# Landau Damping

part 1

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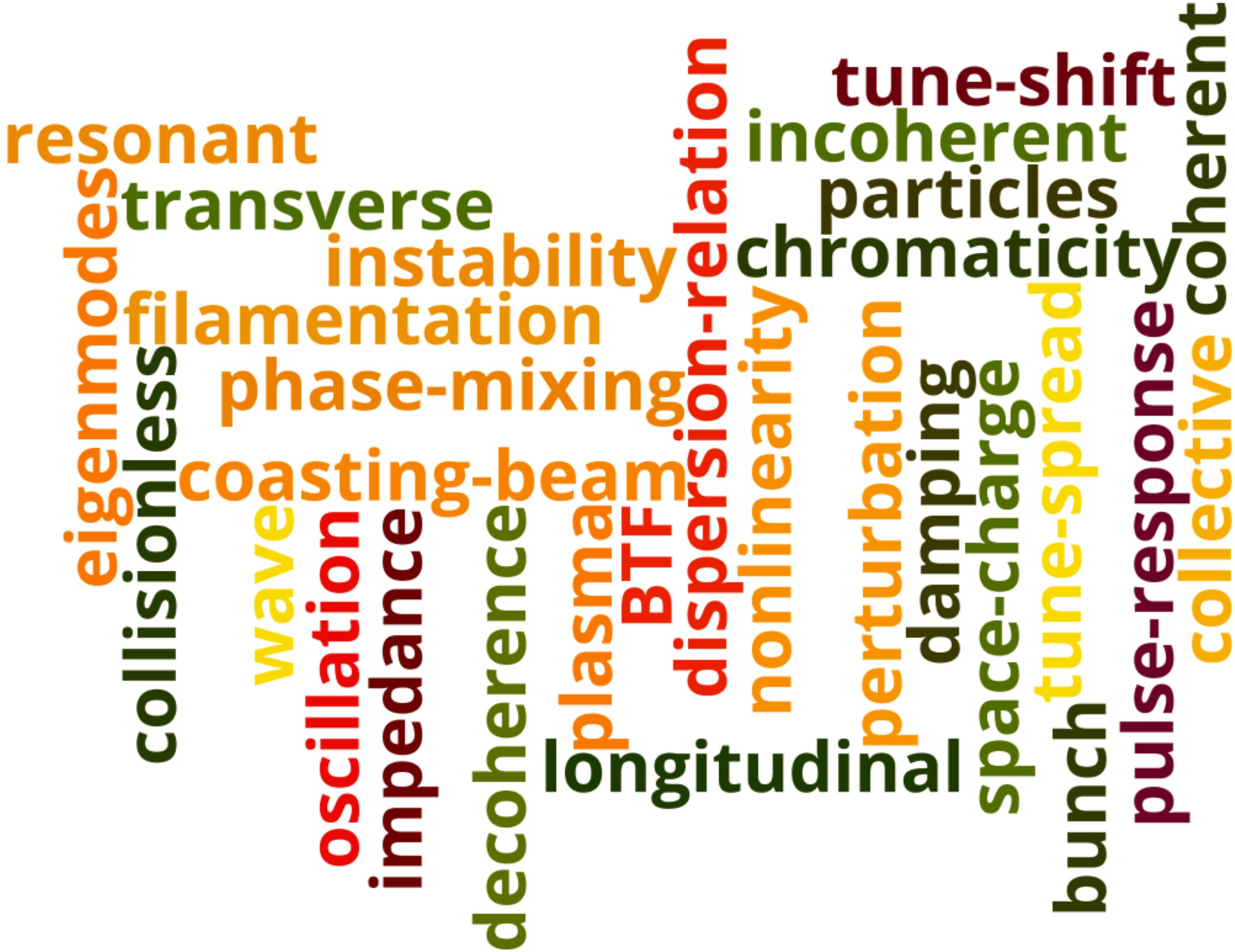
Landau Damping

a basic mechanism of beam dynamics
for all kinds of beams

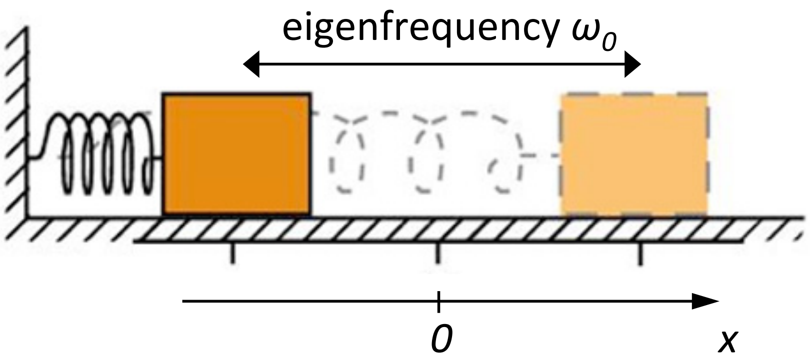


resonant
transverse
eigenmodes
instability
filamentation
phase-mixing
collisionless
coasting-beam
wave
oscillation
impedance
decoherence
plasma
BTF
dispersion-relation
nonlinearity
tune-shift
incoherent
particles
chromaticity
coherent
perturbation
damping
space-charge
tune-spread
pulse-response
collective
longitudinal
bunch

# Landau <u>Damping</u>

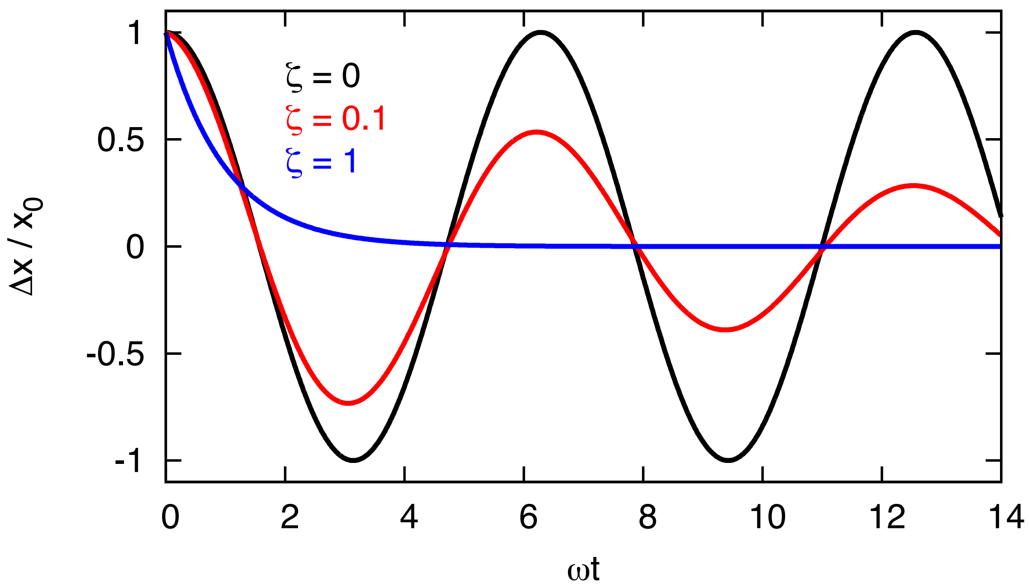
# Damping

A Harmonic Oscillator




With a damping (friction):

$$x'' + 2\zeta\omega_0 x' + \omega_0^2 x = 0$$




no damping
damped
critically damped

# Landau Damping

Lev Landau (1908-1968)
Institute for Physical Problems, Moscow

Nobel Prize Physics 1962
"Theory of Superfluidity"

Discovery of Collisionless Damping:
L. Landau, *On the vibrations of the electronic plasma*, Journal of Physics **10**, 25-34 (1946)

Experimental confirmation:
J. Malmberg, C. Wharton,
Phys Rev Lett **13**, 184 (1964)

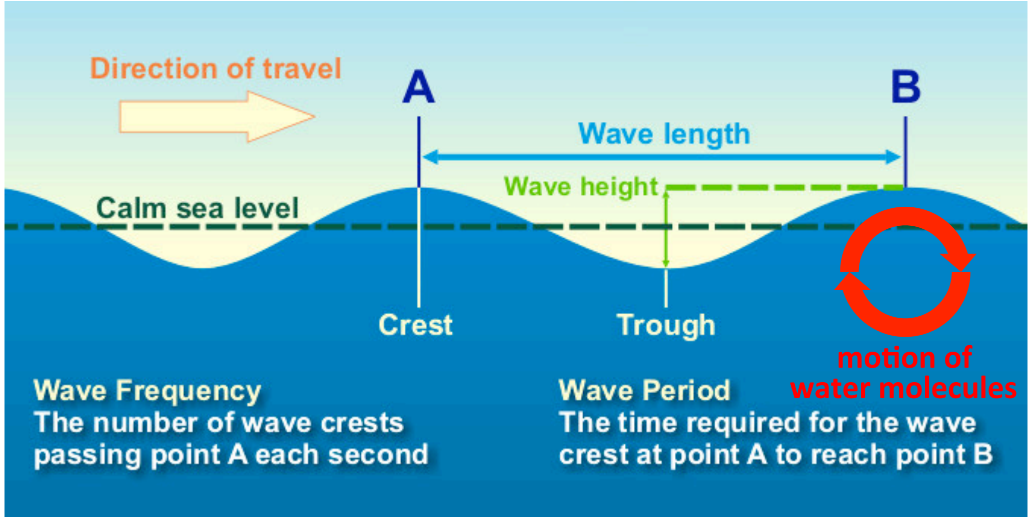
For our damping, "Landau"="collisionless"="frictionless"

# What kind of oscillations?

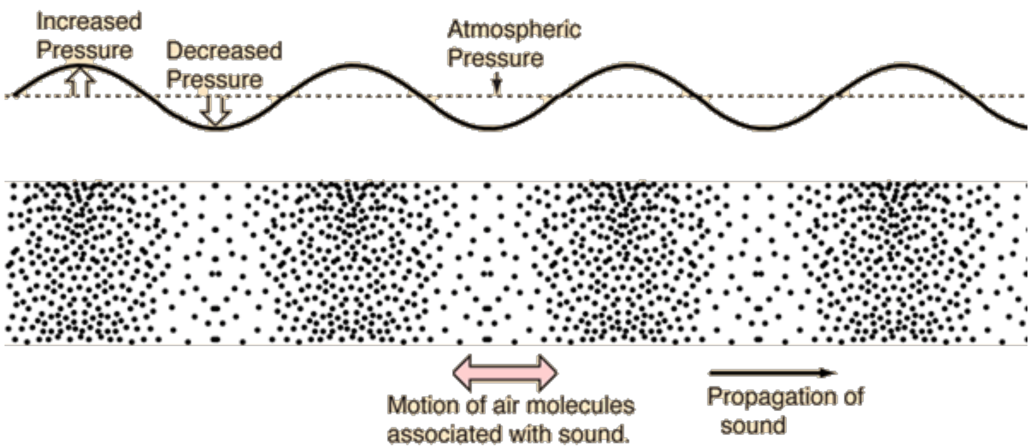
# Waves

# Oscillations: Waves

Water wave



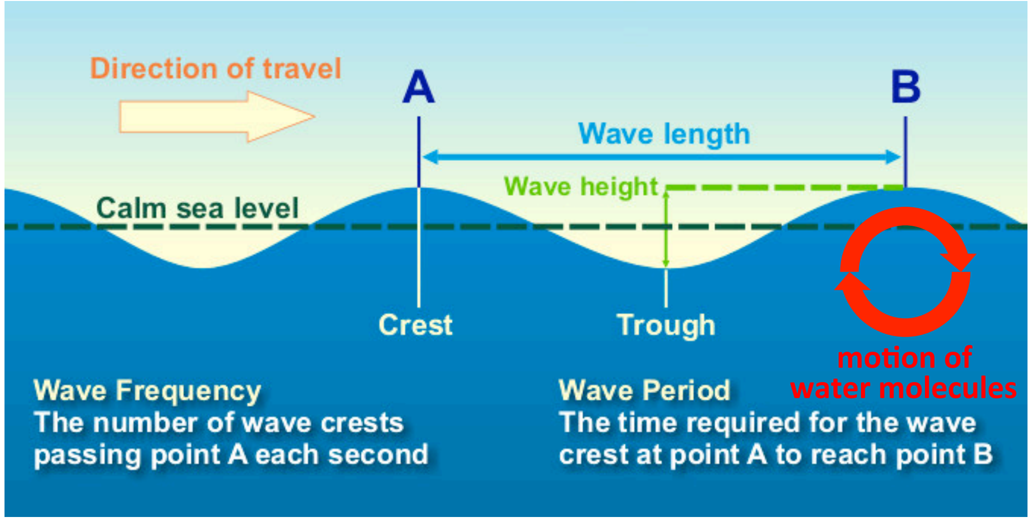

Sound wave



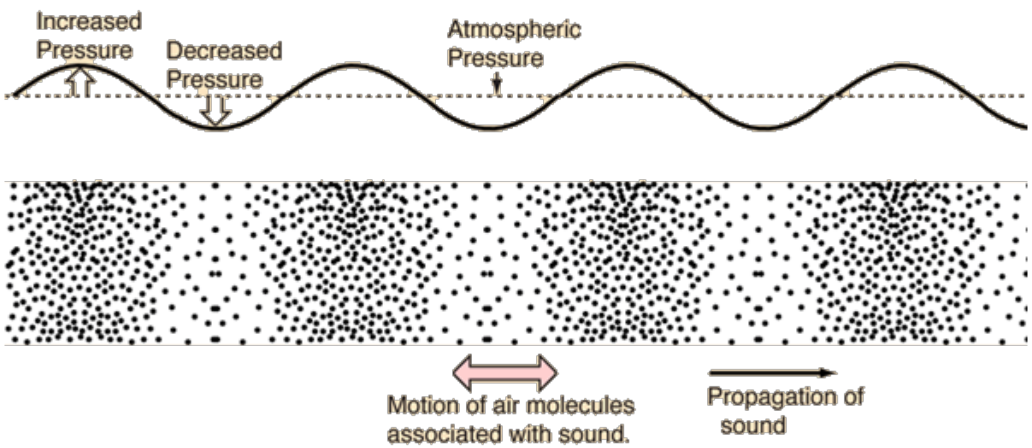

Traveling oscillation in a medium.
Very different from the medium particle motion.

# Oscillations: Waves

Water wave




Sound wave




Landau damping:
wave↔particles collisionless interaction.

# Oscillations: Waves

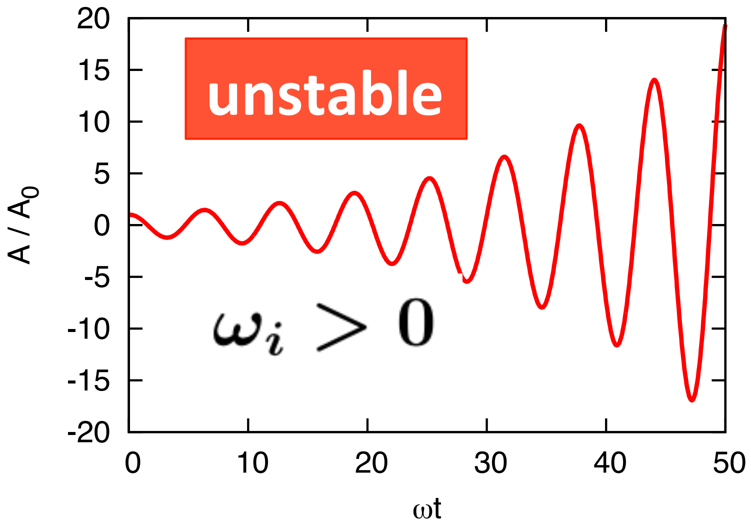
Waves can be
unstable or damped

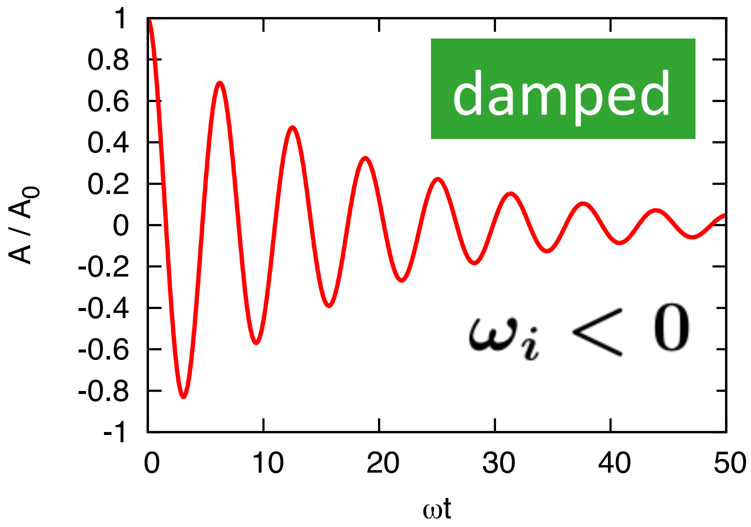
The wave frequency is complex:

$$\omega = \omega_r + i\omega_i$$

The wave physical parameter:

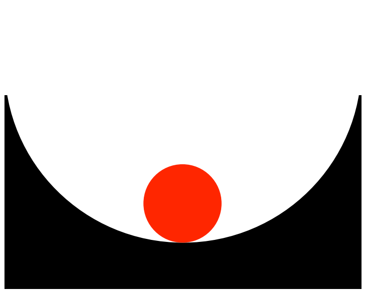
$$A(t) = A_0 \cos(\omega_r t)\, e^{\omega_i t}$$

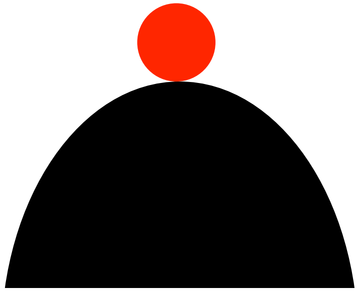


# Stability: the basic idea

| Stable System | Unstable System |
| --- | --- |



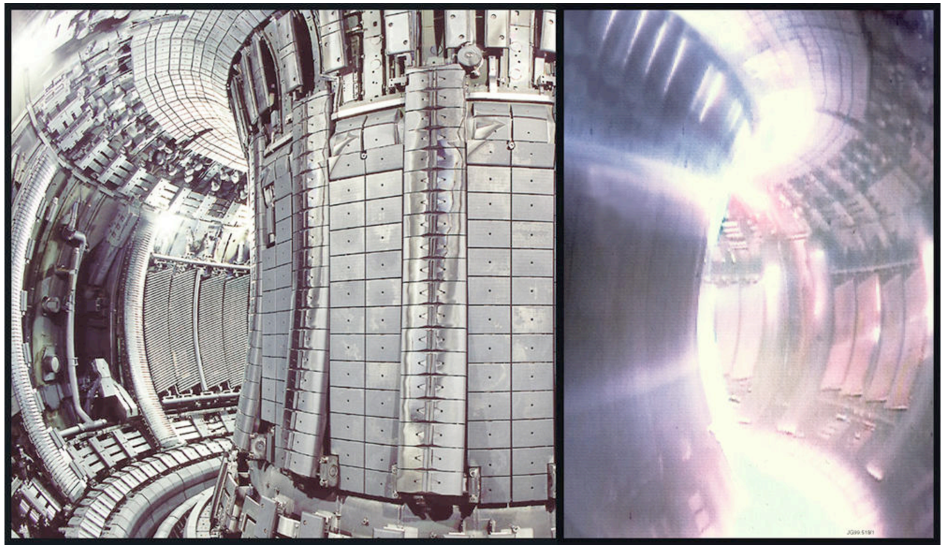


**Here is Landau Damping**

| Stable System | Unstable System |
| --- | --- |
| small perturbation | small perturbation |
| ↓ | ↓ |
| Mechanism to suppress the infinitesimal perturbations (DAMPING) | Mechanism to reinforce the infinitesimal perturbations (INSTABILITY DRIVE) |
| ↓ | ↓ |
| perturbation decays | perturbation grows |
| or $\lvert\gamma_{damping}\rvert > \gamma_{drive}$ | or $\gamma_{drive} > \lvert\gamma_{damping}\rvert$ |

# Landau damping in plasma

# Plasma



Plasma in the JET tokamak

Plasma is a quasi-neutral gas of unbound ions and electrons.

Waves in plasma: collective propagating oscillations of particles and E-M fields.

Electrons are much lighter: oscillations of the electron density

Some waves can be damped.

"Friction" in plasma is collisions.

# Plasma Wave

A basic plasma oscillation:
Langmuir wave

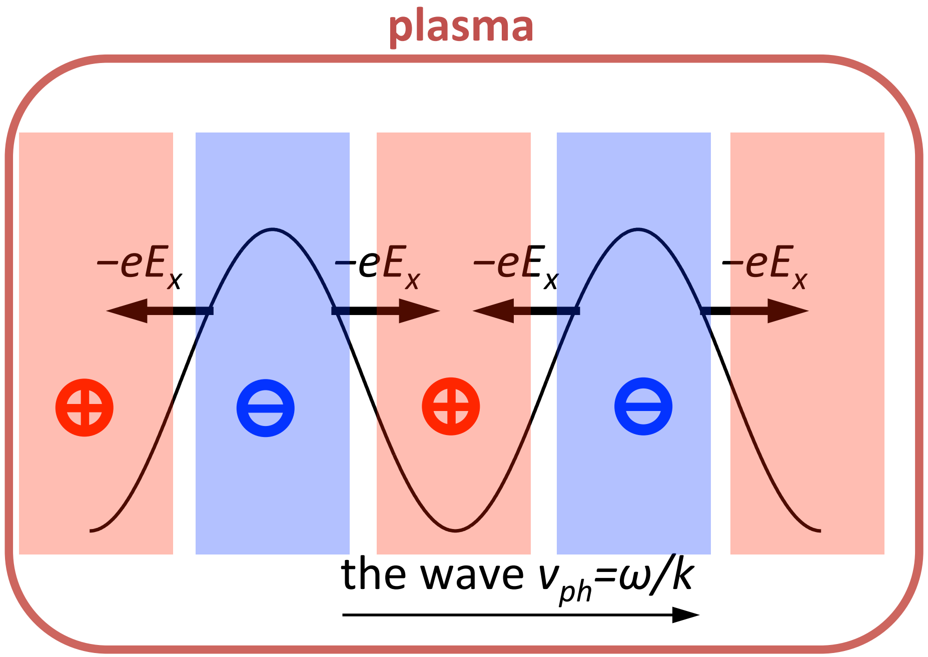
Wave number $k=2\pi/\lambda$

The phase velocity
$v_{ph} = \omega/k$

There are resonant particles $v_x \approx v_{ph}$

The plasma frequency

$$\omega_p^2 = \frac{n_e e^2}{m_e \epsilon_0}$$




The dispersion relation

$$\frac{\omega_p^2}{k^2} \int \frac{\partial \hat{f}_0 / \partial v_x}{v_x - \omega/k} dv_x = 1$$

has a singularity

# Landau Damping In Plasma

The wave frequency is complex

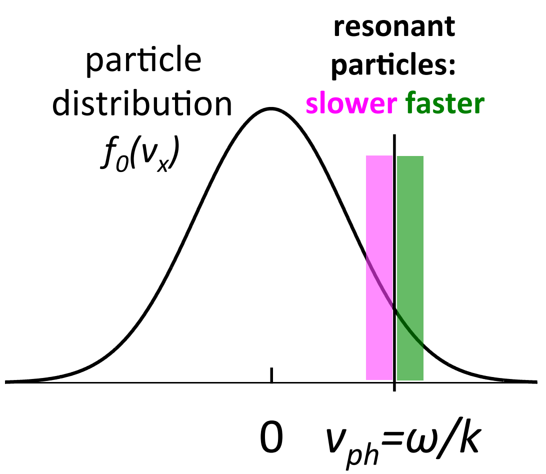
$$\omega = \omega_r + i\omega_i$$

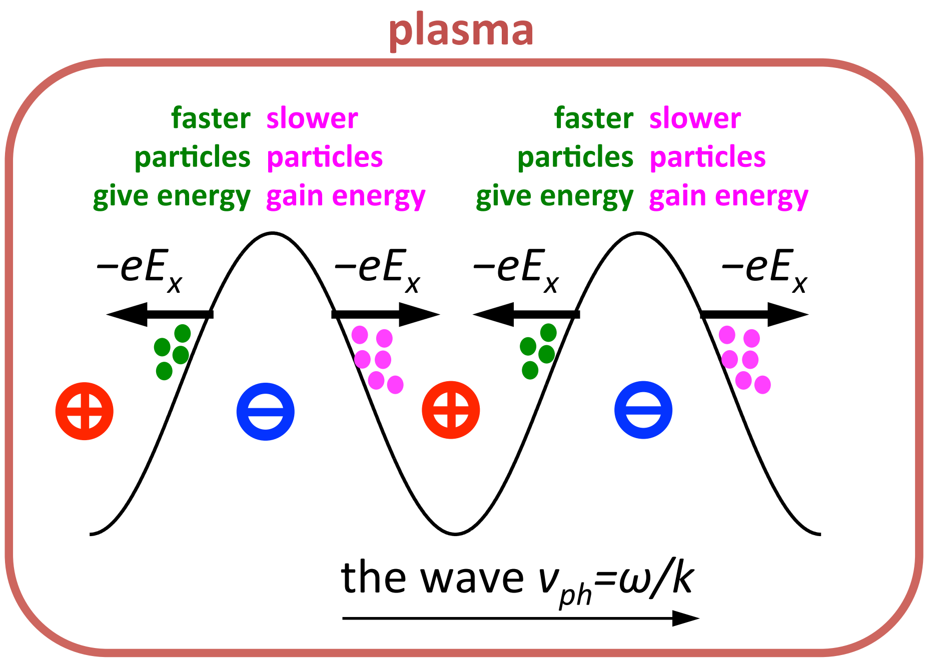
The dispersion relation can be solved,
the integral is calculated as PV + residue

$$\frac{\omega_p^2}{k^2}\left[\mathrm{PV}\int \frac{\partial \hat{f}_0/\partial v_x}{v_x - \omega/k} dv_x + i\pi \frac{\partial \hat{f}_0}{\partial v_x}\bigg|_{v_x=\frac{\omega}{k}}\right] = 1$$

$$\begin{aligned} \omega_r^2 &= \omega_p^2 + 3k^2 v_{th}^2 \\ \omega_i &= -\frac{\pi\omega_r}{2}\frac{\omega_p^2}{k^2}\frac{\partial \hat{f}_0}{\partial v_x}\bigg|_{v_x=\frac{\omega}{k}} \end{aligned}$$

# Landau Damping In Plasma

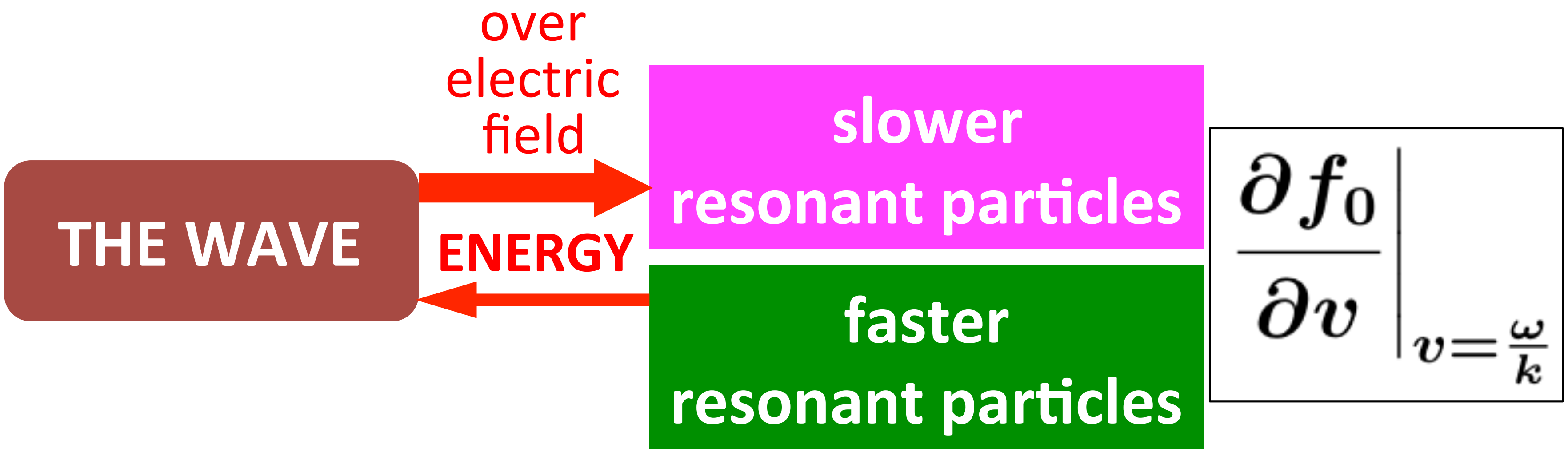



negative $f_0(v_x)$ slope: $N_{gain} > N_{give}$ → the wave decays, **damping**

positive $f_0(v_x)$ slope: $N_{gain} < N_{give}$ → the wave grows, **instability**

# Landau Damping In Plasma



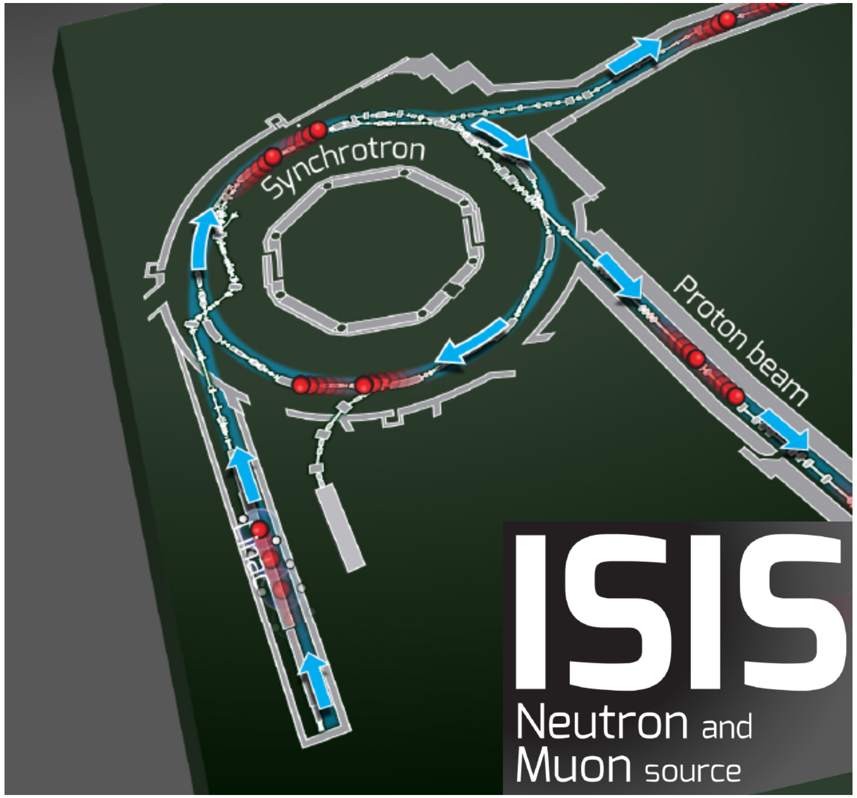

Main ingredients of Landau damping:

- wave−particle collisionless interaction. Here this is the electric field.
- energy transfer: the wave ↔ the (few) resonant particles.

The result is the exponential decay of a small perturbation.

Landau damping is a fundamental mechanism in plasma physics.
Extensively studied in experiment, simulations and theory.

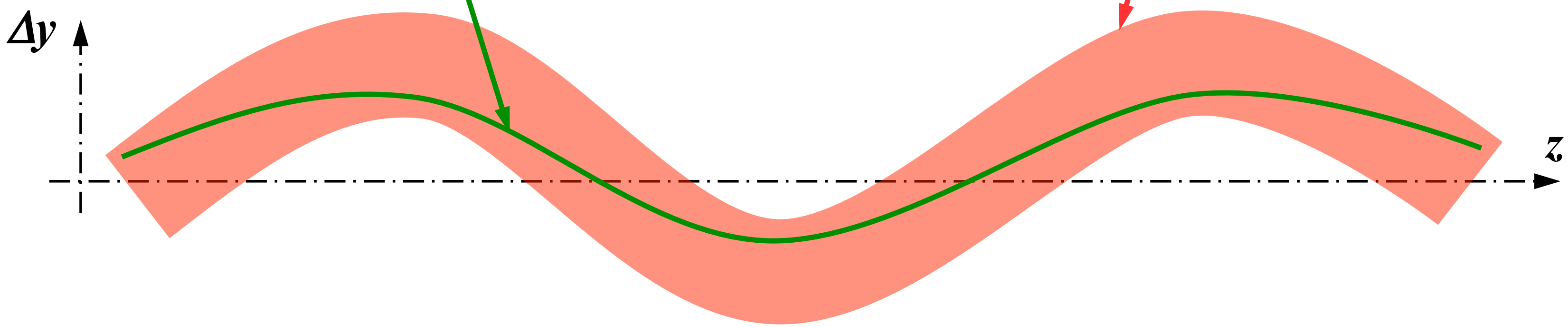
# Waves in particle beams in accelerators?



# Waves in Beams

Incoherent oscillations

Coherent oscillations




Incoherent tune-shifts

Large & important differences

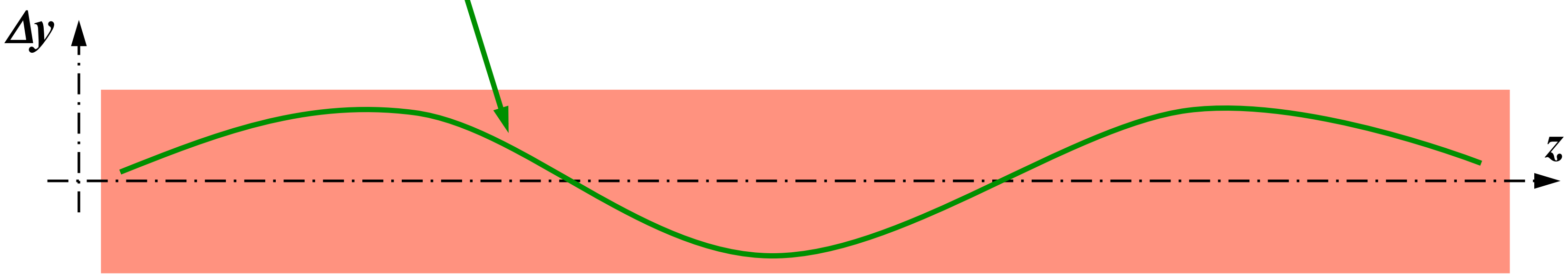
(other lectures)

Coherent tune-shifts

# Waves in Beams

Incoherent oscillations

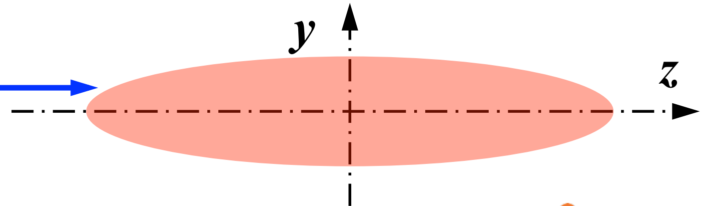
No coherent oscillations
$\langle y \rangle = 0$




coasting beam: L=C, λ(z)=const,
no synchrotron motion, δp=const

Waves in bunches and in coasting beams

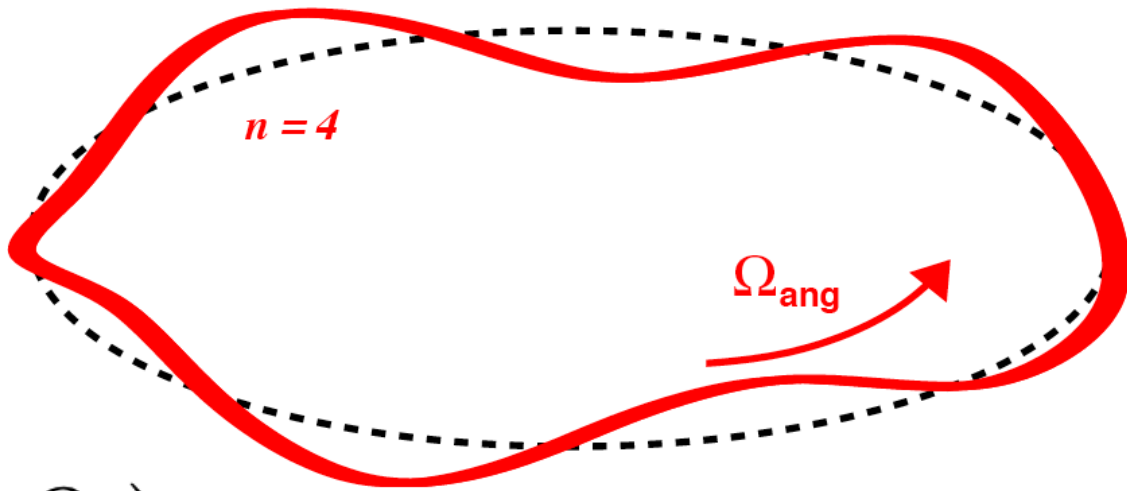
bunched beam: $L_{bunch}$, λ(z) profile,
synchrotron oscillations $Q_s$ : δp−z



# Waves in Beams

## Transverse oscillations in a coasting beam

$$x(s,t) = x_0 e^{ins/R - i\Omega t}$$




*n* is the mode index.

Wave length: *C/n*

Frequencies:

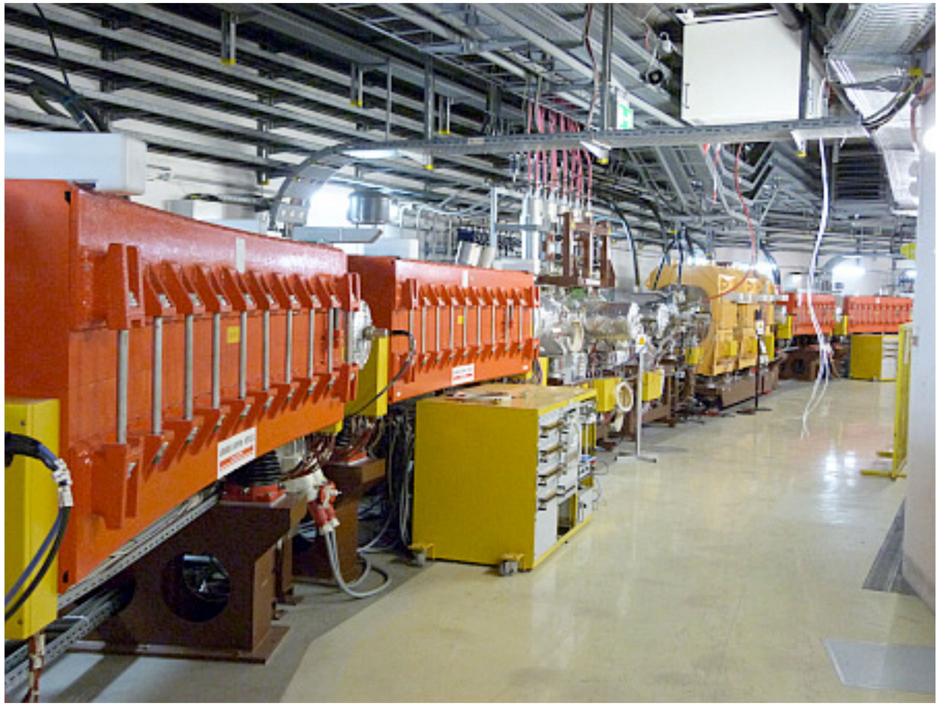
$$\begin{aligned} \text{slow wave} \quad \Omega_s &= (n - Q_\beta)\omega_0 \\ \text{fast wave} \quad \Omega_f &= (n + Q_\beta)\omega_0 \end{aligned}$$

Angular rotation ($\Omega_s$):

$$\Omega_{ang} = \left(1 - \frac{Q_\beta}{n}\right)\omega_0$$

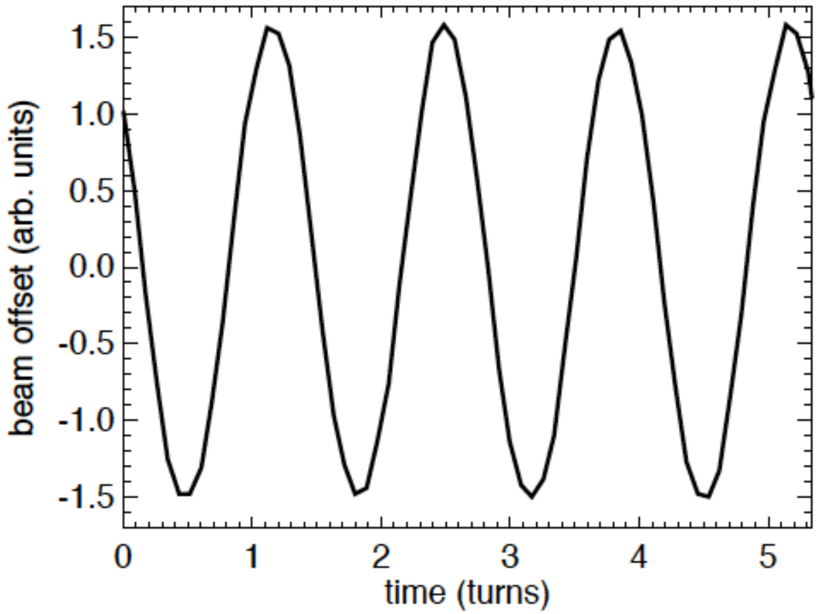
# Waves in Coasting Beams

## Experimental observations of the coasting-beam waves



SIS18 synchrotron at GSI Darmstadt

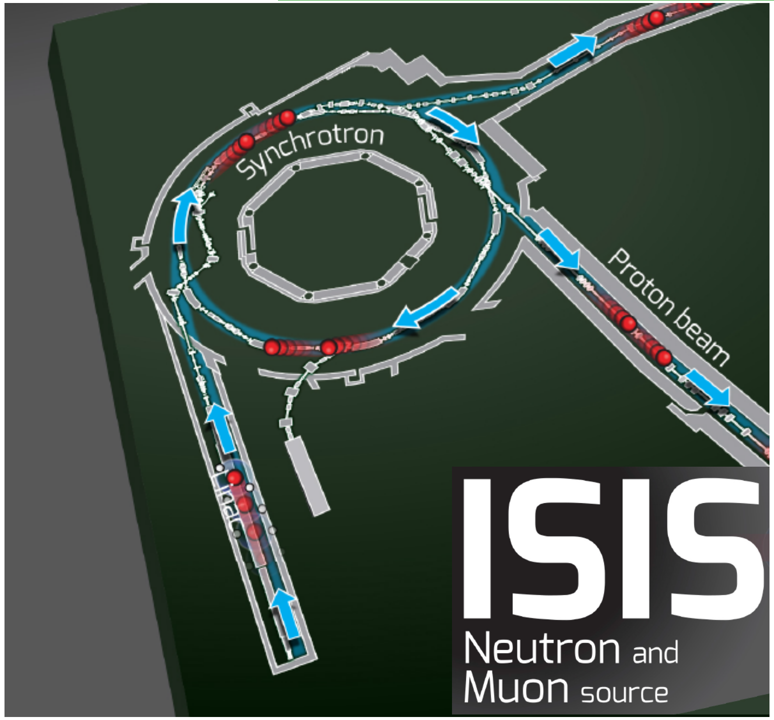
A coasting beam in SIS18.
n=4, as expected for Q=3.25,
with correct $\Omega_s$ and $\Omega_{ang}$




**V. Kornilov, O. Boine-Frankenheim,**
**GSI-Acc-Note-2009-008, GSI Darmstadt (2009)**

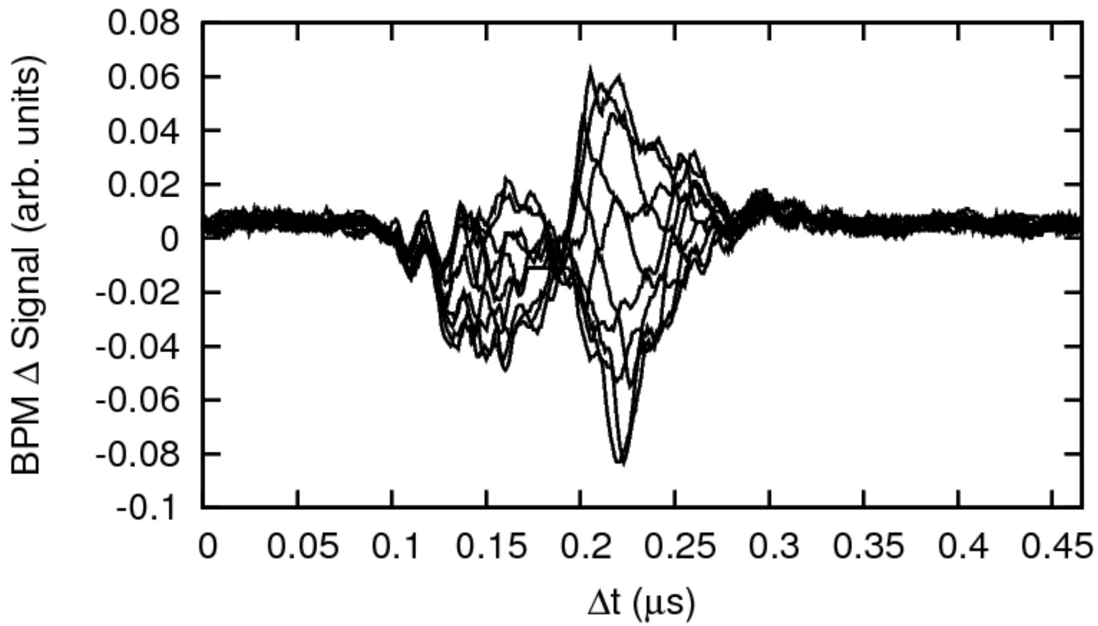
# Waves in Bunched Beams

## Experimental observations of the waves in bunches




ISIS synchrotron at RAL, UK

Unstable head-tail modes in ISIS.
High-intensity beams, 2 bunches,
head-tail mode k=1, τ=0.1 ms.



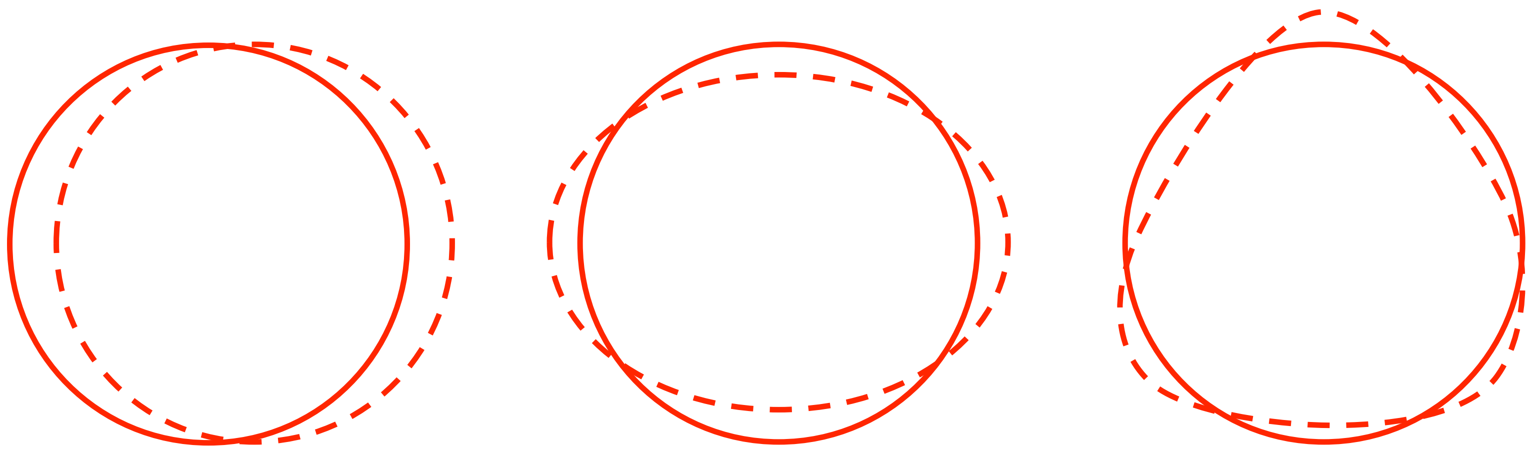

**V. Kornilov, et.al, HB2014 East Lansing, MI, USA, Nov 10-14, 2014**

# Collective oscillations in beams

Different types of coherent oscillations

Transverse, Longitudinal

| Dipolar (m=1) | Quadrupolar (m=2) | Sextupolar (m=3) |
| --- | --- | --- |



Here we consider mostly the dipole transverse oscillations.
For the others: the physics and the formalism are similar.

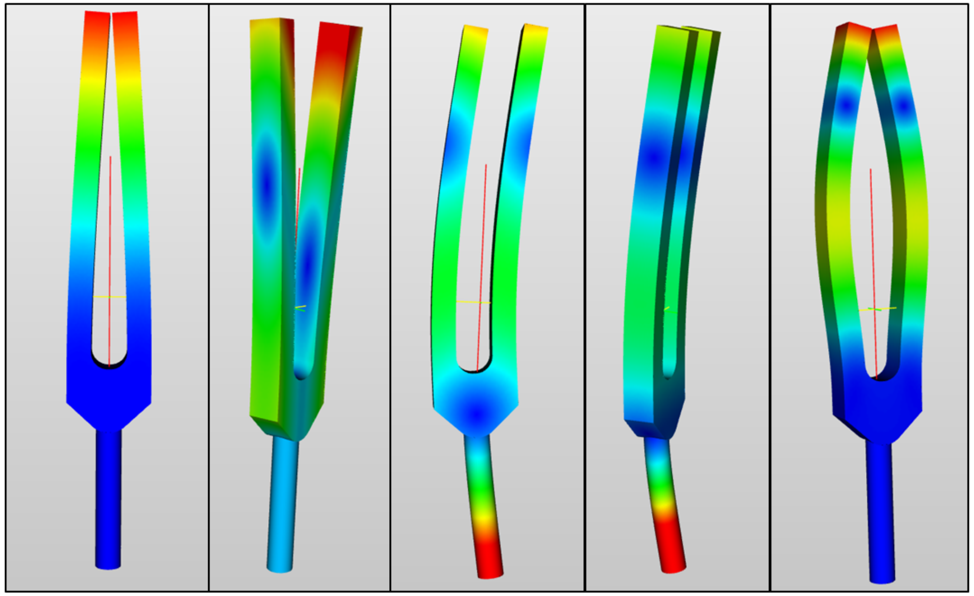
# Special waves: Eigenmodes

# Eigenmodes

Eigenmodes: intrinsic orthogonal oscillations of the dynamical system, with the fixed frequencies (eigenfrequencies)

$$A\vec{x} = \lambda\vec{x}$$

eigenvalue ($\lambda$), eigenmode ($\vec{x}$)



Eigenmodes of a tuning fork.
Pure tone at eigenfrequencies.

We often talk about the shift:

$$\Delta\Omega = \Omega - \Omega_{\mathrm{eigenfrequency}}$$

# Eigenmodes

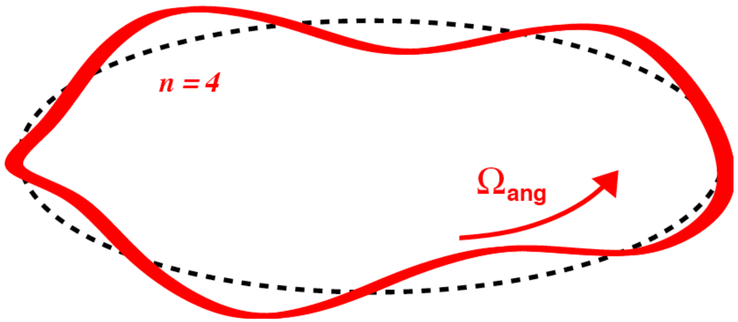
## Transverse eigenmodes in a coasting beam

Eigenmode:

$$x(s,t) = x_0 e^{ins/R - i\Omega t}$$

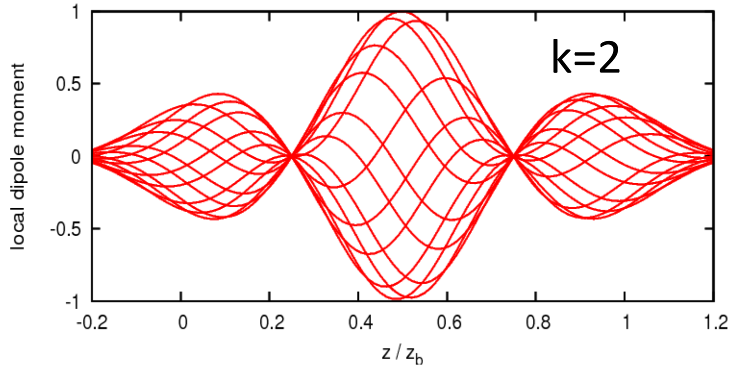
Eigenfrequency:

$$\Omega_s = (n - Q_\beta)\omega_0$$

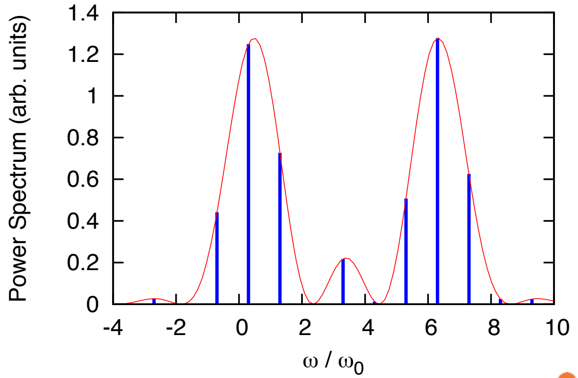



## Transverse eigenmodes in a bunched beam: Head-Tail Modes

Eigenmode:

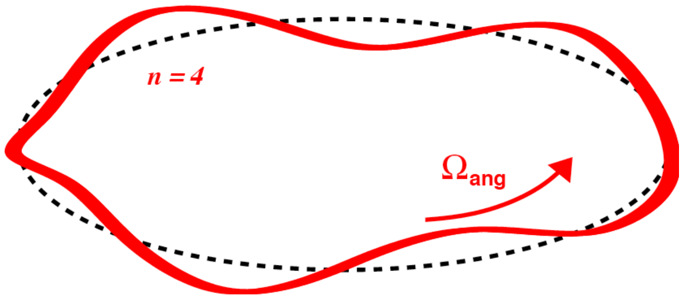



Eigenfrequencies:



# Unstable Oscillations

small initial perturbation ⟨***x***⟩




produces →

excitation force

$$G \propto Z_{\rm ext}^{\perp} I_0 \langle x \rangle$$

forced oscillations (eigenmode)

the perturbation is amplified

⟨***x***⟩ × ***(1+Δ)***

reinforcing mechanism

Especially concerned about the eigenmode perturbations

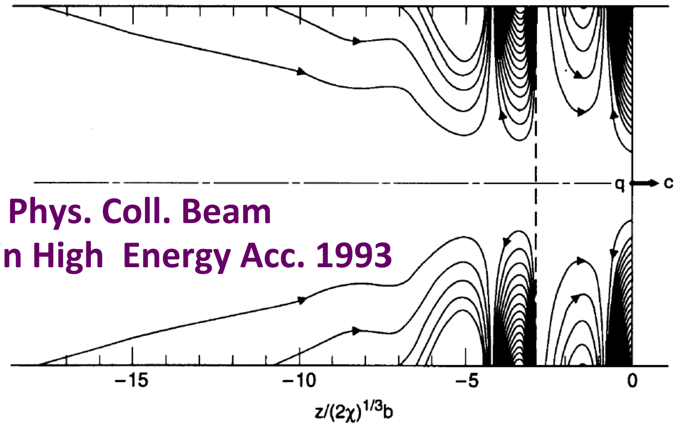
The result is $\Delta Q_{coh}$ and the exponential growth: instability

$$\langle x \rangle (t) = x_0 e^{\mathrm{Im}(\Omega) t}$$

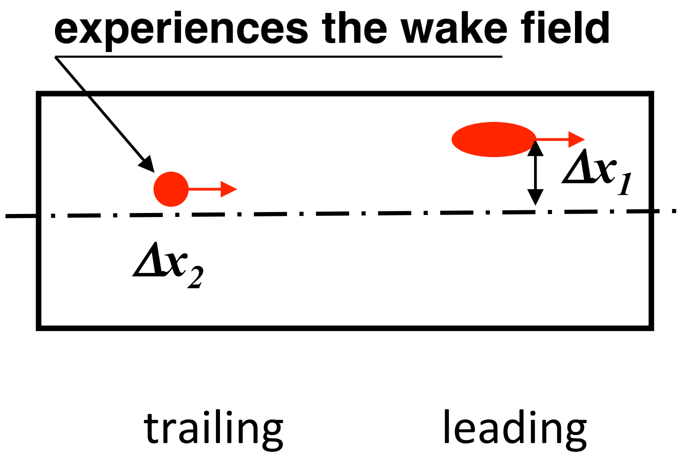
# Wake Fields, Impedances

Dipolar wakes: $F_{x2} \sim \Delta x_1$
(driving) the same for the whole trailing slice: coherent

Quadrupolar wakes: $F_{x2} \sim \Delta x_2$
(detuning) different for individual particles: incoherent




**A.Chao, Phys. Coll. Beam Instab. in High Energy Acc. 1993**




Transverse collective instabilities: Dipolar Wakes $W_1(z)$, Impedances $Z_1(\omega)$

# Beam Transfer Function (BTF)

K.Y.Ng, Physics of Intensity Dependent Beam Instabilities, 2006
A.Hofmann, Proc. CAS 2003, CERN-2006-002
A.Chao, Phys. Coll. Beam Instab. in High Energy Acc. 1993

# Beam Transfer Function

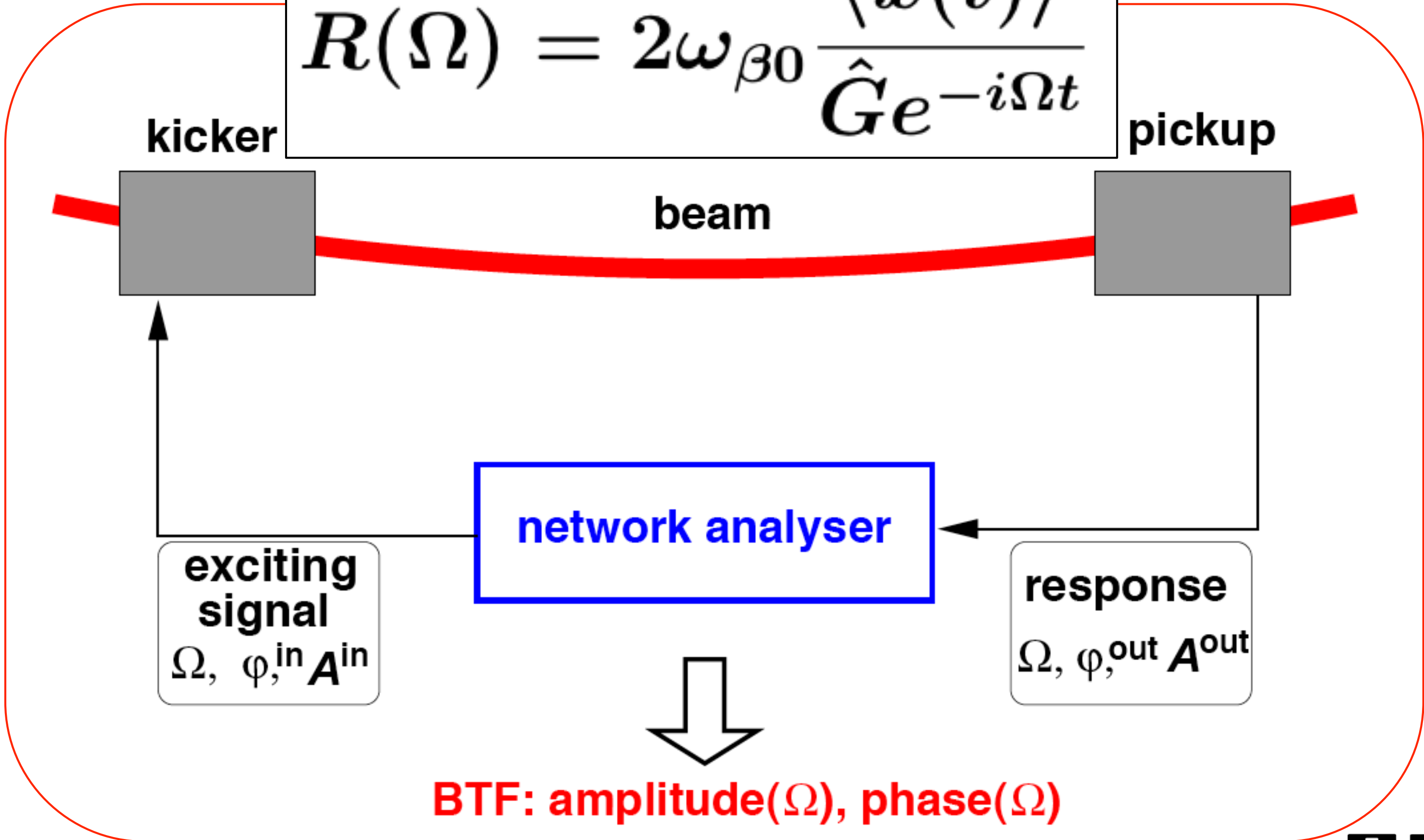
an excitation:

$$x'' + \omega_{\beta i}^2 x = \hat{G} e^{-i\Omega t}$$

beam forced response:

$$\langle x \rangle = A\ e^{-i\Omega t + \Delta\phi}$$

$$R(\Omega) = 2\omega_{\beta 0} \frac{\langle x(t) \rangle}{\hat{G} e^{-i\Omega t}}$$

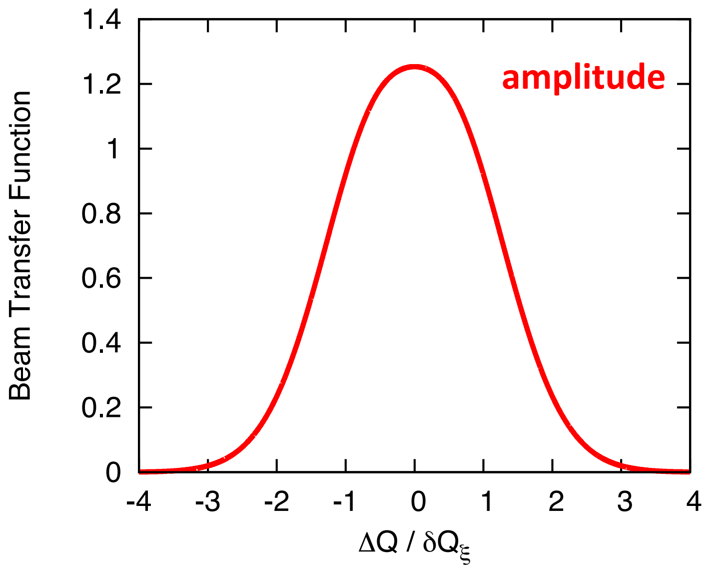


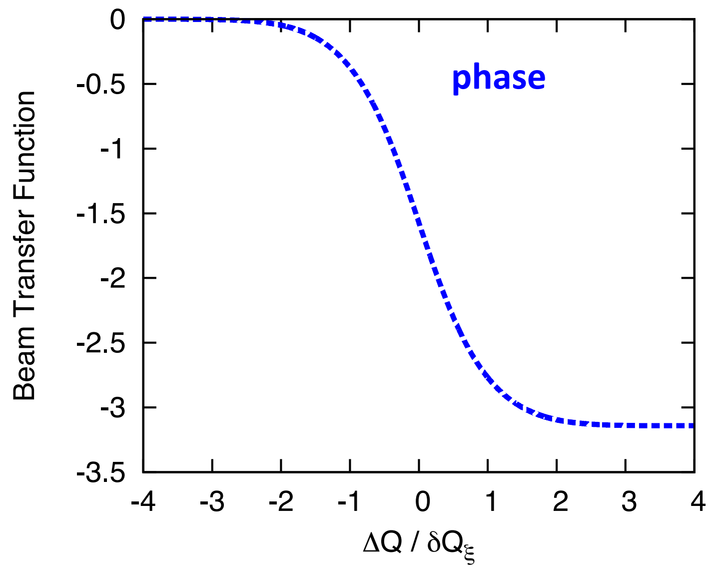
# Beam Transfer Function

BTF is:

- Useful diagnostics; gives the tune, $\delta p$, chromaticity, beam distribution
- A fundamental function in the beam dynamics
- Necessary to describe the beam signals and Landau damping

$$R(\Omega) = \mathrm{PV}\int \frac{f(\omega)d\omega}{\omega - \Omega} + i\pi f(\Omega)$$




$\Delta Q=(\Omega-(m\pm Q_f)f_0)/f_0$

$\delta Q_\xi=|m\eta\pm(Q_{f\eta}\eta-Q_0\xi)|\delta p/p$

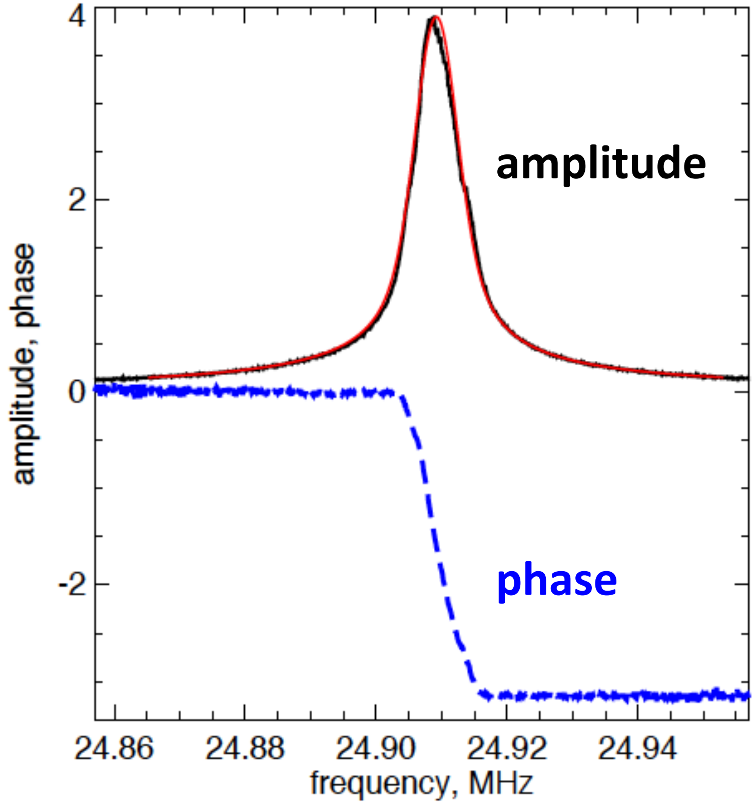
**J.Borer, et al, PAC1979**
**D.Boussard, CAS 1993, CERN 95-06, p.749**
**A.Chao, Phys. Coll. Beam Instab. in High Energy Acc. 1993**
**Handbook of Acc. Physics and Eng. 2013, 7.4.17**

# Beam Transfer Function

**BTF: a standard measurement with a network analyser**

- Collective response to the excitation
- Observe the incoherent spectrum
- Still, the beam is stable: Landau Damping!

A coasting beam $U^{73+}$ in SIS18.
Transverse signal.
Lower side-band of m=24



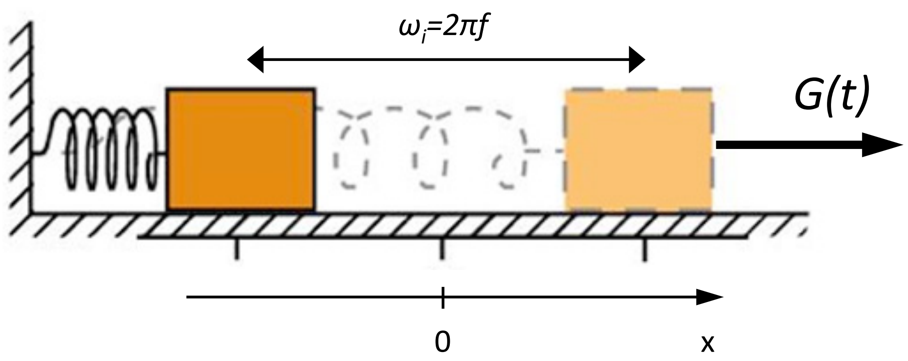

**V.Kornilov, et al, GSI-Acc-Note-2006-12-001, GSI Darmstadt (2006)**

# Landau Damping:

# Interaction
# wave ↔ resonant particles

V.K. Neil and A.M. Sessler, Rev. Sci. Instrum. 6, 429 (1965)
L. J. Laslett, V.K. Neil, and A.M. Sessler, Rev. Sci. Instrum. 6, 46 (1965)
H.G. Hereward, CERN Report 65-20 (1965)
D. Möhl, H. Schönauer, Proc. IX Int. Conf. High Energy Acc., p. 380 (1974)
A.Hofmann, Proc. CAS 2003, CERN-2006-002
A.Chao, Phys. Coll. Beam Instab. in High Energy Acc. 1993
K.Y.Ng, Physics of Intensity Dependent Beam Instabilities, 2006

# Driven Harmonic Oscillator




$$x'' + \omega_i^2 x = \hat{G} e^{-i\Omega t}$$

| The solution | = | homogeneous solution (pulse response) initial conditions | + | particular solution (forced oscillations) |
|---|---|---|---|---|

Off-resonance ($\Omega \neq \omega_i$) and at resonance ($\Omega = \omega_i$), different particular solutions.
Zero initial conditions.

$$x_G(t) = \frac{2\hat{G}}{\omega_i^2 - \Omega^2} \sin\left(\frac{\omega_i - \Omega}{2} t\right) \sin\left(\frac{\omega_i + \Omega}{2} t\right)$$

$$x_G(t) = \frac{\hat{G}}{2\Omega} t \ \sin(\Omega t)$$

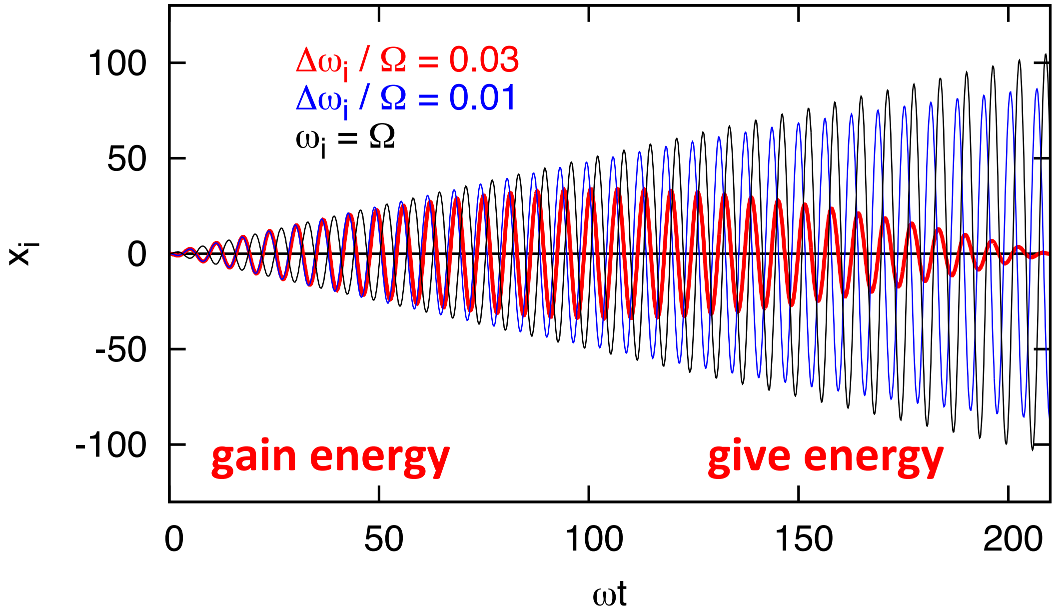
# Driven Harmonic Oscillator

off-resonant beating solution

$$x_G(t) = \frac{2\hat{G}}{\omega_i^2 - \Omega^2} \sin\left(\frac{\omega_i - \Omega}{2} t\right) \sin\left(\frac{\omega_i + \Omega}{2} t\right)$$

resonant solution

$$x_G(t) = \frac{\hat{G}}{2\Omega}\, t\, \sin(\Omega t)$$




wave↔particle energy transfer

# Landau Damping: Dispersion Relation

D. Möhl, H. Schönauer, Proc. IX Int. Conf. High Energy Acc., p. 380 (1974)
A.Hofmann, Proc. CAS 2003, CERN-2006-002
A.Chao, Phys. Coll. Beam Instab. in High Energy Acc. 1993
K.Y.Ng, Physics of Intensity Dependent Beam Instabilities, 2006
W.Herr, Introduction to Landau Damping, CAS2013, CERN-2014-009

# Coherent Oscillations

**An easy derivation of the dispersion relation**

the external drive is INTENSITY × IMPEDANCE × PERTURBATION

$$G = \frac{\langle F_x \rangle}{m\gamma} = \frac{q\beta}{m\gamma C} i Z_{\rm ext}^{\perp} I_0 \langle x \rangle$$

the no-damping complex coherent tune shift is
INTENSITY × IMPEDANCE

$$\Delta Q_{\rm coh} = \frac{I_0 q_{\rm ion}}{4\pi\gamma m c Q_0 \omega_0} i Z_{\rm ext}^{\perp}$$

only the dipole impedance here, no incoherent effects

thus, the external drive is

$$G = 2\omega_{\beta 0}\omega_0 \Delta Q_{\rm coh} \langle x \rangle$$

# Dispersion Relation

**An easy derivation of the dispersion relation**

the external drive is IMPEDANCE TUNE SHIFT × PERTURBATION

$$G = 2\omega_{\beta 0}\omega_0 \Delta Q_{\mathrm{coh}} \langle x \rangle$$

the beam response is the BTF

$$\langle x \rangle = \frac{G}{2\omega_{\beta 0}\sigma_\omega} R(u)$$

combined: the DISPERSION RELATION

$$\boxed{\Delta Q_{\mathrm{coh}} R(\Omega) = 1}$$

provides the resulting Ω for the given impedance and beam

# Stability Diagram

the resulting Ω for the given impedance and beam

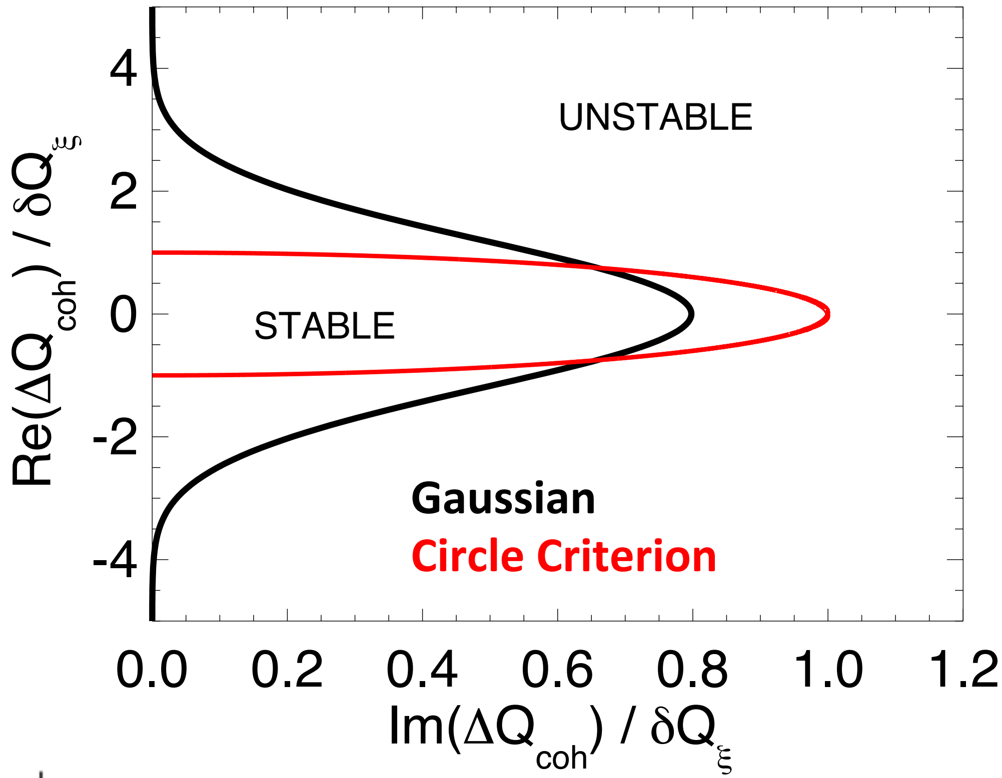
$$\Delta Q_{\mathrm{coh}} R(\Omega) = 1$$

$$\Delta Q_{\mathrm{coh}} \omega_0 \int \frac{f(\omega) d\omega}{\omega - \Omega} = 1$$

Re(Z)>0: the slow wave

$$\omega_s = (n - Q_0)\omega_0$$

$$\delta Q_\xi = \left|\eta(n - Q_0) + Q_0 \xi\right| \delta_p$$




Circle Criterion

$$\frac{|\Delta Q_{\mathrm{coh}}|}{\delta Q_\xi} = 1$$

Circle Criterion: E.Keil, W.Schnell, CERN ISR-TH-RF/69-48 (1969)

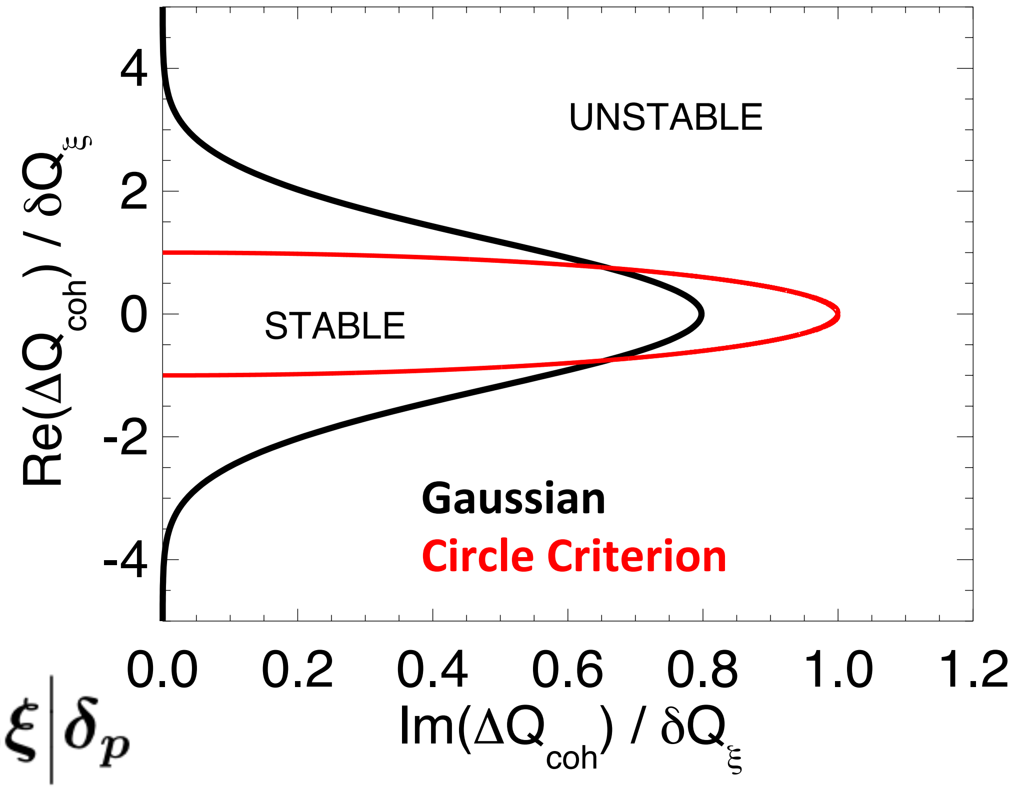
# Stability Diagram

the resulting Ω for the given impedance and beam

$$\Delta Q_{\mathrm{coh}} R(\Omega) = 1$$

$$\frac{|\Delta Q_{\mathrm{coh}}|}{\delta Q_\xi} = 1$$

$$\delta Q_\xi = \left|\eta(n - Q_0) + Q_0 \xi\right| \delta_p$$




Strength of Landau Damping is proportional to the tune-spread

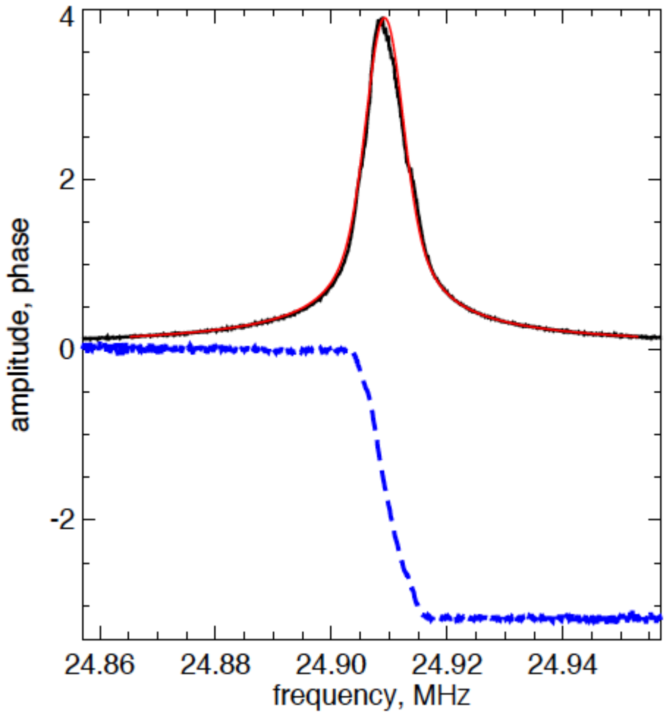
Tune spread provides Landau Damping

# Beam Transfer Function

BTF provides a direct measure of Landau Damping

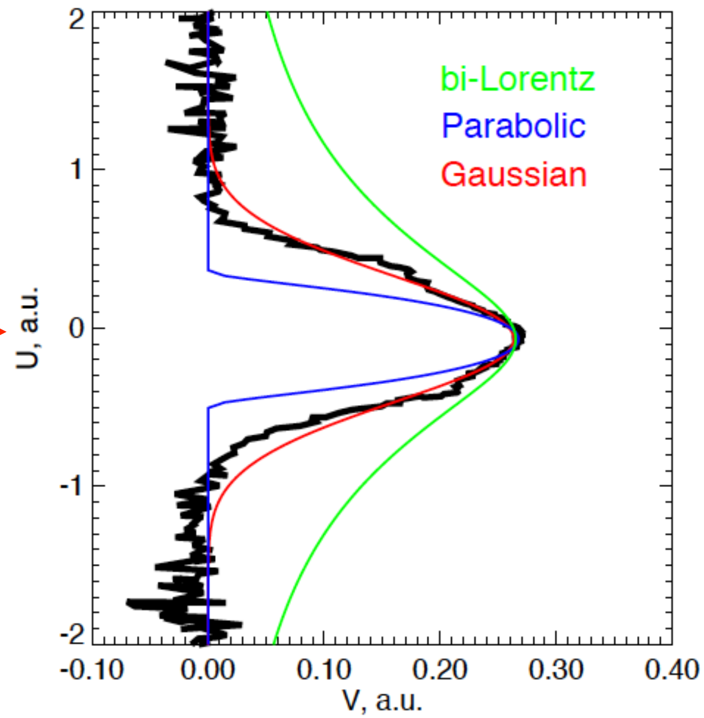
$$\Delta Q_{\text{coh}} R(\Omega) = 1$$

Measured BTF in SIS18




$$\frac{1}{R(\Omega)}$$

Resulting Stability Diagram



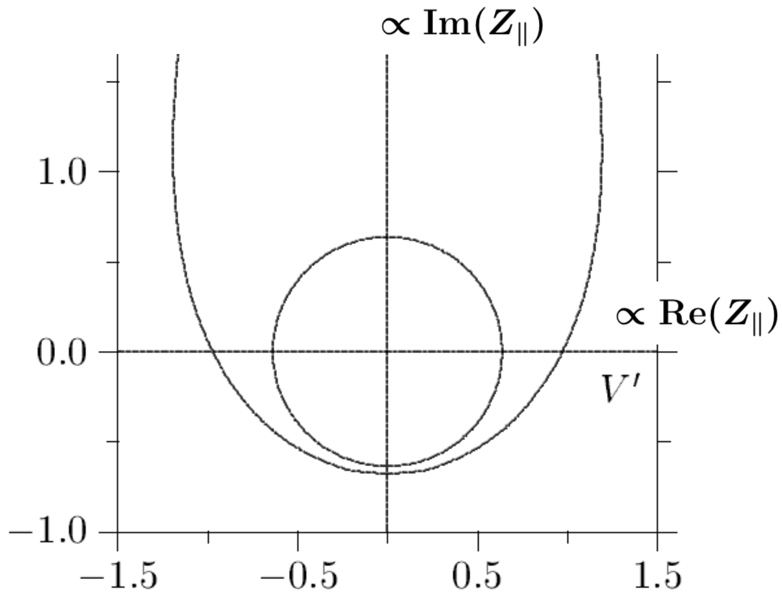

**V.Kornilov, et al, GSI-Acc-Note-2006-12-001, GSI Darmstadt (2006)**

# Longitudinal Stability

Coasting Beam:
Spread in the revolution frequency

$$\mathcal{A}\, I_0 \frac{Z_{\parallel}(\Omega_{\parallel})}{n} \int \frac{\partial f(\omega_0)/\partial \omega_0}{\omega_0 - \Omega_{\parallel}/n} d\omega_0 = 1$$

$$\left|\frac{Z}{n}\right| \leq 0.6 \frac{2\pi\beta^2 E_0 \eta (\Delta p/p)^2}{e I_0}$$




Bunched beams:

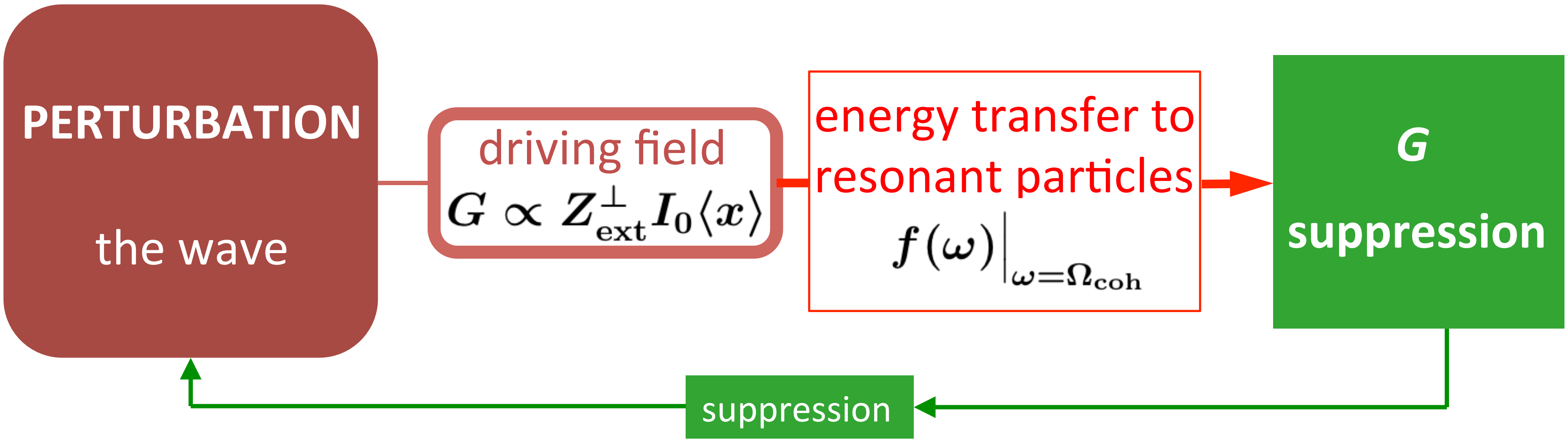
$$\Delta\omega_s^{\mathrm{coh}} \int \frac{f(\omega_s) d\omega_s}{\Omega_{\parallel} - \omega_s} = 1$$

the physics and the formalism are similar


K.Y.Ng, Physics of Intensity Dependent Beam Instabilities, 2006
A.Hofmann, Proc. CAS 2003, CERN-2006-002
E.Keil, W.Schnell, CERN ISR-TH-RF/69-48 (1969)

# Landau Damping

Incomplete (!) mechanism of Landau Damping in beams
for the end of the first part




Main ingredients of Landau damping:

✓ wave−particle collisionless interaction: Impedance driving field

✓ energy transfer: the wave ↔ the (few) resonant particles

# Landau Damping

End of part 1