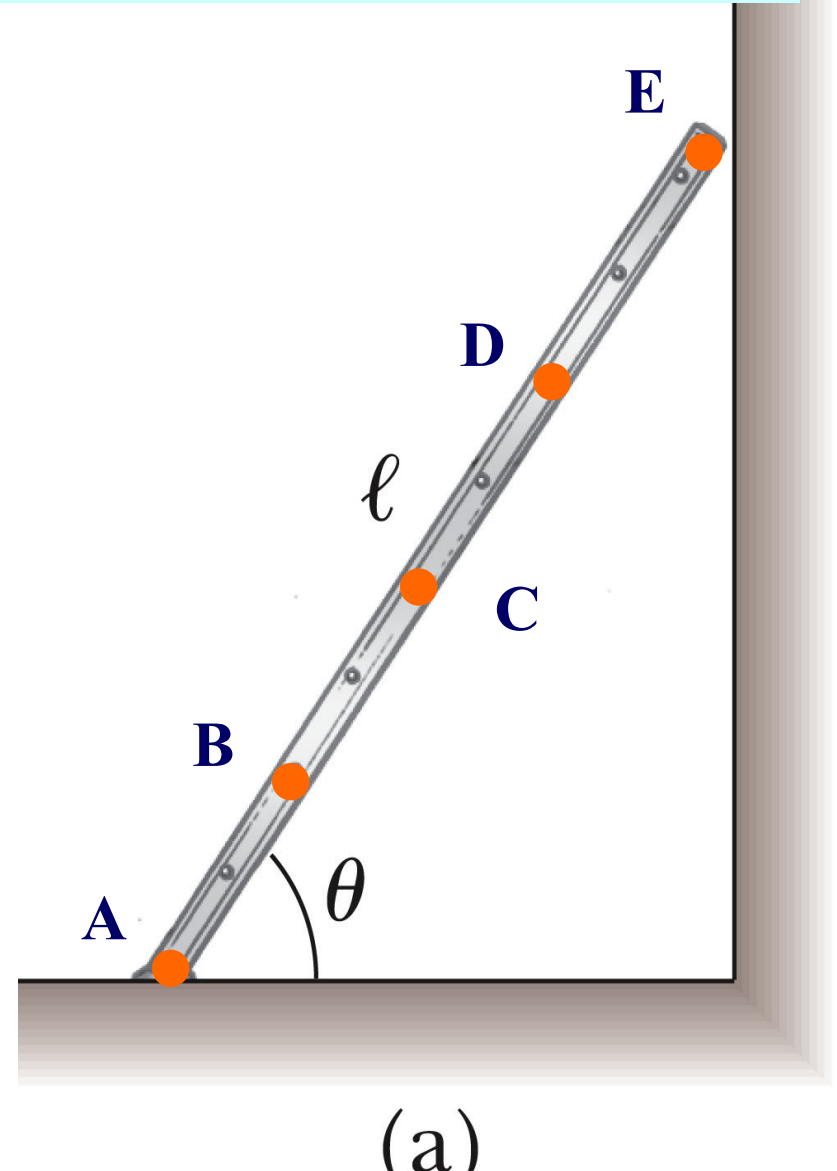
$$x_{CG} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \dots}{m_1 + m_2 + m_3 + \dots} = \frac{\sum m_i x_i}{\sum m_i}$$



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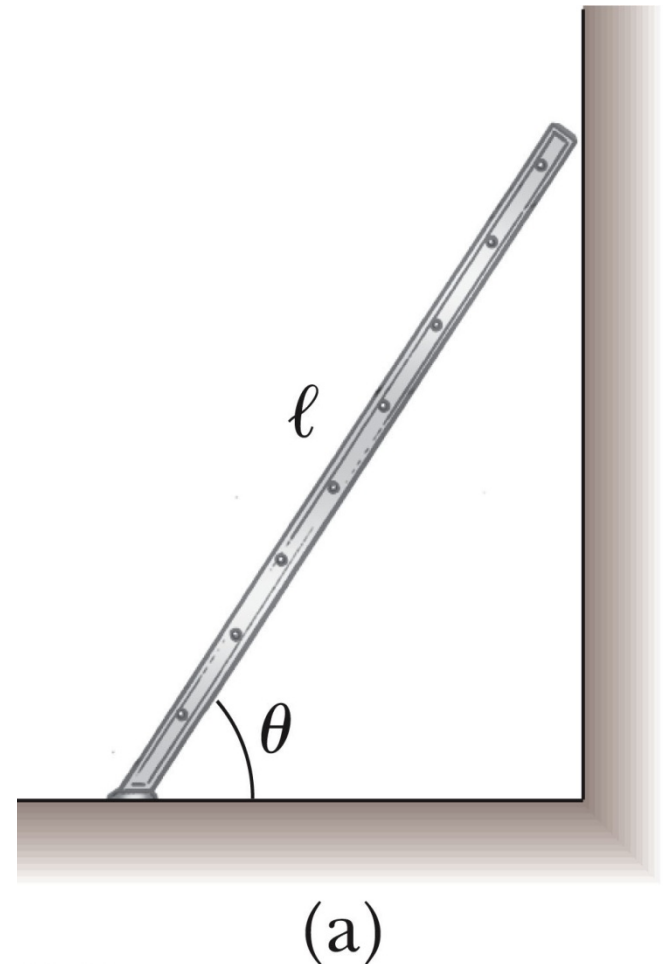
CG of a Ladder

- A uniform ladder of length l rests against a smooth, vertical wall. When you calculate the torque due to the gravitational force, you have to find center of gravity of the ladder. The center of gravity should be located at



Ladder Example

- A uniform ladder of length l rests against a smooth, vertical wall. The mass of the ladder is m , and the coefficient of static friction between the ladder and the ground is $\mu_s = 0.40$. Find the minimum angle θ at which the ladder does not slip.



Problem-Solving Strategy 1

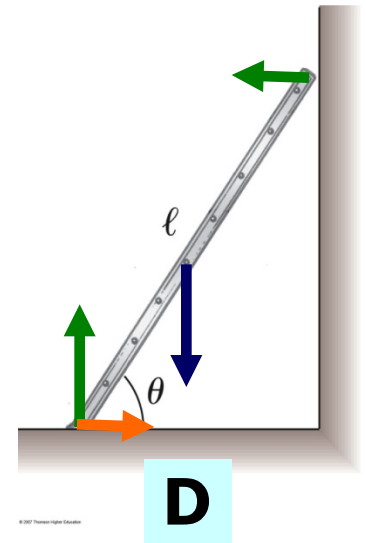
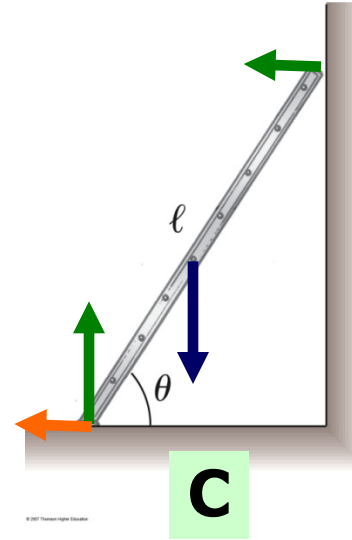
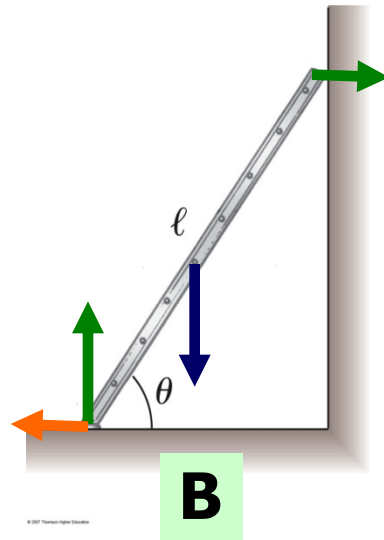
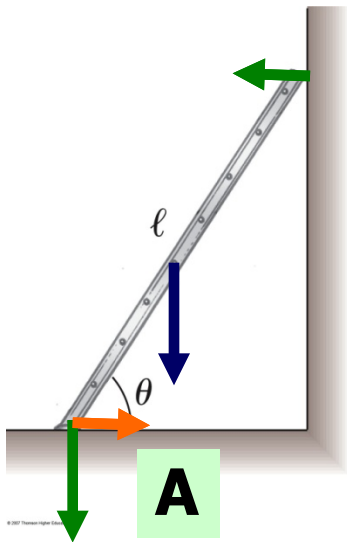
- ❑ Draw sketch, decide what is in or out the system
- ❑ Draw a free body diagram (FBD)
- ❑ Show and label all external forces acting on the object
- ❑ Indicate the locations of all the forces
- ❑ Establish a convenient coordinate system
- ❑ Find the components of the forces along the two axes
- ❑ Apply the first condition for equilibrium
- ❑ Be careful of signs

$$F_{net,x} = \sum F_{ext,x} = 0$$

$$F_{net,y} = \sum F_{ext,y} = 0$$

Which free-body diagram is correct?

- A uniform ladder of length l rests against a smooth, vertical wall. The mass of the ladder is m , and the coefficient of static friction between the ladder and the ground is $\mu_s = 0.40$. gravity: blue, friction: orange, normal: green



- A uniform ladder of length l rests against a smooth, vertical wall. The mass of the ladder is m , and the coefficient of static friction between the ladder and the ground is $\mu_s = 0.40$. Find the minimum angle θ at which the ladder does not slip.

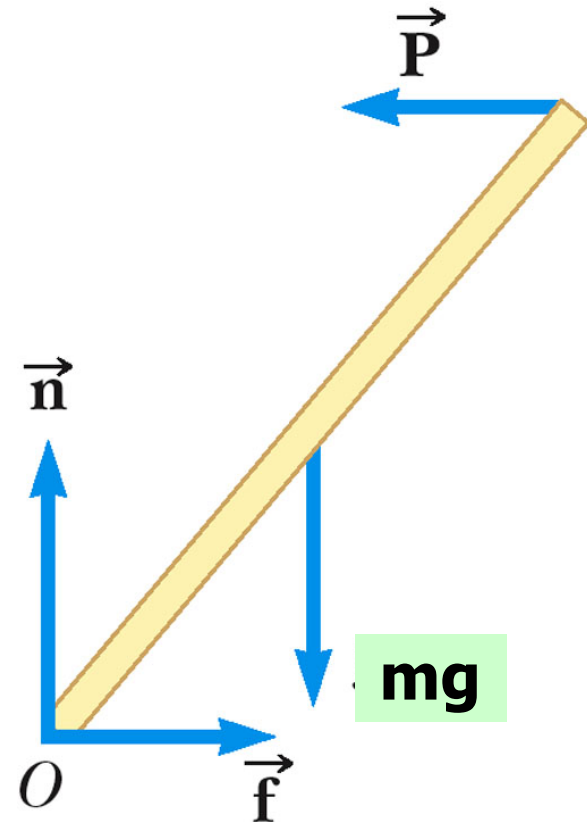
$$\sum F_x = f_x - P = 0$$

$$\sum F_y = n - mg = 0$$

$$P = f_x$$

$$n = mg$$

$$P = f_{x,\max} = \mu_s n = \mu_s mg$$

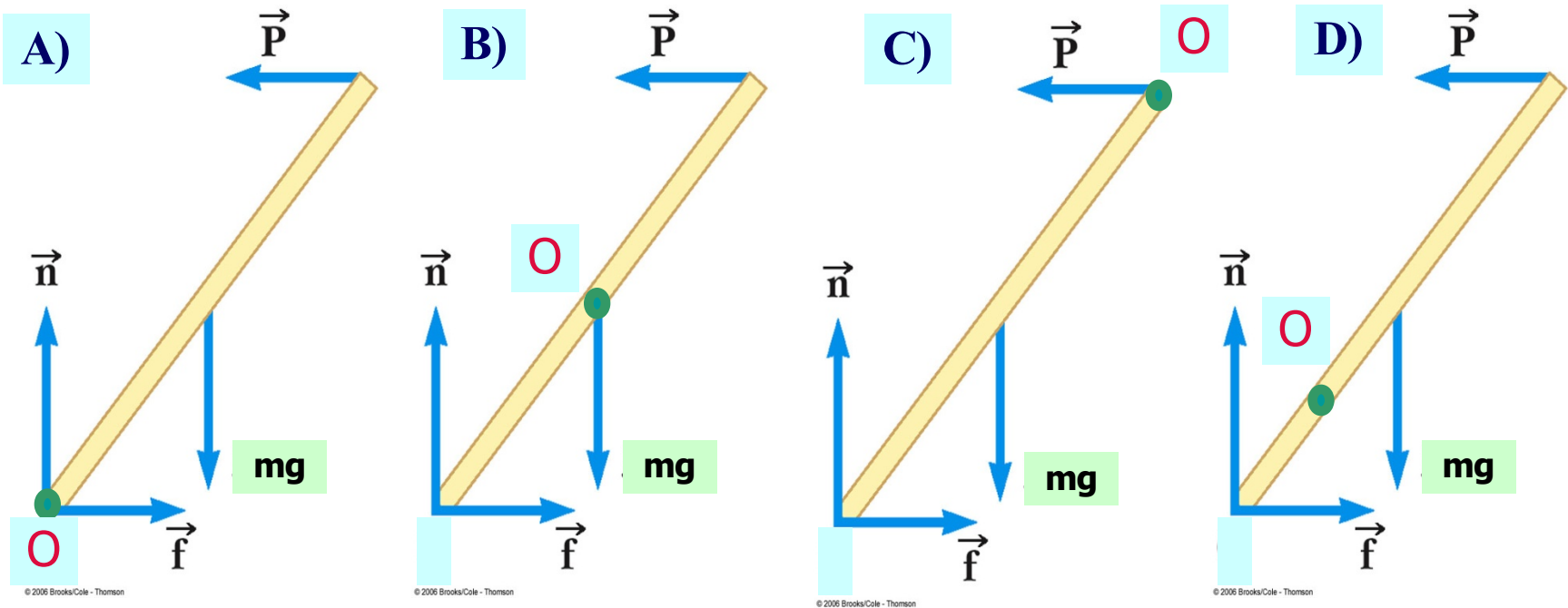


Problem-Solving Strategy 2

- ❑ Choose a convenient axis for calculating the net torque on the object
 - Remember the choice of the axis is arbitrary
- ❑ Choose an origin that simplifies the calculations as much as possible
 - A force that acts along a line passing through the origin produces a zero torque
- ❑ Be careful of sign with respect to rotational axis
 - positive if force tends to rotate object in CCW
 - negative if force tends to rotate object in CW
 - zero if force is on the rotational axis
- ❑ Apply the second condition for equilibrium $\tau_{net,z} = \sum \tau_{ext,z} = 0$

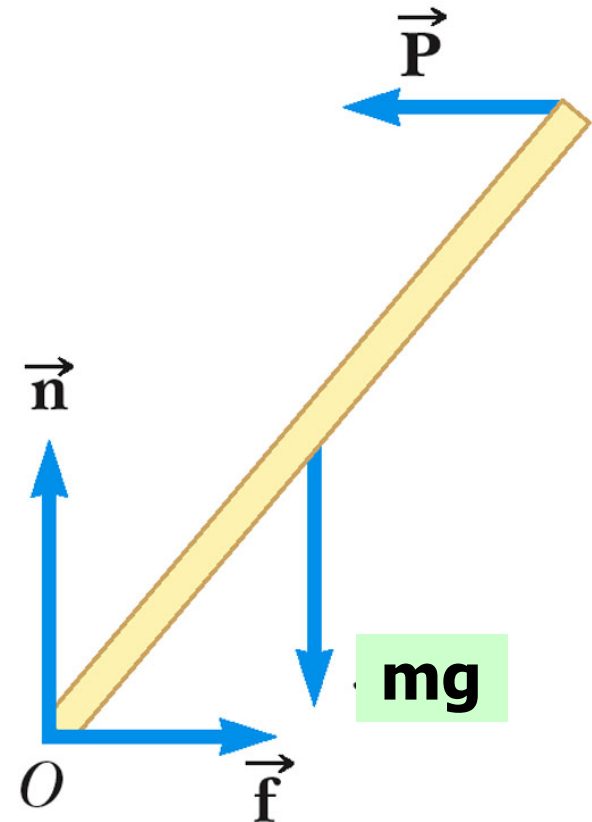
Choose an origin O that simplifies the calculations as much as possible ?

- A uniform ladder of length l rests against a smooth, vertical wall. The mass of the ladder is m , and the coefficient of static friction between the ladder and the ground is $\mu_s = 0.40$. Find the minimum angle.



- A uniform ladder of length l rests against a smooth, vertical wall. The mass of the ladder is m , and the coefficient of static friction between the ladder and the ground is $\mu_s = 0.40$. Find the minimum angle θ at which the ladder does not slip.

$$\begin{aligned}\sum \tau_O &= \tau_n + \tau_f + \tau_g + \tau_P \\ &= 0 + 0 + Pl \sin \theta_{\min} - mg \frac{l}{2} \cos \theta_{\min} = 0 \\ \frac{\sin \theta_{\min}}{\cos \theta_{\min}} &= \tan \theta_{\min} = \frac{mg}{2P} = \frac{mg}{2\mu_s mg} = \frac{1}{2\mu_s} \\ \theta_{\min} &= \tan^{-1}\left(\frac{1}{2\mu_s}\right) = \tan^{-1}\left[\frac{1}{2(0.4)}\right] = 51^\circ\end{aligned}$$



Problem-Solving Strategy 3

- ❑ The two conditions of equilibrium will give a system of equations
- ❑ Solve the equations simultaneously
- ❑ Make sure your results are consistent with your free body diagram
- ❑ If the solution gives a negative for a force, it is in the opposite direction to what you drew in the free body diagram
- ❑ Check your results to confirm

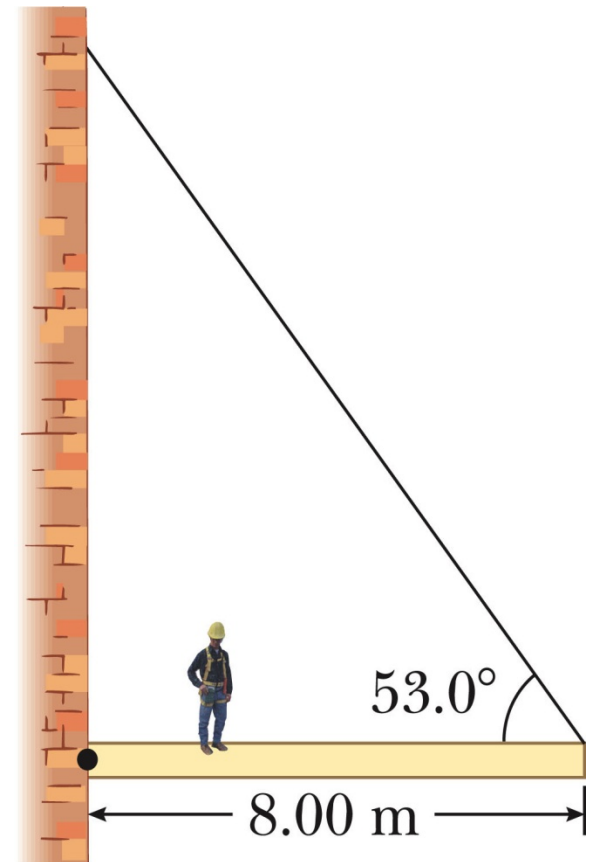
$$F_{net,x} = \sum F_{ext,x} = 0$$

$$F_{net,y} = \sum F_{ext,y} = 0$$

$$\tau_{net,z} = \sum \tau_{ext,z} = 0$$

Horizontal Beam Example

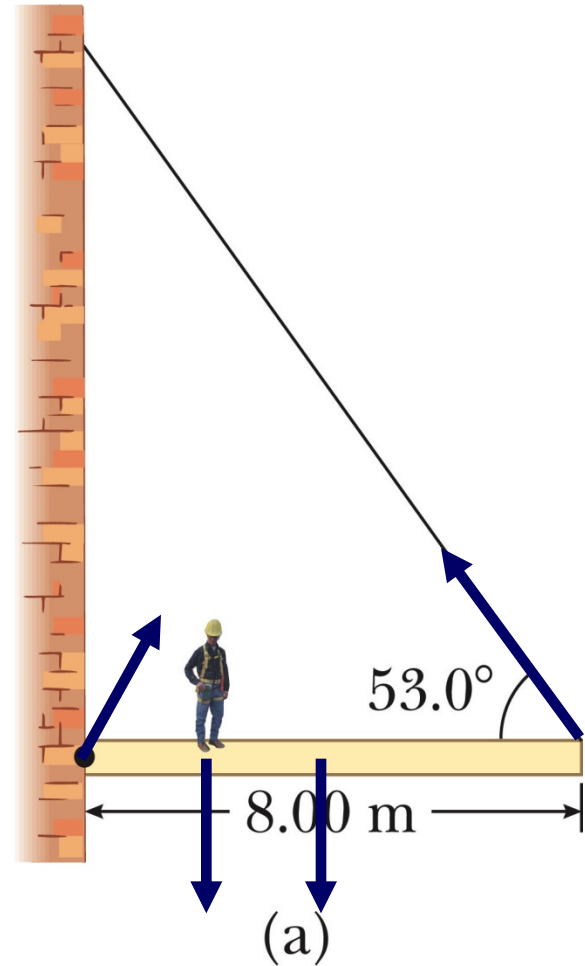
- A uniform horizontal beam with a length of $l = 8.00$ m and a weight of $W_b = 200$ N is attached to a wall by a pin connection. Its far end is supported by a cable that makes an angle of $\phi = 53^\circ$ with the beam. A person of weight $W_p = 600$ N stands a distance $d = 2.00$ m from the wall. Find the tension in the cable as well as the magnitude and direction of the force exerted by the wall on the beam.



(a)

Horizontal Beam Example

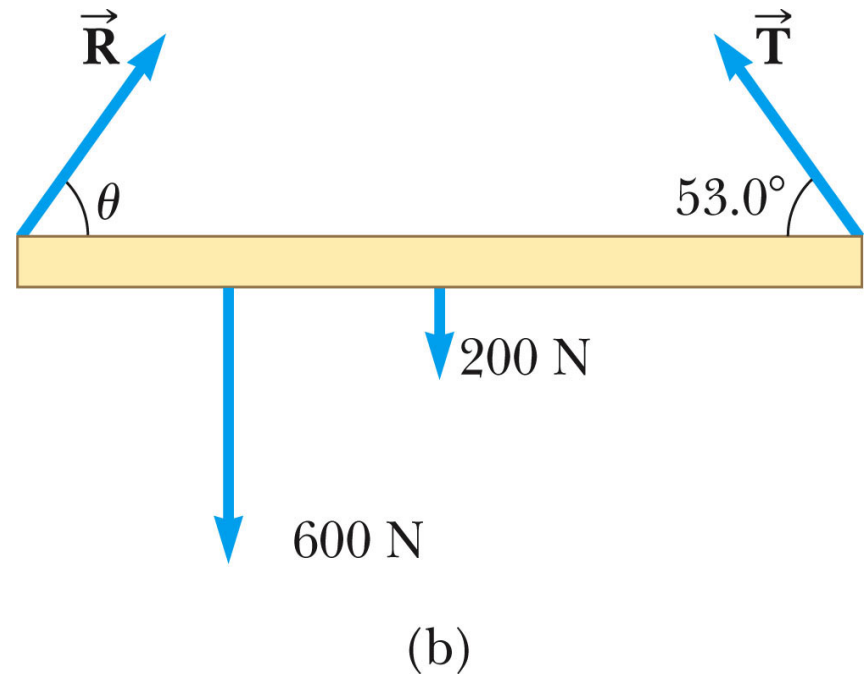
- The beam is uniform
 - So the center of gravity is at the geometric center of the beam
- The person is standing on the beam
- What are the tension in the cable and the force exerted by the wall on the beam?



Horizontal Beam Example, 2

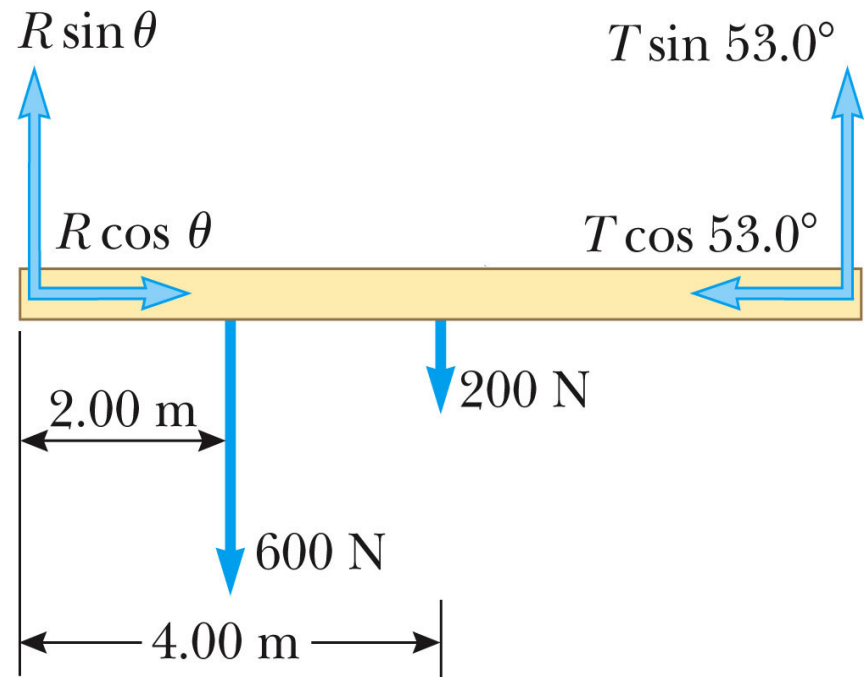
□ Analyze

- Draw a free body diagram
- Use the pivot in the problem (at the wall) as the pivot
 - This will generally be easiest
- Note there are three unknowns (T , R , θ)



Horizontal Beam Example, 3

- The forces can be resolved into components in the free body diagram
- Apply the two conditions of equilibrium to obtain three equations
- Solve for the unknowns



(c)

Horizontal Beam Example, 3

$$\Sigma \tau_z = (T \sin \phi)(l) - W_p d - W_b \left(\frac{l}{2}\right) = 0$$

$$T = \frac{W_p d + W_b \left(\frac{l}{2}\right)}{l \sin \phi} = \frac{(600\text{N})(2\text{m}) + (200\text{N})(4\text{m})}{(8\text{m}) \sin 53^\circ} = 313\text{N}$$

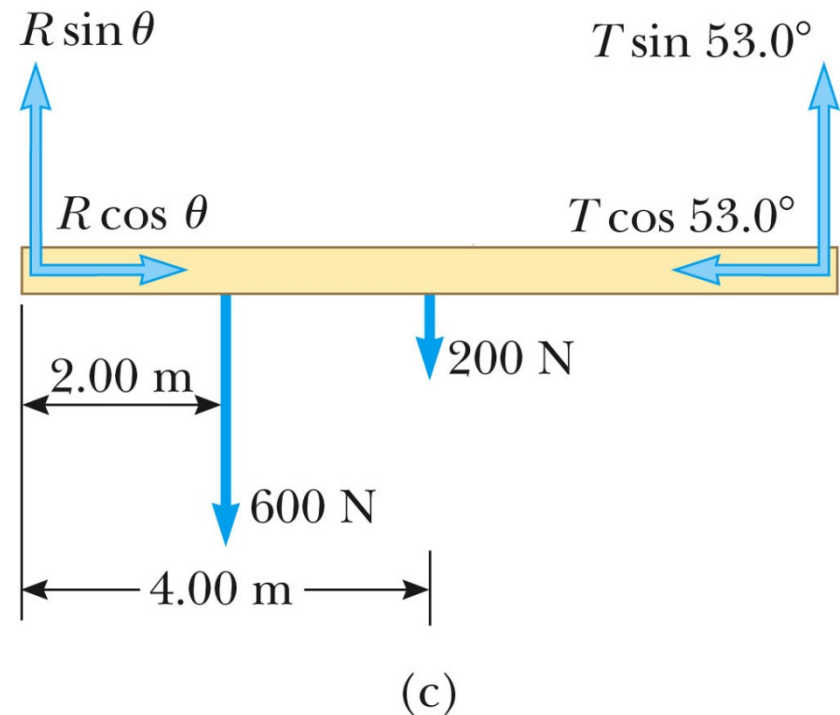
$$\Sigma F_x = R \cos \theta - T \cos \phi = 0$$

$$\Sigma F_y = R \sin \theta + T \sin \phi - W_p - W_b = 0$$

$$\frac{R \sin \theta}{R \cos \theta} = \tan \theta = \frac{W_p + W_b - T \sin \phi}{T \sin \phi}$$

$$\theta = \tan^{-1} \left(\frac{W_p + W_b - T \sin \phi}{T \sin \phi} \right) = 71.7^\circ$$

$$R = \frac{T \cos \phi}{\cos \theta} = \frac{(313\text{N}) \cos 53^\circ}{\cos 71.7^\circ} = 581\text{N}$$



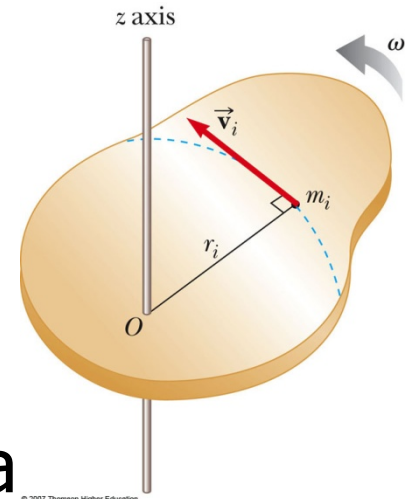
Rotational Kinetic Energy

- An object rotating about z axis with an angular speed, ω , has rotational kinetic energy
- The total rotational kinetic energy of the rigid object is the sum of the energies of all its particles

$$K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i r_i^2 \omega^2$$

$$K_R = \frac{1}{2} \left(\sum_i m_i r_i^2 \right) \omega^2 = \frac{1}{2} I \omega^2$$

- Where I is called the moment of inertia
- Unit of rotational kinetic energy is Joule (J)



Work-Energy Theorem for pure Translational motion

- The work-energy theorem tells us

$$W_{net} = \Delta KE = KE_f - KE_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

- Kinetic energy is for point mass only, ignoring rotation.

- Work

$$W_{net} = \int dW = \int \vec{F} \cdot d\vec{s}$$

- Power

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{s}}{dt} = \vec{F} \cdot \vec{v}$$

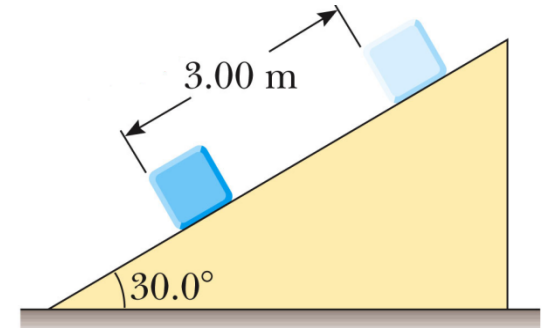
Mechanical Energy Conservation

- Energy conservation

$$W_{nc} = \Delta K + \Delta U$$

- When $W_{nc} = 0$,

$$K_f + U_f = U_i + K_i$$



- The total mechanical energy is conserved and remains the same at all times

$$\frac{1}{2}mv_i^2 + mgy_i = \frac{1}{2}mv_f^2 + mgy_f$$

- Remember, this is for conservative forces, no dissipative forces such as friction can be present

Total Energy of a System

- A ball is rolling down a ramp
- Described by three types of energy

- Gravitational potential energy

$$U = Mgh$$

- Translational kinetic energy

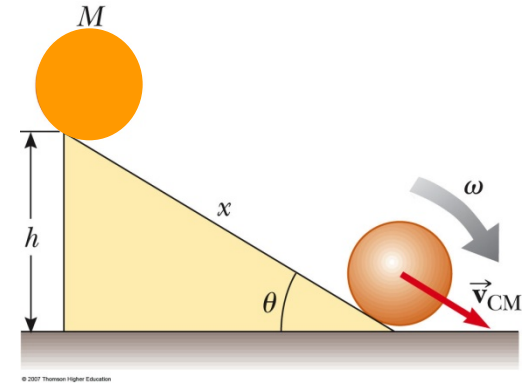
$$K_t = \frac{1}{2} M v_{CM}^2$$

- Rotational kinetic energy

$$K_r = \frac{1}{2} I \omega^2$$

- Total energy of a system

$$E = \frac{1}{2} M v_{CM}^2 + Mgh + \frac{1}{2} I \omega^2$$

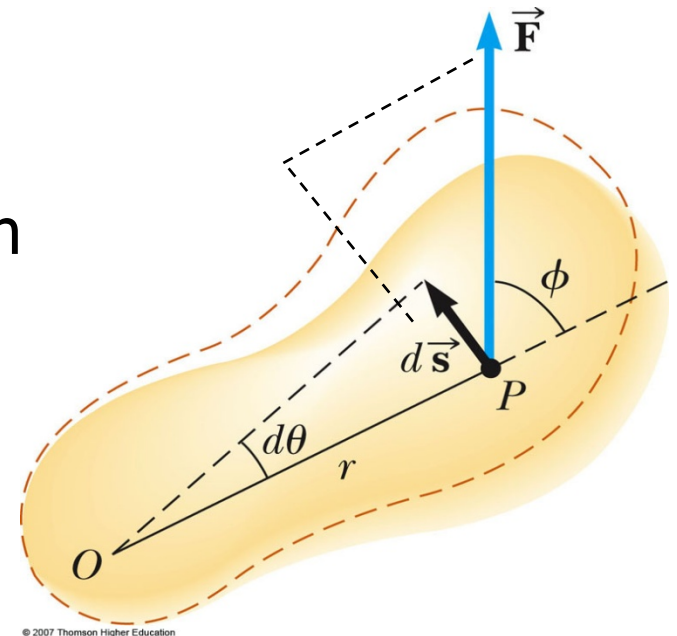


Work done by a pure rotation

- Apply force F to mass at point r , causing rotation-only about axis
- Find the work done by F applied to the object at P as it rotates through an infinitesimal distance ds

$$\begin{aligned}dW &= \vec{F} \cdot d\vec{s} = F \cos(90^\circ - \varphi) ds \\ &= F \sin \varphi ds = Fr \sin \varphi d\theta\end{aligned}$$

- Only transverse component of F does work – the same component that contributes to torque



$$dW = \tau d\theta$$

Work-Kinetic Theorem pure rotation

- As object rotates from θ_i to θ_f , work done by the torque

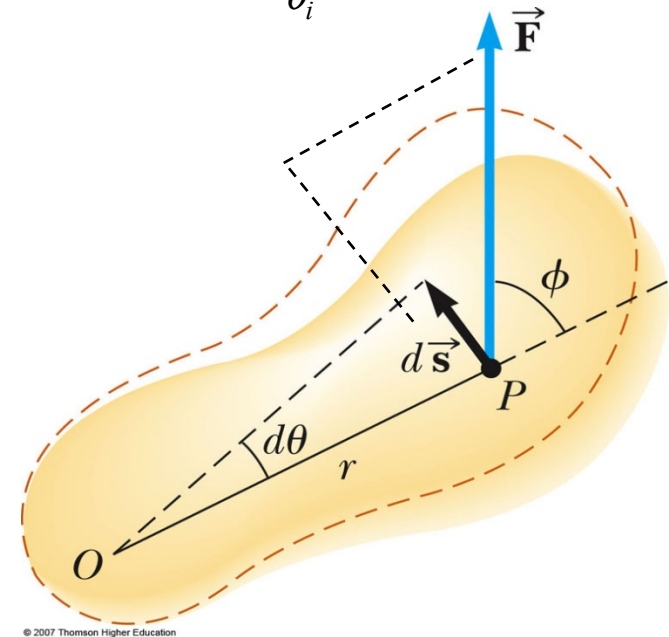
$$W = \int_{\theta_i}^{\theta_f} dW = \int_{\theta_i}^{\theta_f} \tau d\theta = \int_{\theta_i}^{\theta_f} I \alpha d\theta = \int_{\theta_i}^{\theta_f} I \frac{d\omega}{dt} d\theta = \int_{\theta_i}^{\theta_f} I \omega d\omega$$

- I is constant for rigid object

$$W = \int_{\theta_i}^{\theta_f} I \omega d\omega = I \int_{\theta_i}^{\theta_f} \omega d\omega = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2$$

- Power

$$P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau \omega$$



- Ex: An motor attached to a grindstone exerts a constant torque of 10 N-m. The moment of inertia of the grindstone is $I = 2 \text{ kg-m}^2$. The system starts from rest.

- Find the kinetic energy after 8 s

$$K_f = \frac{1}{2} I \omega_f^2 = 1600 \text{ J} \Leftarrow \omega_f = \omega_i + \alpha t = 40 \text{ rad/s} \Leftarrow \alpha = \frac{\tau}{I} = 5 \text{ rad/s}^2$$

- Find the work done by the motor during this time

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta = \tau(\theta_f - \theta_i) = 10 \times 160 = 1600 \text{ J}$$

$$(\theta_f - \theta_i) = \omega_i t + \frac{1}{2} \alpha t^2 = 160 \text{ rad} \quad W = K_f - K_i = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2 = 1600 \text{ J}$$

- Find the average power delivered by the motor

$$P_{\text{avg}} = \frac{dW}{dt} = \frac{1600}{8} = 200 \text{ watts}$$

- Find the instantaneous power at $t = 8 \text{ s}$

$$P = \tau \omega = 10 \times 40 = 400 \text{ watts}$$

Work-Energy Theorem

- For pure translation

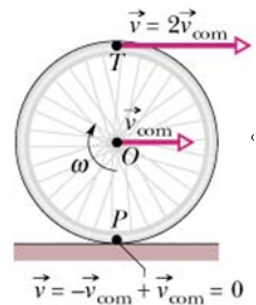
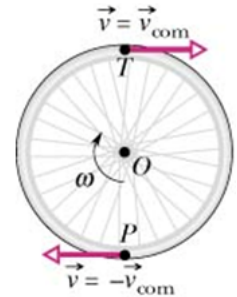
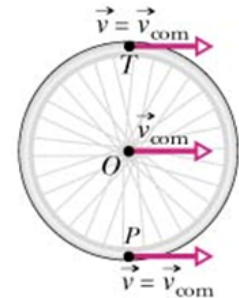
$$W_{net} = \Delta K_{cm} = K_{cm,f} - K_{cm,i} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

- For pure rotation

$$W_{net} = \Delta K_{rot} = K_{rot,f} - K_{rot,i} = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$$

- Rolling: pure rotation + pure translation

$$\begin{aligned} W_{net} &= \Delta K_{total} = (K_{rot,f} + K_{cm,f}) - (K_{rot,i} + K_{cm,i}) \\ &= \left(\frac{1}{2}I\omega_f^2 + \frac{1}{2}mv_f^2 \right) - \left(\frac{1}{2}I\omega_i^2 + \frac{1}{2}mv_i^2 \right) \end{aligned}$$



Energy Conservation

- Energy conservation

- When $W_{nc} = 0$,
$$W_{nc} = \Delta K_{total} + \Delta U$$

$$K_{rot,f} + K_{cm,f} + U_f = K_{rot,i} + K_{cm,i} + U_i$$

- The total mechanical energy is conserved and remains the same at all times

$$\frac{1}{2}I\omega_i^2 + \frac{1}{2}mv_i^2 + mgy_i = \frac{1}{2}I\omega_f^2 + \frac{1}{2}mv_f^2 + mgy_f$$

- Remember, this is for conservative forces, no dissipative forces such as friction can be present

Total Energy of a Rolling System

- A ball is rolling down a ramp
- Described by three types of energy
 - Gravitational potential energy

$$U = Mgh$$

- Translational kinetic energy

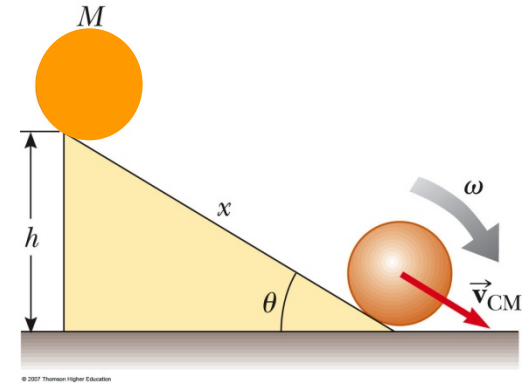
$$K_t = \frac{1}{2} Mv^2$$

- Rotational kinetic energy

$$K_r = \frac{1}{2} I\omega^2$$

- Total energy of a system

$$E = \frac{1}{2} Mv^2 + Mgh + \frac{1}{2} I\omega^2$$



A Ball Rolling Down an Incline

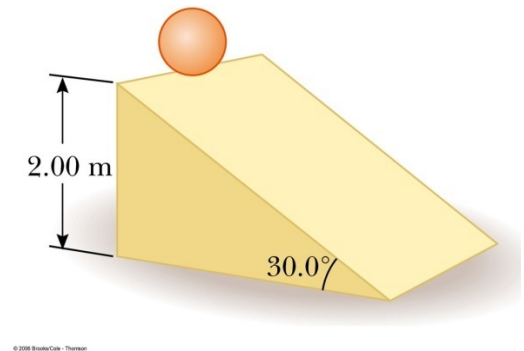
- A ball of mass M and radius R starts from rest at a height of h and rolls down a 30° slope, what is the linear speed of the ball when it leaves the incline? Assume that the ball rolls without slipping.

$$\frac{1}{2}mv_i^2 + mgy_i + \frac{1}{2}I\omega_i^2 = \frac{1}{2}mv_f^2 + mgy_f + \frac{1}{2}I\omega_f^2$$

$$0 + Mgh + 0 = \frac{1}{2}Mv_f^2 + 0 + \frac{1}{2}I\omega_f^2$$

$$I = \frac{2}{5}MR^2 \quad \omega_f = \frac{v_f}{R}$$

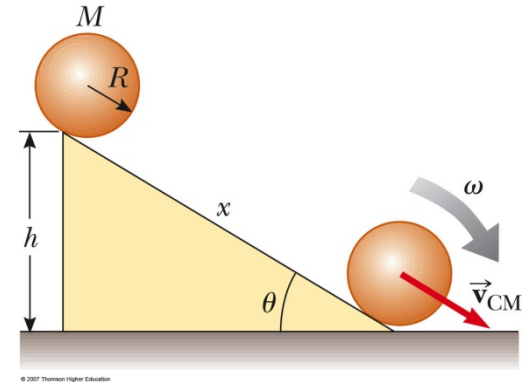
$$Mgh = \frac{1}{2}Mv_f^2 + \frac{1}{2} \frac{2}{5}MR^2 \frac{v_f^2}{R^2} = \frac{1}{2}Mv_f^2 + \frac{1}{5}Mv_f^2$$



$$v_f = \left(\frac{10}{7}gh\right)^{1/2}$$

Rotational Work and Energy

- A ball rolls without slipping down incline A, starting from rest. At the same time, a box starts from rest and slides down incline B, which is identical to incline A except that it is frictionless. Which arrives at the bottom first?



- Ball rolling:

$$\frac{1}{2}mv_i^2 + mgy_i + \frac{1}{2}I\omega_i^2 = \frac{1}{2}mv_f^2 + mgy_f + \frac{1}{2}I\omega_f^2$$

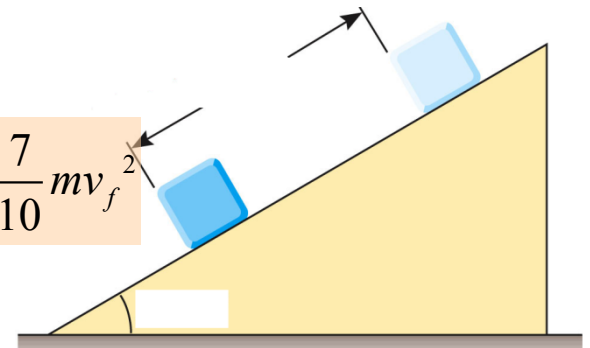
$$mgh = \frac{1}{2}mv_f^2 + \frac{1}{2}I\omega_f^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)(v_f/R)^2 = \frac{7}{10}mv_f^2$$

- Box sliding

$$\frac{1}{2}mv_i^2 + mgy_i = \frac{1}{2}mv_f^2 + mgy_f$$

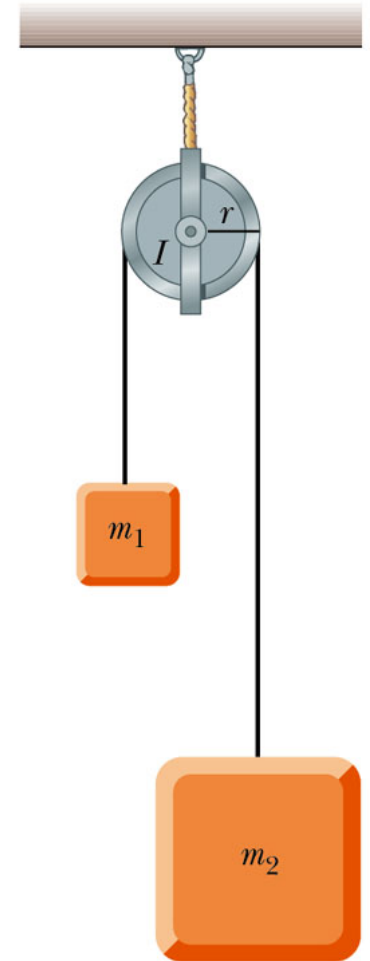
$$\text{sliding: } mgh = \frac{1}{2}mv_f^2$$

$$\text{rolling: } mgh = \frac{7}{10}mv_f^2$$



Blocks and Pulley

- ❑ Two blocks having different masses m_1 and m_2 are connected by a string passing over a pulley. The pulley has a radius R and moment of inertia I about its axis of rotation. The string does not slip on the pulley, and the system is released from rest.
- ❑ Find the translational speeds of the blocks after block 2 descends through a distance h .
- ❑ Find the angular speed of the pulley at that time.



- Find the translational speeds of the blocks after block 2 descends through a distance h .

$$K_{rot,f} + K_{cm,f} + U_f = K_{rot,i} + K_{cm,i} + U_i$$

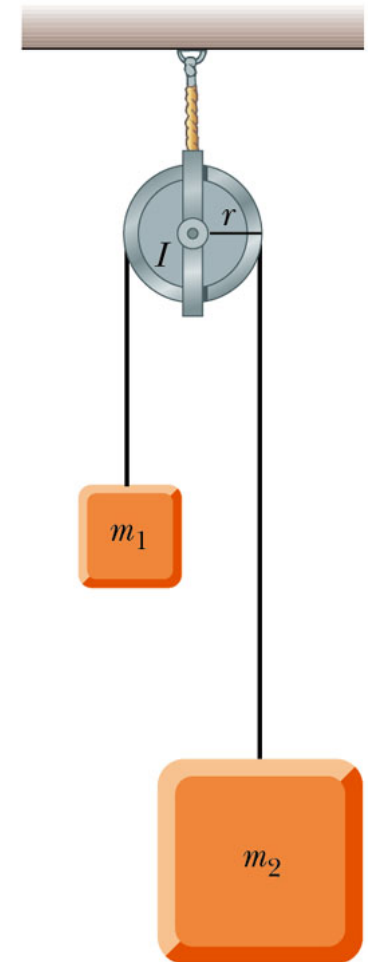
$$\left(\frac{1}{2}m_1v_f^2 + \frac{1}{2}m_2v_f^2 + \frac{1}{2}I\omega_f^2\right) + (m_1gh - m_2gh) = 0 + 0 + 0$$

$$\frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)v_f^2 = m_2gh - m_1gh$$

$$v_f = \left[\frac{2(m_2 - m_1)gh}{m_1 + m_2 + I/R^2} \right]^{1/2}$$

- Find the angular speed of the pulley at that time.

$$\omega_f = \frac{v_f}{R} = \frac{1}{R} \left[\frac{2(m_2 - m_1)gh}{m_1 + m_2 + I/R^2} \right]^{1/2}$$



Angular Momentum

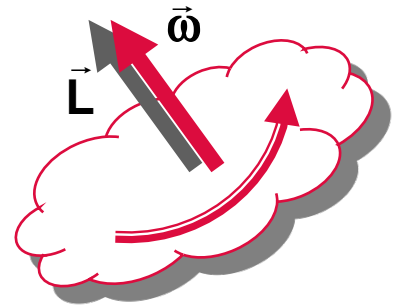
- ❑ Same basic techniques that were used in linear motion can be applied to rotational motion.
 - F becomes τ
 - m becomes I
 - a becomes α
 - v becomes ω
 - x becomes θ
- ❑ Linear momentum defined as $\mathbf{p} = m\mathbf{v}$
- ❑ What if mass of center of object is not moving, but it is rotating?
- ❑ Angular momentum $\mathbf{L} = I\omega$

Angular Momentum I

- Angular momentum of a rotating rigid object

$$\mathbf{L} = I\boldsymbol{\omega}$$

- L has the same direction as $\boldsymbol{\omega}$ *
- L is positive when object rotates in CCW
- L is negative when object rotates in CW



- Angular momentum SI unit: $\text{kg}\cdot\text{m}^2/\text{s}$

Calculate L of a 10 kg disk when $\omega = 320 \text{ rad/s}$, $R = 9 \text{ cm} = 0.09 \text{ m}$

$L = I\omega$ and $I = MR^2/2$ for disk

$$L = \frac{1}{2}MR^2\omega = \frac{1}{2}(10)(0.09)^2(320) = 12.96 \text{ kgm}^2/\text{s}$$

*When rotation is about a principal axis

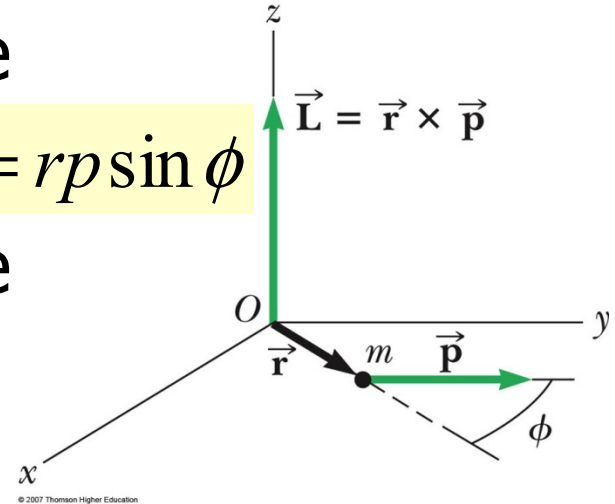
Angular Momentum II

- Angular momentum of a particle

$$L = I\omega = mr^2\omega = mv_{\perp}r = mvr \sin \phi = rp \sin \phi$$

- Angular momentum of a particle

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \mathbf{v})$$

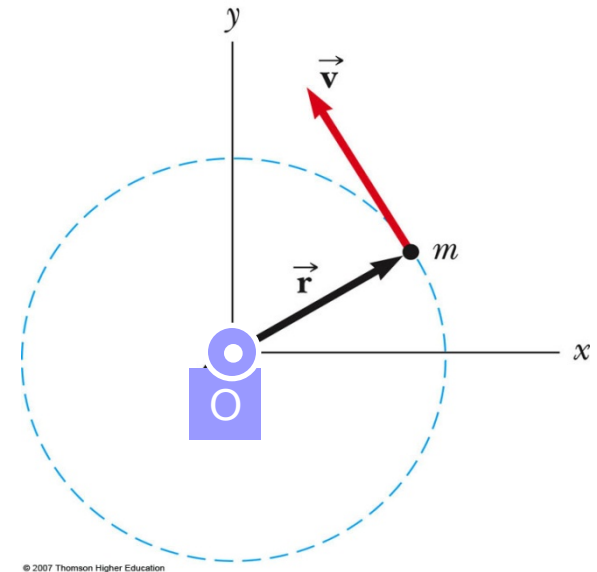


- \mathbf{r} is the particle's instantaneous position vector
- \mathbf{p} is its instantaneous linear momentum
- Only tangential momentum component contribute
- Mentally place \mathbf{r} and \mathbf{p} tail to tail form a plane, \mathbf{L} is perpendicular to this plane

Angular Momentum of a Particle in Uniform Circular Motion

Example: A particle moves in the xy plane in a circular path of radius r . Find the magnitude and direction of its angular momentum relative to an axis through O when its velocity is v .

- The angular momentum vector points out of the diagram
- The magnitude is
$$L = rp \sin\theta = mvr \sin(90^\circ) = mvr$$
- A particle in uniform circular motion has a constant angular momentum about an axis through the center of its path



Angular Momentum and Torque

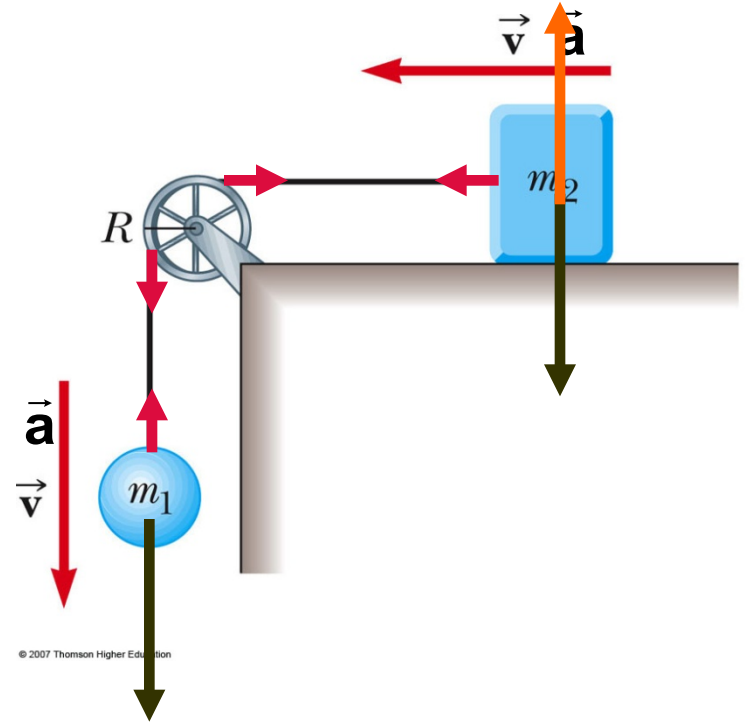
- Rotational motion: apply torque to a rigid body
- The torque causes the angular momentum to change
- The net torque acting on a body is the time rate of change of its angular momentum

$$\mathbf{F}_{\text{net}} = \Sigma \mathbf{F} = \frac{d\mathbf{p}}{dt} \quad \longrightarrow \quad \tau_{\text{net}} = \Sigma \tau = \frac{d\mathbf{L}}{dt}$$

- $\Sigma \tau$ and \mathbf{L} are to be measured about the same origin
- The origin must not be accelerating (must be an inertial frame)

Ex4: A Non-isolated System

A sphere mass m_1 and a block of mass m_2 are connected by a light cord that passes over a pulley. The radius of the pulley is R , and the mass of the thin rim is M . The spokes of the pulley have negligible mass. The block slides on a frictionless, horizontal surface. Find an expression for the linear acceleration of the two objects.

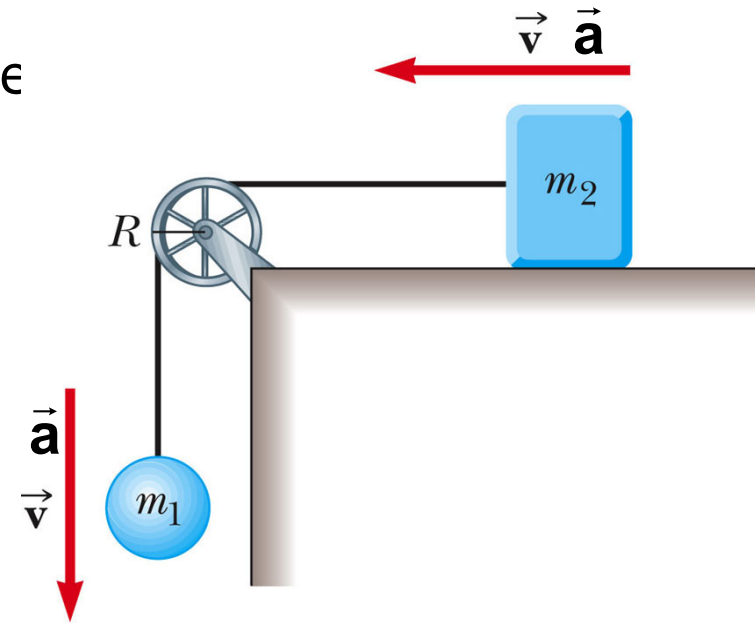


$$\sum \tau_{ext} = m_1 g R$$

Masses are connected by a light cord. Find the linear acceleration a .

- Use angular momentum approach
- No friction between m_2 and table
- Treat block, pulley and sphere as a non-isolated system rotating about pulley axis. As sphere falls, pulley rotates, block slides
- Constraints:

Equal v 's and a 's for block and sphere
 $v = \omega R$ for pulley $\alpha = d\omega / dt$
 $a = \alpha R = dv / dt$



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- Ignore internal forces, consider external forces only
- Net external torque on system: $\tau_{net} = m_1 g R$ about center of wheel
- Angular momentum of system: $L_{sys} = m_1 v R + m_2 v R + I\omega = m_1 v R + m_2 v R + MR^2 \omega$
(not constant)

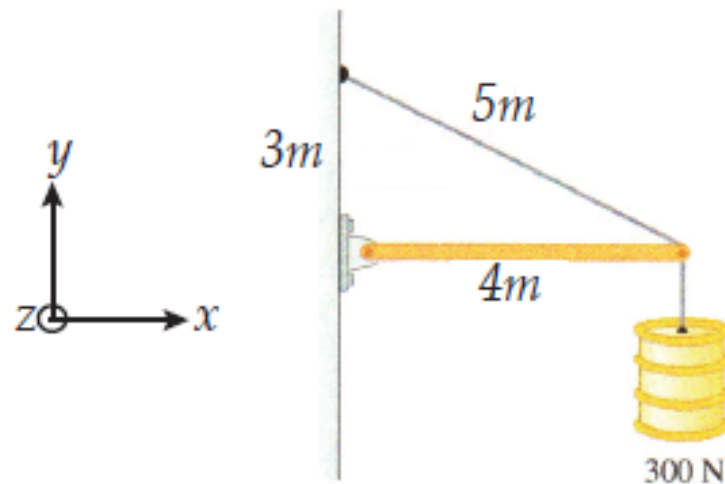
$$\frac{dL_{sys}}{dt} = m_1 a R + m_2 a R + MR^2 \alpha = (m_1 R + m_2 R + MR) a = \tau_{net} = m_1 g R$$

$$\therefore a = \frac{m_1 g}{M + m_1 + m_2}$$

same result followed from earlier
method using 3 FBD's & 2nd law

P1:

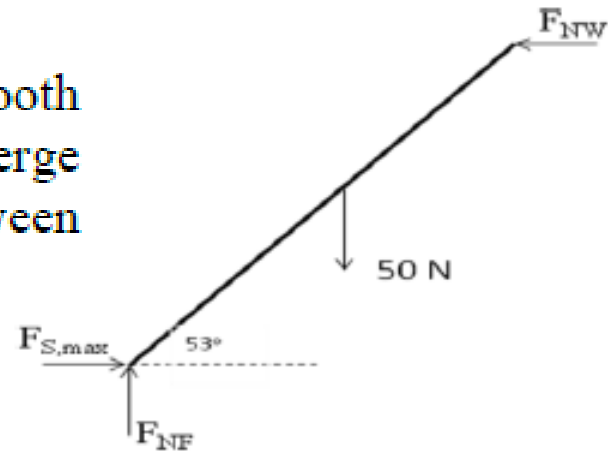
A uniform horizontal beam with a length of $l = 4.00\text{m}$ and a weight of $W_b = 150\text{N}$ is attached to a wall by a pin connection. Its far end is supported by a 5m long cable. Furthermore a weight of $W_A = 300\text{ N}$ is attached at the far end of the beam.



- (a) Find the magnitude of the tension in the cable. (4P)
- (b) Find magnitude and direction of the pin force exerted by the pin on the beam. (6P)

P2:

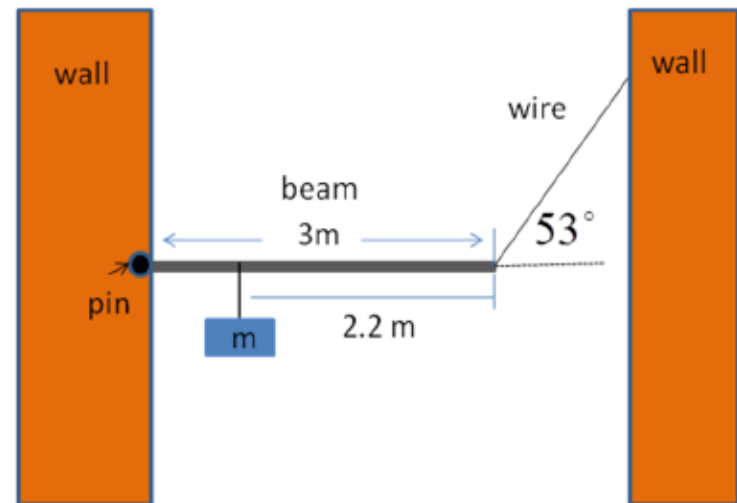
A 10 m ladder weighs 50 N and balances against a smooth wall at 53° to the horizontal. If the ladder is just on the verge of slipping, what is the static coefficient of friction between the floor and the ladder?



P3:

A uniform beam of mass $M = 12\text{ kg}$ and length $L = 3\text{ m}$ is mounted between two walls. The left end is connected to the left wall with a pin and the right end is connected to the right wall with a wire that makes an angle $\theta = 53^\circ$ with the horizontal. A small box of mass $m = 45\text{ kg}$ is hung 2.2 m from the right end of the beam, as shown in the figure. The beam is in static equilibrium.

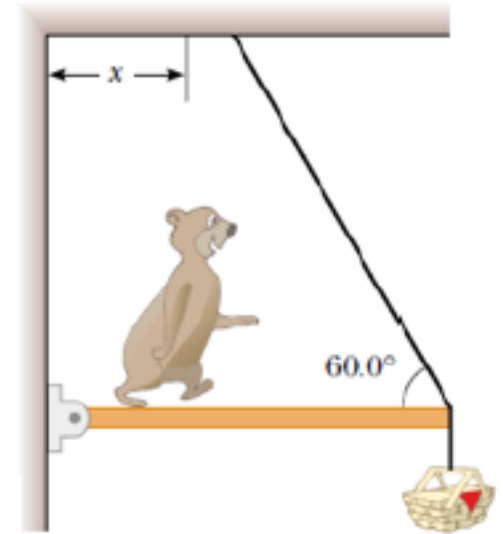
- Draw the FBD for the beam?
- Find the tension in the wire?
- Find the pin force vector exerted on the beam?



P4:

A uniform horizontal beam with length $\ell = 8m$ and weight $W_b = 300N$ is attached to the wall by a pin connection. Its far end is supported by a cable that makes an angle of $\theta = 60^\circ$ with the beam. Furthermore a picnic basket of weight $W_p = 100N$ is hanging from the far end of the beam. The hungry Yogy bear with a weight of $W_Y = 800N$ is standing on a beam at a distance $d = 2m$ from the wall.

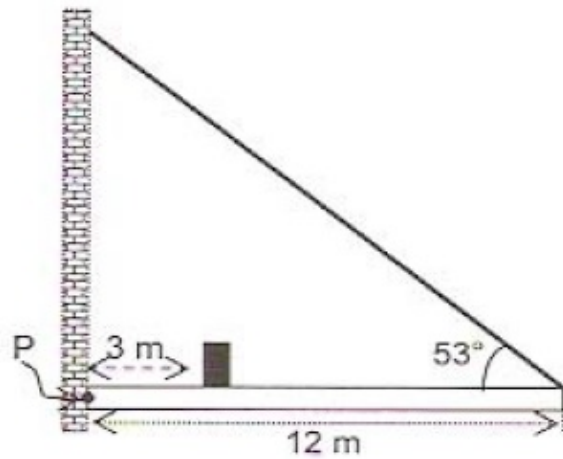
- (a) Calculate the magnitude of the tension in the cable.
- (b) Calculate the force vector exerted by the pin on the beam.



P5:

A 12 m long uniform beam with a weight of 400 N hinged at the wall at point P and supported by a cable as shown in the figure. A box weighing 600 N is set 3 m from the wall.

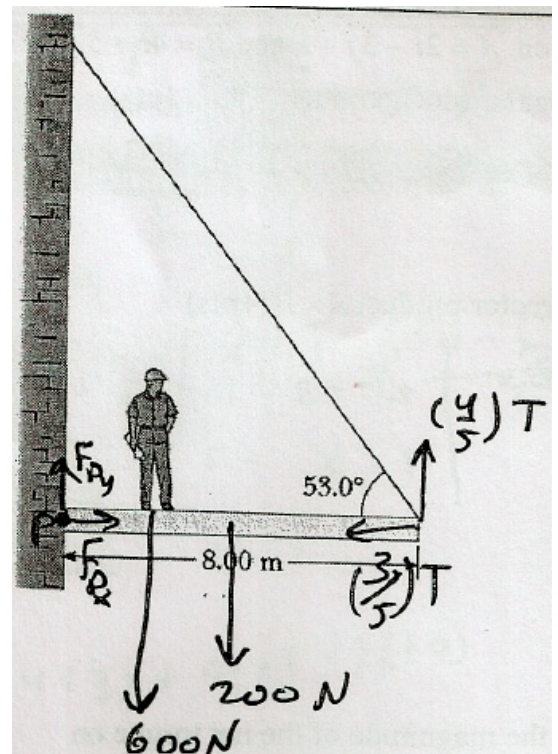
- Draw the free body diagram (FBD) of the beam and find the tension T in the cable.
- Calculate the components F_{Px} and F_{Py} of the reaction force on the beam by the wall at point P.



P6: A uniform horizontal beam of length $\ell = 8\text{ m}$ and weight $W_{\text{beam}} = 200\text{ N}$ is attached to a wall by a pin. Its far end is supported by a massless cable that makes an angle of 53° with the horizontal. If a 600 N man stands $d = 2\text{ m}$ from the wall, find:

a) the tension in the cable (5pts)

b) the forces exerted by the pin on the beam. (5pts)

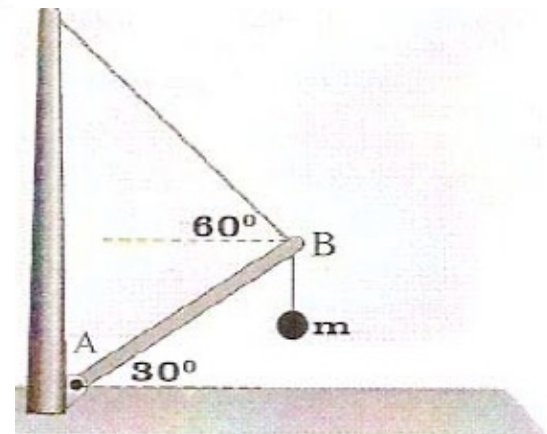


P7: An object of mass $m = 10\text{ kg}$ is supported by a rope attached to a 4 m long rod AB that can rotate at the base point A. If the mass of the rod is $M = 6\text{ kg}$,

(a) Calculate the tension in the cable between the rod and the wall, assuming the cable is holding the system in the position shown in the figure. (5 points)

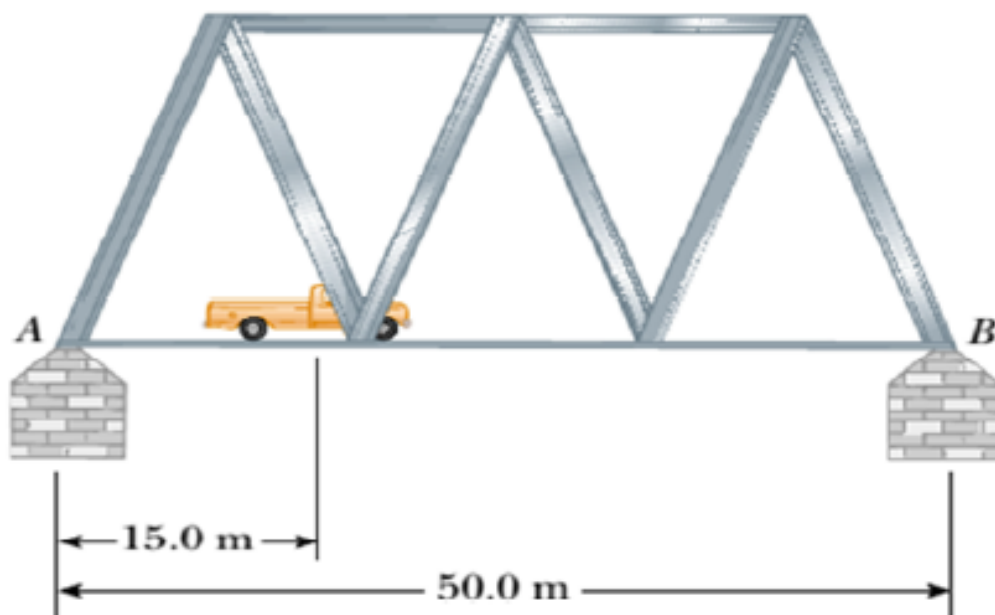
(b) Find the horizontal force, (2 points)

(c) and the vertical force exerted on the base of the rod.



P8:

A bridge of length 50.0 m and mass 8.00×10^4 kg is supported on a smooth pier at each end as in Figure P12.39. A truck of mass 3.00×10^4 kg is located 15.0 m from one end. What are the forces on the bridge at the points of support?

**ANSWERS**

$$5.98 \times 10^5 \text{ N}$$

and

$$4.80 \times 10^5 \text{ N}$$

P9:

A 1 200-N uniform boom is supported by a cable as in Figure P12.46. The boom is pivoted at the bottom, and a 2 000-N object hangs from its top. Find the tension in the cable and the components of the reaction force exerted by the floor on the boom.

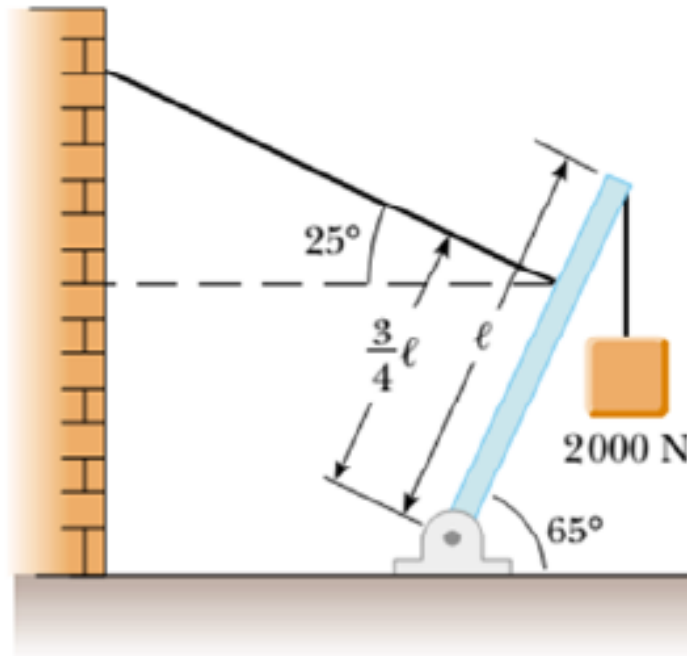


Figure P12.46

ANSWERS

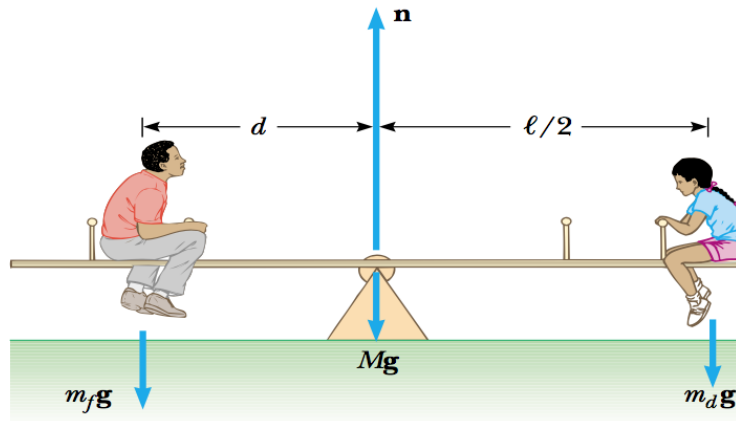
$$T = 1\,465\text{ N} = \boxed{1.46\text{ kN}}$$

$$H = 1\,328\text{ N (toward right)}$$

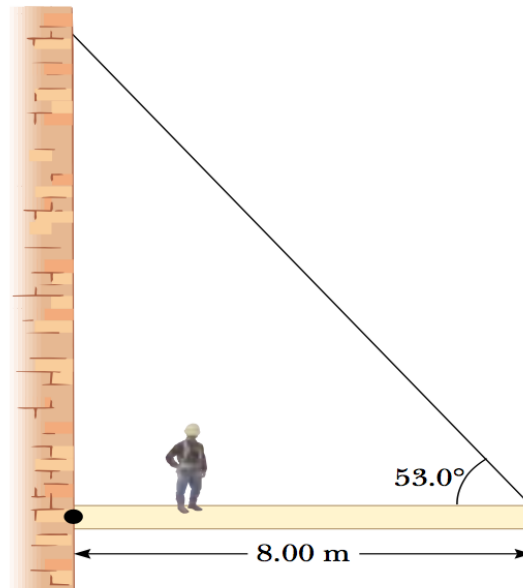
$$V = 2\,581\text{ N (upward)}$$

A seesaw consisting of a uniform board of mass M and length ℓ supports a father and daughter with masses m_f and m_d , respectively, as shown in Figure 12.8. The support (called the *fulcrum*) is under the center of gravity of the board, the father is a distance d from the center, and the daughter is a distance $\ell/2$ from the center.

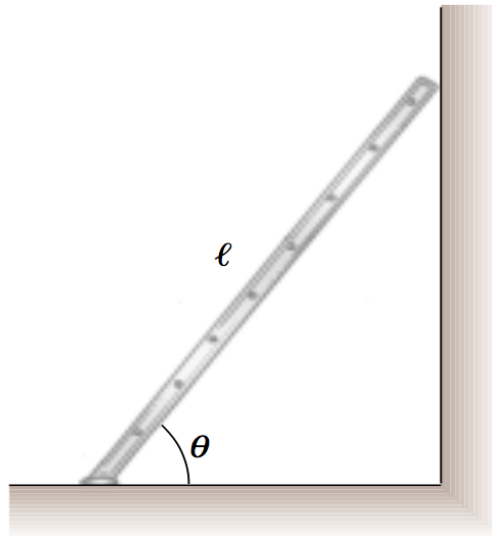
- (A)** Determine the magnitude of the upward force \mathbf{n} exerted by the support on the board.
- (B)** Determine where the father should sit to balance the system.



A uniform horizontal beam with a length of 8.00 m and a weight of 200 N is attached to a wall by a pin connection. Its far end is supported by a cable that makes an angle of 53.0° with the beam (Fig. 12.10a). If a 600-N person stands 2.00 m from the wall, find the tension in the cable as well as the magnitude and direction of the force exerted by the wall on the beam.



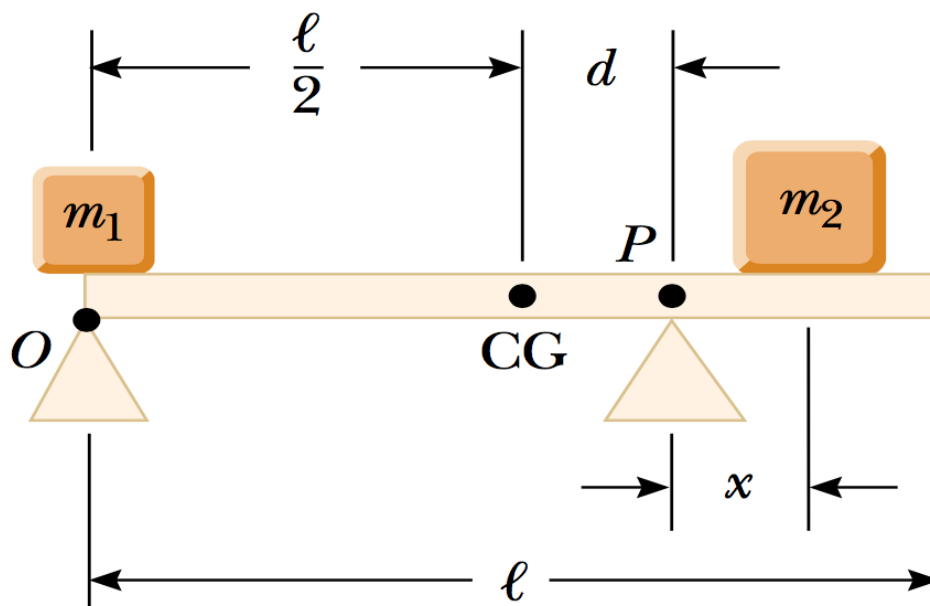
A uniform ladder of length ℓ rests against a smooth, vertical wall (Fig. 12.11a). If the mass of the ladder is m and the coefficient of static friction between the ladder and the ground is $\mu_s = 0.40$, find the minimum angle θ_{\min} at which the ladder does not slip.



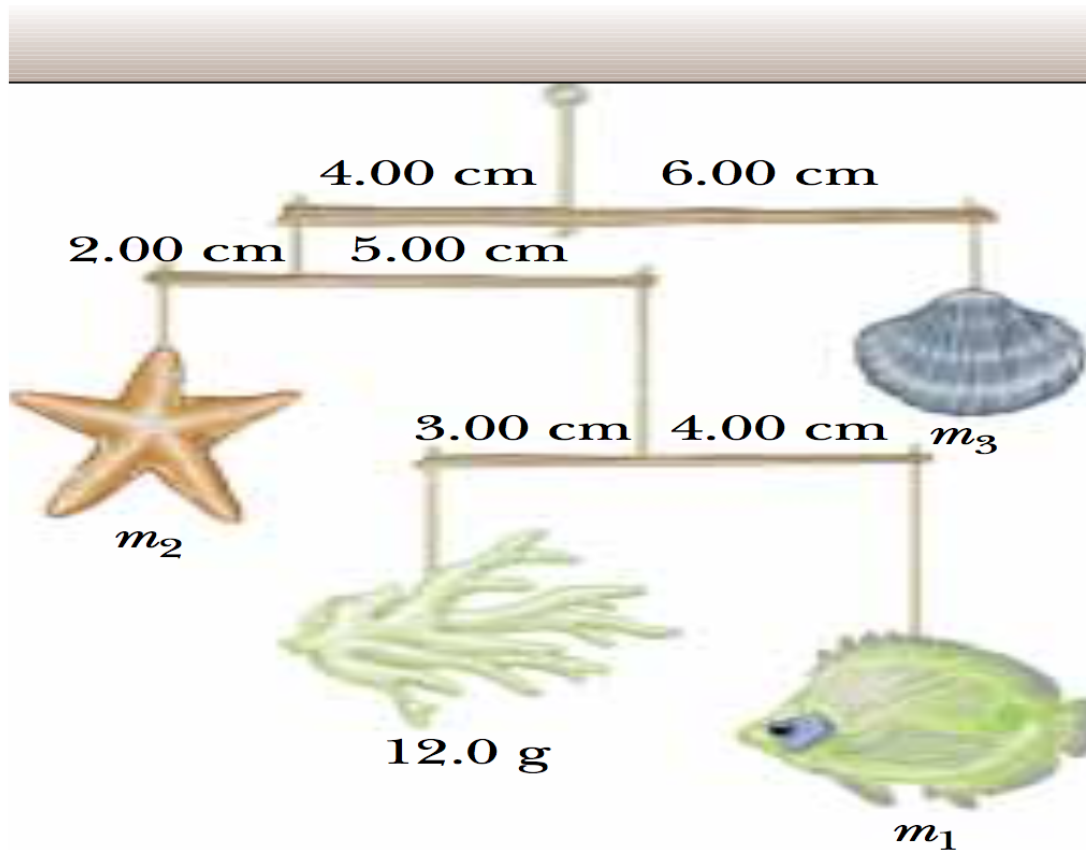
3.



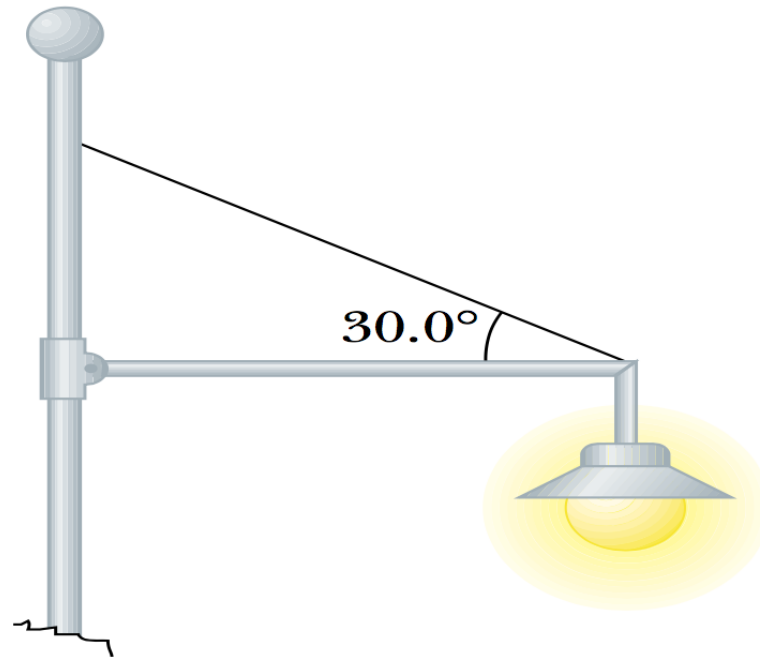
A uniform beam of mass m_b and length ℓ supports blocks with masses m_1 and m_2 at two positions, as in Figure P12.3. The beam rests on two knife edges. For what value of x will the beam be balanced at P such that the normal force at O is zero?



A mobile is constructed of light rods, light strings, and beach souvenirs, as shown in Figure P12.10. Determine the masses of the objects (a) m_1 , (b) m_2 , and (c) m_3 .

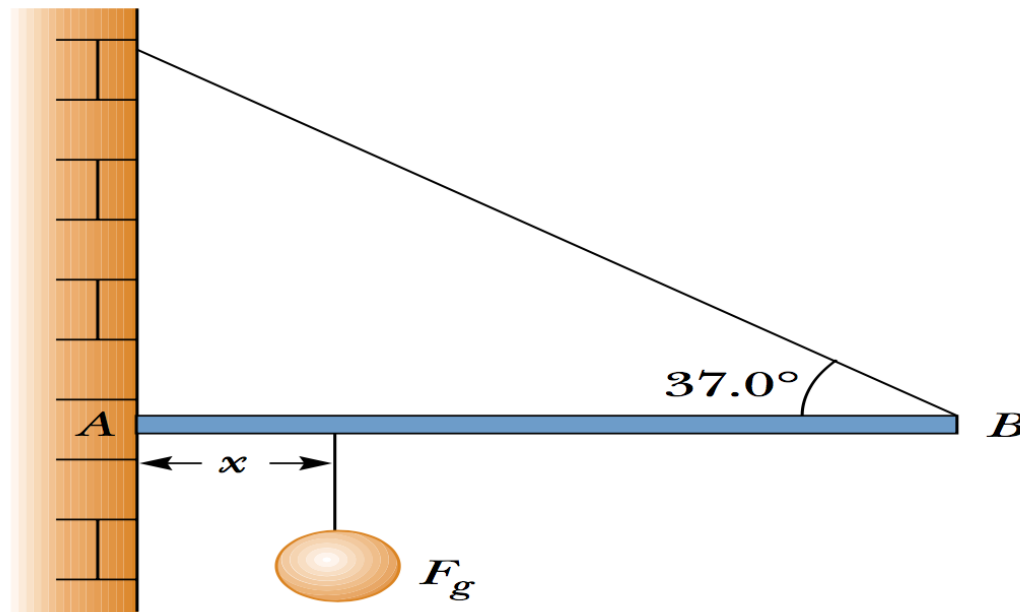


A 20.0-kg floodlight in a park is supported at the end of a horizontal beam of negligible mass that is hinged to a pole, as shown in Figure P12.12. A cable at an angle of 30.0° with the beam helps to support the light. Find (a) the tension in the cable and (b) the horizontal and vertical forces exerted on the beam by the pole.

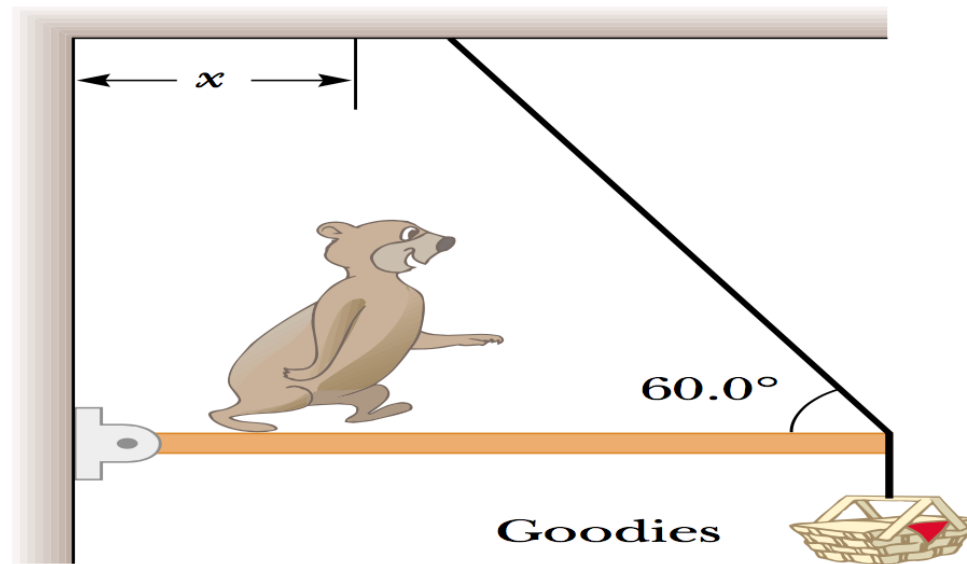


13. A 15.0-m uniform ladder weighing 500 N rests against a frictionless wall. The ladder makes a 60.0° angle with the horizontal. (a) Find the horizontal and vertical forces the ground exerts on the base of the ladder when an 800-N firefighter is 4.00 m from the bottom. (b) If the ladder is just on the verge of slipping when the firefighter is 9.00 m up, what is the coefficient of static friction between ladder and ground?

One end of a uniform 4.00-m-long rod of weight F_g is supported by a cable. The other end rests against the wall, where it is held by friction, as in Figure P12.23. The coefficient of static friction between the wall and the rod is $\mu_s = 0.500$. Determine the minimum distance x from point A at which an additional weight F_g (the same as the weight of the rod) can be hung without causing the rod to slip at point A.



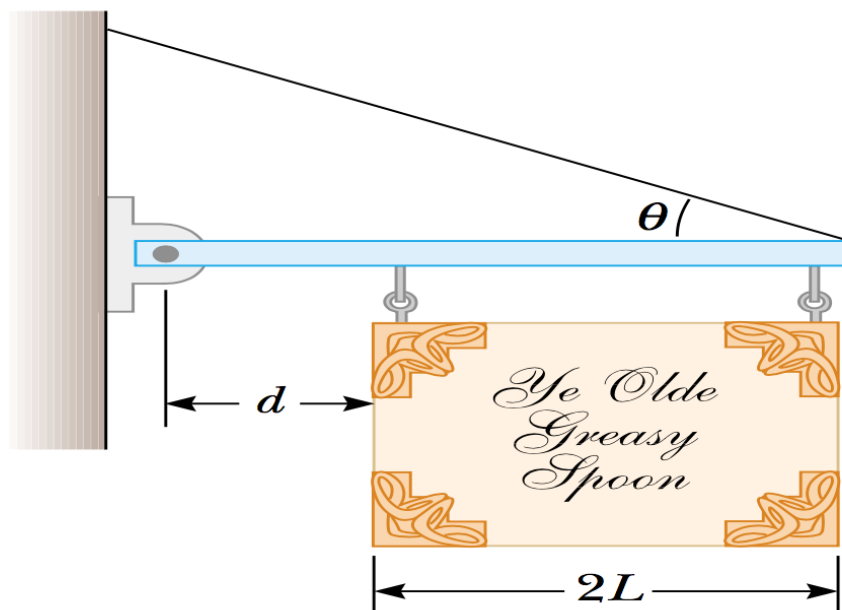
A hungry bear weighing 700 N walks out on a beam in an attempt to retrieve a basket of food hanging at the end of the beam (Fig. P12.43). The beam is uniform, weighs 200 N, and is 6.00 m long; the basket weighs 80.0 N. (a) Draw a free-body diagram for the beam. (b) When the bear is at $x = 1.00$ m, find the tension in the wire and the components of the force exerted by the wall on the left end of the beam. (c) **What If?** If the wire can withstand a maximum tension of 900 N, what is the maximum distance the bear can walk before the wire breaks?



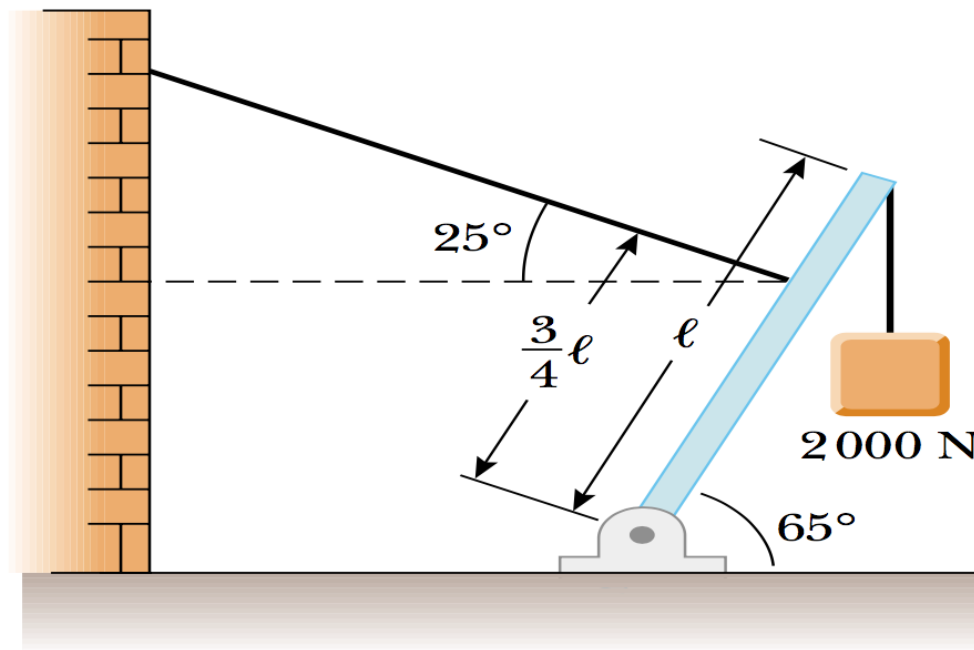
5.




A uniform sign of weight F_g and width $2L$ hangs from a light, horizontal beam, hinged at the wall and supported by a cable (Fig. P12.45). Determine (a) the tension in the cable and (b) the components of the reaction force exerted by the wall on the beam, in terms of F_g , d , L , and θ .



- 46.** A 1 200-N uniform boom is supported by a cable as in Figure P12.46. The boom is pivoted at the bottom, and a 2 000-N object hangs from its top. Find the tension in the cable and the components of the reaction force exerted by the floor on the boom.



]  A uniform sign of weight F_g and width $2L$ hangs from a light, horizontal beam, hinged at the wall and supported by a cable (Fig. P12.45). Determine (a) the tension in the cable and (b) the components of the reaction force exerted by the wall on the beam, in terms of F_g , d , L , and θ .

