

Laplace Transforms - GATE Study Material in PDF

As a student of any stream of Engineering like GATE EC, GATE EE, GATE CS, GATE CE, GATE ME, you will come across one very important concept in Engineering Mathematics – Laplace Transforms. The application of Laplace Transforms is wide and is used in a variety of subjects like Control Systems, Network Theory / Electrical Network and Signals & Systems. If you are preparing for GATE 2017, you should use these free GATE Study Notes, to help you ace the exam. You can also download this GATE Study Material in PDF for revision and reference later. Now let us take a look at the different Laplace Transforms formulas and concepts.

Laplace Transform Concepts

- i. The Laplace transform converts integral and differential equations into algebraic equations.
- ii. It also converts time domain signal into frequency domain signal.
- iii. The Laplace transform of a signal $f(t)$ is denoted by $L\{f(t)\} = F(s)$.

where $F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$ (for Bilateral)

$F(s) = \int_0^{\infty} f(t)e^{-st}dt$ (For Unilateral)

- iv. Inverse Laplace transform converts a frequency domain signal into time domain signal.

$$i. e f(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} f(s)e^{-st}ds$$

Where σ is large enough that $F(s)$ is defined for $\text{Re}\{s\} \geq \sigma$.

Laplace Transforms Formulas

The formulae given below are very useful to solve the many Laplace Transform based problems

$$(1) L\{1\} = \frac{1}{s}$$

$$(2) L\{t^n\} = \frac{n!}{s^{n+1}} \text{ If } n \text{ is integer}$$

$$= \frac{\Gamma(n+1)}{s^{n+1}} \text{ If } n \text{ is a rational number } \left(\frac{p}{q}\right)$$

$$(3) L\{e^{at}\} = \frac{1}{s-a}$$

$$(4) L\{e^{-at}\} = \frac{1}{s+a}$$

$$(5) L\{\sin at\} = \frac{a}{s^2+a^2}$$

$$(6) L\{\cos at\} = \frac{s}{s^2+a^2}$$

$$(7) L\{\sinh at\} = \frac{a}{s^2-a^2}$$

$$(8) L\{\cosh at\} = \frac{s}{s^2-a^2}$$

$$(9) L\{e^{at} \sin bt\} = \frac{b}{(s-a)^2+b^2}$$

$$(10) L\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2+b^2}$$

$$(11) L\{f(at)\} = \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

$$(12) L\{\delta(t)\} = 1$$

$$(13) L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s) \cdot ds$$

$$(14) L\left\{\int_0^t f(t) dt\right\} = \frac{F(s)}{s}$$

$$(15) L\{t^n f(t)\} = (-1)^n \cdot \frac{d^n}{ds^n} F(s)$$

$$(16) \text{ If } f(t+T) = f(t) \text{ then } L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T f(t)e^{-st} dt$$

$$(17) L\{f'(t)\} = sF(s) - f(0)$$

$$(18) L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

$$(19) L\{f^n(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \dots - f^{n-1}(0)$$

$$(20) L\{f(t) * g(t)\} = F(s).G(s) \text{ where } f(t) * g(t) = \int_0^t f(\tau)g(t - \tau)d\tau$$

$$(21) L\{t\} = \frac{1}{s^2}$$

Example Problems of Laplace Transforms

Example 1:

Find the value of $L\{t \cos bt\}$

Solution:

We know that $L\{\cos bt\} = \frac{s}{s^2+b^2}$

$$L\{t.f(t)\} = -\frac{d}{ds}[F(s)]$$

$$\therefore L\{t \cos bt\} = -\frac{d}{ds}\left[\frac{s}{s^2+b^2}\right]$$

$$= -\left[\frac{(s^2+b^2)-s(2s)}{(s^2+b^2)^2}\right]$$

$$L\{t \cos bt\} = \frac{s^2-b^2}{(s^2+b^2)^2}$$

Example 2:

Find the Laplace transform of $f(t) = 1; 0 \leq t < 2$

$$= t-2; 2 \leq t$$

Solution:

$$F(s) = L\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

$$= \int_0^2 1 \cdot e^{-st}dt + \int_2^{\infty} (t-2)e^{-st}.dt$$

$$\begin{aligned}
 &= \left[\frac{e^{-st}}{-s} \right]_0^2 + \left[(t-2) \left(\frac{e^{-st}}{-s} \right) - (1) \left(\frac{e^{-st}}{s^2} \right) \right]_2^\infty \\
 &= -\frac{1}{s} [e^{-2s} - 1] + \left[(0-0) - \left(0 - \frac{1}{s^2} e^{-2s} \right) \right] \\
 &= -\frac{1}{s} e^{-2s} + \frac{1}{s} + \frac{1}{s^2} e^{-2s} = \frac{1}{s} + \frac{(1-s)}{s^2} e^{-2s}
 \end{aligned}$$

Example 3:

Find the inverse Laplace transform of $F(s) = \frac{s+1}{s^2+4}$

Solution:

Given, $F(s) = \frac{s+1}{s^2+4}$

$$= \frac{s}{s^2+4} + \frac{1}{s^2+4}$$

$$F(s) = \frac{s}{s^2+2^2} + \frac{1}{2} \cdot \frac{2}{s^2+2^2}$$

Apply inverse Laplace transform then we get

$$f(t) = \cos 2t + \frac{1}{2} \sin 2t$$

Example 4:

Find the inverse Laplace transform of the function $F(s) = \log \left(\frac{s+3}{s+4} \right)$

Solution:

Given, $F(s) = \log \left(\frac{s+3}{s+4} \right)$

$$F(s) = \log(s+3) - \log(s+4)$$

$$\frac{d}{ds} F(s) = \frac{1}{s+3} - \frac{1}{s+4}$$

Apply inverse Laplace transform then we get

$$-t \cdot f(t) = e^{-3t} - e^{-4t}$$

$$\therefore f(t) = \frac{e^{-4t} - e^{-3t}}{t}$$

Initial Value Theorem

$$\text{If } L\{f(t)\} = F(s) \text{ then } \lim_{t \rightarrow 0} x(t) = \lim_{s \rightarrow \infty} s \cdot X(s)$$

Note:

Initial value theorem is applicable only if the function is strictly proper function i.e. Number of poles of $X(s)$ is greater than the number of zeros of $X(s)$.

Final Value Theorem

$$\text{If } L\{f(t)\} = F(s) \text{ then } \lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} s \cdot X(s)$$

Note:

- i. Final value theorem is applicable only when the system is stable i.e. all the poles of $X(s)$ should be lie on the Left Half of s -plane.
- ii. If there are pairs of complex conjugate poles on the imaginary axis $x(t)$ will contain sinusoidal components and final value is not defined.

Example Problems of Inverse Value Theorem & Final Value Theorem

Example 1:

Given that $y'' - 5y' + 6y = 0, y(0) = 2$ and $y'(0) = 2$. Then find the value of $y(t)$ at $t = 1$

Solution:

$$\text{Given, } y'' - 5y' + 6y = 0$$

Apply Laplace transform on both sides then, we get

$$[s^2Y(s) - sy(0) - y'(0)] - 5[sY(s) - y(0)] + 6Y(s) = 0$$

$$(s^2 - 5s + 6)Y(s) - 2s - 2 + 5(2) = 0$$

$$Y(s) = \frac{(2s-8)}{s^2-5s+6}$$

$$Y(s) = \frac{(2s-8)}{(s-3)(s-2)}$$

$$Y(s) = \frac{(6-8)}{(s-3)(3-2)} + \frac{2(2)-8}{(s-2)(2-3)}$$

$$Y(s) = \frac{-2}{s-3} + \frac{4}{s-2}$$

$$\therefore y(t) = 4e^{2t} - 2e^{3t}$$

$$y(1) = 4e^2 - 2e^3 = -10.615$$

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