EECS 142 Laboratory #3: Background Reading

Ladder Filter Design, Fabrication, and Measurement

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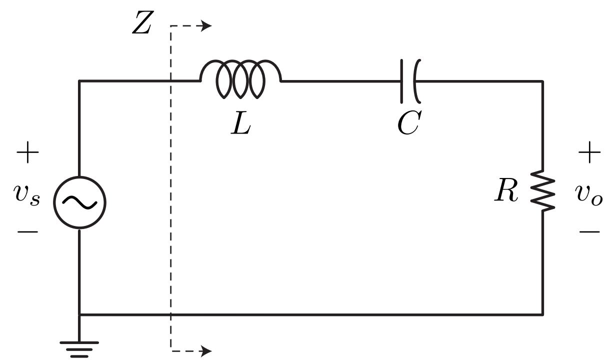

Figure 1: A series RLC circuit.

1 Introduction

In this laboratory you will design, simulate, and build LC low-pass and band-pass filters. You will characterize your fabricated filters using a *Network Anayzer* and an oscilloscope by measuring the two-port parameters of the filter. The fabricated filter will be compared against your simulations. The non-ideal effects of the components, such as loss and self-resonance, as well as the board parasitic inductance, capacitance, and delay, will be highlighted in the measurements.

1.1 Resonance

Because RLC circuits and resonance play such a crucial in filters, we begin with a review of this topic. These circuits are simple enough to allow full analysis, and yet rich enough to form the basis for most of the circuits we will study.

Series RLC Circuits

The RLC circuit shown in Fig. 1 is deceptively simple. The impedance seen by the source is simply given by

$$Z = j\omega L + \frac{1}{j\omega C} + R = R + j\omega L\left(1 - \frac{1}{\omega^2 LC}\right) \tag{1}$$

The impedance is purely real at at the *resonant frequency* when $\Im(Z) = 0$, or $\omega = \pm\frac{1}{\sqrt{LC}}$. At resonance the impedance takes on a minimal value. It's worthwhile to investigate the cause of resonance, or the cancellation of the reactive components due to the inductor and capacitor. Since the inductor and capacitor voltages are always 180° out of phase, and one reactance is dropping while the other is increasing, there is clearly always a frequency when the magnitudes are equal. Thus resonance occurs when $\omega L = \frac{1}{\omega C}$. A phasor diagram, shown in Fig. 2, shows this in detail.

So what's the magic about this circuit? The first observation is that at resonance, the voltage across the reactances can be larger, in fact much larger, than the voltage across the

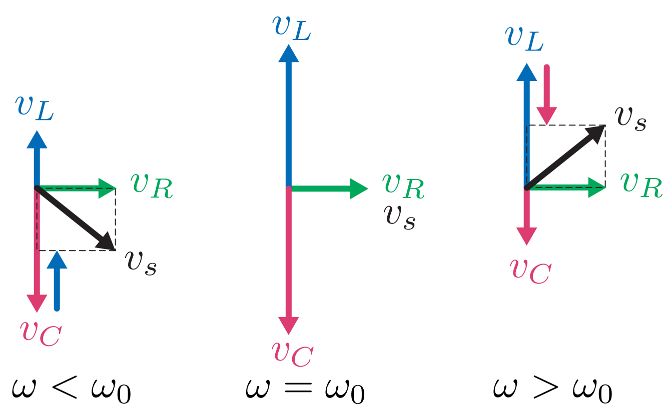

Figure 2: The phasor diagram of voltages in the series RLC circuit (a) below resonance, (b) at resonance, and (c) beyond resonance.

resistors R. In other words, this circuit has voltage gain. Of course it does not have power gain, for it is a passive circuit. The voltage across the inductor is given by

$$v_L = j\omega_0 L i = j\omega_0 L \frac{v_s}{Z(j\omega_0)} = j\omega_0 L \frac{v_s}{R} = jQ \times v_s \tag{2}$$

where we have defined a circuit Q factor at resonance as

$$Q = \frac{\omega_0 L}{R} \tag{3}$$

It's easy to show that the same voltage multiplication occurs across the capacitor

$$v_C = \frac{1}{j\omega_0 C} i = \frac{1}{j\omega_0 C} \frac{v_s}{Z(j\omega_0)} = \frac{1}{j\omega_0 C} \frac{v_s}{R} = -jQ \times v_s \tag{4}$$

This voltage multiplication property is the key feature of the circuit that allows it to be used as an impedance transformer.

It's important to distinguish this Q factor from the intrinsic Q of the inductor and capacitor. For now, we assume the inductor and capacitor are ideal. We can re-write the Q factor in several equivalent forms owing to the equality of the reactances at resonance

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C} \frac{1}{R} = \frac{\sqrt{LC}}{C} \frac{1}{R} = \sqrt{\frac{L}{C}} \frac{1}{R} = \frac{Z_0}{R} \tag{5}$$

where we have defined the $Z_0 = \sqrt{\frac{L}{C}}$ as the characteristic impedance of the circuit.

1.1.1 Circuit Transfer Function

Let's now examine the transfer function of the circuit

$$H(j\omega) = \frac{v_o}{v_s} = \frac{R}{j\omega L + \frac{1}{j\omega C} + R} \tag{6}$$

$$H(j\omega) = \frac{j\omega RC}{1 - \omega^2 LC + j\omega RC} \tag{7}$$

Obviously, the circuit cannot conduct DC current, so there is a zero in the transfer function. The denominator is a quadratic polynomial. It's worthwhile to put it into a standard form that quickly reveals important circuit parameters

$$H(j\omega) = \frac{j\omega \frac{R}{L}}{\frac{1}{LC} + (j\omega)^2 + j\omega \frac{R}{L}} \tag{8}$$

Using the definition of Q and ω_0 for the circuit

$$H(j\omega) = \frac{j\omega \frac{\omega_0}{Q}}{\omega_0^2 + (j\omega)^2 + j\frac{\omega\omega_0}{Q}} \tag{9}$$

Factoring the denominator with the assumption that $Q > \frac{1}{2}$ gives us the complex poles of the circuit

$$s^{\pm} = -\frac{\omega_0}{2Q} \pm j\omega_0 \sqrt{1 - \frac{1}{4Q^2}} \tag{10}$$

The poles have a constant magnitude equal to the resonant frequency

$$|s| = \sqrt{\cancel{\frac{\omega_0^2}{4Q^2}} + \omega_0^2 \left(1 - \cancel{\frac{1}{4Q^2}}\right)} = \omega_0 \tag{11}$$

A root-locus plot of the poles as a function of Q appears in Fig. 3. As $Q \to \infty$, the poles move to the imaginary axis. In fact, the real part of the poles is inversely related to the Q factor.

1.1.2 Circuit Bandwidth

As shown in Fig. 4, when we plot the magnitude of the transfer function, we see that the selectivity of the circuit is also related inversely to the Q factor. In the limit that $Q \to \infty$, the circuit is infinitely selective and only allows signals at resonance ω_0 to travel to the load. Note that the peak gain in the circuit is always unity, regardless of Q, since at resonance the L and C together disappear and effectively all the source voltage appears across the load.

The selectivity of the circuit lends itself well to filter applications. To characterize the peakiness, let's compute the frequency when the magnitude squared of the transfer function drops by half

$$|H(j\omega)|^2 = \frac{\left(\omega\frac{\omega_0}{Q}\right)^2}{\left(\omega_0^2 - \omega^2\right)^2 + \left(\omega\frac{\omega_0}{Q}\right)^2} = \frac{1}{2} \tag{12}$$

This happens when

$$\left(\frac{\omega_0^2 - \omega^2}{\omega_0\omega/Q}\right)^2 = 1 \tag{13}$$

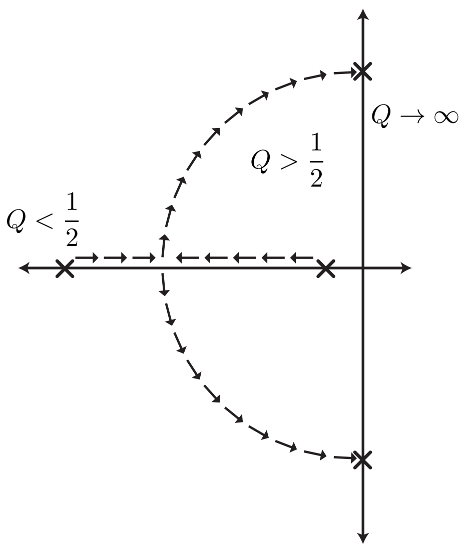

Figure 3: The root locus of the poles for a second-order transfer function as a function of Q. The poles begin on the real axis for $Q < \frac{1}{2}$ and become complex, tracing a semi-circle for increasing Q.

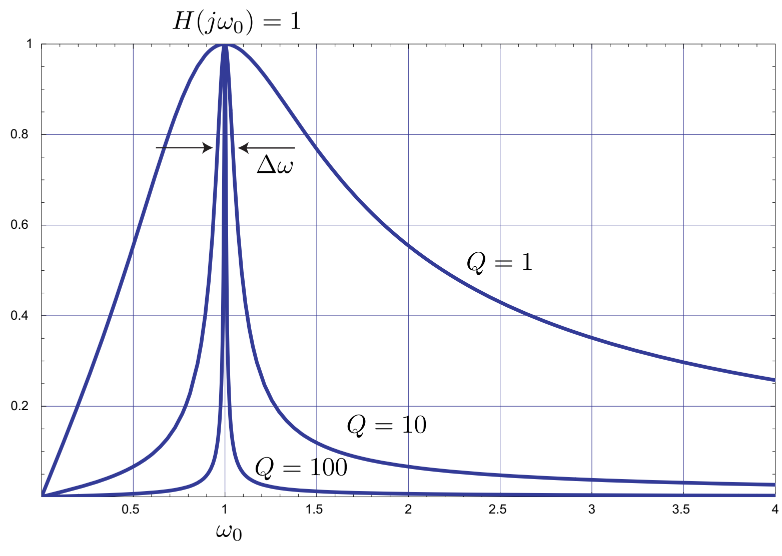

Figure 4: The transfer function of a series RLC circuit. The output voltage is taken at the resistor terminals. Increasing Q leads to a more peaky response.

Solving the above equation yields four solutions, corresponding to two positive and two negative frequencies. The peakiness is characterized by the difference between these frequencies, or the bandwidth, given by

$$\Delta\omega = \omega_+ - \omega_- = \frac{\omega_0}{Q} \tag{14}$$

which shows that the normalized bandwidth is inversely proportional to the circuit Q

$$\frac{\Delta\omega}{\omega_0} = \frac{1}{Q} \tag{15}$$

You can also show that the resonance frequency is the geometric mean frequency of the 3 dB frequencies

$$\omega_0 = \sqrt{\omega_+\omega_-} \tag{16}$$

Energy Storage in RLC "Tank"

Let's compute the ebb and flow of the energy at resonance. To begin, let's assume that there is negligible loss in the circuit. The energy across the inductor is given by

$$w_L = \tfrac{1}{2}Li^2(t) = \tfrac{1}{2}LI_M^2\cos^2\omega_0 t \tag{17}$$

Likewise, the energy stored in the capacitor is given by

$$w_C = \tfrac{1}{2}Cv_C^2(t) = \tfrac{1}{2}C\left(\frac{1}{C}\int i(\tau)d\tau\right)^2 \tag{18}$$

Performing the integral leads to

$$w_C = \tfrac{1}{2}\frac{I_M^2}{\omega_0^2 C}\sin^2\omega_0 t \tag{19}$$

The total energy *stored* in the circuit is the sum of these terms

$$w_s = w_L + w_C = \tfrac{1}{2}I_M^2\left(L\cos^2\omega_0 t + \frac{1}{\omega_0^2 C}\sin^2\omega_0 t\right) = \tfrac{1}{2}I_M^2 L \tag{20}$$

which is a constant! This means that the reactive stored energy in the circuit does not change and simply moves between capacitive energy and inductive energy. When the current is maximum across the inductor, all the energy is in fact stored in the inductor

$$w_{L,\max} = w_s = \tfrac{1}{2}I_M^2 L \tag{21}$$

Likewise, the peak energy in the capacitor occurs when the current in the circuit drops to zero

$$w_{C,\max} = w_s = \tfrac{1}{2}V_M^2 C \tag{22}$$

Now let's re-introduce loss in the circuit. In each cycle, a resistor R will dissipate energy

$$w_d = P \cdot T = \tfrac{1}{2}I_M^2 R \cdot \frac{2\pi}{\omega_0} \tag{23}$$

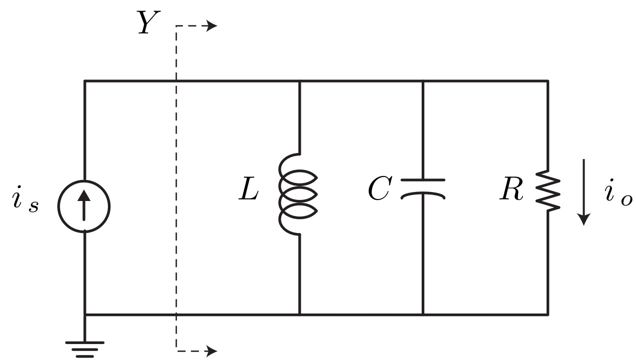

Figure 5: A parallel RLC circuit.

The ratio of the energy stored to the energy dissipated is thus

$$\frac{w_s}{w_d} = \frac{\frac{1}{2}LI_M^2}{\frac{1}{2}I_M^2 R \frac{2\pi}{\omega_0}} = \frac{\omega_0 L}{R}\frac{1}{2\pi} = \frac{Q}{2\pi} \tag{24}$$

This gives us the physical interpretation of the *Quality Factor* Q as 2π times the ratio of energy stored per cycle to energy dissipated per cycle in an RLC circuit

$$Q = 2\pi \frac{w_s}{w_d} \tag{25}$$

We can now see that if $Q \gg 1$, then an initial energy in the tank tends to slosh back and forth for many cycles. In fact, we can see that in roughly Q cycles, the energy of the tank is depleted.

Parallel RLC Circuits

The parallel RLC circuit shown in Fig. 5 is the dual of the series circuit. By "dual" we mean that the role of voltage and current are interchanged. Hence the circuit is most naturally probed with a current source i_s. In other words, the circuit has current gain as opposed to voltage gain, and the admittance minimizes at resonance as opposed to the impedance. Finally, the role of capacitance and inductance are also interchanged. In principle, therefore, we don't have to repeat all the detailed calculations we just performed for the series case, but in practice it's worthwhile exercise.

The admittance of the circuit is given by

$$Y = j\omega C + \frac{1}{j\omega L} + G = G + j\omega C\left(1 - \frac{1}{\omega^2 LC}\right) \tag{26}$$

which has the same form as Eq. 1. The resonant frequency also occurs when $\Im(Y) = 0$, or when $\omega = \omega_0 = \pm\frac{1}{\sqrt{LC}}$. Likewise, at resonance the admittance takes on a minimal value. Equivalently, the impedance at resonance is maximum. This property makes the parallel

RLC circuit an important element in tuned amplifier loads. It's also easy to show that at resonance the circuit has a current gain of Q

$$i_C = j\omega_0 C v_o = j\omega_0 C \frac{i_s}{Y(j\omega_0)} = j\omega_0 C \frac{i_s}{G} = jQ \times i_s \tag{27}$$

where we have defined the circuit Q factor at resonance by

$$Q = \frac{\omega_0 C}{G} \tag{28}$$

in complete analogy with Eq. 3. Likewise, the current gain through the inductor is also easily derived

$$i_L = -jQ \times i_s \tag{29}$$

The equivalent expressions for the circuit Q factor are given by the inverse of the relations of Eq. 5

$$Q = \frac{\omega_0 C}{G} = \frac{R}{\omega_0 L} = \frac{R}{\frac{1}{\sqrt{LC}} L} = \frac{R}{\sqrt{\frac{L}{C}}} = \frac{R}{Z_0} \tag{30}$$

The phase response of a resonant circuit is also related to the Q factor. For the parallel RLC circuit the phase of the admittance is given by

$$\angle Y(j\omega) = \tan^{-1}\left(\frac{\omega C \left(1 - \frac{1}{\omega^2 LC}\right)}{G}\right) \tag{31}$$

The rate of change of phase at resonance is given by

$$\left.\frac{d\angle Y(j\omega)}{d\omega}\right|_{\omega_0} = \frac{2Q}{\omega_0} \tag{32}$$

A plot of the admittance phase as a function of frequency and Q is shown in Fig. 6.

1.1.3 Circuit Transfer Function

Given the duality of the series and parallel RLC circuits, it's easy to deduce the behavior of the circuit. Whereas the series RLC circuit acted as a filter and was only sensitive to voltages near resonance ω_0, likewise the parallel RLC circuit is only sensitive to currents near resonance

$$H(j\omega) = \frac{i_o}{i_s} = \frac{v_o G}{v_o Y(j\omega)} = \frac{G}{j\omega C + \frac{1}{j\omega L} + G} \tag{33}$$

which can be put into the same canonical form as before

$$H(j\omega) = \frac{j\omega \frac{\omega_0}{Q}}{\omega_0^2 + (j\omega)^2 + j\frac{\omega\omega_0}{Q}} \tag{34}$$

where we have appropriately re-defined the circuit Q to correspond the parallel RLC circuit. Notice that the impedance of the circuit takes on the same form

$$Z(j\omega) = \frac{1}{Y(j\omega)} = \frac{1}{j\omega C + \frac{1}{j\omega L} + G} \tag{35}$$

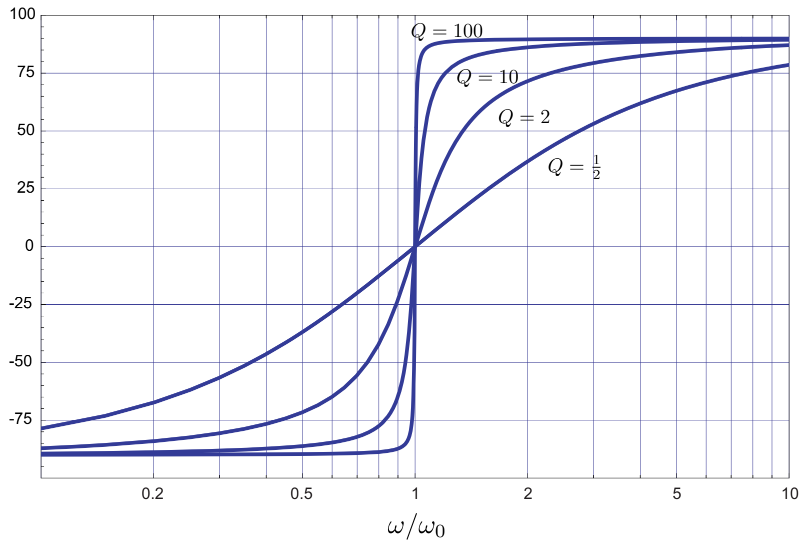

Figure 6: The phase of a second order admittance as function of frequency. The rate of change of phase at resonance is proportional to the Q factor.

which can be simplified to

$$Z(j\omega) = \frac{j\frac{\omega}{\omega_0}\frac{1}{GQ}}{1 + \left(\frac{j\omega}{\omega_0}\right)^2 + j\frac{\omega}{\omega_0 Q}} \tag{36}$$

At resonance, the real terms in the denominator cancel

$$Z(j\omega_0) = \frac{j\frac{R}{Q}}{\underbrace{1 + \left(\frac{j\omega_0}{\omega_0}\right)^2}_{=0} + j\frac{1}{Q}} = R \tag{37}$$

It's not hard to see that this circuit has the same half power bandwidth as the series RLC circuit, since the denominator has the same functional form

$$\frac{\Delta\omega}{\omega_0} = \frac{1}{Q} \tag{38}$$

A plot of this impedance versus frequency has the same form as Fig. 4 multiplied by the resistance R.

Energy storage in a parallel RLC circuit is completely analogous to the series RLC case and in fact the general equation relating circuit Q to energy storage and dissipation also holds in the parallel RLC circuit.

2 The Many Faces of Q

As we have seen, in RLC circuits the most important parameter is the circuit Q and resonance frequency ω_0. Not only do these parameters describe the circuit in a general way, but

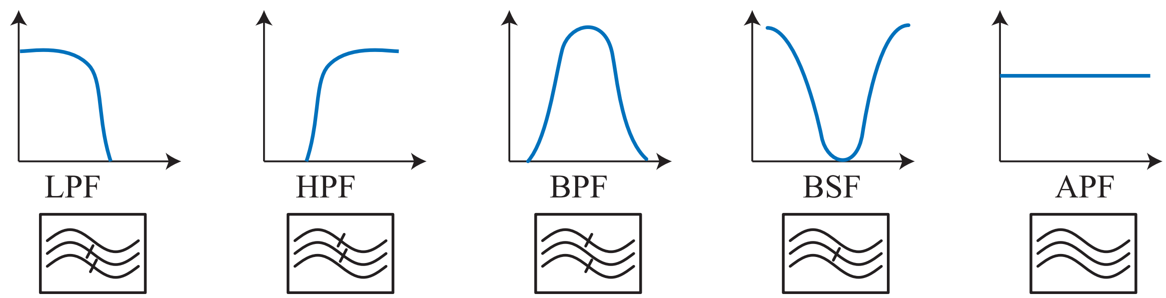

Figure 7: Standard filter type include the (a) low-pass filter, (b) high-pass filter, (c) band-pass filter, (d) band reject filter, and (e) all-pass filter.

they also give us immediate insight into the circuit behavior.

The Q factor can be computed several ways, depending on the application. For instance, if the circuit is designed as a filter, then the most important Q relation is the half-power bandwidth

$$Q = \frac{\omega_0}{\Delta\omega} \tag{39}$$

We shall also find many applications where the phase selectivity of these circuits is of importance. An example is a resonant oscillator where the noise of the system is rejected by the tank based on the phase selectivity. In an oscillator any "excess phase" in the loop tends to move the oscillator away from the natural resonant frequency. It is therefore desirable to maximize the rate of change of phase of the circuit impedance as a function of frequency. For the parallel RLC circuit we derived the phase of the admittance (Eq. 32) which gives us another way to interpret and compute Q

$$Q = \frac{\omega_0}{2}\frac{d\angle Y(j\omega)}{d\omega} \tag{40}$$

For applications where the circuit is used as a voltage or current multiplier, the ratio of reactive voltage (current) to real voltage (current) is most relevant. For a series case we found

$$Q = \frac{v_L}{v_R} = \frac{v_C}{v_R} \tag{41}$$

and for the parallel case

$$Q = \frac{i_L}{i_R} = \frac{i_C}{i_R} \tag{42}$$

The last and one of the most important interpretations of Q is in the definition of energy, relating the energy storage and losses in a RLC "tank" circuit. We can define the of Q a circuit at frequency ω as the energy stored in the tank W divided by the rate of energy loss

$$Q = W/\frac{dW}{d\phi} = \omega W/\frac{dW}{dt} \tag{43}$$

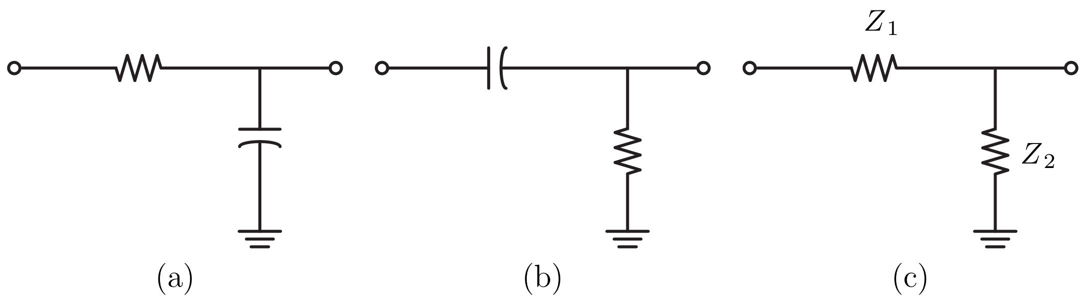

Figure 8: (a) A simple RC low-pass filter. (b) A simple RC high-pass filter. (c) An arbitrary filter viewed as a voltage divider.

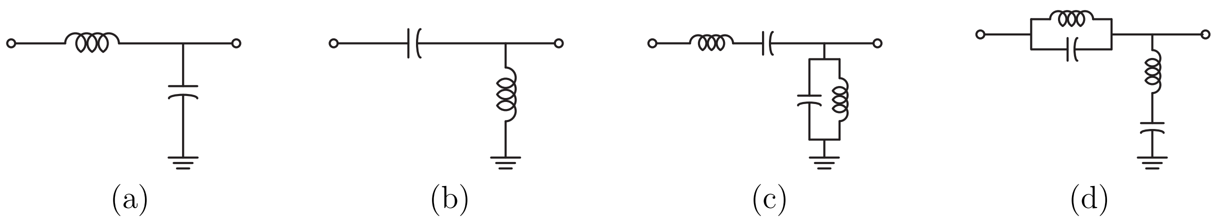

Figure 9: (a) An LC low-pass filter. (b) An LC high-pass filter. (c) An LC pass band filter. (d) An LC band reject filter.

2.1 Filter Frequency Response

Filters are key building blocks in communication systems. Common filters that you are no doubt familiar with include low-pass filters (LPF), high-pass filters (HPF), and band-pass filters (BPF). Less common, but equally important, include band-reject filters and all-pass filters. Common notation and ideal filter magnitude responses for these various filters are shown in Fig. 7. The all-pass filter may seem useless at first, but is actually quite useful when we examine the filter's phase transfer characteristic. The filter can be used to equalize the phase response of a distorted signal.

A LPF passes the lower frequencies to the output and attenuates the high frequency components beyond the cut-off frequency. The simplest low-pass filter consist of an RC circuit shown in Fig. 8a, which attenuates the signal at 20 dB/decade beyond the cutoff, a rather gentle roll-off. To improve the roll-off, the LC filter shown in Fig. 9a can be used. Intuitively, at low frequencies the inductor is a short and the capacitor is open, so the signal is coupled to the output. At resonance the voltage across the inductor/capacitor equal, which sets the 3-dB frequency. At high frequencies, though, the inductor reactance increases and the capacitor reactance decreases, decoupling the input from the output and tending to short the output signals to ground. The transfer function of this filter is given by analyzing the circuit as a voltage divider (Fig. 8c), where $Z_1 = Z_C||R_L$ and $Z_2 = j\omega L + R_S$

$$\frac{V_2}{V_1} = \frac{Z_C||R_L}{Z_C||R_L + j\omega L + R_S} \tag{44}$$

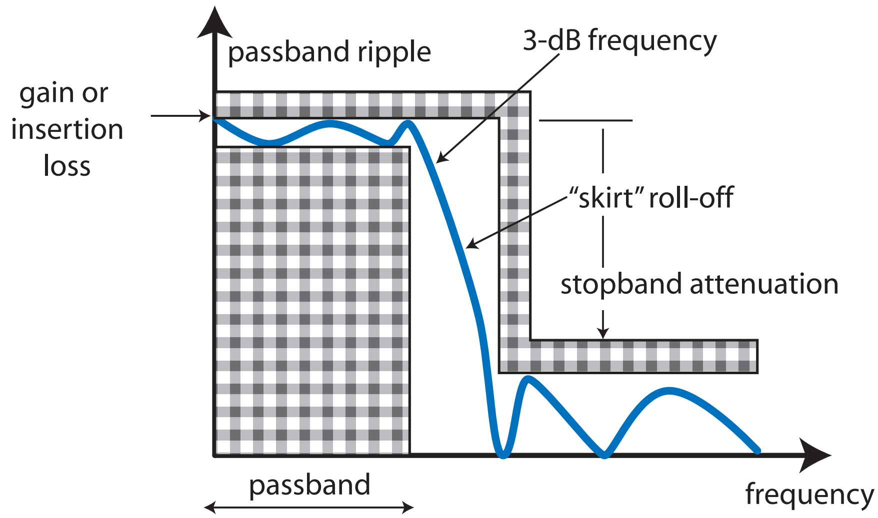

Figure 10: The desired filter response "mask" and the transfer characteristics of a filter.

Assume that $R_0 = R_L = R_S$

$$= \frac{1}{1 + (1 + j\omega L/R_0)(1 + j\omega R_0 C)} \tag{45}$$

which shows that the filter has two complex poles, causing a roll-off of 40 dB/dec. The steepness and bandwidth of the filter near the cutoff frequency is controlled quality factor Q of the transfer function.

This filter can be converted to a high-pass filter by interchanging the inductor and the capacitor, which rejects the low frequencies due to the capacitor coupling and the inductor shunting the signal to ground, and passes the high frequencies, as shown in Fig. 9b. How do we realize a band-pass filter? If we view the circuit as an arbitrary voltage divider, notice that in the passband of the filter the impedance Z_2 is shorted and the impedance Z_1 is opened whereas the opposite occurs ideally in the stop-band. A short circuit is realized at an arbitrary frequency using a series LC circuit and an open circuit is realized with a parallel LC circuit. Putting these ideas together leads to Fig. 9c, a passband filter.

By the same argument, a band reject filter is realized by interchanging the role of the Z_1 and Z_2, as shown in Fig. 9d. We see that simple LC resonant circuits are extremely versatile and form the core of an entire family of filters. As we shall see this powerful insight can be extended even further.

How do we select the various component values to realize a given filter response? Well, the answer depends on what you are trying to achieve. A filter is typically characterized by specifying the filter mask shown in Fig. 10. The mask is characterized by the following parameters: corner frequency or 3-dB bandwidth, pass-band ripple, which measures how much the in-band signal gain varies (which leads to distortion), the stop-band rejection, and the steepness of the "skirt" of the filter (the transition region of the filter). The filter roll-off is related to the filter order, or the number of poles. Other important metrics include the group delay (see below), insertion loss of the filter (rather than the ripple), and the stop-band

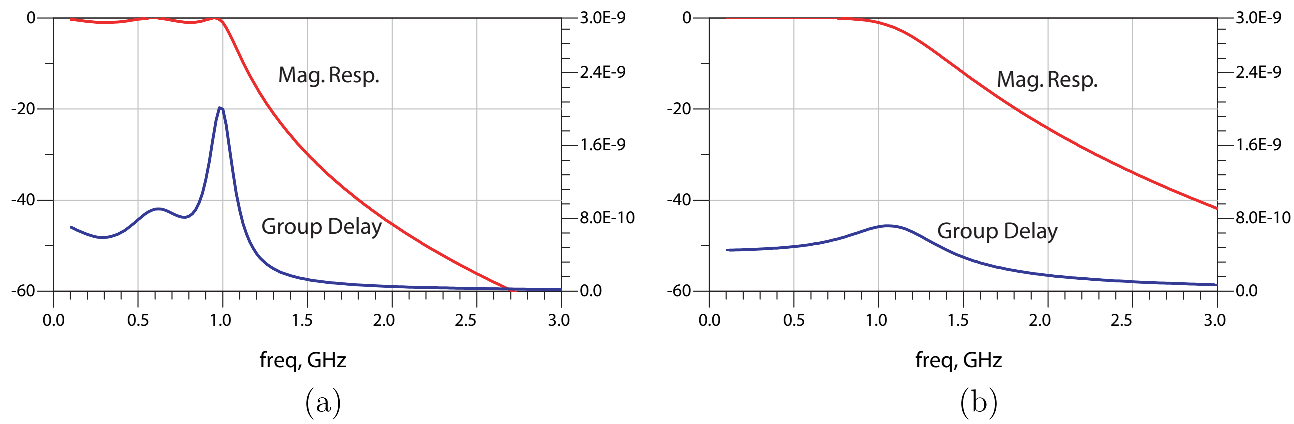

Figure 11: (a) A sharp roll-off filter has more than 40-dB attenuation at 2 GHz and poor group delay (Chebyshev order 5) whereas (b) a soft roll-off filter has much smaller group delay (Butterworth order 5).

ripple (only for certain filter types). Some filters have transmission zeros, which cause the filter response to go to zero at specific frequencies in the stop-band. Insertion loss is related to the fact that real inductors/capacitors have loss, and some of the input energy is absorbed by the network and converted to heat.

An ideal filter would delay all frequency components of the signal by the same amount, i.e. the filter would have a constant *group delay*

$$\tau_g = -\frac{d\angle H}{d\omega} \tag{46}$$

Any variation in the group delay leads to distortion since different components of the signal arrive at different times. Notice that an ideal filter should therefore have a linear phase response or a flat (constant) group delay. To see this, notice that a distortionless filter should preserve the waveform shape, which means that the output can only be a scaled and delayed version of the input signal (for the band of interest). The transfer function for such a filter takes the form

$$H(s) = |H_0|e^{-sT} \tag{47}$$

where H_0 is the scaling factor and T is the delay of the filter. It is therefore important for the filter approximate this response in the band of interest, which means minimizing the ripple in the amplitude response and realizing a linear phase response (or constant group delay).

In general there is a trade-off in the filter attenuation characteristics and the group delay, which means that more out-of-band magnitude attenuation results in more phase distortion in-band and vice-versa. In Fig. 11 we compare two filters which differ in their attenuation rate; notice that the filter with higher attenuation has considerably worse group delay variation. An all-pass filter can be used to compensate for the phase distortion of a given filter, or to compensate for the phase distortion of a transmission medium (such as a cable or device with poor frequency response).

A simple way to observe the distortion caused by the non-constant group delay is to plot the step response of the filter. Since the step transition has high frequency components

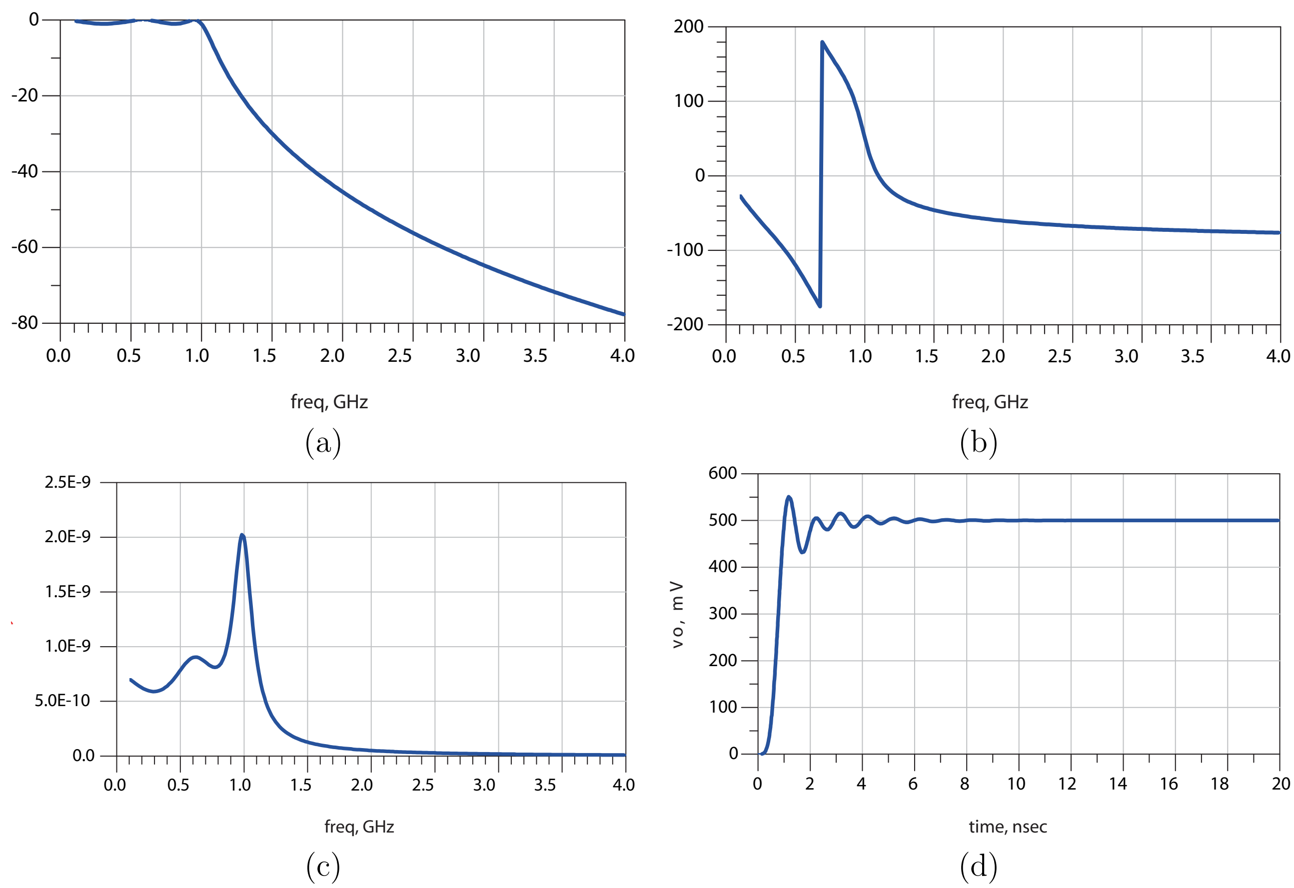

Figure 12: (a) The magnitude, (b) phase response, (c) group delay and (d) step response of a filter.

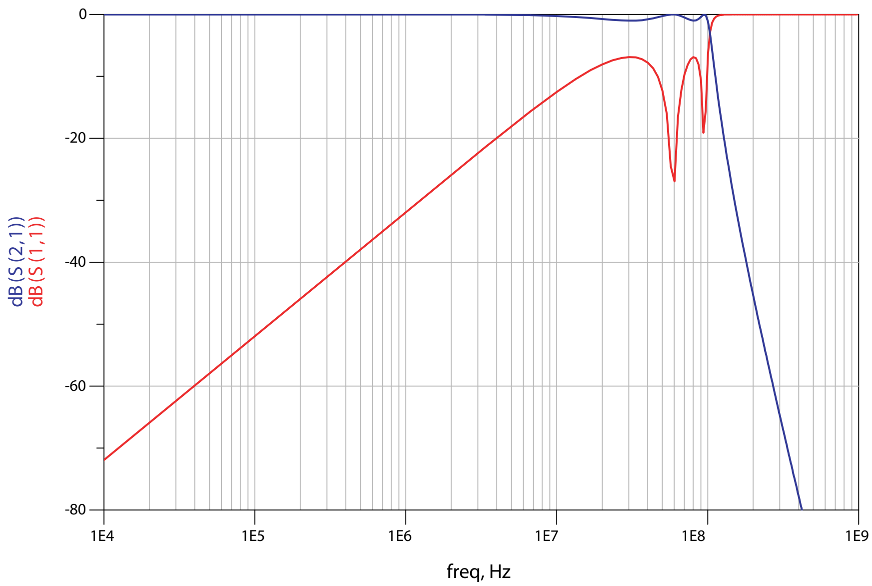

Figure 13: The input reflection coefficient s_{11} and transfer coefficient s_{21} for a fifth-order Chebyshev filter.

which must all arrive at the same instant, any deviation from a linear phase response leads to distortion in the waveform, as shown in Fig. 12.

Filters are two-port elements and thus a full characterization requires the specification of four complex parameters. If a filter is realized with only passive elements, then the two-port is reciprocal and $z_{12} = z_{21}$. Many filters are also symmetric so $z_{11} = z_{22}$. In these cases the filter is fully characterized by two complex frequency dependent parameters. Most commonly filters are characterized in terms of their scattering parameters since this is how the filters are measured. The input reflection coefficient, or s_{11}, is particulary important since the energy transferred into the filter is given by

$$P_{in} = 1 - |\Gamma(\omega)|^2 = 1 - |s_{11}|^2 \tag{48}$$

which means in the pass-band we desire $|s_{11}| \approx 0$ (no power reflected) whereas in the stop-band we desire $|s_{11}| \approx 1$, which means that all the incident power is reflected. If the filter is realized with lossless or low-loss elements, then the input power is actually the power delivered to the load. A filter characterized in this way is shown in Fig. 13, where the input reflection s_{11} and transmission s_{21} are plotted versus frequency.

The filter transfer characteristics are measured or calculated through s_{21}. For an ideal filter $|s_{21}| \approx 1$ in the passband and $|s_{21}| \approx 0$ in the stop-band. For any real filter, there is some insertion loss due to the inevitable resistance in the components, and the magnitude of $|s_{21}|$ in the passband indicates this loss (under matched conditions).

2.2 Ladder Filters

The concept of a voltage divider can be extended, as shown in Fig. 14a, to even realize higher out-of-band attenuation. The buffer is used to isolate the two filters and so the overall

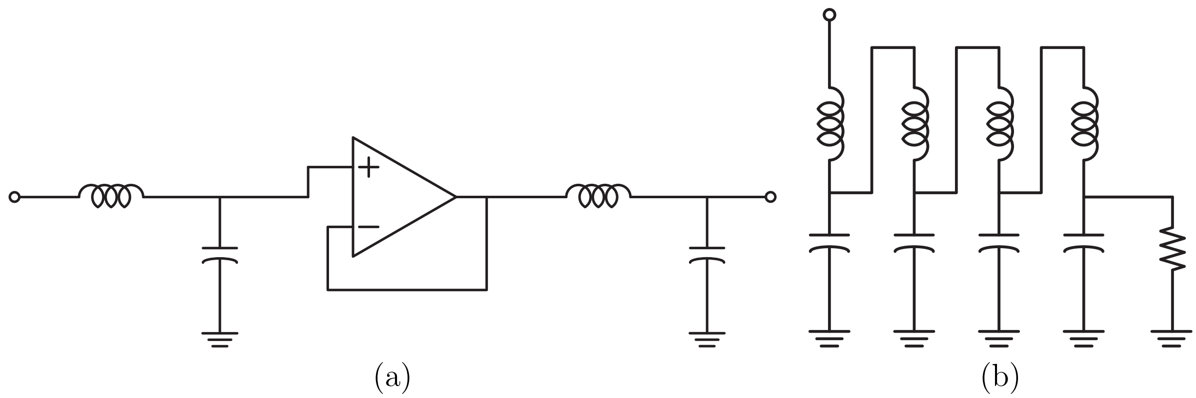

Figure 14: (a) Cascading simple filters results in higher order roll-off characteristics. (b) Cascaded filters without the buffer.

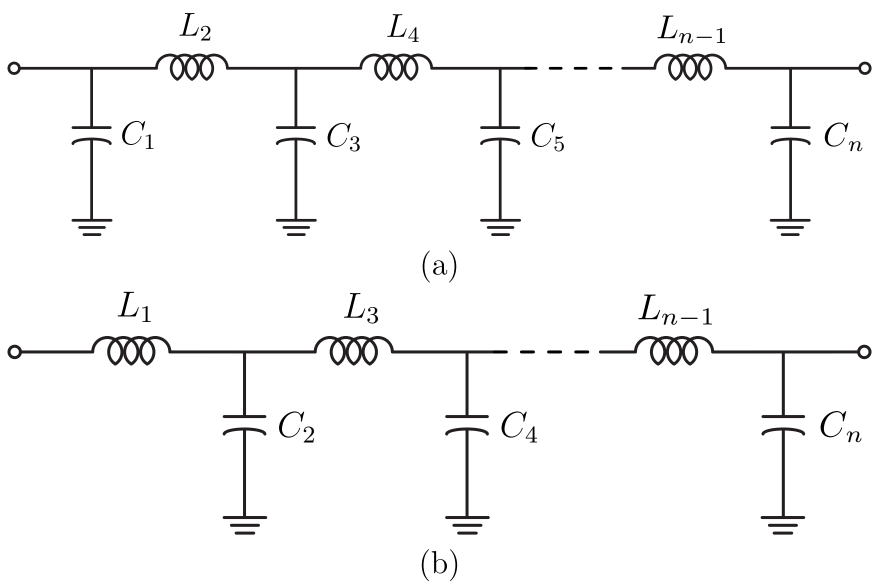

Figure 15: (a) The canonical ladder low-pass filter of order n (odd). (b) The canonical ladder filter of order n (even).

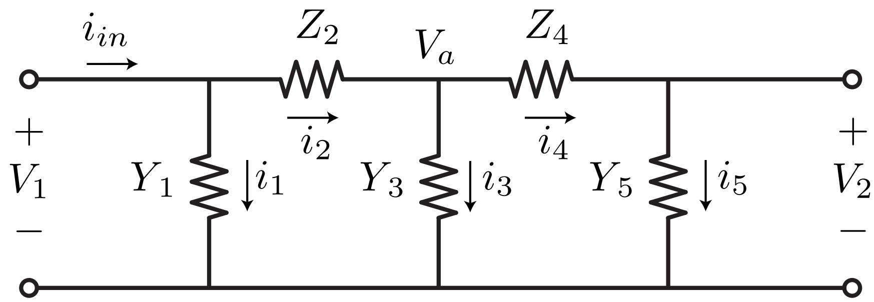

Figure 16: (a) An arbitrary fifth-order ladder filter structure.

transfer function is a cascade of the individual transfer functions, doubling the attenuation from 40 dB/dec to 80 dB/dec, a significant improvement. In actual practice, the same effect can be realized without the buffer, as shown in Fig. 14b, except now the transfer function is more complicated but has the same order. All of the filters discussed up to this point can in fact be extended in this fashion to realize higher order filters. The order of the filter correspond to the number of "rungs" in the ladder filter, redrawn in standard form in Fig. 15. This canonical ladder filter structure can be converted from low-pass to high-pass, band-pass, or band-stop by the following simple transformations:

- LP $\rightarrow$ HP: $L \rightarrow C, C \rightarrow L$
- LP $\rightarrow$ BP: $L \rightarrow$ series $LC, C \rightarrow$ parallel LC
- LP $\rightarrow$ BS: $L \rightarrow$ parallel $LC, C \rightarrow$ series LC

The two-port parameters of an arbitrary ladder filter shown in Fig. 16 can be calculated by noting that the input impedance is given by[1]

$$Z_{11} = \cfrac{1}{y_1 + \cfrac{1}{z_2 + \cfrac{1}{y_3 + \cfrac{1}{z_4 + \cdots}}}} \tag{49}$$

To calculate the transfer characteristic Z_{21}, leave the output open-circuited and assume the voltage $V_2 = 1\text{V}$ and find the input current step by step [1]. Note that $i_5 = y_5 V_2 = y_5$ and $i_4 = i_5$ so we have

$$V_a = i_4 z_4 + V_2 = 1 + z_4 y_5 \tag{50}$$

Repeating this calculation and using $i_2 = i_3 + i_4$

$$i_3 = y_3 V_a = y_3 + y_3 z_4 y_5 \tag{51}$$

$$V_1 = (i_3 + i_4) z_2 + V_a = z_2 y_3 + z_2 y_3 z_4 y_5 + z_2 y_5 + z_4 y_5 + 1 \tag{52}$$

The input current is given by $I_1 = y_1 V_1 + i_2$, which leads to

$$y_{21} = y_1 z_2 y_3 + y_1 z_2 y_3 z_4 y_5 + y_1 z_2 y_5 + y_1 z_4 y_5 + y_1 + y_3 + y_3 z_4 y_5 + y_5 \tag{53}$$

In practice carrying out the algebra is unnecessary since filters have been studied extensively and canonical filter structures have been tabulated and are widely available [2]. Filter design tools are also abundant on the web and through specialized software packages (such as ADS). Nevertheless you are encouraged to play around with a few simple filters to gain intuition before using the tools.

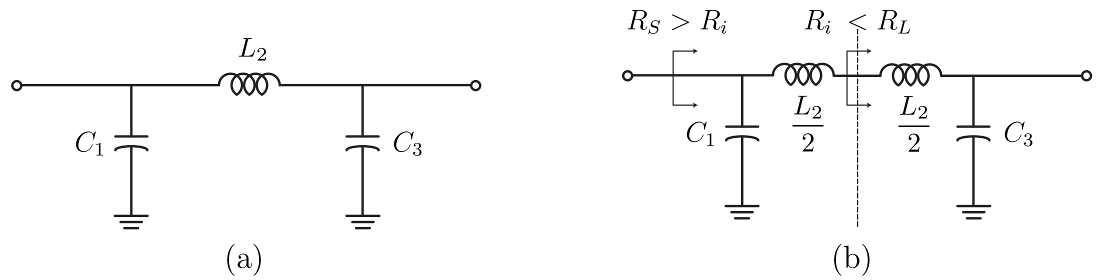

Figure 17: (a) A three section low-pass filter. (b) The series element is bisected to form two "L" matching networks.

2.2.1 Impedance Matching

It is now clear that in a standard band-pass filter structure, all the LC component values are chosen to resonate at the center frequency of the filter response, which constrains the product of LC. The ratio of these components must be chosen carefully to produce the desired filter properties. For instance, in order to obtain an impedance match, the input impedance of the filter should equal the desired load impedance across the passband. If we view a simple three section low-pass matching network as to front-to-back "L" matching networks (bisecting the series element as shown in Fig. 17), then we view this as a classic impedance down transformation by the factor $(1+Q_2^2)^{-1}$ followed by an up transformation of $(1+Q_1^2)$ resulting in an overall transformation of (at resonance)

$$Z_{in} = \frac{1+Q_1^2}{1+Q_2^2} \cdot R_L \tag{54}$$

Which shows that the filter can amplify or attenuate the voltage, or in other words change the impedance seen by the source. This impedance matching property of the filter is useful if the source and load impedance are different. In many applications, though, these are the same so ideally $Z_{in} = R_L$ at resonance, which means we should choose the filter component values to satisfy this constraint, or $Q_1 = Q_2$. The overall transfer characteristics of the filter come down to one free parameter, the quality factor Q.

2.3 Standard Filter Families

Filters have been thoroughly analyzed and classified into families of filters with specific filter characteristics. These various filters trade-off passband ripple for sharper attenuation or slightly worse group delay. The terminology behind these filters is widely known and standardized (although the spelling *Cheybchev* varies) and the names of the filters derive from the original mathematicians who studied the functions that underlie these filter transfer characteristics.

[1] In fact, this form if very useful for synthesis of a ladder filter of a given transfer function. Using long division, a transfer function can be written in continued fraction form, and the element values are readily calculated.

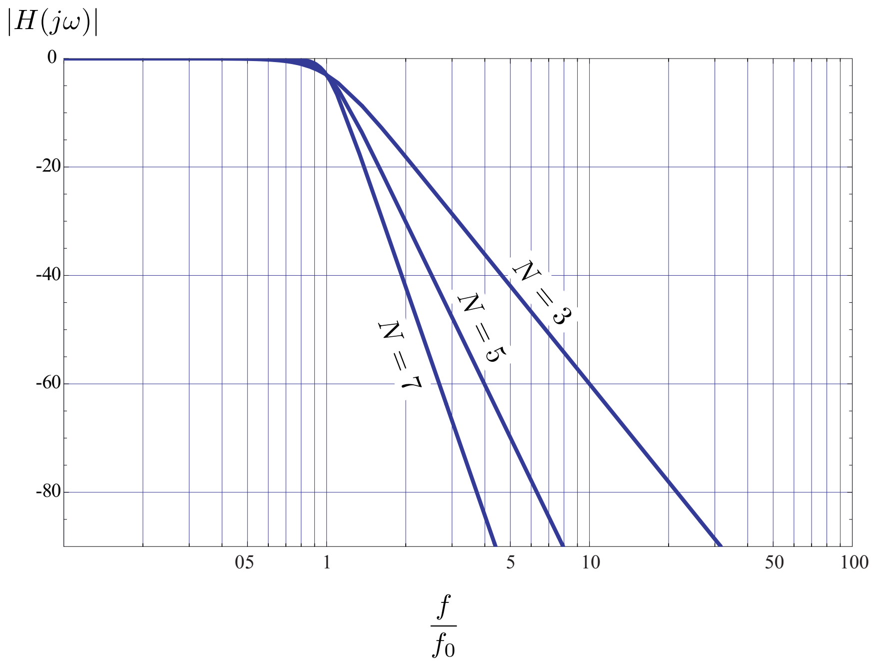

Figure 18: The voltage transfer characteristics of a Butterworth filter. The roll-off slope varies with the filter order n.

2.3.1 Butterworth Filters

The simplest of the family of filters, the Butterworth filters are also known as "maximally flat", since the transfer function

$$\left|\frac{V_2}{V_1}\right| = \frac{1}{\sqrt{1+(f/f_0)^{2n}}} \tag{55}$$

has n zero derivatives at the origin. This means that the filter response remains as flat as possible in the passband. As evident in Fig. 18, the order of the filter n determines the stop-band rolloff. To realize such a filter requires exactly n elements in the ladder structure. It is relatively easy to show that the poles of this filter lie uniformly on a half-circle in the left-hand plane with radius ω_0.

2.3.2 Chebyshev Filters

The Chebyshev filter has a sharper roll-off compared to the Butterworth filter in the stop-band, as shown in Fig. 19. The trade-off is that the Chebyshev filter introduces ripple in the passband. The transfer function for this filter is given by

$$\left|\frac{V_2}{V_1}\right| = \frac{1}{\sqrt{1+\epsilon^2 T_n^2(f/f_0)}} \tag{56}$$

where $T_n(x)$ is a Chebyshev polynomial of order n. This filter is realized with n elements. The in-band ripple is controlled by adjusting the factor ϵ. For a given value of ϵ, the in-band

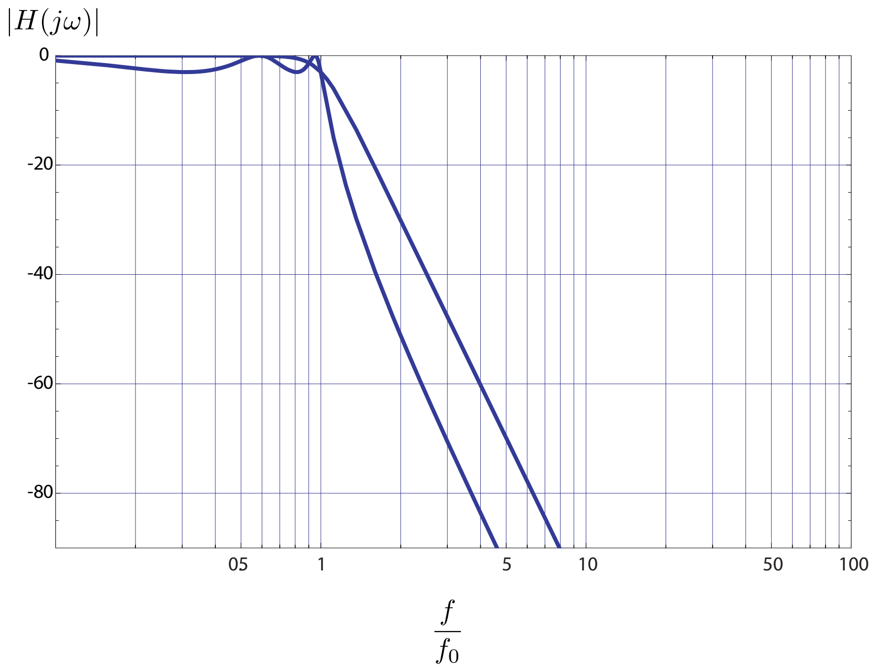

Figure 19: The voltage transfer characteristics of a Chebyshev and Butterworth filter. The Chebyshev has faster roll-off and in-band ripple whereas the Butterworth filter has a flat in-band response.

ripple is given by

$$\text{ripple in dB} = 20 \log_{10} \frac{1}{1+\epsilon^2} \tag{57}$$

The Chebyshev polynomial is given by

$$T_n(x) = \cosh(n \text{arccosh}(x)) \tag{58}$$

It is not too difficult to show that the poles of this filter are distributed on the left-hand side of the s-plane along an ellipse[3].

2.3.3 Other Filter Families

In the literature you will encounter other filter families that display various other trade-offs. For instance, the Bessel filter has a maximally flat group delay, which is ideal for applications intolerant to phase distortion in the passband. Inverse Chebyshev filters have no ripple in the passband but have ripple in the stop band instead, as shown in Fig. 20. These filters have both poles and zeros in their transfer characteristics. Elliptical filters allow one to specify passband and stopband ripple and achieve very good stop-band attenuation, but require high-Q poles. For all filter families it's important to remember that they are all realized by using the ladder filter structure. Only the components values vary to change the transfer function from one filter type to another.

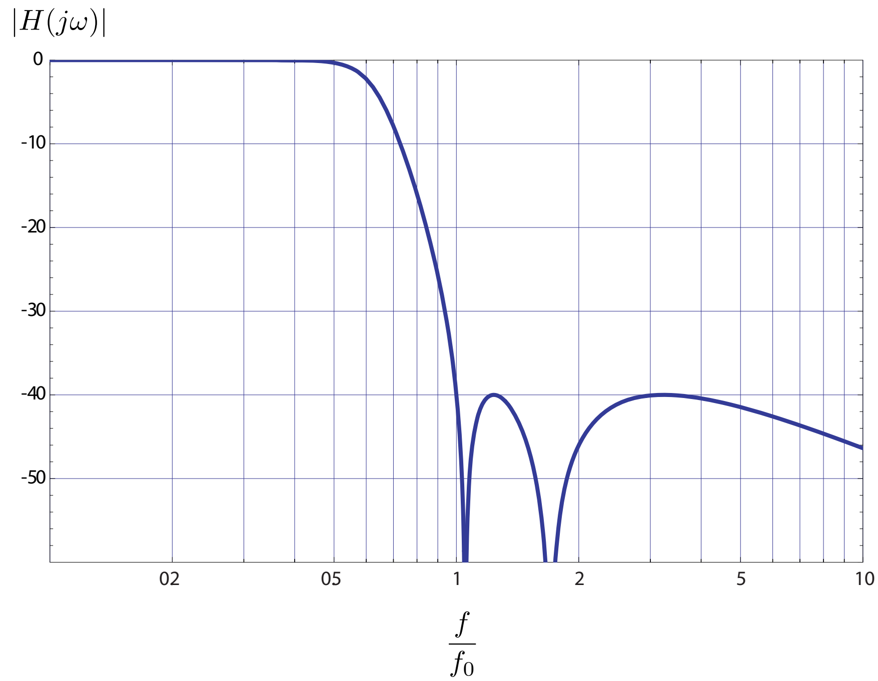

Figure 20: The voltage transfer characteristics of a type-II or Inverse Chebyshev filter. The Inverse Chebyshev has good out-of-band rejection.

2.4 Filter Transformations

Most filters families are tabulated as low-pass filters and normalized for a cutoff frequency of $\omega_c = 1$ radian/sec. Two types of filters can be used, one beginning with a shunt capacitor or series inductor (Fig. 15). The component values are given as g_n (Farads/Henrys), assuming a source impedance of $R_s = 1\Omega$ and load impedance $R_L = 1\Omega$ (odd-order filters). For even-order filters, $R_L = 1/g_{n+1}$.

2.4.1 Frequency and impedance scaling

The low-pass filter cutoff frequency can be scaled to ω_c and scaled to a source impedance of R_0 by modifying g_n in the following way

$$L_n = \frac{R_0 g_n}{\omega_c} \tag{59}$$

$$C_n = \frac{g_n}{R_0 \omega_c} \tag{60}$$

2.4.2 Low-pass to high-pass transformation

The frequency substitution $-\omega_c/\omega \to \omega'$ converts the filter prototype from low-pass to high-pass. The new component values are given by

$$C'_n = \frac{1}{g_n} \tag{61}$$

$$L'_n = \frac{1}{g_n} \tag{62}$$

2.4.3 Band-pass and band-stop transformation

The frequency substitution $\frac{\omega_0}{\omega_2-\omega_1}\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right) \rightarrow \omega'$ is used to tranform the low-pass filter to a bandpass filter. The fractional bandwidth Δ of the filter is given by

$$\Delta = \frac{\omega_2 - \omega_1}{\omega_0} \tag{63}$$

The center frequency is the geometric (not arithmetic) mean of the 3-dB frequencies

$$\omega_0 = \sqrt{\omega_1 \omega_2} \tag{64}$$

Carrying out the arithmetic means that a series inductor is transformed into a series LC circuit with component values

$$L'_n = \frac{L_n R_0}{\omega_0 \Delta} \tag{65}$$

$$C'_n = \frac{\Delta}{\omega_0 L_n R_0} \tag{66}$$

and the shunt capacitors are transformed into a parallel LC circuit with components given by

$$L'_n = \frac{R_0 \Delta}{\omega_0 C_n} \tag{67}$$

$$C'_n = \frac{C_n}{\omega_0 R_0 \Delta} \tag{68}$$

The inverse transformation $\Delta\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^{-1} \rightarrow \omega'$ is used to realize a bandstop filter. The series inductors are converted to parallel LC branches

$$L'_n = \frac{R_0 L_n \Delta}{\omega_0} \tag{69}$$

$$C'_n = \frac{1}{\omega_0 L_n R_0 \Delta} \tag{70}$$

whereas the shunt capacitors are converted into series LC branches

$$L'_n = \frac{R_0}{\omega_0 C_n \Delta} \tag{71}$$

$$C'_n = \frac{C_n \Delta}{\omega_0 R_0} \tag{72}$$

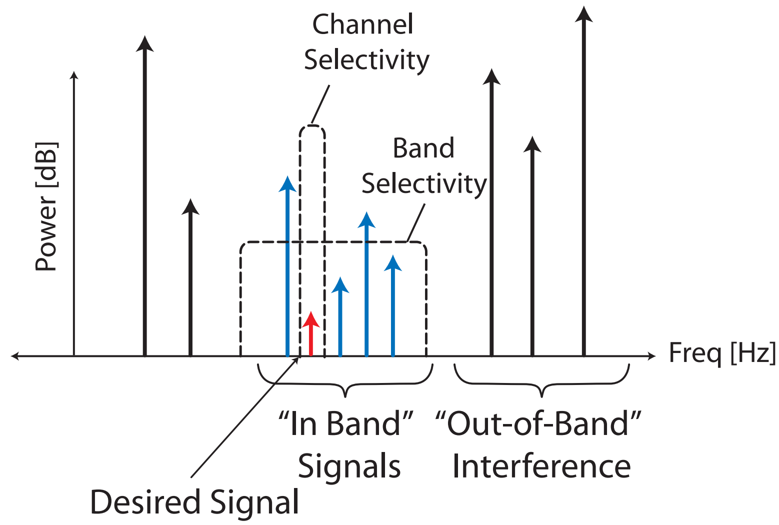

Figure 21: A radio receiver must be able to discriminate between many undesired and large interfering signals in order to "hear" a weak desired signal. Each arrow respresents a modulated signal at a given frequency.

2.5 Filters in Communication Systems

In a wireless communication system filters are used to "pick a needle out of a haystack," or in other words to select the desired signal of interest in a sea of interfering signals. This is shown schematically in Fig. 21, where the desired signal is at channel 2, or 910 MHz, in a radio band that can support many different channels. The first filter in this example is a fixed filter that selects all the channels of interest while rejecting "out of band" signals, such as the multitude of in other frequency bands, such as UHF television up to 600 MHz and other cellular and wireless communication signals in the 2-5 GHz spectrum. The second filter is used to further isolate the desired channel from the rest of the signals to provide channel selectivity. A difficult, but typical situation for a receiver is the "near-far" problem, shown in Fig. 22, where a multitude of interferers appear around the desired signal which is weak. If the receiver does not have sufficient sensitivity and linearity, these interfering signals can jam a receiver.

Notice that the second narrow filter needs to have a variable center frequency unless we use static channel assignments, which is very unrealistic in practice. Since in practice it is much easier to build a high quality fixed center frequency filter rather than a tunable filter, the most popular way to realize the same effect is to downconvert the signal of interest to a fixed intermediate frequency (IF), where channel selection and blocker attenuation can be performed. This is the basis of the superheterodyne receiver architecture, shown in Fig. 23. Notice that there are three filters used in this architecture, an RF band select filter, an image-reject filter, the importance of which will be highlighted later, and the a second IF filter. In this lab you will design these filters, simulate them, and then build and test the filters.

In a high-speed communication system using amplitude modulation, the time-domain

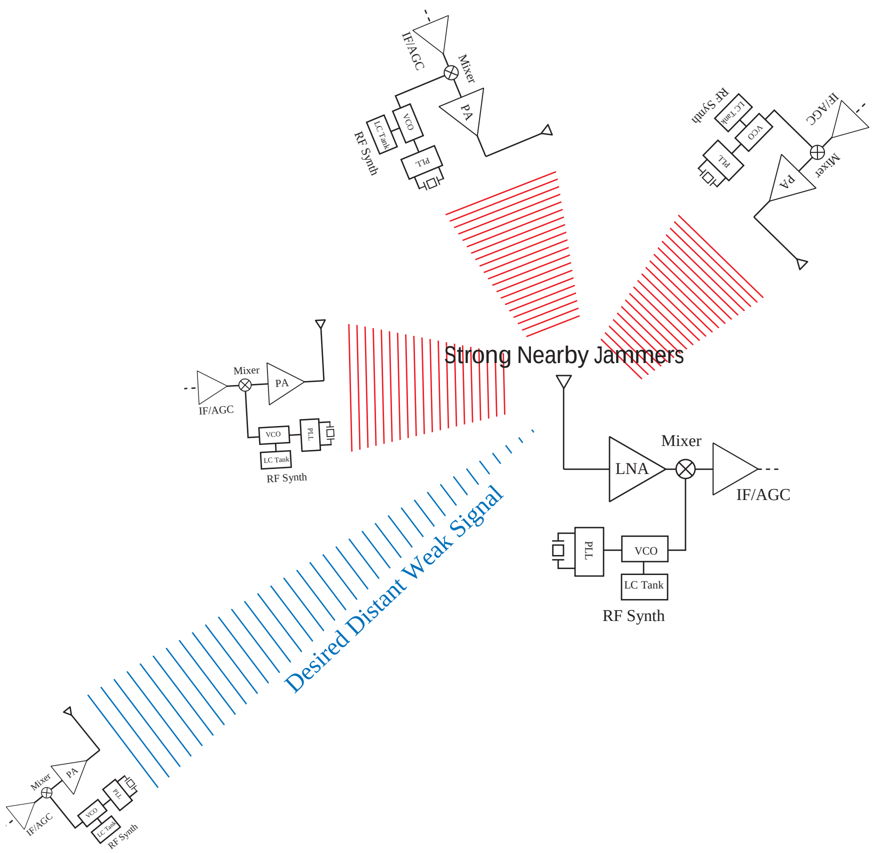

Figure 22: A radio receiver must contend with a near-far scenario where the desired transmitter is far away resulting in a weak received signal in the presence of many nearby unwanted interferers.

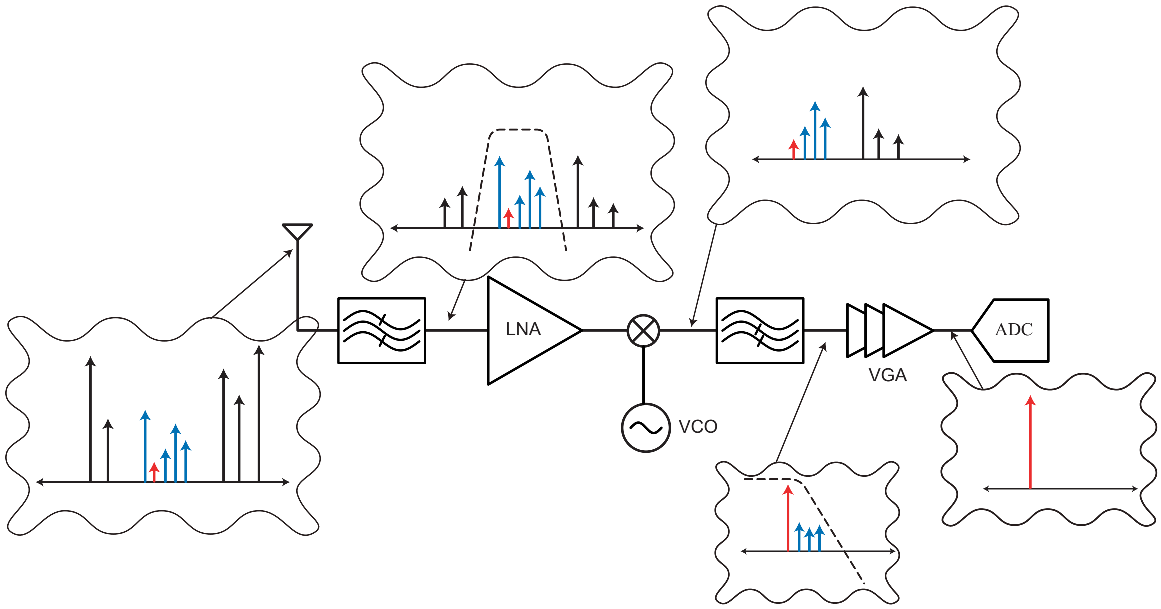

Figure 23: A superheterodyne radio receiver architecture uses several band-pass, band-reject, and low-pass filters.

ripple at the amplitude transitions leads to inter-symbol interference (ISI) and "eye closure". In Fig. 24 we compare the pulse response of a Bessel and Chebyshev filter. Note that in both cases we lose the sharp edges due to filtering, but the Bessel filter does not ring, which prevents "leakage" of one bit onto another.

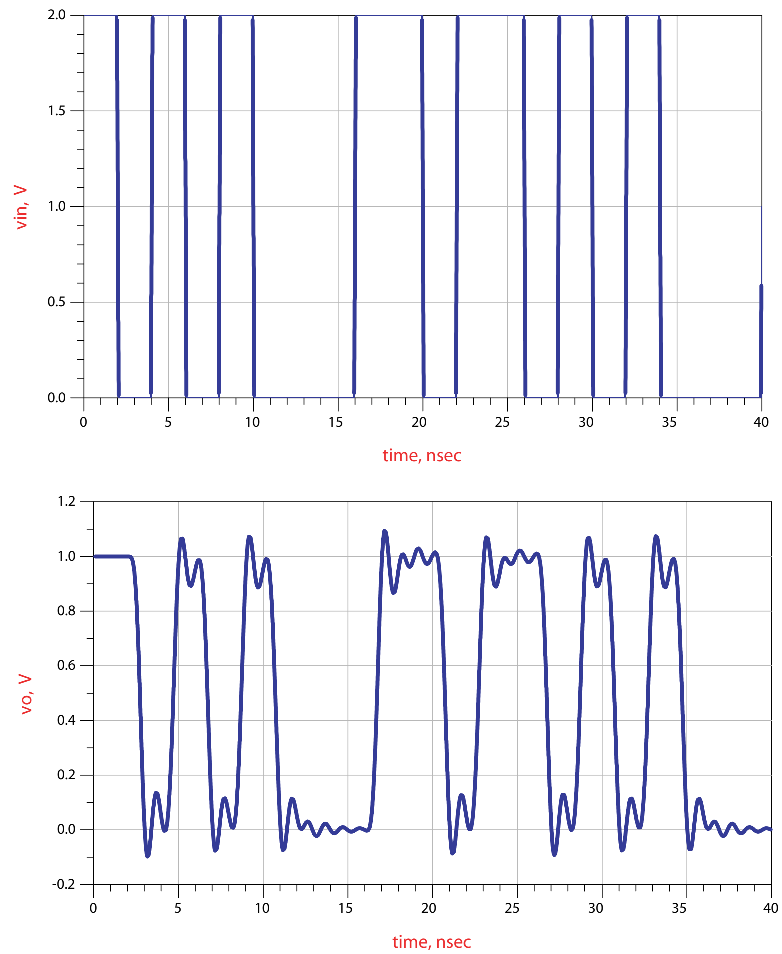

Figure 24: Input pulse data waveform (top) and the filtered response (bottom) shows significant inter-symbol interference (ISI).

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