Landau-Lifshitz Formulation of GR

Post-Newtonian and post-Minkowskian theory start with the Landau-Lifshitz formulation

Define the "gothic" metric density $\mathfrak{g}^{\alpha\beta} \equiv \sqrt{-g} g^{\alpha\beta}$

Then Einstein's equations can be written in the form

$$\partial_{\mu\nu} H^{\alpha\mu\beta\nu} = \frac{16\pi G}{c^4}(-g)\left(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL}\right)$$

$$H^{\alpha\mu\beta\nu} \equiv \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu}$$

$$t^{\alpha\beta}_{\rm LL} \sim \partial\mathfrak{g} \cdot \partial\mathfrak{g}$$

Antisymmetry of $H^{\alpha\mu\beta\nu}$ implies the conservation equation

$$\partial_\beta \left[(-g)\left(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL}\right)\right] = 0 \iff \nabla_\beta T^{\alpha\beta} = 0$$

Landau-Lifshitz Formulation of GR

The Landau-Lifshitz pseudotensor

$$
\begin{aligned}
(-g)t^{\alpha\beta}_{\mathrm{LL}} := \frac{c^4}{16\pi G}\Big\{ & \partial_\lambda \mathfrak{g}^{\alpha\beta}\partial_\mu \mathfrak{g}^{\lambda\mu} - \partial_\lambda \mathfrak{g}^{\alpha\lambda}\partial_\mu \mathfrak{g}^{\beta\mu} + \frac{1}{2} g^{\alpha\beta} g_{\lambda\mu}\partial_\rho \mathfrak{g}^{\lambda\nu}\partial_\nu \mathfrak{g}^{\mu\rho} \\
& - g^{\alpha\lambda} g_{\mu\nu}\partial_\rho \mathfrak{g}^{\beta\nu}\partial_\lambda \mathfrak{g}^{\mu\rho} - g^{\beta\lambda} g_{\mu\nu}\partial_\rho \mathfrak{g}^{\alpha\nu}\partial_\lambda \mathfrak{g}^{\mu\rho} + g_{\lambda\mu} g^{\nu\rho}\partial_\nu \mathfrak{g}^{\alpha\lambda}\partial_\rho \mathfrak{g}^{\beta\mu} \\
& + \frac{1}{8}\left(2\mathfrak{g}^{\alpha\lambda}\mathfrak{g}^{\beta\mu} - \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\lambda\mu}\right)\left(2\mathfrak{g}_{\nu\rho}\mathfrak{g}_{\sigma\tau} - \mathfrak{g}_{\rho\sigma}\mathfrak{g}_{\nu\tau}\right)\partial_\lambda \mathfrak{g}^{\nu\tau}\partial_\mu \mathfrak{g}^{\rho\sigma} \Big\}
\end{aligned}
$$

Landau-Lifshitz Formulation of GR

Conservation equation allows the formulation of global conservation laws:

$$E \equiv \int (-g)\big(T^{00} + t^{00}_{\rm LL}\big)\, d^3x$$

$$\frac{dE}{dt} = \oint (-g) t^{0j}_{\rm LL}\, d^2S_j$$

Similar conservation laws for linear momentum, angular momentum, and motion of a center of mass, with

$$P^j \equiv \frac{1}{c}\int (-g)\big(T^{j0} + t^{j0}_{\rm LL}\big)\, d^3x$$

$$J^j \equiv \frac{1}{c}\epsilon^{jkl}\int (-g) x^k \big(T^{l0} + t^{l0}_{\rm LL}\big)\, d^3x$$

$$X^j \equiv \frac{1}{E}\int (-g) x^j \big(T^{00} + t^{00}_{\rm LL}\big)\, d^3x$$

The "relaxed" Einstein equations

Define potentials $$h^{\alpha\beta} \equiv \eta^{\alpha\beta} - \mathfrak{g}^{\alpha\beta}$$

Impose a coordinate condition (gauge): Harmonic or deDonder gauge

$$\partial_\beta h^{\alpha\beta} = 0 \qquad \Box_g x^{(\alpha)} = 0$$

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}$$

$$\Box \equiv -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\tau^{\alpha\beta} \equiv (-g)\left(T^{\alpha\beta}[\mathsf{m}, g] + t^{\alpha\beta}_{\mathrm{LL}}[h] + t^{\alpha\beta}_{\mathrm{H}}[h]\right)$$

$$g) t^{\alpha\beta}_{\mathrm{H}} := \frac{c^4}{16\pi G}\left(\partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} - h^{\mu\nu}\partial_{\mu\nu}h^{\alpha\beta}\right)$$

$$\partial_\beta \tau^{\alpha\beta} = 0$$

Matter tells spacetime how to curve

Spacetime tells matter how to move

Still equivalent to the exact Einstein equations

The "relaxed" Einstein equations

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}$$

$$\partial_\beta \tau^{\alpha\beta} = 0$$

Solve for h as a functional of matter variables

Solve for evolution of matter variables to give h(t,x)

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Iterating the "Relaxed" Einstein Equations

Assume that $h^{\alpha\beta}$ is "small", and iterate the relaxed equation:

$$\Box h^{\alpha\beta}_{N+1} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}(h_N)$$

$$h^{\alpha\beta}_{N+1} = \frac{4G}{c^4}\int \frac{\tau^{\alpha\beta}(h_N)(t-|\boldsymbol{x}-\boldsymbol{x'}|/c, \boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|} d^3x'$$

Start with $h_0 = 0$ and truncate at a desired N

Yields an expansion in powers of G, called a post-Minkowskian expansion

Find the motion of matter using

$$\partial_\beta \tau^{\alpha\beta}(h_N) = 0$$

Solving the "Relaxed" Einstein Equations

$$\Box\psi = -4\pi\mu \Longrightarrow \psi = \int_{\mathcal{C}} \frac{\mu(t - |\boldsymbol{x} - \boldsymbol{x'}|/c, \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|} d^3x'$$

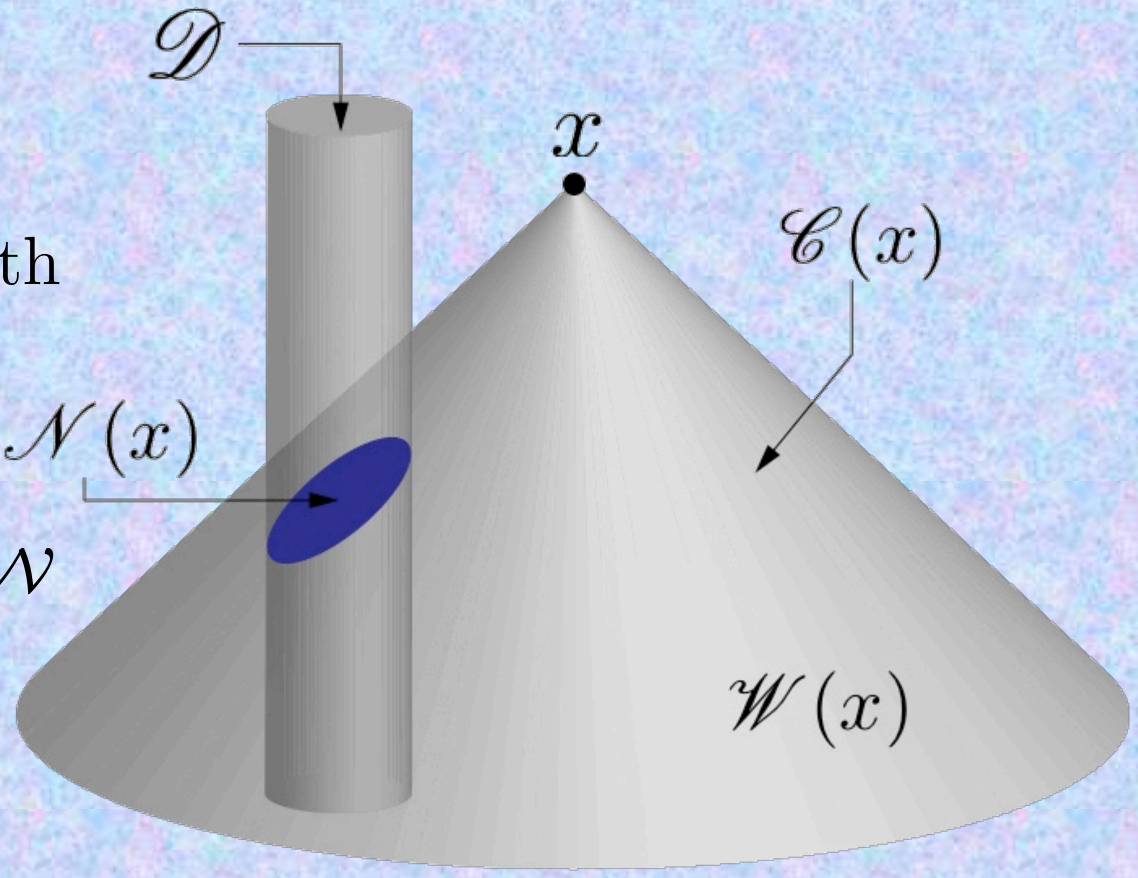
$\mathcal{N} : r' < \mathcal{R}\,,$

$\mathcal{W} : r' > \mathcal{R}$

$\mathcal{R} \sim$ wavelength

$\sim s/v$

$$\psi = \psi_{\mathcal{N}} + \psi_{\mathcal{W}}$$



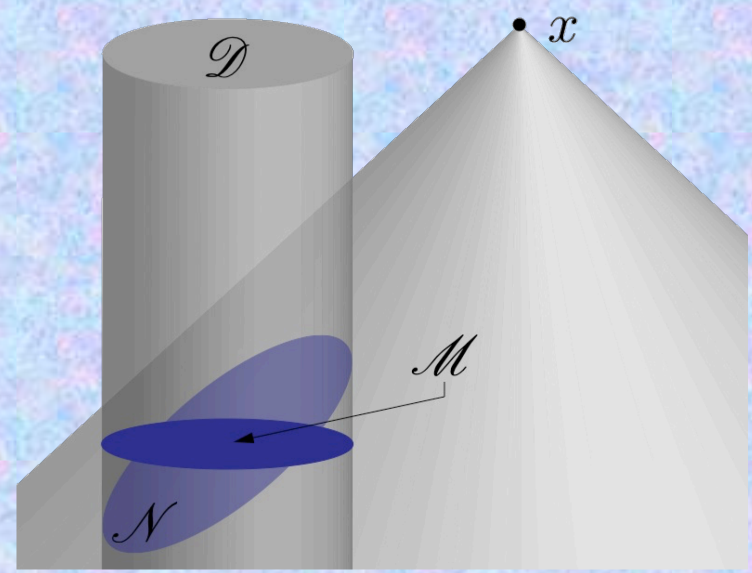
Solving the "Relaxed" Einstein Equations: Far zone

Near zone integral: $\psi_{\mathcal{N}}$

For x >> x′, Taylor expand |x-x′|

$$\frac{\mu(t-|\boldsymbol{x}-\boldsymbol{x'}|/c,\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{x'}|} = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} x'^L \partial_L \frac{\mu(t-r/c,\boldsymbol{y})}{r}.$$

$$\psi_{\mathcal{N}}(t,\boldsymbol{x}) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \partial_L \left[\frac{1}{r} \int_{\mathcal{M}} \mu(\tau,\boldsymbol{x'}) x'^L \, d^3x' \right]$$



A multipole expansion

$$\tau = t - R/c$$

Integrals depend on $\mathcal{R}$

Solving the "Relaxed" Einstein Equations: Far zone

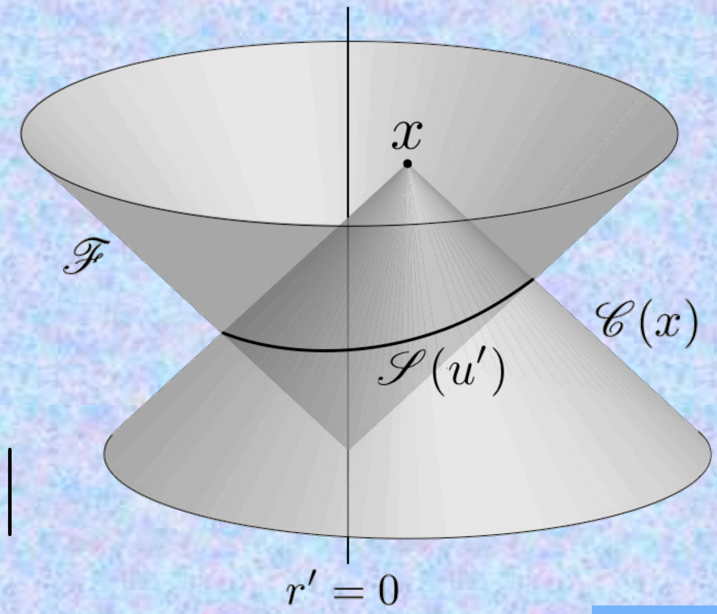
Far zone integral: $\psi_{\mathcal{W}}$

Since contributions to μ in the far zone come from retarded fields, they have the generic form

$$\mu \sim f(\tau', \theta', \phi')/r'^n$$

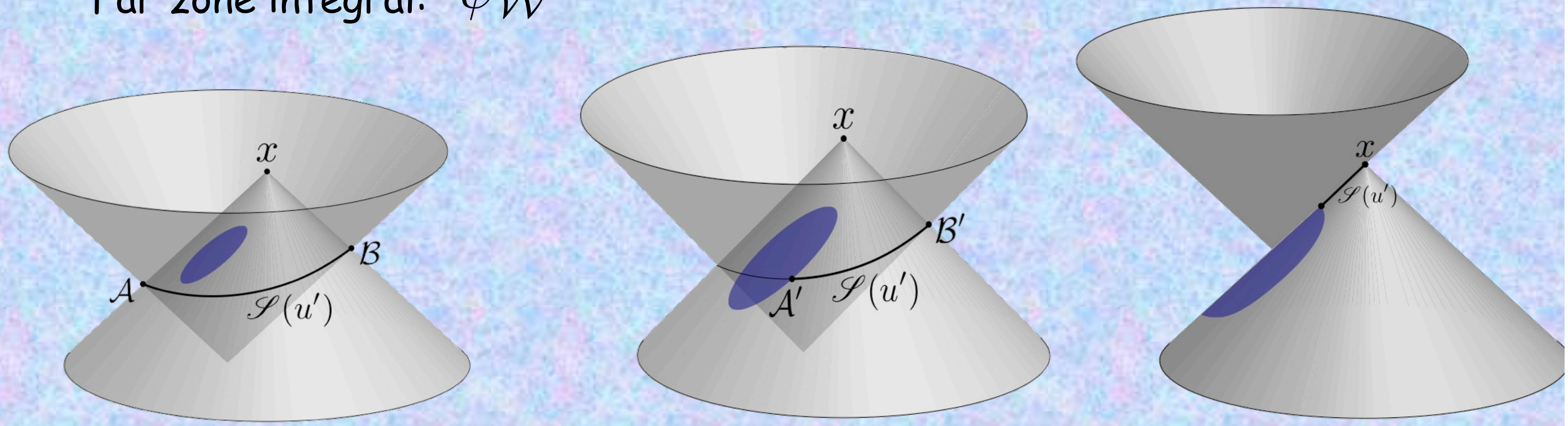
Change variables from (r', θ', φ') to (u', θ', φ'), where u' = cτ' = ct'-r'

$$u' + r' = ct - |\boldsymbol{x} - \boldsymbol{x}'|$$



Solving the "Relaxed" Einstein Equations: Far zone

Far zone integral: $\psi_{\mathcal{W}}$

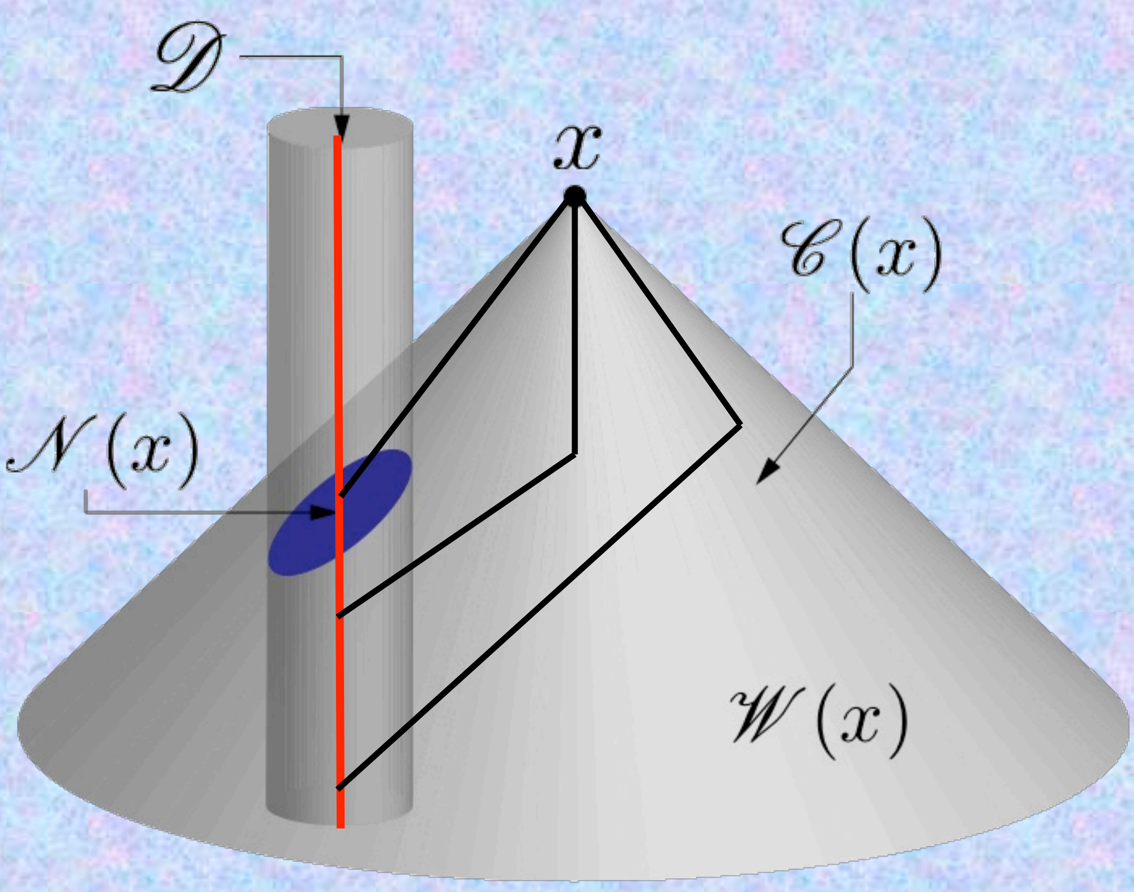


$$\psi_{\mathcal{W}} = \frac{1}{4\pi}\int_{-\infty}^{u} du' \oint_{\mathcal{S}(u')} \frac{f(u'/c,\theta',\phi')}{r'(u',\theta',\phi')^{n-2}} \frac{d\Omega'}{ct - u' - \boldsymbol{n'}\cdot\boldsymbol{x}}$$

Integral also depends on $\mathcal{R}$

But $\psi = \psi_{\mathcal{N}} + \psi_{\mathcal{W}}$ is independent of $\mathcal{R}$

Gravity as a source of gravity and gravitational "tails"



Solving the "Relaxed" Einstein Equations: Near zone

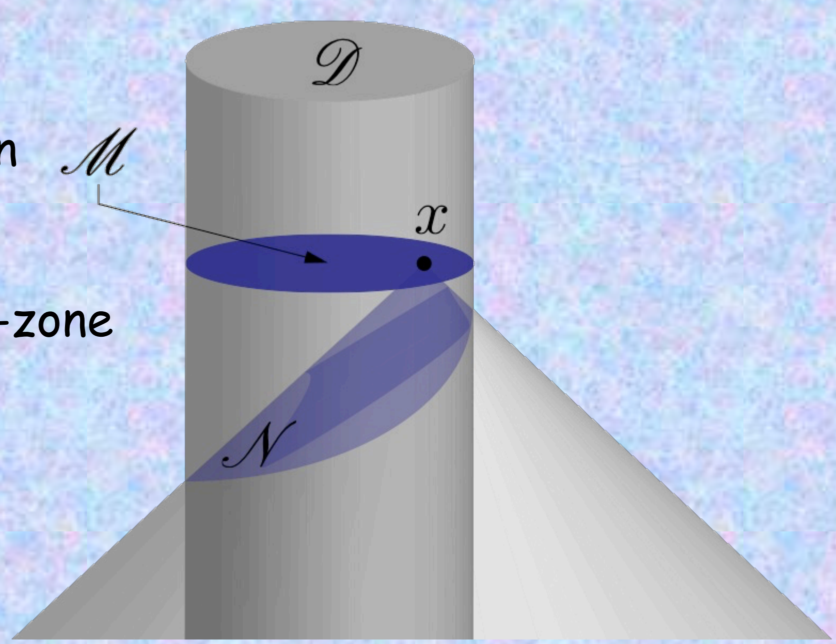
Near zone integral: $\psi_{\mathcal{N}}$

For x ~ x′, Taylor expand about t

$$\mu(t - |\boldsymbol{x} - \boldsymbol{x'}|/c) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell! c^\ell} \left(\frac{\partial}{\partial t} \right)^\ell \mu(t, \boldsymbol{x'}) |\boldsymbol{x} - \boldsymbol{x'}|^\ell$$

$$\psi_{\mathcal{N}}(t, \boldsymbol{x}) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell! c^\ell} \left(\frac{\partial}{\partial t} \right)^\ell \int_{\mathcal{M}} \mu(t, \boldsymbol{x'}) |\boldsymbol{x} - \boldsymbol{x'}|^{\ell - 1} \, d^3 x'$$

- A post-Newtonian expansion in powers of 1/c
- Instantaneous potentials
- Must also calculate the far-zone integral $\psi_{\mathcal{W}}$



Post-Newtonian approximation: Near zone

…pand h^{00} in the near zone:

Newtonian plus corrections up to 2.5 PN order within τ^{00}

No 0.5 PN term: conservation of M

1 PN correction d^2X/dt^2

$$h^{00}_{\mathcal{N}} = \frac{4G}{c^4}\Big\{ \int_{\mathcal{M}} \frac{\tau^{00}}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x' + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int_{\mathcal{M}} \tau^{00}|\boldsymbol{x}-\boldsymbol{x}'|\, d^3x'$$

$$- \frac{1}{6c^3}\frac{\partial^3}{\partial t^3}\int_{\mathcal{M}} \tau^{00}|\boldsymbol{x}-\boldsymbol{x}'|^2\, d^3x' + \frac{1}{24c^4}\frac{\partial^4}{\partial t^4}\int_{\mathcal{M}} \tau^{00}|\boldsymbol{x}-\boldsymbol{x}'|^3\, d^3x'$$

$$- \frac{1}{120c^5}\frac{\partial^5}{\partial t^5}\int_{\mathcal{M}} \tau^{00}|\boldsymbol{x}-\boldsymbol{x}'|^4\, d^3x'\Big] + O(c^{-6})\Big\}$$

2 PN term

Pure function of time – a coordinate effect

2.5 PN term

$$\frac{Gm}{rc^2} \sim \frac{v^2}{c^2} \sim \epsilon$$

Near zone physics; Motion of extended fluid bodies

Matter variables:

$$\begin{aligned} \text{rescaled mass density} &: \rho^* \equiv \rho\sqrt{-g}(u^0/c) \\ \text{proper pressure} &: p \\ \text{internal energy per unit mass} &: \Pi \\ \text{four} - \text{velocity of fluid element} &: u^\alpha = u^0(1, \boldsymbol{v}/c) \end{aligned}$$

$$\nabla_\alpha(\rho u^\alpha) = 0 \Longleftrightarrow \frac{\partial \rho^*}{\partial t} + \boldsymbol{\nabla}(\rho^* \boldsymbol{v}) = 0$$

Slow-motion assumption v/c << 1:

$$T^{0j}/T^{00} \sim v/c\,, \qquad T^{jk}/T^{00} \sim (v/c)^2$$

$$h^{0j}/h^{00} \sim v/c\,, \qquad h^{jk}/h^{00} \sim (v/c)^2$$

Post-Newtonian approximation: Near zone

Recall the action for a geodesic

$$\frac{Gm}{rc^2} \sim \frac{v^2}{c^2} \sim \epsilon$$

$$S = -mc^2 \int_1^2 d\tau$$

$$= -mc \int_1^2 \sqrt{-g_{\alpha\beta} \frac{dr^\alpha}{dt} \frac{dr^\beta}{dt}}\, dt$$

$$= -mc \int_1^2 \left(1 - 2\frac{U}{c^2} + \delta g_{00} + 2\frac{v^j}{c}\delta g_{0j} - \frac{v^2}{c^2} + \frac{v^i v^j}{c^2}\delta g_{ij}\right)^{1/2} dt$$

ε $\quad$ ε^2 $\quad$ ε^2 $\quad$ ε $\quad$ ε^2

We need to calculate

δg_{00} to $O(\epsilon^2)$

δg_{0j} to $O(\epsilon^{3/2})$

δg_{ij} to $O(\epsilon)$

Two iterations of the relaxed equations required

Post-Newtonian approximation: Near zone

Conversion between h and g

$$(-g) = 1 - h + \frac{1}{2}h^2 - \frac{1}{2}h^{\mu\nu}h_{\mu\nu} + O(G^3)\,,$$

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{2}h\eta_{\alpha\beta} + h_{\alpha\mu}h^{\mu}{}_{\beta} - \frac{1}{2}hh_{\alpha\beta}$$
$$+ \left(\frac{1}{8}h^2 - \frac{1}{4}h^{\mu\nu}h_{\mu\nu}\right)\eta_{\alpha\beta} + O(G^3)\,,$$

To 1PN order:

$$g_{00} = -1 + \frac{1}{2}h^{00} + \frac{1}{2}h^{kk} - \frac{3}{8}\left(h^{00}\right)^2 + O(c^{-6})\,,$$

$$g_{0j} = -h^{0j} + O(c^{-5})\,,$$

$$g_{jk} = \delta_{jk}\left[1 + \frac{1}{2}h^{00}\right] + O(c^{-4})\,,$$

Post-Newtonian limit of general relativity

$$g_{00} = -1 + \frac{2}{c^2}U + \frac{2}{c^4}\left(\psi + \frac{1}{2}\partial_{tt}X - U^2\right) + O(c^{-6}) \,,$$

$$g_{0j} = -\frac{4}{c^3}U_j + O(c^{-5}) \,,$$

$$g_{jk} = \delta_{jk}\left(1 + \frac{2}{c^2}U\right) + O(c^{-4}) \,,$$

$$U(t,\boldsymbol{x}) := G\int \frac{\rho^{*\prime}}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x' \,,$$

$$\psi(t,\boldsymbol{x}) := G\int \frac{\rho^{*\prime}\left(\frac{3}{2}v'^2 - U' + \Pi' + 3p'/\rho^{*\prime}\right)}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x' \,,$$

$$X(t,\boldsymbol{x}) := G\int \rho^{*\prime}|\boldsymbol{x}-\boldsymbol{x'}|\, d^3x' \,,$$

$$U^j(t,\boldsymbol{x}) := G\int \frac{\rho^{*\prime}v'^j}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x'$$

Post-Newtonian Hydrodynamics

From $\nabla_\beta T^{\alpha\beta} = 0$

Post-Newtonian equation of hydrodynamics

$$
\begin{aligned}
\rho^* \frac{dv^j}{dt} = &-\partial_j p + \rho^* \partial_j U \\
&+ \frac{1}{c^2}\left[\left(\frac{1}{2}v^2 + U + \Pi + \frac{p}{\rho^*}\right)\partial_j p - v^j \partial_t p\right] \\
&+ \frac{1}{c^2}\rho^*\Big[(v^2 - 4U)\partial_j U - v^j\left(3\partial_t U + 4v^k \partial_k U\right) \\
&\qquad + 4\partial_t U_j + 4v^k\left(\partial_k U_j - \partial_j U_k\right) + \partial_j \Psi\Big] \\
&+ O(c^{-4})
\end{aligned}
$$

$$\Psi = \psi + \frac{1}{2}\partial_{tt} X$$

N-body equations of motion

Main assumptions:

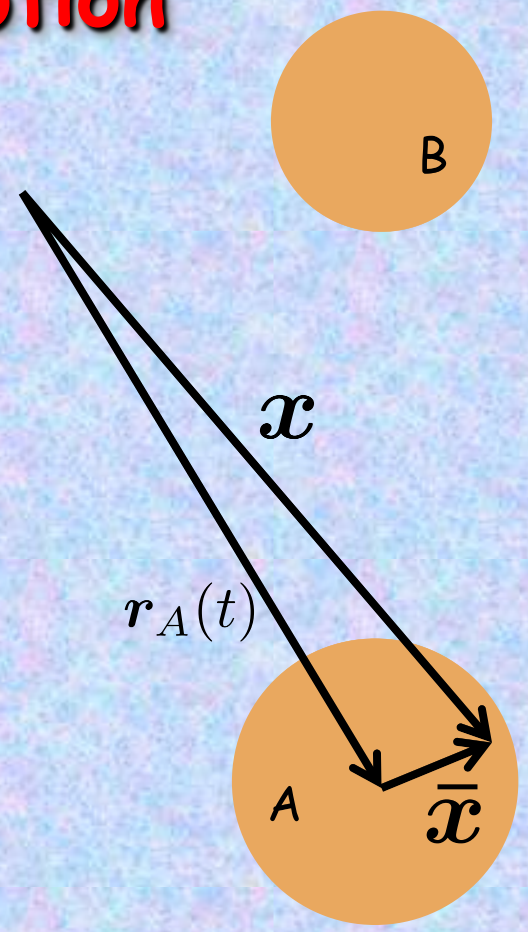
- Bodies small compared to typical separation (R << r)
- "isolated" -- no mass flow
- ignore contributions that scale as R^n
- assume bodies are reflection symmetric

$$\text{mass}: \quad m_A \equiv \int_A \rho^* d^3x$$

$$\text{position}: \quad \boldsymbol{r}_A(t) \equiv \frac{1}{m_A}\int_A \rho^* \boldsymbol{x} d^3x$$

$$\text{velocity}: \quad \boldsymbol{v}_A(t) \equiv \frac{1}{m_A}\int_A \rho^* \boldsymbol{v} d^3x = \frac{d\boldsymbol{r}_A}{dt}$$

$$\text{acceleration}: \quad \boldsymbol{a}_A(t) \equiv \frac{1}{m_A}\int_A \rho^* \boldsymbol{a} d^3x = \frac{d\boldsymbol{v}_A}{dt}$$



$$\boldsymbol{x} \equiv \boldsymbol{r}_A(t) + \bar{\boldsymbol{x}}$$