Laplace Transforms and Integral Equations

Bernd Schröder

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Original DE & IVP

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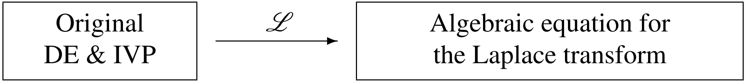

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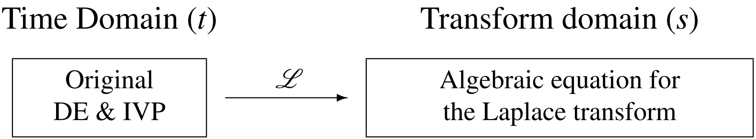

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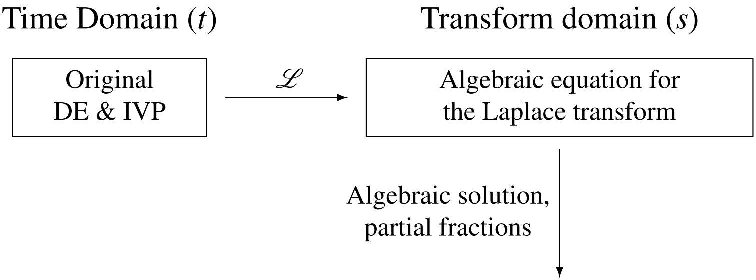

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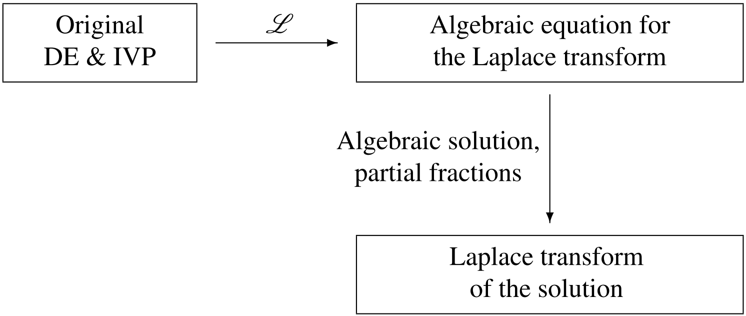
Time Domain (t) Transform domain (s)



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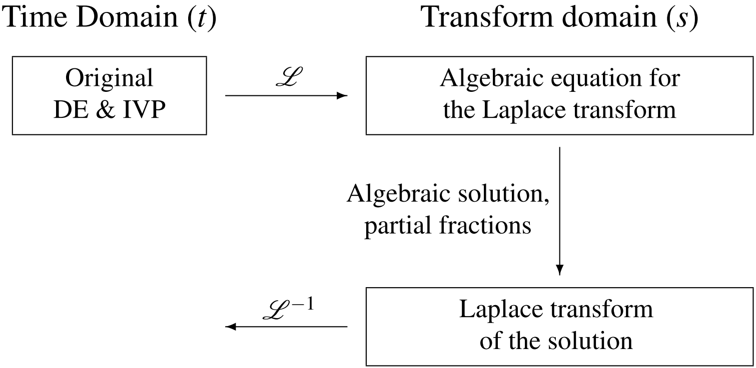

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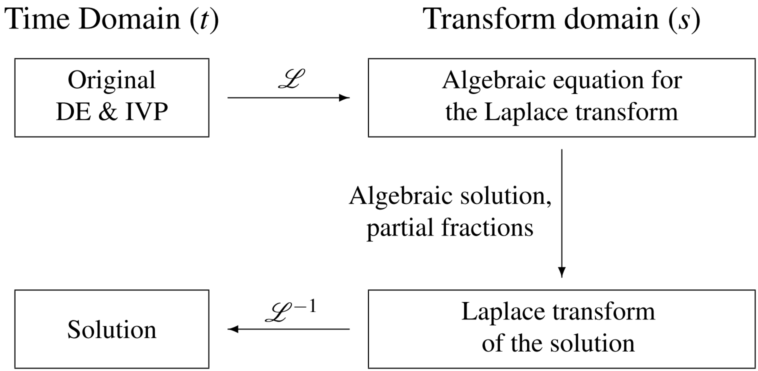

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1. Definite integrals of the form $\int_0^t f(\tau)\,d\tau$ arise in circuit theory: The charge of a capacitor is the integral of the current over time.
   (We assume the capacitor is initially uncharged.)
2. $\mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$

Solve the Initial Value Problem

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&= e^{t}\left(\frac{1}{3} + \frac{1}{3} - \frac{2}{3}\right) + e^{-2t}\left(-\frac{1}{3} + \frac{1}{6} + \frac{1}{6}\right) + t + \left(-\frac{1}{2} + \frac{2}{3} - \frac{1}{6}\right)
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&= \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t} - \frac{1}{2} + \frac{1}{3}e^{t} + \frac{1}{6}e^{-2t} - 2\left[-\frac{1}{2}z + \frac{1}{3}e^{z} - \frac{1}{12}e^{-2z}\right]_0^t \\
&= \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t} - \frac{1}{2} + \frac{1}{3}e^{t} + \frac{1}{6}e^{-2t} + t - 0 - \frac{2}{3}e^{t} + \frac{2}{3} + \frac{1}{6}e^{-2t} - \frac{1}{6} \\
&= e^{t}\left(\frac{1}{3} + \frac{1}{3} - \frac{2}{3}\right) + e^{-2t}\left(-\frac{1}{3} + \frac{1}{6} + \frac{1}{6}\right) + t + \left(-\frac{1}{2} + \frac{2}{3} - \frac{1}{6}\right) \\
&= t
\end{aligned}$$

Does $f(t) = -\frac{1}{2} + \frac{1}{3}e^t + \frac{1}{6}e^{-2t}$ Really Solve the Initial Value Problem $f'(t) + f(t) - 2\int_0^t f(z)\,dz = t, f(0) = 0$?

Initial value: Look at f!

$$\left[-\frac{1}{2} + \frac{1}{3}e^t + \frac{1}{6}e^{-2t}\right]' + \left[-\frac{1}{2} + \frac{1}{3}e^t + \frac{1}{6}e^{-2t}\right] - 2\int_0^t -\frac{1}{2} + \frac{1}{3}e^z + \frac{1}{6}e^{-2z}\,dz$$

$$\begin{aligned}
&= \frac{1}{3}e^t - \frac{1}{3}e^{-2t} - \frac{1}{2} + \frac{1}{3}e^t + \frac{1}{6}e^{-2t} - 2\left[-\frac{1}{2}z + \frac{1}{3}e^z - \frac{1}{12}e^{-2z}\right]_0^t \\
&= \frac{1}{3}e^t - \frac{1}{3}e^{-2t} - \frac{1}{2} + \frac{1}{3}e^t + \frac{1}{6}e^{-2t} + t - 0 - \frac{2}{3}e^t + \frac{2}{3} + \frac{1}{6}e^{-2t} - \frac{1}{6} \\
&= e^t\left(\frac{1}{3} + \frac{1}{3} - \frac{2}{3}\right) + e^{-2t}\left(-\frac{1}{3} + \frac{1}{6} + \frac{1}{6}\right) + t + \left(-\frac{1}{2} + \frac{2}{3} - \frac{1}{6}\right) \\
&= t \qquad \surd
\end{aligned}$$