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# Lagrangian Mechanics and Rigid Body Motion

Group 2

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# Abstract

In our report we will discuss Lagrangian Mechanics and the Motion of Rigid Bodies. Lagrangian Mechanics is a reformulation of Classical Mechanics, first introduced by the famous mathematician Joseph-Louis Lagrange, in 1788. We shall discuss the uses of Lagrangian Mechanics and include two examples - the Spherical Pendulum and the Double Pendulum. In each case we will derive the equations of motion, and then try to solve these numerically and/or analytically. We will investigate the effect of removing the gravitational field (in the case of the Spherical Pendulum) and discuss any links between the two, as well as any implications of the solutions.

A rigid body is a collection of N points such that the distance between any two of them is fixed regardless of any external forces they are subject to. We shall look at the kinematics, the Inertia Tensor and Euler's Equation and use this to explain about the dynamical stability of rigid bodies. Symmetric tops are the main example that we will investigate and discuss. We will look into the precession rate and the spinning rate and discuss two examples, Feynman's wobbling plate and the hula hoop. A more complicated rigid body we shall then explore is the heavy symmetric top, in which we take into account the forces exerted by a gravitational field.

## Acknowledgements

We would like to thank our supervisor Dr. Ryan Barnett for giving us his valuable time and effort in helping us complete this project. His passion and commitment to the subject was very clear to us throughout our collaboration with him and he gave us a lot of insight and inspiration on the topic. We really appreciate the support and guidance he has given us and it has been a pleasure working with him.

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# Chapter 1

# Introduction

What is Lagrangian Mechanics? Lagrangian Mechanics is a reformulation of Newtonian Mechanics, first introduced by the famous mathematician Joseph-Louis Lagrange, in 1788. Newtonian Mechanics is very convenient in Cartesian coordinates, but as soon as we change the coordinate system (e.g. polar coordinates), the equations of motion can become very complicated to find. Lagrangian mechanics uses the energies (scalars) of the system, unlike Newtonian mechanics which uses the forces (vectors). Hence it is much simpler to use the Lagrangian formulation when dealing with a non-Cartesian coordinate system.

There are many applications of Lagrangian Mechanics. In both the examples we look at, we carry out a similar analysis - after choosing a suitable coordinate system, first we derive the Lagrangian. We next use Lagrange's equation to derive the equations of motion for the masses. The examples we look at are the Spherical Pendulum and the Double Pendulum. We also consider the effect of removing the gravitational field, and in each case we try to solve the equations of motion - mostly this is done numerically. Throughout we will try and link the solutions to the idea of chaotic systems - in which a small change in initial solutions leads to a large change in the solutions themselves. Finally we discuss any further implications and consider links between the two examples.

Given a rigid body, we will determine some key equations of the motion, particularly those revolving around rotation. From these more basic principles, we will then derive the Euler equations and apply this knowledge to real life situations and explain mathematically real life occurrences.

One of the most important applications of Euler equations and Euler's angles is the motion of free tops. In section 3.2, we will derive the relation between the precession and the spin of free symmetric tops. In one of Richard Feynman's books, "'Surely you're joking, Mr. Feynman':

adventures of a curious character", Feynman mentioned the interesting ratio of wobbling rate and spinning rate of a plate. This ratio can be mathematically verified. In addition, we will discuss the wobble-spin ratio for a hula hoop, which is slightly more complicated than the wobbling plate.

In the last section, we look into heavy symmetric top problems, which come from the popular toy of spinning tops. For this typical problem that a spinning top rotating on a horizontal plane. We first derive the equation of motions in general, using Euler's angles and the Lagrangian. Based on these differential equations, we use MATLAB ode45 to solve numerically for three specific scenarios which the top behaves differently depending on initial momentum of inertia. Further, we show that it is possible to solve of these three cases analytically when we assume that the top is spinning "fast" enough.

# Chapter 2

# Foundations of Lagrangian Mechanics and Rigid Body Motion

Suppose we have a system of $N$ particles. The position of each particle is defined by 3 coordinates. Therefore, to define the configuration of the entire system we require $3N$ coordinates. Let us rewrite these coordinates as $x^A, A = 1, ..., 3N$. These coordinates parametrize a space called the **configuration space** of the system. The parameters that define the configuration of a system are called **generalized coordinates**. (1)

## 2.1 Lagrangian Mechanics

### 2.1.1 The Lagrangian

The core of Lagrangian Mechanics is the **Lagrangian**, a function of positions $x^A$ and velocities $\dot{x}^A$ of all the particles, which summarizes the dynamics of a system. Any function which generates the correct equations of motion can be taken as a Lagrangian - so there is no single expression for all physical systems. The non-relativistic Lagrangian for a system of particles is defined to be

$$\mathcal{L}(x^A, \dot{x}^A) = T(\dot{x}^A) - V(x^A),$$

where $T(\dot{x}^A) = \frac{1}{2}\sum_A m_A(\dot{x}^A)^2$ is the kinetic energy and $V(x^A)$ is the potential energy. For the rest of this report we will be referring to the Lagrangian in Newtonian Mechanics $\mathcal{L}(x^A, \dot{x}^A) = T(\dot{x}^A) - V(x^A)$.

### 2.1.2 The Principle of Least Action

To describe The Principle of Least Action we first need to consider all smooth paths between a fixed starting point and a fixed end point. Of all the possible paths, only one is the true path taken by the system. Let us define the **Action**, $S$, as

$$S = \int_{t_i}^{t_f} \mathcal{L}(x^A, \dot{x}^A)dt,$$

where $t_i$ is the time at the starting point and $t_f$ is the time at the end point. (4)

The **Principle of Least Action** states that the system follows a path which minimizes the Action.

*Proof.* Let's suppose we vary a true path by $\delta x^A(t)$. We get

$$x^A(t) \to x^A(t) + \delta x^A(t),$$

where we fixed the end points of the path by demanding $\delta x^A(t) + \delta x^A(t_i) = \delta x^A(t_f) = 0$. Then the change in the action is

$$\begin{aligned} \delta S &= \delta\Big[\int_{t_i}^{t_f} \mathcal{L}dt\Big] \\ &= \int_{t_i}^{t_f} \delta\mathcal{L}dt \\ &= \int_{t_i}^{t_f} (\frac{\partial \mathcal{L}}{\partial x^a}\delta x^A + \frac{\partial \mathcal{L}}{\partial \dot{x}^A}\delta \dot{x}^A)dt. \end{aligned}$$

Now integrating the second term by parts gives

$$\delta S = \int_{t_i}^{t_f} (\frac{\partial \mathcal{L}}{\partial x^A} - \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{x}^A}))\delta x^A dt + \Big[\frac{\partial \mathcal{L}}{\partial \dot{x}^A}\delta x^A\Big]_{t_i}^{t_f}.$$

Since we have fixed the end points of the path, $\delta x^A(t_i) = (\delta t_f) = 0$ and so the last term vanishes. The action is a minimum implies that $\delta S = 0$ for all changes in the path $\delta x^A(t)$. We can see that this holds if and only if the integral is zero and hence

$$\frac{\partial \mathcal{L}}{\partial x^A} - \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{x}^A}) = 0, \qquad A = 1, ..., 3N.$$

These are known as **Lagrange's Equations**. To conclude the proof we need to show that Lagrange's equations are equivalent to Newton's. From the definition of the Lagrangian, we have $\frac{\partial \mathcal{L}}{\partial x^a} = -\frac{\partial V}{\partial x^A}$, and $\frac{\partial \mathcal{L}}{\partial \dot{x}^A} = \frac{\partial T}{\partial \dot{x}} = m\dot{x} = p_A$. Substituting these into Lagrange's Equations gives Newton's Equation, $\dot{p}_A = -\frac{\partial V}{\partial x^A}$. □

**Example.** Suppose we have a system with one particle. Let the particle moves from some point to another point by free motion in a certain amount of time. The true path is the path shown in Figure 2.1. Now suppose the particle moves in a different path in the same amount of time, as shown in Figure 2.2. If you integrate the Lagrangian over the different path, you'll find that the Action is larger than that for the actual motion.

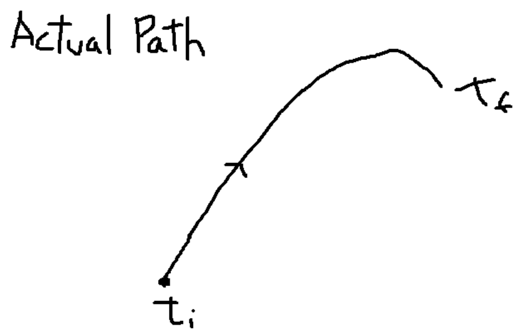


Figure 2.1: Actual Path

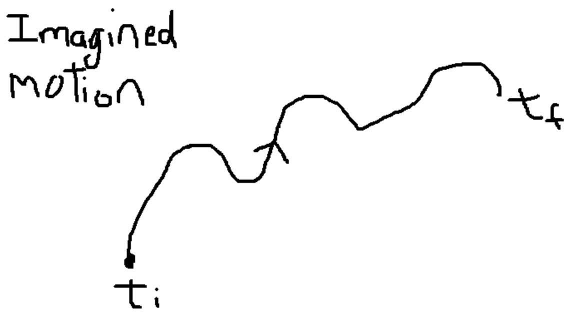


Figure 2.2: Imagined Motion

Using the Principle of Least Action we could informally restate Newtons Law of Motion as "The average kinetic energy minus the average potential energy is as small as possible for the path of an object going from one point to another".

### 2.1.3 Lagrange's Equation

From the Principle of Least Action, we derived **Lagrange's Equation.**

$$\frac{\partial \mathcal{L}}{\partial x^A} - \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{x}^A}) = 0, \qquad A = 1, ..., 3N.$$

Just as we saw in the proof of the Principle of Least Action, we can obtain the equations of motion by substituting in the Lagrangian.

**Example.** Suppose we have a simple spring-mass system with kinetic energy $T = \frac{1}{2}m\dot{x}^2$ and potential energy $V = \frac{1}{2}kx^2$. Here the Lagrangian is

$$\mathcal{L} = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2.$$

Finding the necessary equations to input into Lagrange's Equation we get

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}, \qquad \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{x}}) = m\ddot{x}, \qquad \frac{\partial \mathcal{L}}{\partial \dot{x}} = -kx.$$

Now by substituting these back into Lagrange's Equation we get

$$m\ddot{x} + kx = 0 \implies F = m\ddot{x} = -kx,$$

which is our required equation of motion.

### 2.1.4 Lagrange's Equation in a Generalized Coordinate System

The parameters that define the configuration of a system are called **generalized coordinates**. By expressing each position vector as functions of the generalized coordinates and time, every position vector can be written in terms of generalized coordinates, $\mathbf{q} = (q_1, q_2, ..., q_n)$, where the vector $\mathbf{q}$ is a point in the configuration space of the system. (5) We shall now show that Lagrange's Equations hold in a generalized coordinate system. Let

$$q_a = q_a(x_1, ..., x_{3N}, t), \qquad a = 1, ..., n$$

be a generalized coordinate. Differentiating using the chain rule we get

$$\dot{q}_a = \frac{dq_a}{dt} = \frac{\partial q_a}{\partial x^A}\dot{x^A} + \frac{\partial q_a}{\partial t}.$$

We are able to invert the generalized coordinates back to it orginal coordinates, $x^A = x^A(q_a, t)$, as long as $det(\frac{\partial x^A}{\partial q_a}) \neq 0$. Then we have

$$\dot{x}^A = \frac{\partial x}{\partial q_a}\dot{q}_a + \frac{\partial x^A}{\partial t}.$$

Now differentiating the Lagrangian with respect to the generalized coordinates and substituting for $\frac{\partial \dot{x}^A}{\partial q_a}$, we get

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial q_a} &= \frac{\partial \mathcal{L}}{\partial x^A}\frac{\partial x^A}{\partial q_a} + \frac{\partial \mathcal{L}}{\partial \dot{x}^A}\frac{\partial \dot{x}^A}{\partial q_a} \\ &= \frac{\partial \mathcal{L}}{\partial x^A}\frac{\partial x^A}{\partial q_a} + \frac{\partial \mathcal{L}}{\partial \dot{x}^A}\left(\frac{\partial^2 x^A}{\partial q_a \partial q_b}\dot{q}_b + \frac{\partial^2 x^A}{\partial t \partial q_a}\right),\end{aligned}$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_a} = \frac{\partial \mathcal{L}}{\partial \dot{x}^A}\frac{\partial x^A}{\partial q_a}.$$

Using the fact that $\frac{\partial \dot{x}^A}{\partial \dot{q}_a} = \frac{\partial x^A}{\partial q_a}$ we get

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_a}\right) = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^A}\right)\frac{\partial x^A}{\partial q_a} + \frac{\partial \mathcal{L}}{\partial \dot{x}^A}\left(\frac{\partial^2 x^A}{\partial q_a \partial q_b}\dot{q}_b + \frac{\partial^2 x^A}{\partial q_a \partial t}\right).$$

Combining we get

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_a}\right) - \frac{\partial \mathcal{L}}{\partial q_a} = \left(\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^A}\right) - \frac{\partial \mathcal{L}}{\partial x^A}\right)\frac{\partial x^A}{\partial q_a}.$$

We can see that if the Lagrangian equation is solved in the $x^A$ coordinate system then the RHS vanishes and hence it is also solved in the $q_a$ coordinate system. This shows that Lagrange's Equation holds in a generalized coordinate system.

### 2.1.5 Noether's Theorem

We say that the Lagrangian is **symmetric** over q if

$$\frac{\partial \mathcal{L}}{\partial q} = 0.$$

A quantity H is **conserved** if

$$\frac{\partial H}{\partial t} = 0.$$

**Noether's Theorem** states that whenever we have a continuous symmetry of Lagrangian, then there is an associated conservation law. (6)

Here is some brief intuition behind this theorem. Given Lagrange's Equation in generalized coordinates

$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}}) = \frac{\partial \mathcal{L}}{\partial q}.$$

Now suppose the Lagrangian is symmetric over q, then

$$\frac{\partial \mathcal{L}}{\partial q} = 0 \implies \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}}) = 0.$$

This implies that the rate of change of the **generalized momentum**, defined by $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$, is zero and so p = constant for all time. Hence it is a conserved quantity.

Here are some basic examples. Time translation symmetry gives conservation of energy; space translation symmetry gives conservation of momentum; rotation symmetry gives conservation of angular momentum.

## 2.2 Rigid Body Motion

Before getting into the motion of rigid bodies, we need to define what a rigid body is. A rigid body is a collection of $N$ points such that the distance between any two of them is fixed regardless of any external forces they are subject to i.e. $|\boldsymbol{r}_i - \boldsymbol{r}_j| = constant$. Every rigid body has 6 degrees of freedom, three translations along each of the axes, and three rotations about each of the axes.

### 2.2.1 Kinematics

In this section, we consider 2 frames of reference, the body frame and the space frame, both of which can be depicted from Figure 2.3.

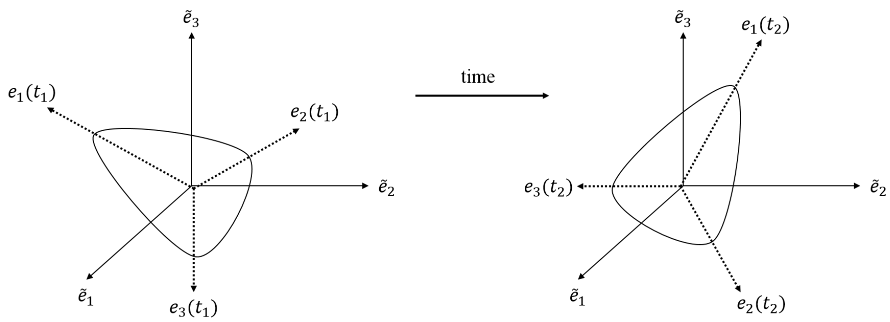


Figure 2.3: Space and body frame.

In the figure, we see that the space frame is fixed from our point of view and the body frame is fixed from the body's point of view. Hence over time, the body frame moves relative to the object and the space frame maintains its position. Each axis on the body frame can be written in terms of the fixed axis on the space frame using an orthogonal matrix $R(t)$ with components $R_{ab}(t)$.This means that the matrix R represents a time dependent rotation. Now we can define angular velocity in each respective frame. Given a point $\boldsymbol{r}$ and taking the time derivative we get:

$$\frac{d\boldsymbol{r}}{dt} = \frac{d\tilde{r}_a}{dt}\tilde{\boldsymbol{e}}_{\boldsymbol{a}}$$

in the space frame, and

$$\frac{d\boldsymbol{r}}{dt} = r_a \frac{d\boldsymbol{e}_{\boldsymbol{a}}(t)}{dt} = r_a \frac{dR_{ab}}{dt}\tilde{\boldsymbol{e}}_{\boldsymbol{b}}$$

in the body frame.

Alternatively, we can consider how the axis of the body frame changes with respect to time,

$$\frac{d\boldsymbol{e}_{\boldsymbol{a}}}{dt} = \frac{dR_{ab}}{dt}\tilde{\boldsymbol{e}}_{\boldsymbol{b}} = (\frac{dR_{ab}}{dt}R^{-1})_{bc}\boldsymbol{e}_{\boldsymbol{c}} \equiv w_{ac}\boldsymbol{e}_{\boldsymbol{c}}. \tag{2.1}$$

### 2.2.2 The Moment Of Inertia

The **Moment of Inertia** is a tensor representing the resistance of a body to angular acceleration. To derive the tensor we need to firstly consider the kinetic energy for a rotating body.

$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i \dot{\boldsymbol{r}}_{\boldsymbol{i}}^2 \\ &= \frac{1}{2}\sum_i m_i(\boldsymbol{w} \times \boldsymbol{r}_{\boldsymbol{i}}) \cdot (\boldsymbol{w} \times \boldsymbol{r}_{\boldsymbol{i}}) \\ &= \frac{1}{2}\sum_i m_i((\boldsymbol{w} \cdot \boldsymbol{w})(\boldsymbol{r}_{\boldsymbol{i}}) \cdot \boldsymbol{r}_{\boldsymbol{i}}) - (r_i \cdot \boldsymbol{w})^2), \end{aligned}$$

which can be rewritten as $T = \frac{1}{2}w_a I_{ab} w_b$ where $I_{ab}$ are components of the inertia tensor measured in the body frame. The moments of inertia matrix can be found by:

$$I = \int dV \rho(\boldsymbol{r}) \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}. \tag{2.2}$$

The matrix $I$ is symmetric and real, so it is diagonalisable resulting in the matrix,

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}.$$

Here, the eigenvalues $I_a$ are called the **Principal Moments of Inertia** which represents the moment of inertia about each of its principal axes which can be defined as 3 mutually perpendicular axes where the moment of inertia is at a maximum.

### 2.2.3 Angular Momentum

Angular momentum, $\boldsymbol{L}$, can be described as the cross product of the particle's position vector and it's momentum vector and can be expressed in terms of the inertia tensor as shown below

$$\begin{aligned} \boldsymbol{L} &= \sum_i m_i \boldsymbol{r_i} \times \dot{\boldsymbol{r}}_{\boldsymbol{i}} \\ &= \sum_i m_i \boldsymbol{r_i} \times (\boldsymbol{w} \times \boldsymbol{r_i}) \\ &= \sum_i m_i (r_i^2 \boldsymbol{w} - (\boldsymbol{w} \cdot \boldsymbol{r_i}) \boldsymbol{r_i}) \\ &= I\boldsymbol{w}. \end{aligned}$$

In the body frame, we can write $\boldsymbol{L} = L_a e_a$ to get $L_a = I_{ab} w_b$.(17)

### 2.2.4 Euler's Equations

Consider the rotation of a rigid body, then as the body is free angular momentum is conserved implying $\frac{d\boldsymbol{L}}{dt} = 0$ and expanding this out using the body frame gives the following:

$$\begin{aligned} 0 = \frac{d\boldsymbol{L}}{dt} &= \frac{dL_a}{dt}\boldsymbol{e_a} + L_a \frac{d\boldsymbol{e_a}}{dt} \\ &= \frac{dL_a}{dt}\boldsymbol{e_a} + L_a \boldsymbol{w} \times \boldsymbol{e_a}. \end{aligned}$$

From this we can derive 3 non-linear coupled first order differential equations known as Euler's Equations.

$$\begin{aligned} I_1 \dot{w}_1 + w_2 w_3 (I_3 - I_2) &= 0, \\ I_2 \dot{w}_2 + w_3 w_1 (I_1 - I_3) &= 0, \\ I_3 \dot{w}_3 + w_1 w_2 (I_2 - I_1) &= 0. \end{aligned}$$

### 2.2.5 Dynamical Stability of Rigid Body Motion

Considering a book a rigid body and try tossing it up about all three of it's axis as demonstrated in the images below:

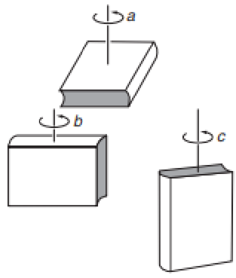


Figure 2.4: Tossing a book by three axis. (14)

we find that when rotating about the minimum and maximum edge, the book merely rotates and lands similarly to how it started (shown by a and c in Figure 2.4). However, if we take the third end (shown as b in Figure 2.4), which is the intermediate edge, then the book begins to twist and turn mid air before landing. This shows that the rotational motion is stable about two of its axis and unstable about the third one. This can be explained using Euler's equations and by considering the different moments of inertia rotating about each axis creates. Suppose that $w_1 = constant \neq 0$, $w_2 = 0$ and $w_3 = 0$ and then Euler's equations become:

$$I_1\dot{w}_1 + w_2w_3(I_3 - I_2) = 0, \tag{2.3}$$
$$I_2\dot{w}_2 + w_3w_1(I_1 - I_3) = 0, \tag{2.4}$$
$$I_3\dot{w}_3 + w_1w_2(I_2 - I_1) = 0. \tag{2.5}$$

Because both $w_2$ and $w_3$ are assumed to be 0, we can disregard the second term from Equation (2.3) and therefore determine that $w_1$ is a constant. Following on from this we can differentiate Equation (2.4) with respect to time and substitute in for $\dot{w}_3$ from Equation (2.5) to get:

$$I_2\ddot{w}_2 - \frac{(I_1 - I_3)(I_2 - I_1)}{I_3}w_1^2w_2 = 0.$$

Then by dividing through by $I_2$ gives:

$$\ddot{w}_2 + Aw_2 = 0,$$

where

$$A = \frac{(I_1 - I_3)(I_1 - I_2)}{I_2I_3}w_1^2.$$

Now consider three cases for $I_1$ where it is the largest, smallest and intermediate value in comparison to the other moments of inertia $I_2$ and $I_3$. If $I_1$ is at its largest or smallest then $A > 0$ so let $A = k^2$ and the corresponding solution for $w_2$ is:

$$w_2 = a\cos(kt) + b\sin(kt),$$

and this oscillates with frequency $\sqrt{A}$ with bounded amplitude for all t. Then taking $I_1$ to be the intermediate value implies that $A < 0$ and so by setting $A = -k^2$, we get the following general solution:

$$w_2 = ae^{kt} + be^{-kt}.$$

In this case, $w_2$ increases exponentially with time and therefore the motion is unstable. Similarly, by differentiating equation (2.5) with respect to time and substituting in for $\dot{w}_2$, you get the same second order differential equation, except that it is in terms of $w_3$ instead. Therefore we can draw the same conclusion that if $I_1$ either the largest or the smallest, $w_3$ is stable and unstable if $I_1$ is the intermediate value. (14)

### 2.2.6 Euler Angles

Euler's theorem states that every rotation can be expressed as a product of 3 successive rotations about 3 different axes. The angles about these axes are known as **Euler's angles** and can be used to describe the rotation from the space frame to the body frame.
Let $\{\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3\}$ be axes of the space frame and $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be axes of the body frame then we can find a matrix $R$ that rotates from $\{\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3\}$ to $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. $R$ can be expressed as the product of the following 3 matrices.

$$R_3(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{2.6}$$

$$R_3(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Each of these rotation are depicted below in the figures.

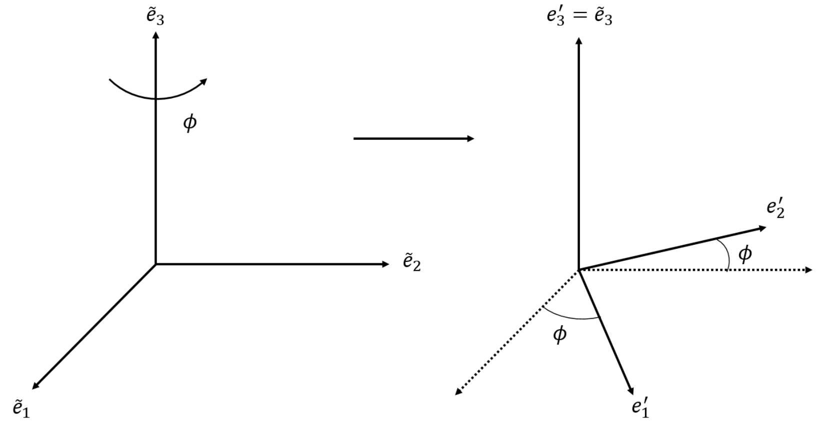


Figure 2.5: Rotation about $\tilde{\boldsymbol{e}}_3$ axis of the space frame for angle $\phi$.

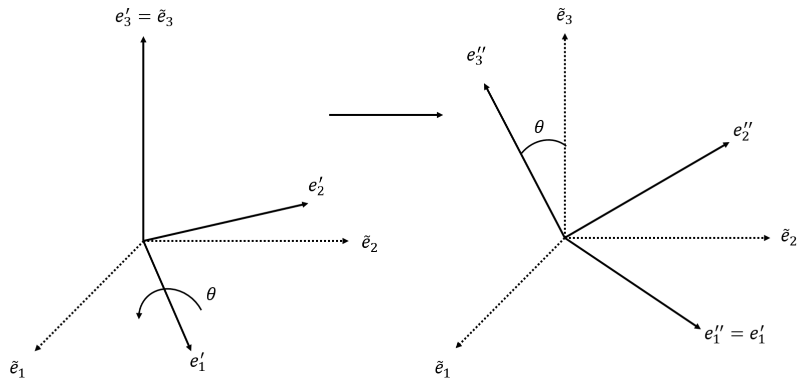


Figure 2.6: Rotation about $\boldsymbol{e}_1'$ axis for angle $\theta$.

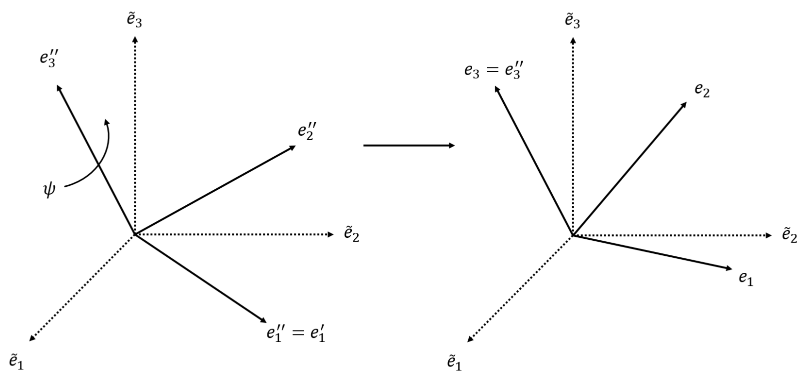


Figure 2.7: Rotation about $\boldsymbol{e}_3''$ axis for angle $\psi$.

Multiplying the 3 matrices together gives the following matrix for $R$.

$$R = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \sin\phi\cos\psi + \cos\theta\sin\phi\cos\phi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \cos\theta\cos\psi\sin\phi & -\sin\psi\sin\phi + \cos\theta\cos\psi\cos\phi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} \tag{2.7}$$

### 2.2.7 Angular Velocity Using Euler Angles

Using our knowledge of Euler Angles, we can express the angular velocity $\boldsymbol{\omega}$ in terms of $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ axes of the body frame. First, we know that

$$\boldsymbol{\omega} = \dot{\phi}\tilde{\boldsymbol{e}}_3 + \dot{\theta}\boldsymbol{e}'_1 + \dot{\psi}\boldsymbol{e}_3. \tag{2.8}$$

All we need to do is to find $\tilde{\boldsymbol{e}}_3$ and $\boldsymbol{e}'_1$ in terms of $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. This can be done by inverse matrix and matrix multiplication.

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = R \begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \tilde{e}_3 \end{pmatrix} \implies \begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \tilde{e}_3 \end{pmatrix} = R^{-1} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

where matrix R as in Equation (2.7).

$$\implies \tilde{\boldsymbol{e}}_3 = \sin\theta \sin\psi \boldsymbol{e}_1 + \sin\theta\cos\psi \boldsymbol{e}_2 + \cos\theta \boldsymbol{e}_3 \tag{2.9}$$

We can find $\boldsymbol{e}'_1$ in a similar way.

$$\begin{aligned} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = R_3(\psi)R_1(\theta)R_3(\phi) \begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \tilde{e}_3 \end{pmatrix} &= R_3(\psi)R_1(\theta) \begin{pmatrix} e'_1 \\ e'_2 \\ e'_3 \end{pmatrix} \\ \implies \begin{pmatrix} e'_1 \\ e'_2 \\ e'_3 \end{pmatrix} &= R_1^{-1}(\theta)R_3^{-1}(\psi) \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \end{aligned}$$

where $R_1(\theta)$ and $R_3(\psi)$ as in Equation (2.6).

$$\implies \boldsymbol{e}'_1 = \cos\psi \boldsymbol{e}_1 - \sin\psi \boldsymbol{e}_2 \tag{2.10}$$

Putting Equation (2.8)-(2.10) together, we obtain the angular velocity:

$$\boldsymbol{\omega} = (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\boldsymbol{e}_1 + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\boldsymbol{e}_2 + (\dot{\psi} + \dot{\phi}\cos\theta)\boldsymbol{e}_3.(17) \tag{2.11}$$

# Chapter 3

# Applications and Examples

## 3.1 Examples and Applications of Lagrangian Mechanics

In this section, we investigate in detail two extended applications of the Lagrangian - the Spherical Pendulum and the Double Pendulum. We aim to consider the idea of chaos in the solutions to the equations of motion for each pendulum.

### 3.1.1 The Spherical Pendulum

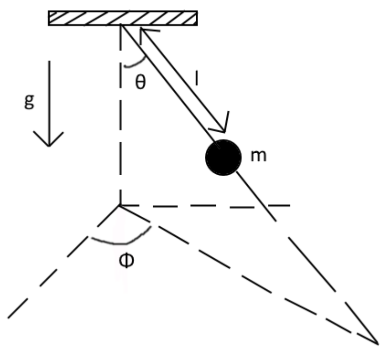


Figure 3.1: The Spherical Pendulum.

The spherical pendulum involves a mass m being suspended from the ceiling by a string of length $l$. The mass is allowed to move in 3D space. To start us off, we look first briefly at an example in the book "Lagrangian and Hamiltonian Mechanics" (10). This example goes far as deriving the equations of motion, which we do first as well. Next we go a step further and investigate the solutions to these equations, and finally discuss some of their implications.

Choosing our generalized coordinates to be spherical polar coordinates, let the position of the mass at any time be given by $(x, y, z)$. Then we derive the expressions for $\dot{x}$, $\dot{y}$ and $\dot{z}$ in terms of our chosen coordinate system as follows:

$$\begin{aligned} x = l\sin\theta\cos\phi &\implies \dot{x} = l\dot{\theta}\cos\theta\cos\phi - l\dot{\phi}\sin\theta\sin\phi \\ y = l\sin\theta\sin\phi &\implies \dot{y} = l\dot{\theta}\cos\theta\sin\phi + l\dot{\phi}\sin\theta\cos\phi \\ z = -l\cos\theta &\implies \dot{z} = l\dot{\theta}\sin\theta. \end{aligned}$$

Now substituting the above expressions into $T = \dfrac{1}{2}m\dot{\mathbf{r}}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ and simplifying yields:

$$T = \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta).$$

Next, the potential energy relative to the ceiling is just the same as would be for a regular pendulum, simply just:

$$V = mgz = -mgl\cos\theta.$$

Hence we have our Lagrangian, $\mathcal{L} = \dfrac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mgl\cos\theta$. Then we consider Lagrange's Equation for our chosen coordinate system - i.e. we need to evaluate $\dfrac{\partial\mathcal{L}}{\partial x^A} - \dfrac{d}{dt}(\dfrac{\partial\mathcal{L}}{\partial\dot{x}^A}) = 0$ for $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$. Now, $x^1$ is fixed, since r = l, constant - so this just gives us 0. Next, the equation of motion for $x^2 = \theta$ (after dividing through by $ml^2$) is:

$$\ddot{\theta} = \dot{\phi}^2\sin\theta\cos\theta - \frac{g}{l}\sin\theta. \tag{3.1}$$

Lastly considering $x^3 = \phi$ in Lagrange's Equation yields (again after eliminating $ml^2$): $\dfrac{d}{dt}(\dot{\phi} - \dot{\phi}\cos 2\theta) = 0 \implies \dfrac{d}{dt}(\dot{\phi}\sin^2\theta) = 0$, which gives the second equation of motion (here C is an arbitrary constant - notice if C = 0, this is just a simple pendulum moving in the vertical plane) as:

$$\dot{\phi}\sin^2\theta = C. \tag{3.2}$$

Note that the generalised momentum associated with $\phi$ here is $ml^2\dot{\phi}\sin^2\theta$. Physically, this is the angular momentum in the vertical plane (1) and by Noether's Theorem, we expect this to be constant and hence also we see above that $\dot{\phi}\sin^2\theta$ is a conserved quantity. Finally we note that substituting (3.2) into (3.1) yields (note K = C$^2$):

$$\ddot{\theta} = \frac{K\cos\theta}{\sin^3\theta} - \frac{g}{l}\sin\theta. \tag{3.3}$$

We now look to find numerical solutions for the ODE (3.3) so obtaining a solution $\theta(t)$ and then use (3.2) to obtain $\phi(t)$. To do so, we define a system of ODEs which are solvable in MATLAB:

$$u_1 = \theta(t), \quad u_2 = \dot{\theta}(t), \quad v_1 = \phi(t), \quad v_2 = \dot{\phi}(t).$$

Next, we rewrite the terms in (3.3) as follows:

$$a = a(u_1) = K\frac{\cos u_1}{\sin^3 u_1}, \quad b = b(u_1) = -g\frac{\sin u_1}{l}, \quad d = d(u_1) = \frac{C}{\sin^2 u_1}.$$

The above substitution transforms our system into one MATLAB can solve with ode45:

$$\frac{du_1}{dt} = u_2(t), \quad \frac{du_2}{dt} = a + b, \quad \frac{dv_1}{dt} = v_2(t), \quad v_2 = d.$$

Now we may begin to consider solutions for the above system. In the analysis below, we have fixed our constant C and the length l to be 1, because we are more interested in how the initial conditions effect the solutions, so the analyses in this section apply for a constant length. To observe solutions numerically, we look at plots of $\phi$ and $\theta$ against time, and also a plot of how the mass moves, i.e. how $(x, y, z)$ vary. (Note that for all plots below against time, $\phi$ is red and $\theta$ is blue).

To start off, it would be good to understand the idea of Chaos - Lorenz defined the concept of Chaos as: "Chaos - When the present determines the future, but the approximate present does not approximately determine the future(13)." What this means in terms of a system of ODEs is that even a slight change in the initial conditions can lead to significant changes in the solutions of the system. The reason this is relevant here is because for any set of initial conditions where $\theta_0$ is near $\pi$ or 0, our system is chaotic, so that even a small change to the value of $\theta_0$ will lead to a huge change in the solution. As an example, compare the system with the initial conditions $\theta_0 = 0.1, \dot{\theta_0} = 1, \phi_0 = \pi, \dot{\phi_0} = 1$ to that with initial conditions $\theta_0 = 0.05, \dot{\theta_0} = 1, \phi_0 = \pi, \dot{\phi_0} = 1$:

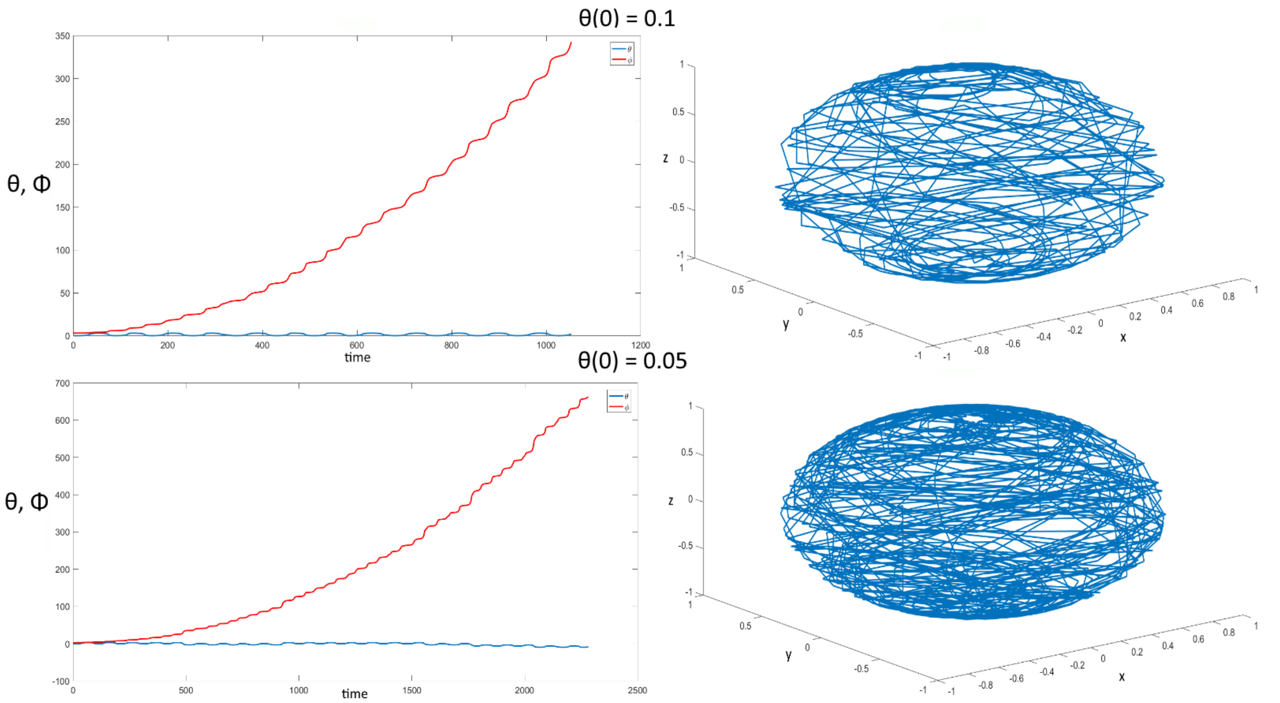


Figure 3.2: $\theta$ and $\phi$ against time on the left and $(x, y, z)$ on the right, for the two values of $\theta_0$

Clearly in the two plots on the right above, there is a fair distinction, even though $\theta_0$ was only decreased by 0.05. For values of $\theta_0$ altered close to $\pi$, an almost identical phenomenon is observed. The reasoning behind this is the fact that in our equation (3.3), there is a $\sin^3\theta$ in the denominator in the first term, which approaches $\infty$ as $\theta$ approaches 0 or $\pi$, i.e. the angular acceleration $\ddot{\theta}$ becomes arbitrarily large. This makes the system chaotic for values of $\theta_0$ near $\pi$ and 0.

Next, we alter the values for the other initial conditions to investigate the effect this has on our solutions. Firstly, note that the value of $\phi_0$ does not greatly affect the solutions on its own. We observe this below - the plot below in blue depicts the solution with initial conditions $\theta_0 = \frac{\pi}{4}, \dot{\theta}_0 = 1, \phi_0 = \frac{\pi}{2}, \dot{\phi}_0 = 1$ and the one in red shows solutions with initial conditions $\theta_0 = \frac{\pi}{4}, \dot{\theta}_0 = 1, \phi_0 = \frac{7\pi}{4}, \dot{\phi}_0 = 1$. The two have been superimposed to show the similarity of the solutions.

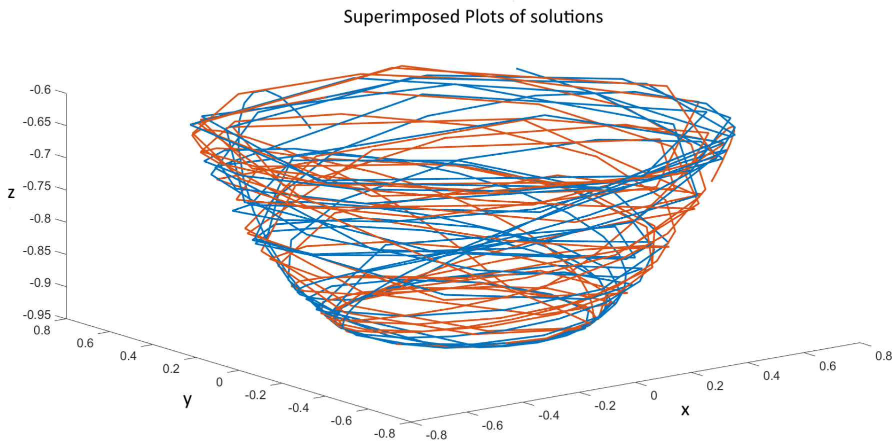


Figure 3.3: A large change in $\phi_0$ does not affect the solutions of the system greatly

Below, we consider the effect of adjusting the values of $\dot{\phi}_0$ and $\dot{\theta}_0$ on the system; throughout we have fixed $\theta_0$ to be $\frac{\pi}{4}$ and $\phi_0$ as $\pi$.

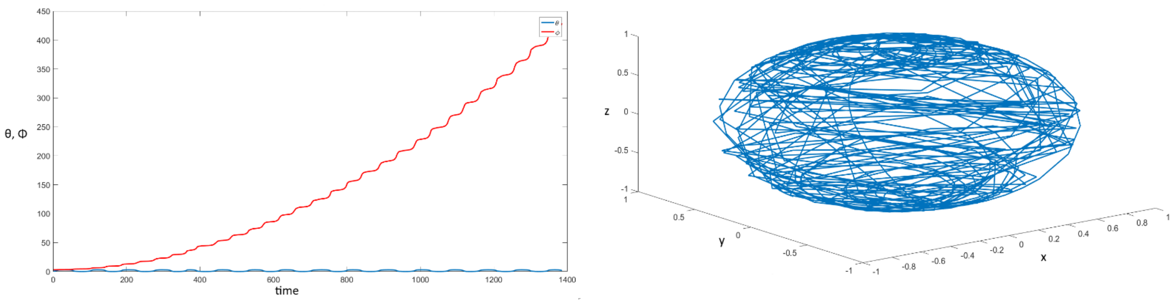


Figure 3.4: Solution with $\dot{\theta}_0 = 12\text{rad}s^{-1}$

Since changing $\phi_0$ does not affect the solutions greatly, we can compare the plots above with those in Figure 3.3, in which we fixed both $\dot{\theta}_0$ and $\dot{\phi}_0$ to be 1. Hence, we know that increasing the magnitude of $\dot{\theta}_0$ makes the system more chaotic. Changing the sign of $\dot{\theta}_0$ has little effect on the solution for larger values and so for $\dot{\theta}_0$ = -12rads$^{-1}$ we get similar plots.

Lastly, with the same initial conditions as before, but taking $\dot{\phi}_0$ to be 12, we observe that the solution is very similar to that in Figure 3.3 - i.e. increasing $\dot{\phi}_0$ does not affect the system heavily. Changing the sign of $\dot{\phi}_0$, however, produces a very different solution - though it is worth noting for values of $\dot{\phi}_0$ of large magnitude, regardless of sign, the solutions are similar. Overall, it seems that $\phi_0$ and $\dot{\phi}_0$ do not affect the solutions greatly.

Now, we consider the effect of removing the gravitational field - so that g = 0. We see the solution for the initial conditions $\theta_0 = \dfrac{\pi}{4}, \dot{\theta}_0 = 1, \phi_0 = \pi, \dot{\phi}_0 = 1$ below:

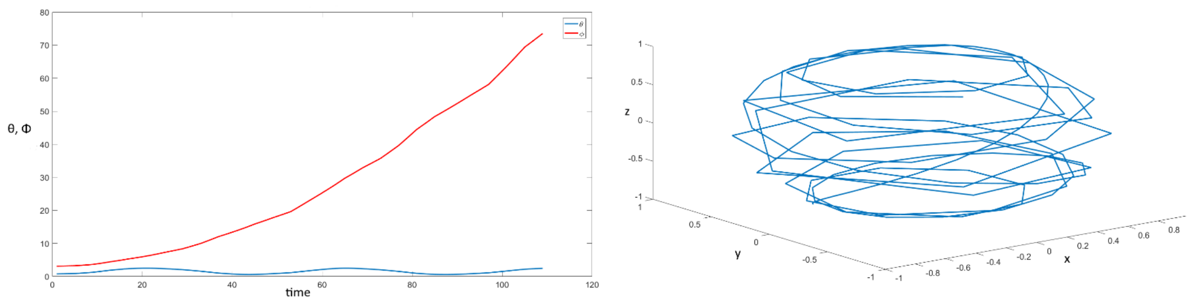


Figure 3.5: Solution for initial conditions as above, with g = 0

The solution above appears to have multiple horizontal circles from top to bottom, with diameter equal to that of the sphere in the middle. This suggests it may be worth considering the idea of great circles - Kern and Bland defined a great circle to be a section of a sphere which "contains the diameter of the sphere"(22). Although we do not have sufficient information on the solutions here to conclude that the route taken by the mass under no gravitational field is that of the great circles, it would certainly be an interesting idea for further research. Initial conditions could be varied greatly and a more precise conclusion could then be reached.

### 3.1.2 Double Pendulum

Double Pendulums are a classic example of a Chaotic System. Here we first look at a question from "Lagrangian and Hamiltonian Mechanics" (10) which is essentially an analysis of the Double Pendulum. The question involves deriving the equations of motion for a double pendulum. Next, we investigate the solutions to these equations, further exploring ideas of chaos in our results, as well as comparing our two examples and finally discussing further implications and ideas.

A Double Pendulum "consists of two simple pendulums, with one pendulum suspended from the bob of the other" (10). Both pendulums move in the same vertical plane. For our generalized coordinate system, we use plane polar coordinates.

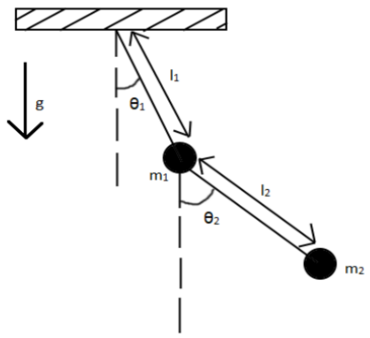


Figure 3.6: The Double Pendulum.

Firstly consider the position of the mass $m_1$ - call this $(x_1, y_1)$. From the above picture, clearly $(x_1, y_1) = (l_1 \sin\theta_1, -l_1 \cos\theta_1)$ so that $(\dot{x_1}, \dot{y_1}) = (\dot{\theta}_1 l_1 \cos\theta_1, \dot{\theta}_1 l_1 \sin\theta_1)$. Substituting our expressions into $T_1 = \frac{1}{2}m\dot{\mathbf{r}}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ returns the K.E. as

$$T_1 = \frac{1}{2}m_1 {l_1}^2 \dot{\theta_1}^2.$$

The potential energy is simply

$$V_1 = m_1 g l_1 y_1 \implies V_1 = -m_1 g l_1 \cos\theta_1.$$

Next, we consider the position, $(x_2, y_2)$ of the mass $m_2$. This is clearly seen from the diagram to be:

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2, y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2.$$

Taking the time derivative then gives us:

$$\dot{x_2} = \dot{\theta}_1 l_1 \cos\theta_1 + \dot{\theta}_2 l_2 \cos\theta_2, \dot{y_2} = \dot{\theta}_1 l_1 \sin\theta_1 + \dot{\theta}_2 l_2 \sin\theta_2.$$

Substituting the above into $T_2 = \frac{1}{2}m_2(\dot{x_2}^2 + \dot{y_2}^2)$ and simplifying (as well as using the identity $\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 = \cos(\theta_1 - \theta_2))$ returns:

$$T_2 = \frac{1}{2}m_2({l_1}^2 \dot{\theta_1}^2 + {l_2}^2 \dot{\theta_2}^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)).$$

The P.E. is simply $V_2 = m_2 g y_2 = -m_2 g(l_1 \cos\theta_1 + l_2 \cos\theta_2)$. Now because $\mathcal{L} = \Sigma T - \Sigma V$, the Lagrangian for the system is:

$$\mathcal{L} = \frac{1}{2}(m_1+m_2){l_1}^2{\dot{\theta}_1}^2 + \frac{1}{2}m_2{l_2}^2{\dot{\theta}_2}^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + (m_1+m_2)gl_1\cos\theta_1 + m_2 g l_2 \cos\theta_2.$$

Now, as before, to derive the equations of motion for the system, we must consider Lagrange's Equation, $\frac{\partial\mathcal{L}}{\partial x^A} - \frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial\dot{x}^A}) = 0$, where $x^1 = \theta_1$ and $x^2 = \theta_2$, since we used plane polar coordinates as our generalized system. Evaluating this for $x^1 = \theta_1$ gives us the first equation of motion (after simplifying) as:

$$l_1(m_1+m_2)\ddot{\theta}_1 + l_2 m_2\cos(\theta_1-\theta_2)\ddot{\theta}_2 + l_2 m_2 \sin(\theta_1-\theta_2){\dot{\theta}_2}^2 + (m_1+m_2)g\sin\theta_1 = 0. \quad (3.4)$$

Now we turn our attention to $x^2 = \theta_2$ - the other equation of motion is:

$$l_2 m_2\ddot{\theta}_2 + l_1 m_2\cos(\theta_1-\theta_2)\ddot{\theta}_1 - l_1 m_2\sin(\theta_1-\theta_2){\dot{\theta}_1}^2 + m_2 g\sin\theta_2 = 0. \quad (3.5)$$

Now, clearly (3.4) and (3.5) are too complicated to even consider analytical solutions, so we turn to numerical solutions. We proceed by defining the above as a system of 4 ODEs, which we can then use ode45 to solve in MATLAB. Note that the method that follows is an adaptation of that seen here (11). Firstly, redefine our variables like so:

$$f_1 = \theta_1(t), \quad f_2 = \dot{\theta}_1(t), \quad h_1 = \theta_2(t), \quad h_2 = \dot{\theta}_2(t).$$

Next, looking at (3.4) and (3.5), we wish to rewrite the equations more compactly. One way to do this is to define the functions:
$P = l_1(m_1+m_2), \quad Q = Q(f_1,h_1) = l_2 m_2\cos(f_1-h_1),$
$R = R(f_1,h_1,h_2) = -l_2 m_2\sin(f_1-h_1){h_2}^2 - (m_1+m_2)g\sin f_1, \quad S = l_2 m_2,$
$T = T(f_1,h_1) = l_1 m_2\cos(f_1-h_1), \quad U = U(f_1,f_2,h_1) = l_1 m_2\sin(f_1-h_1){f_2}^2 - m_2 g\sin h_1.$

Now, with the above substitutions, the equations (3.4) and (3.5) transform into:

$$P\frac{df_2}{dt} + Q(f_1,h_1)\frac{dh_2}{dt} = R(f_1,h_1,h_2)$$

$$S\frac{dh_2}{dt} + T(f_1,h_1)\frac{df_2}{dt} = U(f_1,f_2,h_1).$$

Finally, eliminating one derivative from each equation and simplifying gives us our transformed system as:

$$\frac{df_1}{dt} = f_2, \quad \frac{dh_1}{dt} = h_2, \quad \frac{df_2}{dt} = \frac{RS-QU}{PS-QT}, \quad \frac{dh_2}{dt} = \frac{PU-TR}{PS-QT}.$$

Now we may use ode45 to solve the above system of differential equations numerically. As before, since we are more interested in the effects of the initial conditions rather than the parameters, we will fix $m_1, m_2, l_1$ and $l_2$ to be equal to 1. Below, we observe and discuss solutions associated with various initial conditions. Note that in the solutions, the red plot shows how the mass $m_2$ moves and the blue plot displays the movement of $m_1$, whilst in the $\theta$ against time plots, red is $\theta_2$ and blue is $\theta_1$).

First, we consider the case where $\theta_1(0) = \dfrac{3\pi}{4}$ and $\theta_2(0) = \dfrac{\pi}{2}$. In this situation, the general shape of the solutions to the equations of motion stays relatively similar as $\dot{\theta}_1(0)$ and $\dot{\theta}_2(0)$ vary between -4.3rad$s^{-1}$ and 4.4rad$s^{-1}$ (though of course the masses cover a greater distance). The case where $\dot{\theta}_1(0), \dot{\theta}_2(0) = 4.4$rad$s^{-1}$ is shown below:

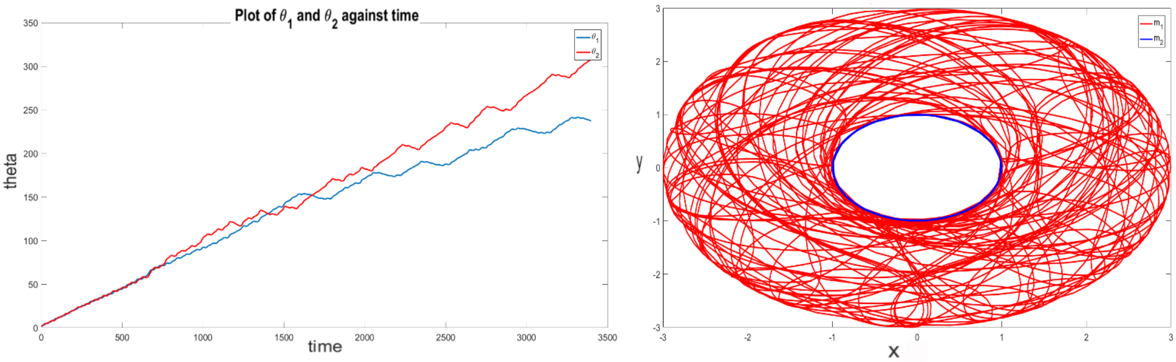


Figure 3.7: Solution for initial conditions as given above

The solutions appear relatively similar for all $\dot{\theta}_1(0)$ and $\dot{\theta}_2(0)$ in the range above - what is surprising is the sudden change in solution when $\dot{\theta}_1(0), \dot{\theta}_2(0) = 4.5$rad$s^{-1}$ (also at -4.4rad$s^{-1}$):

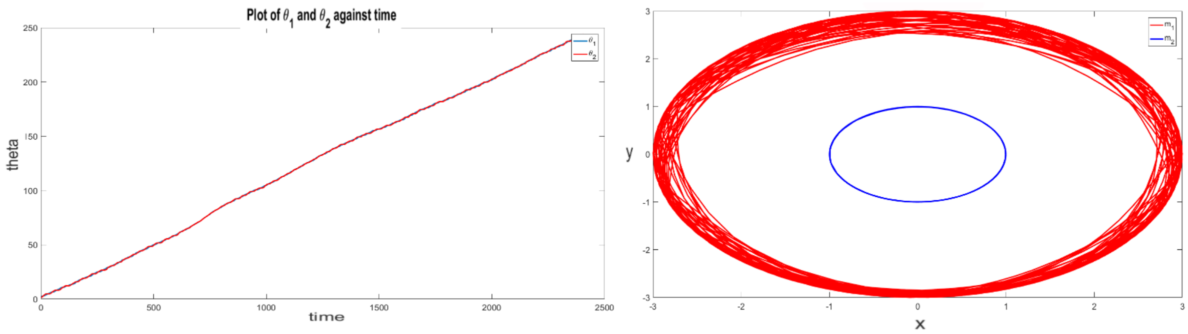


Figure 3.8: Very rapid change in solution after small change in initial conditions

The above is a clear example of the Double Pendulum being chaotic - a small change in $\dot{\theta}_1(0)$ and $\dot{\theta}_2(0)$ has caused the solution to appear completely changed - note how in the first instance the entire region between the paths of the two masses is covered, but in the second, only a small region near the boundary is covered. This means the dependence on these initial conditions for the system is extremely sensitive. Although the system is chaotic everywhere, there would be some critical initial conditions (e.g. above) where it is much more sensitive to small changes.

Next, we look at what happens for very small values of $\theta_1(0)$. Below is the solution plot for $\theta_1(0) = \dfrac{\pi}{20}$, $\dot{\theta}_1(0) = 0$, $\theta_2(0) = \dfrac{\pi}{2}$, $\dot{\theta}_2(0) = 0$:

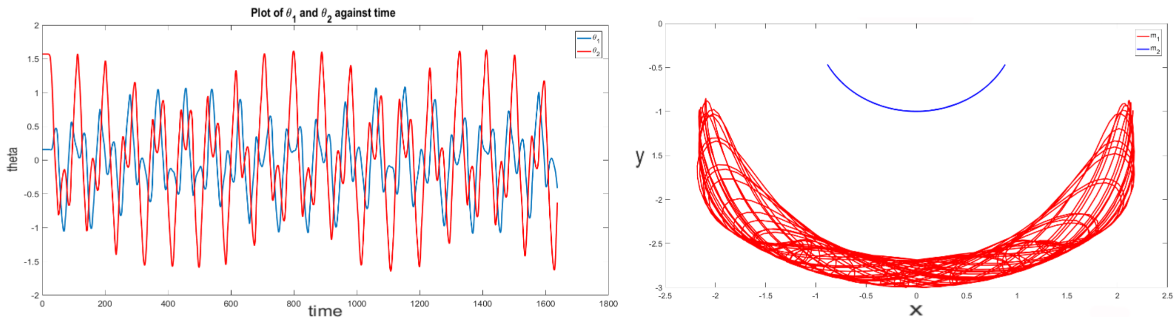


Figure 3.9: Solution for small $\theta_1(0)$

In this case, clearly $\theta_2$ oscillates with a very high frequency, whilst $\theta_1$ also with a moderately high frequency. The mass $m_1$ does not move around a lot, however $m_2$ appears to have covered a lot more distance (although it is not heavily displaced and remains mostly in one region). The system is again chaotic with these initial conditions, although of course not as greatly as in the last case.

Finally, we consider solutions to the system with initial conditions $\theta_1(0) = \dfrac{3\pi}{4}$, $\dot{\theta}_1(0) = 0$, $\theta_2(0) = \pi$, $\dot{\theta}_2(0) = 0$. Because the system is not greatly effected by $\dot{\theta}_1(0)$ and $\dot{\theta}_2(0)$ as long as they are between -4.3 and 4.4, we can compare this to Figure 3.7 and conclude what effect increasing $\theta_2(0)$ has.

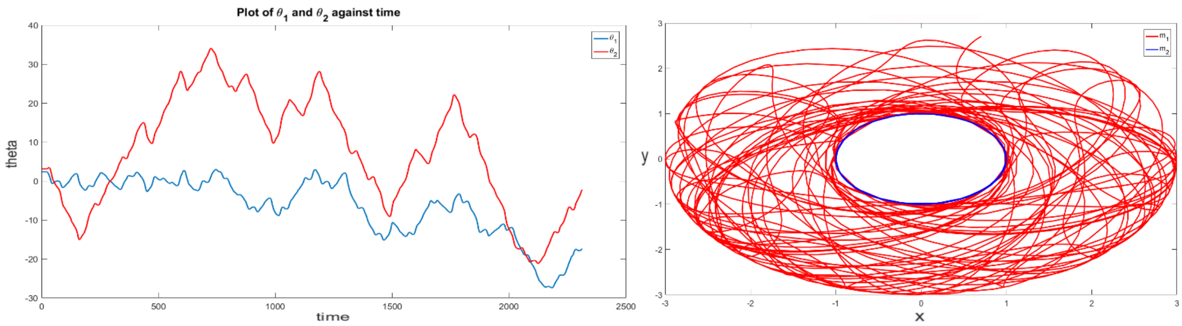


Figure 3.10: Large increase in $\theta_2(0)$ as compared to Figure 3.7

Observe that the general shape of the solution in Figure 3.10 is the same as that in Figure 3.7 - the differences are largely due to the fact that $\dot{\theta}_1(0)$ and $\dot{\theta}_2(0)$ have been reduced to 0 for the solution depicted in Figure 3.10. This implies that, although the system is entirely chaotic, changes to $\theta_2(0)$ and $\theta_1(0)$ (similar case) alone do not affect the system as heavily as if we altered the other initial conditions too.

Lastly, we contrast the two examples and discuss their importance in a more general context. Note that the Spherical Pendulum only displayed chaotic behaviour at singularities of the function $\ddot{\theta}$; however, the Double Pendulum is chaotic everywhere. Comparing Figure 3.2 to Figures 3.7 and 3.8 also suggests that for certain initial conditions, the Double Pendulum is far more chaotic than the Spherical Pendulum with its critical initial conditions. Now - why are these pendulums important? The Spherical Pendulum is a generalization of the regular pendulum, since it allows an extra degree of freedom. This is hence very useful in many contexts, because so many everyday objects utilise the pendulum, such as clocks. The Double Pendulum, as we saw above, is an example of a very chaotic system - therefore it is very important to Chaos Theory as well as Dynamical Systems. All in all, both examples of the Lagrangian we have looked at above are quite fundamental to many areas of Applied Mathematics.

## 3.2 Rigid Bodies - Free Tops

### 3.2.1 The Spherical Top

If you have watched a basketball game, you might notice that the player always puts a back spin on the basketball when he or she is shooting. This can increase chance of scoring.(15)(16) But have you wondered why the spin is always about the same axis? Here is a mathematical explanation.
Suppose the basketball is a solid sphere without deformation. From the definition of principal moments of inertia, we can easily see that $I_1 = I_2 = I_3$, so the Euler's Equations become

$$I_1\dot{\omega}_1 = I_2\dot{\omega}_2 = I_3\dot{\omega}_3 = 0.$$

This implies that he angular velocity $\boldsymbol{\omega}$ is time independent, i.e. a constant.
The angular momentum is

$$\boldsymbol{L} = I\boldsymbol{\omega} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \boldsymbol{\omega} = I_1\boldsymbol{\omega}$$

where $I_1$ is the principal moment of inertia of the sphere which is a constant scalar. The angular momentum is a constant vector in the same direction as the angular velocity $\boldsymbol{\omega}$. Therefore the sphere will keep spinning about the fixed axis in the body frame.

### 3.2.2 The Symmetric Top

**Symmetric Top in the Body Frame**

Now, let's think about a symmetric top with $I_1 = I_2 \neq I_3$ and so the Euler's Equations become

$$\begin{aligned} I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_1) &= 0 \\ I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3) &= 0 \\ I_3\dot{\omega}_3 &= 0 \end{aligned} \tag{3.6}$$

From the third equation of (3.6), we can see that $\omega_3$, which is the angular speed around the symmetric axis, is a constant. If we introduce a new constant, $\Omega$, defined by

$$\Omega = \frac{I_1 - I_3}{I_1}\omega_3, \tag{3.7}$$

then we obtain a system of ordinary differential equations

$$\begin{aligned} \dot{\omega}_1 &= \Omega\omega_2 \\ \dot{\omega}_2 &= -\Omega\omega_1 \end{aligned} \tag{3.8}$$

Take the time derivative of the first equation in (3.8), and substitute in the value of $\dot{\omega}_2$, we have

$$\ddot{\omega}_1 + \Omega^2\omega_1 = 0.$$

A solution to this differential equation is

$$\omega_1 = \omega_0 \mathrm{sin}\Omega t$$

where $\omega_0$ is a constant. One can easily find a similar expression for $\omega_2$.
The angular velocity can be written as

$$\boldsymbol{\omega} = (\omega_0\mathrm{sin}\Omega t, \omega_0\mathrm{cos}\Omega t, \omega_3) \tag{3.9}$$

Recall that $\omega_3$ is a constant, so we can conclude that the angular velocity is a constant in magnitude and the symmetric top rotates around the $\boldsymbol{e}_3$ axis of the body frame with a constant frequency $\Omega$. The direction of rotation depends on the sign of $\Omega$.
Consider $\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$, if we integrate it with respect to time, we get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{\omega_0}{\Omega}\mathrm{cos}\Omega t + A \\ \frac{\omega_0}{\Omega}\mathrm{sin}\Omega t + B \end{pmatrix}$$

where $A$ and $B$ are integration constants. Without loss of generality, set $A = B = 0$.
If $I_1 > I_3$, $\Omega$ is positive, the direction of rotation is clockwise. Similarly, if $I_1 < I_3$, the rotation is counter-clockwise (Figure 3.11). Such rotation is known as the precession.(14)

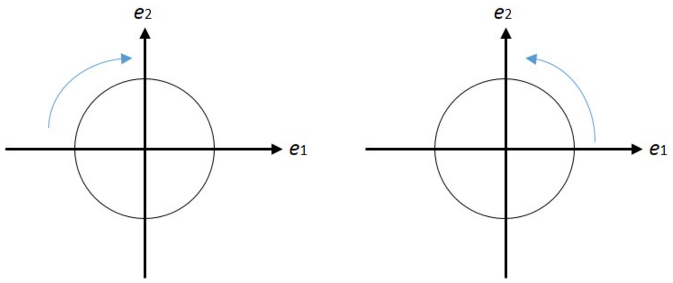


Figure 3.11: The direction of precession when $I_1 < I_3$ (left) and $I_1 > I_3$ (right)

**Symmetric Top in the Space Frame**
With the help of Euler angles, we can describe the motion of free symmetric top in the space frame.
Since the body is free, the angular momentum is conserved, i.e. $\frac{d\boldsymbol{L}}{dt} = 0$, and the direction of the angular momentum is fixed. Let's choose the angular momentum to be in the direction of $\tilde{\boldsymbol{e}}_3$ axis in the space frame, as shown in Figure 3.12.

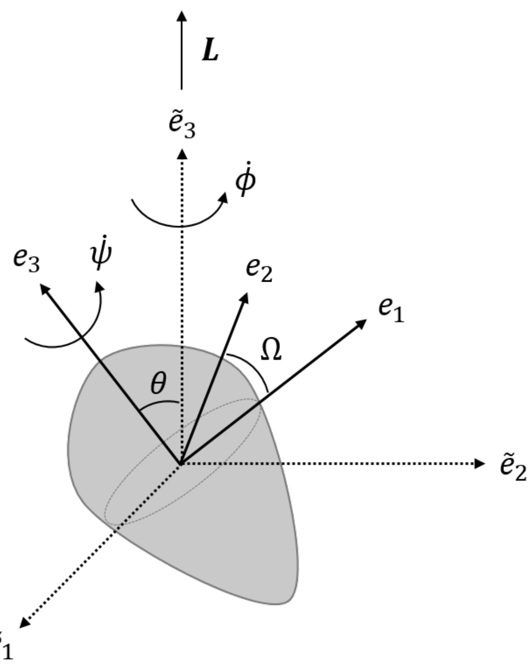


Figure 3.12: Euler angles for free symmetric top with angular momentum in the direction of $\tilde{\boldsymbol{e}}_3$

**Claim**: the angle between $e_3$ axis of the body frame and the angular momentum $\boldsymbol{L}$ is fixed.

*Proof.* The claim can be proved using the formula for calculating the angle between two vectors.

$$\begin{aligned}
\cos\theta &= \frac{\boldsymbol{e_3}\cdot\boldsymbol{L}}{\|\boldsymbol{e_3}\|\|\boldsymbol{L}\|} \\
&= \frac{(0,0,1)^t\cdot(I_1\omega_1, I_1\omega_2, I_3\omega_3)^t}{[(I_1\omega_1)^2+(I_1\omega_2)^2+(I_3\omega_3)^2]^{1/2}} \\
&= \frac{I_3\omega_3}{[(I_1\omega_0)^2+(I_3\omega_3)^2]^{1/2}}
\end{aligned}$$

This is a constant because $I_1$, $I_3$, $\omega_0$ and $\omega_3$ are all constants.
Therefore the angle between $\boldsymbol{e}_3$ axis and $\boldsymbol{L}$, $\theta$, is fixed. Hence, $\dot{\theta} = 0$. □

From subsection 2.2.7, we know the angular velocity can be expressed in terms of $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of the body frame by Equation (2.11):

$$\boldsymbol{\omega} = \dot{\phi}\sin\theta[\sin(\psi)\boldsymbol{e}_1 + \cos(\psi)\boldsymbol{e}_2] + (\dot{\psi} + \dot{\phi}\cos\theta)\boldsymbol{e}_3 \tag{3.10}$$

According to Figure 3.8, together with Equation (3.7), we have

$$\dot{\psi} = \Omega = \frac{I_1 - I_3}{I_1}\omega_3. \tag{3.11}$$

Equation (3.9) and (3.10) are both angular velocity of the free symmetric top, so each components should be equal. In particular, $\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$. We can get the precession frequency:

$$\dot{\phi} = \frac{\omega_3 - \dot{\psi}}{\cos\theta} = \frac{I_3\omega_3}{I_1\cos\theta} \tag{3.12}$$

This equation holds for all free symmetric tops. In the next two subsections, we will use this equation to find the wobbling-spinning ratio of specific symmetric shapes.

### 3.2.3 Feynman's Wobbling Plate

In the book "'Surely you're joking, Mr. Feynman' : adventures of a curious character", the famous physicist Richard Feynman tells a story which influenced his career in physics.

"Within a week I was in the cafeteria and some guy, fooling around, throws a plate in the air. As the plate went up in the air I saw it wobble, and I noticed the red medallion of Cornell on the plate going around. It was pretty obvious to me that the medallion went around faster than the wobbling.
I had nothing to do, so I start to figure out the motion of the rotating plate. I discover that when the angle is very slight, the medallion rotates twice as fast as the wobble rate - two to

one. It came out of a complicated equation!"(18)

Mr Feynman made a "joke" with the readers: the ratio of wobbling rate to spinning rate is actually approximately two to one. The plate wobbles faster than it rotates, which can be easily verified by throwing a plate in the air. The one to two ratio can be proved using our knowledge on symmetric top. For simplicity, let's consider the plate as a uniform disc with radius $r$ and mass $m$. The plate is a free symmetric top, so we can use our result derived in the previous part. As shown in Figure 3.13, $\dot{\psi}$ is the precession frequency, or the "wobble rate",

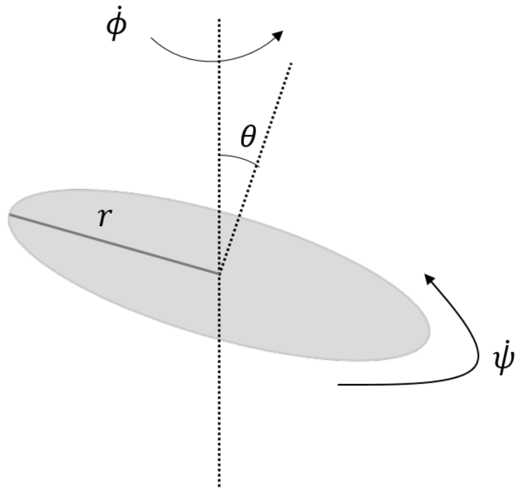


Figure 3.13: Feynman's wobble plate illustration

and

$$\dot{\psi} = -\omega_3 \tag{3.13}$$

is the spinning speed. If we can find the principal moments of inertia, $I_1$ and $I_3$, of the disc, we can find the relation between the wobble rate and the spinning speed. (17)
Define the $\boldsymbol{e}_3$ axis of the body frame to be perpendicular to the disc. The moments of inertia are about the centre of mass of the plate, then

$$I_1 = \int \rho y^2 dxdy,\ I_2 = \int \rho x^2 dxdy,$$
$$I_3 = \int \rho x^2 + y^2 dxdy = I_1 + I_2.$$

where $\rho = \frac{m}{\pi r^2}$. Since $I_1 = I_2$ for the disc, we have

$$I_3 = 2I_1. \tag{3.14}$$

Substitute Equation (3.13) and (3.14) into equation (3.12):

$$\dot{\phi} = \frac{-2\dot{\psi}}{\cos\theta} \approx -2\dot{\psi}.$$

Above holds when the angle $\theta$ is small. Therefore, the plate's wobble rate is twice as fast as the spinning speed.

### 3.2.4 The Hula Hoop

Here comes a question: how about the wobbling-spinning ratio of other symmetric tops? In what follows, we will try to find the ratio of a hula hoop.

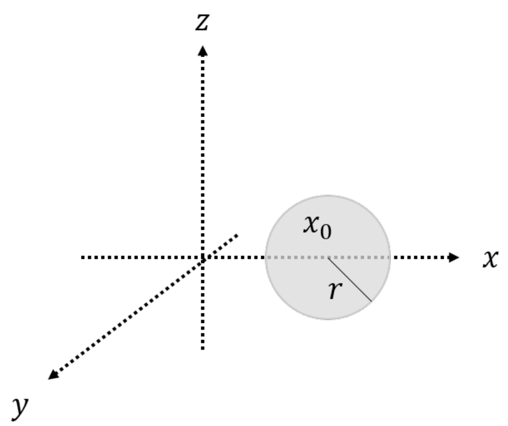


Figure 3.14: Rotation of the circle about the external axis $z$

Mathematically, we can treat the hula hoop as a uniform torus obtained by the rotating the circle about an external axis, as shown in the Figure 3.14. Initially, the circle is in the $x$-$z$ plane centered at $(x_0, 0, 0)$ with radius $r$ $(r < x_0)$. The circle is rotated about $z$-axis for $2\pi$ and a torus is obtained. Any point on the surface of torus can be expressed by two angles. To distinguish these angles from Euler angles, we name these two angles $\alpha$ and $\beta$.

$$\boldsymbol{r}(\alpha, \beta) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 + r\cos\alpha \\ 0 \\ r\sin\alpha \end{pmatrix}. \tag{3.15}$$

Using the formula for change of variables in surface integration, a surface element is

$$\begin{aligned} dS &= \|\boldsymbol{J}\|\, d\alpha\, d\beta \\ &= \|\frac{\partial \boldsymbol{r}}{\partial \alpha} \times \frac{\partial \boldsymbol{r}}{\partial \beta}\|\, d\alpha\, d\beta \\ &= r(r\cos\alpha + x_0)\, d\alpha\, d\beta \end{aligned}$$

Here, $\boldsymbol{J}$ is known as the vector Jacobian. A volume element of the torus can be obtained by multiplying the surface element by a small amount of the radius $r$, $dr$.

$$dV = dx\,dy\,dz = dS\,dr = r(r\cos\alpha + x_0)\,d\alpha\,d\beta\,dr \tag{3.16}$$

The volume of the torus, $V$, can be found either by direct integration or by volume of revolution. If the mass of the torus is $m$, then the density is

$$\rho = \frac{m}{V} = \frac{m}{\iiint dV} = \frac{m}{2\pi^2 r^2 x_0}. \tag{3.17}$$

Let the $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ axes of the body frame be defined as shown in Figure 3.15. Now we can work out the principal moments of inertia using Equation (2.2) and change of variable formula Equation (3.16):

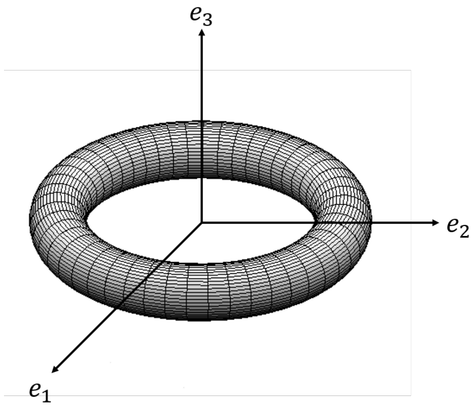


Figure 3.15: A torus with axes $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of the body frame.

$$\begin{aligned} I &= \rho \int\int\int dx\,dy\,dz \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} \\ &= \rho \int_0^r \int_0^{2\pi} \int_0^{2\pi} r'(r'\cos\alpha + x_0) \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} d\alpha\,d\beta\,dr' \end{aligned} \tag{3.18}$$

where $r'$ is a dummy variable. $x$, $y$ and $z$ can be expressed in terms of $\alpha$, $\beta$ and $r'$ using equation (3.15). Here is an example on how to find $I_1$:

$$\begin{aligned} I_1 &= \rho \int_0^r \int_0^{2\pi} \int_0^{2\pi} r'(r'\cos\alpha + x_0)(y^2+z^2)d\alpha\,d\beta\,dr' \\ &= \rho \int_0^r \int_0^{2\pi} \int_0^{2\pi} r'(r'\cos\alpha + x_0)[\sin^2\beta(x_0 + r'\cos\alpha)^2 + r'^2\sin^2\alpha]d\alpha\,d\beta\,dr' \\ &= \frac{m}{8}(4x_0^2 + 5r^2) \end{aligned} \tag{3.19}$$

In a similar way, we can find the value of $I_3$:

$$I_3 = \frac{m}{4}(4x_0^2 + 3r^2) \tag{3.20}$$

Substitute Equation (3.19) and (3.20) into Equation (3.12), the wobbling rate of the torus is:

$$\dot{\phi} = \frac{I_3\omega_3}{I_1 \cos\theta} = \frac{(4x_0^2 + 3r^2)}{(4x_0^2 + 5r^2)} \frac{2\omega_3}{\cos\theta} \approx 2\omega_3 \frac{(4x_0^2 + 3r^2)}{(4x_0^2 + 5r^2)}$$

The above holds when the angle $\theta$ is small. The wobble-spin ratio depends on the shape of the torus, i.e. the major radius $x_0$ and the minor radius $r$. If we take a very "thin" hula hoop, that is, $\frac{r}{x_0} \approx 0$, the wobbling rate becomes

$$\dot{\phi} \approx 2\omega_3 \frac{(4x_0^2 + 3r^2)}{(4x_0^2 + 5r^2)} = 2\omega_3 \frac{4 + 3\frac{r^2}{x_0^2}}{4 + 5\frac{r^2}{x_0^2}} \approx 2\omega_3 = -2\dot{\psi}.$$

Therefore, wobbling-spinning ratio of a wobbling hula hoop is also approximately 2 as a wobbling plate.

## 3.3 Rigid Bodies - The Heavy Symmetric Top

In previous sections, we have discussed motions of free tops in various scenarios. Here in this section the aim is to explore motions of heavy symmetric tops interacting in the gravitational field. We will first derive general equations of motion in Lagrange Formalism using Euler's angles and principal axis, then we will consider some concrete cases of heavy tops and simulations using MATLAB. Finally we try to solve those equations analytically under one of the conditions.

### 3.3.1 The Heavy Symmetric Top Problem

Spinning tops have long been used as entertainment gadgets and some of its modern variations are still attracting a lot of people. For example the Gyro-top (Gyroscope) designed by Tiger & Co. was strikingly popular in the 1960s and 70s with an annual production of about 200,000 to 300,000 units(19).

Formally speaking, a **heavy symmetric top** is a body with a point fixed at the pivot and is subject to gravitational field, but is free to rotate (as shown in Figure 3.16)(3).

Generally, questions of this kind involve a inertial frame $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$ fixing at the origin and a body frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. By Euler's Theorem, the body frame can be viewed as transformation

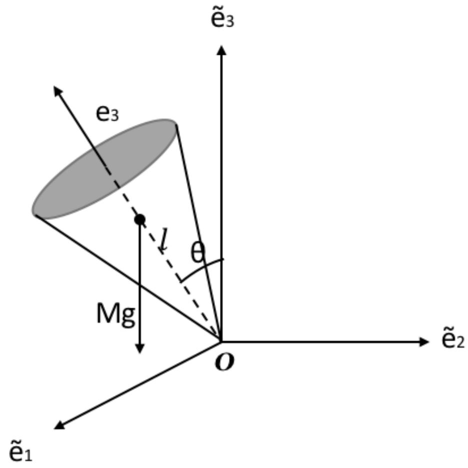


Figure 3.16: Heavy symmetric top

of inertial frame by Euler's angles $(\phi, \theta, \psi)$. Note that $I_1 = I_2$ because of symmetry. Also as in the diagram, the distance between $O$ and center of mass is $l$. The top has a weight of $Mg$ since it is "Heavy" under gravitational field(17).

#### Finding the Lagrangian

Considering right hand rule and orientations of Euler's angles $(\phi, \theta, \psi)$, it is clear that $\dot{\phi}$ is directed in the same direction as $\tilde{e}_3$ and $\dot{\psi}$ is pointed along $e_3$. The Lagrangian is:

$$\mathcal{L} = T - V \tag{3.21}$$

$$V = Mgl\cos\theta. \tag{3.22}$$

But kinetic energy remains unclear because of the **angular velocity** $\boldsymbol{\omega}$: $\omega = \dot{\phi} + \dot{\theta} + \dot{\psi}$. Considering directions mentioned above, we have $\dot{\psi} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$, $\dot{\theta} = \begin{pmatrix} \dot{\theta}\cos\psi \\ -\dot{\theta}\sin\psi \\ 0 \end{pmatrix}$,

$$\dot{\phi} = R_3(\phi)R_1(\theta)R_3(\psi)e_3\dot{\phi} = \dot{\phi}\begin{pmatrix} \sin\theta\sin\phi \\ \sin\theta\cos\phi \\ \cos\theta \end{pmatrix}.$$

Hence, the angular velocity is $\omega = \begin{pmatrix} \dot{\theta}\cos\psi + \sin\theta\sin\psi\cdot\dot{\phi} \\ -\dot{\theta}\sin\psi + \sin\theta\cos\phi\cdot\dot{\phi} \\ \dot{\psi} + \cos\theta\cdot\dot{\phi} \end{pmatrix}$.

Using $I_1 = I_2$, $\omega$ and equation (3.21), (3.22), we get the **Lagrangian**:

$$\mathcal{L} = \frac{1}{2} I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3\omega_3^2 - Mgl\cos\theta \tag{3.23}$$

$$= \frac{1}{2} I_1(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \frac{1}{2} I_3(\dot{\psi}^2 + \cos\theta\dot{\phi})^2 - Mgl\cos\theta \tag{3.24}$$

**Using Invariant Quantities**

Because $\omega_3$ remains invariant, angular momentum about $e_3$ is also invariant:

$$\frac{\partial\mathcal{L}}{\partial\dot{\psi}} = I_3(\dot{\psi} + \cos\theta\dot{\phi}) = I_3\omega_3 = p_\psi. \tag{3.25}$$

Similarly, another invariant quantity is $p_\phi$:

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = I_1\sin^2\theta\dot{\phi} + I_3\cos\theta(\dot{\psi} + \cos\theta\dot{\phi}) = p_\phi. \tag{3.26}$$

Hence, total energy $E$ is also invariant:

$$E = T + V = \frac{1}{2} I_1(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \frac{1}{2} I_3(\dot{\psi}^2 + \cos\theta\dot{\phi})^2 + Mgl\cos\theta.$$

The invariant property of these three quantities is of great importance to our future analysis. The invariant $p_\psi$ and $p_\phi$, with equation (3.25) allow us to express $\dot{\phi}$ and $\dot{\psi}$ in terms of $\theta$:

$$\dot{\psi} = \frac{p_\psi - I_3\cos\theta\dot{\phi}}{I_3}. \tag{3.27}$$

Taking it back to equation (3.26), we have:

$$\dot{\phi} = \frac{p_\phi - I_3\cos\theta p_\psi}{I_1\sin^2\theta}, \tag{3.28}$$

and again taking back to (3.28) gives:

$$\dot{\psi} = \frac{p_\psi}{I_3} - \frac{\cos\theta p_\phi - I_3\cos\theta p_\psi}{I_1\sin^2\theta}. \tag{3.29}$$

At this stage, one could use Euler-Lagrange equation to solve equation (3.29), (3.30) for $\theta(t)$. But here we explore another interesting way of solving them. Define "reduced energy"(3) to be $E' = E - \frac{1}{2}I_3\omega_3^2$, then $E'$ is also invariant due to the fact that both E and $\omega_3$ are constant

$$E' = \frac{1}{2} I_1\dot{\theta}^2 + V_{eff}(\theta).$$

where $V_{eff}$ is the effective potential, which is

$$V_{eff} = \frac{(p_\phi - p_\psi \cos\theta)^2}{2I_1 sin^2\theta} + Mgl\cos\theta.$$

**Equations of Motion**

Here we have expressed $E'$ solely in terms of $\theta$. Let $u = \cos\theta, \alpha = \frac{2E'}{I_1}, \beta = \frac{2Mgl}{I_1}, a = \frac{p_\psi}{I_1}, b = \frac{p_\phi}{I_1}$ we finally have:

$$\dot{u}^2 = (1-u^2)(\alpha - \beta u) - (b - au)^2 \tag{3.30}$$

$$\dot{\phi} = \frac{b - au}{1-u^2} \tag{3.31}$$

$$\dot{\psi} = \frac{I_1 a}{I_3} - \frac{u(b-au)}{1-u^2} * \tag{3.32}$$

We can then solve equation (3.30) by taking the square root followed by integration, then take into (3.31) and (3.32) to find solution for $\phi$ and $\psi$. This will give the elliptic integral:

$$t = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{(1-u^2)(\alpha - \beta u) - (b-au)^2}}.$$

Note that if we are considering a top spinning on a horizontal plane, then $u = \cos(\theta)$ should be greater than zero.

Let $f(u)$ be the right hand side of (3.30):

$$f(u) = (1-u^2)(\alpha - \beta u) - (b-au)^2 = \beta u^3 - (\alpha + a^2)u^2 + (2ab - \beta)u + (\alpha - b^2) \tag{3.33}$$

it is then a cubic function, we can roughly locate the value of roots of u by plotting polynomial $f(u)$. Also cos takes value between -1 and 1. In between these $u_1$ and $u_2$, $\dot{\phi}$ has three possibilities. Here we depict the three scenarios in terms of the motion of the top by tracing the curve of the intersection of the figure axis on a sphere of unit radius . (This trace is known as the **locus** of the figure axis.)

As we can see from figure (3.17), $\theta$ varies between $\theta_1 = \arccos u_1$ and $\theta_2 = \arccos u_2$, which give the following three scenarios:

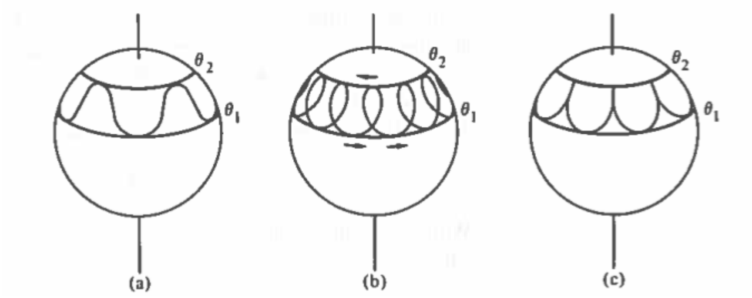


Figure 3.17: Possible roots of $f(u)$ which describe the turning angle $\theta(2)$.

Note that the values of roots $u_1$ and $u_2$ hence reflects the extent of periodical shaking, which is known as **nutation** of this spinning top.

Note also that we call the spinning top a **sleeping top** if it spins upright in the direction of $\tilde{e}_3$, with $\theta = \dot{\theta} = 0$

### 3.3.2 Examples with Matlab Simulation

Instead of considering the heavy top above, we now consider a heavy spinning plate of mass $m$ attached on the top of a stick of length $l$.

Equations of motion in section (3.3.1) is applicable in this case, but tops moves in different locus as we vary initial conditions. In this subsection, we are going to deal with different initial **precession** $\phi$, then solve numerically and see what will be the resulting locus using MATLAB. All the code in this chapter will be included by the appendix in the end(20). First we have $l = 0.15m, m = 0.1kg, \dot{\psi}_0 = 100\pi, \phi_0 = \frac{\pi}{3}, \dot{\phi} = 3$. Hence, $f(u)$ this will give the wave-like locus correspondent to the case shown in figure3.17 (a), with nutation between 1.0472 and 1.1993:

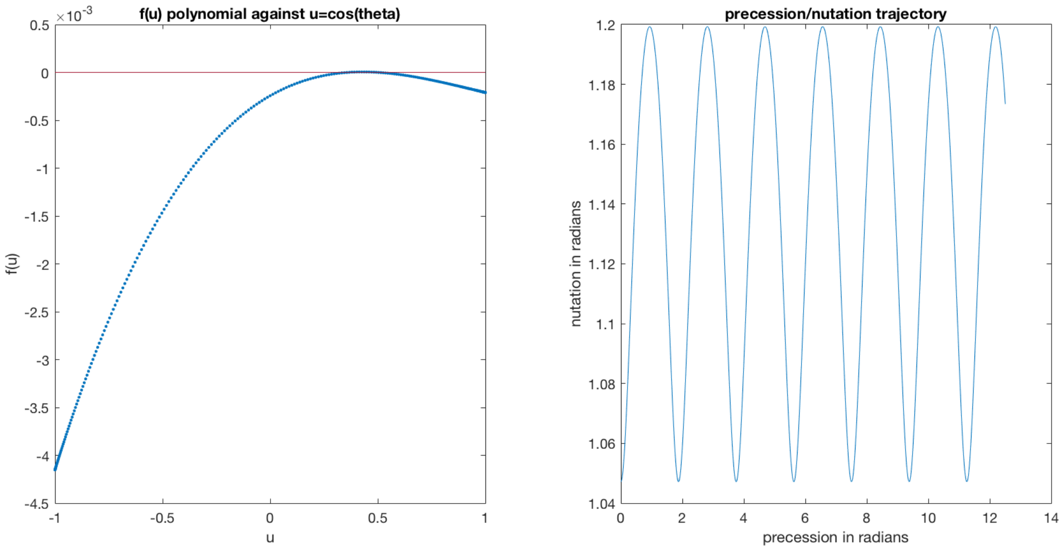


Figure 3.18: The first case, reflecting case (a) in the previous figure.

The second observation occurs when we reset $\dot{\phi}$ to be -3, resulting $\theta$ varies between with $\theta$ varies between 1.0472 and 1.7776:

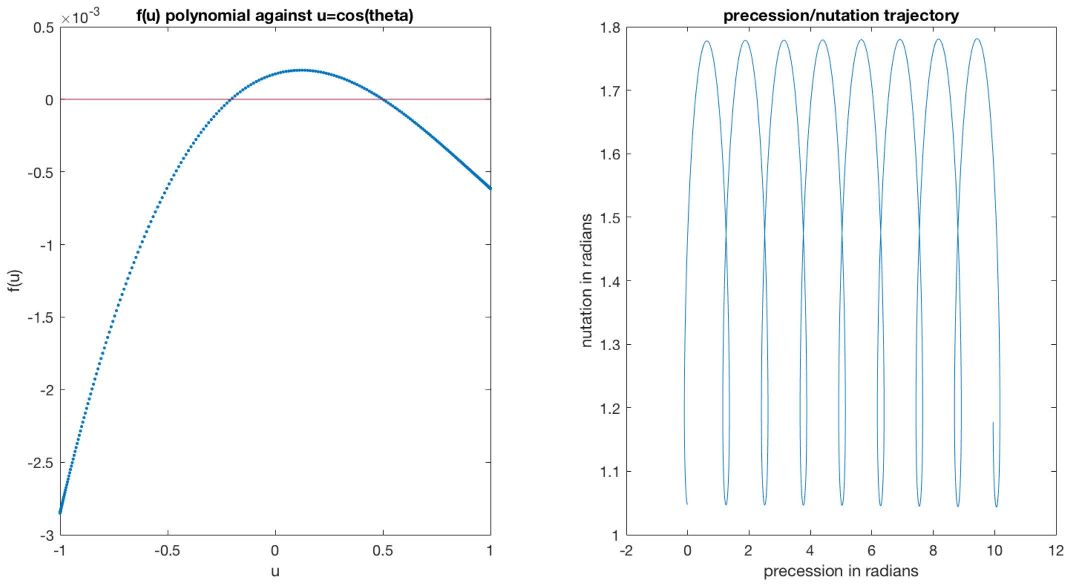


Figure 3.19: The second case, reflecting case (b) in the previous figure.

The third scenario is realized by setting $l = 0.15m, m = 0.1kg, \dot{\psi}_0 = 100\pi, \phi_0 = \frac{\pi}{3}, \dot{\phi}$ =0, this will give the cusp-like shape (right hand side figure 3.20) with $\theta$ varies between 1.0472 and 1.5140.

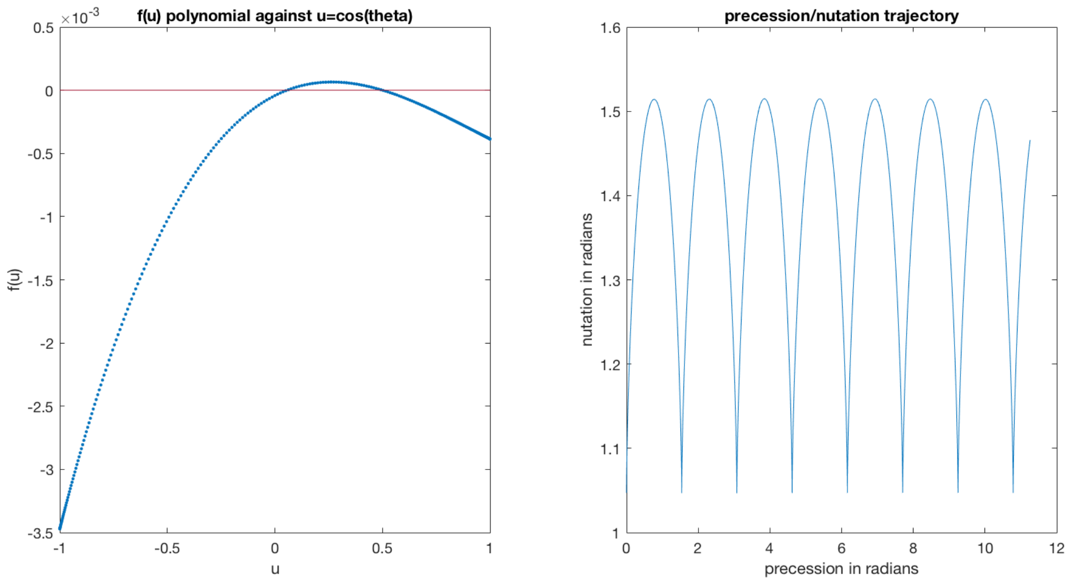


Figure 3.20: The Third case, reflecting case (c) in the previous figure.

### 3.3.3 Analytically solve for the Third Motion

So how exactly does the third motion work? This third motion is generated by spinning the top and let it go at a specific angle $\theta$, so we have

$$\dot{\theta}_{t=0} = 0 \tag{3.34}$$

$$\dot{\phi}_{t=0} = 0 \tag{3.35}$$

Recall that the quantity $p_\phi = I_1\dot{\phi}\sin^2\theta + I_3\omega_3\cos\theta$ is constant through t, then taking value at $t = 0$, $p_\phi = I_3\omega_3\cos\theta_{t=0}$

This cusp-like motion is produced in the following way: As the top starts to fall under gravity, $\theta$ increases by time, giving an increasing $\dot{\phi}$ because that $p_\phi$ is constant.

In this scenario, it is actually possible for us to solve equations of motion analytically:

Assuming that kinetic energy generated by rotation about $\tilde{e}_3$ is much greater than the potential energy, which is:

$$\frac{1}{2}I_3\omega_3^2 >> 2Mgl \tag{3.36}$$

In such situation, we call the top a "Fast Top" and $E'$ reduces to $Mgl\cos\theta_0$, which means $\alpha = \beta u_0$. Hence, now equation(3.35) becomes $f(u) = (u_0 - u)[\beta(1 - u^2) - a^2(u_0 - u)]$, with roots $u_0$ and $u_1$ such that:

$$(1 - u_1^2) - \frac{a^2}{\beta}(u_0 - u_1) = 0 \tag{3.37}$$

writing $x = u_0 - u$, $x_1 = u_0 - u_1$, $p = \frac{a^2}{\beta} = -2\cos\theta_0$ and $q = \sin^2\theta_0$, this is then:

$$x_1^2 + px_1 - q = 0 \tag{3.38}$$

Except for the occasion that $I_3 << I_1$, using (3.39) assumption, the ratio $\frac{a^2}{\beta}$ is much greater than 1. The only root of (3.40) is then $x_1 = \frac{q}{p} \approx \frac{\beta \sin^2\theta_0}{a^2}$

Hence, the magnitude of nutation decreases for large spinning velocity.

# Chapter 4

# Conclusion

Lagrangian Mechanics is a reformulation of Newtonian Mechanics which uses the energies of a physical system. It makes solving problems much easier when using a non-Cartesian coordinate system. We have introduced several important definitions and theorems, including The Principle of Least Action, Lagrange's Equation and Noether's Theorem. We further discussed some examples, such as the Spherical Pendulum and the Double Pendulum, in which we derived the equations of motion and solved them numerically, exploring ideas of chaos within solutions.

We have introduced several key ideas within Rigid Body Motion, including: The Moment of Inertia, Angular Momentum, Euler's Equation, Dynamics and Euler Angles. We have looked into several examples of free tops, such as The Spherical Top, The Symmetric Top, Feynmann's Wobbling Plate and The Hula Hoop. A more interesting problem we looked at is the Heavy Symmetric Top in which we include the effects of gravity. We analyzed the Heavy Symmetric Top problem and solved the equations numerically and analytically.

The assumption that a body is rigid greatly simplifies the analysis of a physical system and gives us a great approximation for ordinary solids subject to typical stress. This makes Rigid Body Dynamics key part of modern day engineering. It can be applied to many real world bodies, such as planes, cars, trains and boats. Very often, non-Cartesian coordinates will be used to solve these problems and hence Lagrangian Mechanics plays a deep role within the dynamics of rigid bodies.

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# Appendices

# Appendix A

# MATLAB code

Please note that all techniques discussed in this report have been coded from scratch in MATLAB, unless otherwise stated. All code, as well as comments explaining exactly how it works, can be found here in the Appendix.

## Spherical Pendulum Code

Listing A.1: Matlab Code for numerically solving equations of the Spherical Pendulum

```
%parameters, initial conditions
l=1; C=1; K=C^2; g=9.8;
tspan=10;

y0=[theta theta_prime phi phi_prime];
[t,u]=ode45(@S_pend, [0 ,tspan],[1 0 0 (2*g/l)^1/2]);

%position of mass
x=l.*sin(u(:,1)).*cos(u(:,3));
y=l.*sin(u(:,1)).*sin(u(:,3));
z=-l.*cos(u(:,1));

%plots
figure(1)
plot3(x,y,z,'linewidth',2)
hold on
h=gca;
get(h,'fontSize')
set(h,'fontSize',14)
xlabel('x','fontSize',14);
ylabel('y','fontSize',14);
zlabel('z','fontSize',14);
title('Spherical Pendulum','fontsize',14)
```

```
fh = figure(1);
set(fh, 'color', 'white');

figure(2)
plot(u(:,1),'linewidth',2)
hold on
plot(u(:,3),'r','linewidth',2)
h=gca;
get(h,'fontSize')
set(h,'fontSize',14)
legend('\theta','\phi')
xlabel('time','fontSize',14);
ylabel('theta/phi','fontSize',14);
title('Plot of \theta and \phi against time','fontsize',14)
fh = figure(2);
set(fh, 'color', 'white');

%substitutions, system of ODEs
function [uprime] = S_pend(t, u)

l=1; C=1; K=C^2; g=9.8;

a = K*cos(u(1))./(sin(u(1))).^3;
b = -g*sin(u(1))/l;
d = C./(sin(u(1))).^2 ;

uprime(1) = u(2);
uprime(3)= u(4) ;
uprime(2)= a + b ;
uprime(4)= d ;

uprime=uprime';

end
```

[Note that in the above, we will vary the values for g, the length and the initial conditions. The code has been adapted from (12).]

# Double Pendulum Code

Listing A.2: Matlab Code for numerically solving equations of the Double Pendulum

```
%call ode45 to solve function
tspan = 50;
[t,y]=ode45(@pend, [0 ,tspan],[ 3*pi/4 0 pi/3 0]);

%position of the masses
```

```
x1=l1*sin(y(:,1));
y1=-l1*cos(y(:,1));
x2=l1*sin(y(:,1))+l2*sin(y(:,3));
y2=-l1*cos(y(:,1))-l2*cos(y(:,3));

%plots
figure(1)
plot(x1,y1,'linewidth',2)
hold on
plot(x2,y2,'r','linewidth',2)
h=gca;
get(h,'fontSize')
set(h,'fontSize',14)
xlabel('X','fontSize',14);
ylabel('Y','fontSize',14);
title('Chaotic Double Pendulum','fontsize',14)
fh = figure(1);
set(fh, 'color', 'white');

figure(2)
plot(y(:,1),'linewidth',2)
hold on
plot(y(:,3),'r','linewidth',2)
h=gca;
get(h,'fontSize')
set(h,'fontSize',14)
legend('\theta_1','\theta_2')
xlabel('time','fontSize',14);
ylabel('theta','fontSize',14);
title('\theta_1(t=0)=2.5 and \theta_2(t=0)=1.0','fontsize',14)
fh = figure(2);
set(fh, 'color', 'white');

%pendulum function
function [yprime] = pend(t, y)

%parameters
l1=1; l2=2; m1=1; m2=1; g=9.8;

%substitutions
P = (m1+m2)*l1;
Q = m2*l2*cos(y(1)-y(3));
T = m2*l1*cos(y(1)-y(3));
S = m2*l2;
R = -m2*l2*y(4)* y(4)*sin(y(1)-y(3))-g*(m1+m2)*sin(y(1));
U = m2*l1*y(2)*y(2)*sin(y(1)-y(3))-m2*g*sin(y(3));

%system of odes
yprime(1) = y(2);
yprime(3)= y(4);
yprime(2)= (R*S-Q*U)/(P*S-Q*T);
```

```
yprime(4)= (P*U-R*T)/(P*S-Q*T);

yprime=yprime';
end
```

[Note that in the above, for the solutions we vary the lengths, masses and initial conditions (i.e. $\theta_i$ and $\dot{\theta}_i$ i = 1, 2). The code has been adapted from (12).]

# Heavy Symmetric Top Code

Listing A.3: Matlab Code for Simulations of Heavy Symmetric Top Problem

```

clear all;
% Start with a clean slate
global p_psi p_phi I0 mgl;
% Make these variables visible to function

phi_dot0 = 0;
% Initial condition on azimuthal velocity

psi_dot0 = 100 * pi;
% Initial wheel spinning speed

theta0 = pi/3;
% Initial wheel elevation angle

I0 = 2.33e-3;
% Moment of inertia arount bottom pivot point

I_psi = 1.25e-4;
% Moment of inertia for heavy wheel around Z? axis

m = 0.1;
% Mass of top

g = 9.81;
% Acceleration due to gravity

l = 0.15;
% Length of stem

mgl = m * g * l;

p_psi = I_psi * (psi_dot0 + phi_dot0*cos(theta0));
%Compute and print p_phi precess

```

```
p_phi = p_psi * cos(theta0) + I0 * phi_dot0 * sin(theta0)^2;
%Complete system energy

precess = mgl/p_psi;
%Compute slow precession speed value

E = 0.5 * p_psi * p_psi / I_psi + 0.5*I0*phi_dot0^2*sin(theta0)^2+mgl*cos(theta0);
%Complete system engergy

%Compute and print p_psi % compute polynomial coefficients to solve for
% roots to cubic eqn for nutation excursion
c0 = 2.0 * I0 * (E - p_psi^2 / (2.0 * I_psi)) - p_phi^2;
c1 = 2.0 * p_phi * p_psi - 2.0 * I0 * mgl;
c2 = -2.0 * I0 * (E - p_psi^2 / (2.0 * I_psi)) - p_psi^2;
c3 = 2.0 * I0 * mgl;

% points for plotting restoring force function
a = linspace(0,pi,251);

% Restoring force function for plotting
ke = c0 + c1 * cos(a) + c2* cos(a).^2 + c3 * cos(a).^3;
%plot1
subplot(1,2,1)
plot(cos(a),ke,'.')
hold on;
plot(cos(a),zeros(251));
title('f(u) polynomial against u=cos(theta)')

% Find roots that give nutation extremes
r = roots([c3, c2, c1, c0])

for i = 1:3
    if(abs(r(i)) <= 1.0)
      acos(r(i))
    end
end

% Generate time sequence for solving ODE
t = linspace(0,3,1001);

% Solve ODE for full theta/phi motion
x0 = [0 ;theta0; 0];

[t,x] = ode45(@(t,x) xdot(t,x,p_phi,p_psi,I0,mgl),t, x0);

%plot2
subplot(1,2,2)
plot(x(1:1001,3), x(1:1001,2)); % Plot theta/phi trajectory grid

```

```
%Equations of motion
function dxdt = xdot(t,x,p_phi,p_psi,I0,mgl)
    dxdt = zeros(3,1);
    dxdt(1)= (p_phi * cos(x(2)) - p_psi) * (p_psi * cos(x(2)) - p_phi) / (I0 * sin(x(2))^3)- mgl * sin(x(2));
    dxdt(2) = -x(1) / I0;
    dxdt(3) = (p_phi - p_psi * cos(x(2))) / (I0 * sin(x(2))^2);
end
```