Large deviations for the range of a simple random walk.

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Large Deviation Background

Rate function The function $I : \mathcal{X} \mapsto [0, \infty]$ is called rate function if

- I is lower semicontinuous
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Large deviation Principle

A sequence of random variables $X_1, X_2, \dots$ is said to satisfy a LDP with speed a_n and rate function I if for all Borel sets $A \subset M$,

- $\limsup_{n\to\infty} \frac{1}{a_n} \log \mathbb{P}\{X_n \in A\} \leq - \inf_{X \in cl(A)} I(x)$
- $\liminf_{n\to\infty} \frac{1}{a_n} \log \mathbb{P}\{X_n \in A\} \geq - \inf_{X \in int(A)} I(x)$

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In easy word, $\lim_{n\to\infty} \frac{1}{a_n} \log \mathbb{P}\{X_n \in A\} \approx -I(x)$.

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Contraction Principle

If $X_1, X_2, \dots$ satisfy LDP with speed a_n and rate function I, and $f : M \to M'$ is a continuous mapping, then sequences $f(X_1), f(X_2), \dots$ satisfies LDP with speed a_n and rate function J given by

$$J(y) = \inf_{x \in f^{-1}(\{y\})} I(x)$$

Introduction

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Let $R_n = \sharp\{S_i : 1 \leq i \leq n\}$ be the number of points visited by the random walk up to time n.

Expected size of Range

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Theorem (Dvoretzky and Erdős, 1950)

$$\mathbb{E}R_n = \begin{cases} \kappa n + O(n^{1/2}), & \text{if } d = 3; \\ \kappa n + O(\log n), & \text{if } d = 4; \\ \kappa n + c_d + O(n^{2-d/2}), & \text{if } d \geq 5 \end{cases}$$

where

- $\kappa = \mathbb{P}(S_i \neq 0 \text{ for } i > 0)$ is the exit probability from the origin.
- $c_d (d = 5, 6, \ldots)$ are positive constants.

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where

- $\kappa = \mathbb{P}(S_i \neq 0 \text{ for } i > 0)$ is the exit probability from the origin.
- $c_d (d = 5, 6, \ldots)$ are positive constants.

Further, the strong law of large numbers holds

$$\lim_{n \to \infty} \frac{R_n}{\mathbb{E}R_n} = 1 \quad a.s. \text{ if } d \geq 2.$$

The Large Deviation Behaviour

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Theorem (Kesten and Hamana, 2001)

For $n > 0$,

$$\psi(\theta) := \lim_{n\to\infty} \frac{-1}{n} \log \mathbb{P}(R_n \geq \theta n) \text{ exists}$$

for all θ (but $\psi(\theta)$ may equal $+\infty$). Moreover,

$$\begin{aligned} \psi(\theta) = 0, &\quad \text{for } \theta \leq \kappa \\ 0 < \psi(\theta) < \infty, &\quad \text{for } \kappa < \theta \leq 1 \\ \psi(\theta) = \infty, &\quad \text{for } \theta > 1. \end{aligned}$$

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Question: What can we say about the probability of unusually small range ($\{R_n \leq \theta n\}$ for $\theta < \kappa$)?

Result

Theorem 1

Let $d \geq 3$. For every $b > 0$,

$$\lim_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}(R_n \leq bn) = -\frac{1}{d} I^\kappa(b),$$

where

$$I^\kappa(b) = \inf_{\phi\in\Phi^\kappa(b)} [\frac{1}{2}\int_{\mathbb{R}^d} |\nabla\phi|^2(x)dx]$$

with

$$\Phi^\kappa(b) = \{\phi \in H^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} \phi^2(x)dx = 1, \int_{\mathbb{R}^d}(1 - e^{-\kappa\phi^2(x)})dx \leq b\}.$$

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- The size of the torus **depends** on timescale.

Random walk on torus

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Obvious remark: $\mathcal{R}_n \leq R_n$.

We will use this fact to prove the upper bound of Theorem 1.

Proposition 2

$\frac{1}{n}\mathcal{R}_n$ satisfies LDP on $\mathbb{R}^+$ with speed $n^{\frac{d-2}{d}}$ and rate function

$$J_N^\kappa(b) = \inf_{\phi \in \partial\Phi_N^\kappa(b)} [\frac{1}{2}\int_{\Lambda_N} |\nabla\phi|^2(x)dx]$$

with

$$\partial\Phi_N^\kappa(b) = \{\phi \in H^1(\Lambda_N) : \int_{\Lambda_N} \phi^2(x)dx = 1, \int_{\Lambda_N} (1 - e^{-\kappa\phi^2(x)})dx = b\}.$$

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- Approximate $\mathcal{S}_n$ by its conditional walk (skeleton).
- Proof LDP of its skeleton using Donsker-Varadhan theory for empirical pair measure.
- Get appropriate rate function for LDP.

Approximate random walk

Set $\mathbb{S}_{n,\epsilon} = \{\mathcal{S}_{i\epsilon n^{2/d}}\}_{1 \leq i \leq \frac{1}{\epsilon} n^{\frac{d-2}{d}}}$ and let $\mathbb{E}_{n,\epsilon}$ be the conditional expectation given $\mathbb{S}_{n,\epsilon}$.

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Proposition 3

For all $\delta > 0$,

$$\lim_{\epsilon \downarrow 0} \limsup_{n \to \infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}\left(\frac{1}{n}|\mathcal{R}_n - \mathbb{E}_{n,\epsilon}\mathcal{R}_n| \ge \delta\right) = -\infty.$$

LDP of $\frac{1}{n}\mathbb{E}_{n,\epsilon}\mathcal{R}_n$

We aim to represent $\mathbb{E}_{n,\epsilon}\frac{1}{n}\mathcal{R}_n$ as a function of the empirical measure in order to apply Donsker-Varadhan theory. We will scale the torus from size $Nn^{1/d}$ to size N to remove $n-$dependence.

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Define empirical pair measure:

$$L^2_{n,\epsilon} = \epsilon n^{-\frac{d-2}{d}} \sum_{i=1}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}} \delta_{(n^{-1/d}\mathcal{S}_{(i-1)\epsilon n^{2/d}}, n^{-1/d}\mathcal{S}_{i\epsilon n^{2/d}})}.$$

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Lemma 4

$$\lim_{n\to\infty} ||\frac{1}{n}\mathbb{E}_{n,\epsilon}R_n - \Phi_{1/\epsilon}(L^2_{n,\epsilon})||_\infty = 0 \text{ for all } \epsilon > 0.$$

where,

$$\Phi_\eta(\mu) = \int_{\Lambda_N} dx\Big(1-\exp[-\eta\kappa\int_{\Lambda_N\times\Lambda_N}\varphi_\epsilon(y-x,z-x)\mu(dy,dz)]\Big),$$

with

$$\varphi_\epsilon(y,z) = \int_0^\epsilon ds\frac{p^\pi(s/d,-y)p^\pi\big((\epsilon-s)/d,z\big)}{p^\pi(\epsilon/d,z-y)}.$$

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We first define:

$$W_i = \{\mathcal{S}_j : (i-1)\epsilon n^{2/d} \leq j \leq i\epsilon n^{2/d}\} \qquad (1 \leq i \leq \frac{1}{\epsilon} n^{\frac{d-2}{d}})$$

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to be the range of random walk between each skeleton.

Note that $\frac{1}{n}\mathcal{R}_n = \frac{1}{n}\sharp\{\bigcup_{i=1}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}} W_i\}$.

Now, we can see that:

$$\mathbb{E}_{n,\epsilon}\frac{1}{n}\mathcal{R}_n = \frac{1}{n}\sum_{x\in\Lambda_{Nn^{\frac{1}{d}}}}\left(1-\mathbb{P}_{n,\epsilon}\{x\notin\bigcup_{i=1}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}}W_i\}\right)$$

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$$= \frac{1}{n}\sum_{x\in\Lambda_{Nn^{\frac{1}{d}}}}\left(1-\mathbb{P}_{n,\epsilon}(\bigcap_{i=1}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}}\{x\notin W_i\})\right)$$

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$$= \frac{1}{n} \sum_{x \in \Lambda_{Nn^{\frac{1}{d}}}} \left(1 - \exp\Big(\sum_{i=1}^{\frac{1}{\epsilon} n^{\frac{d-2}{d}}} \log[1 - \mathbb{P}_{n,\epsilon}\{x \in W_i\}]\Big)\right)$$

$$= \frac{1}{n} \sum_{x \in \Lambda_{Nn^{\frac{1}{d}}}} \Big(1 - \exp\big(\sum_{i=1}^{\frac{1}{\epsilon} n^{\frac{d-2}{d}}} \log[1 - \mathbb{P}_{n,\epsilon}\{x \in W_i\}]\big)\Big)$$

$$\approx \int_{\Lambda_N} dx \Big(1 - \exp\big(\frac{1}{\epsilon} n^{\frac{d-2}{d}} \int_{\Lambda_N \times \Lambda_N} L^2_{n,\epsilon}(dy, dz) \log[1 - q_{n,\epsilon}(y - x, z - x)]\big)\Big).$$

$$= \frac{1}{n} \sum_{x \in \Lambda_{Nn^{\frac{1}{d}}}} \Big(1 - \exp\Big(\sum_{i=1}^{\frac{1}{\epsilon} n^{\frac{d-2}{d}}} \log[1 - \mathbb{P}_{n,\epsilon}\{x \in W_i\}]\Big)\Big)$$

$$\approx \int_{\Lambda_N} dx \Big(1 - \exp\Big(\frac{1}{\epsilon} n^{\frac{d-2}{d}} \int_{\Lambda_N \times \Lambda_N} L^2_{n,\epsilon}(dy, dz) \log[1 - q_{n,\epsilon}(y-x, z-x)]\Big)\Big).$$

where

- the last equality come from scaling the torus and adding empirical measure.
- $q_{n,\epsilon}(y,z) = \mathbb{P}(\sigma \leq \epsilon n^{\frac{2}{d}} | S_0 = y, S_{\epsilon n^{2/d}} = z)$, where $\sigma = \min\{n : S_n = 0\}$.

By using Donsker-Varadhan theory, along with the previous lemma, we can deduce that:

Proposition 5

$\frac{1}{n}\mathbb{E}_{n,\epsilon}\mathcal{R}_n$ satisfies LDP on $\mathbb{R}^+$ with speed $n^{\frac{d-2}{d}}$ and rate function:

$$J_{\epsilon/d}(b) = \inf\{\frac{1}{\epsilon} I^{(2)}_{\epsilon/d}(\mu) : \mu \in \mathcal{M}_1^+(\Lambda_N \times \Lambda_N), \Phi_{1/\epsilon}(\mu) = b\}.$$

Performing the limit

This step is to derive appropriate limit result for rate function, via functional analysis. The result of this step is that we obtain the required rate function in Proposition 2.

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The proof can be seen on van den Berg, Bolthausen and den Hollander(2001) "Moderate deviation for the volume of the Wiener sausage ".

Increasing the size of torus

Proposition 6

$\lim_{N\to\infty} I_N^\kappa(b) = I^\kappa(b)$ for all $b > 0$. where I^κ is the rate function defined in Theorem 1.

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$\lim_{N\to\infty} I_N^\kappa(b) = I^\kappa(b)$ for all $b > 0$. where I^κ is the rate function defined in Theorem 1.

Combine this proposition with the previous result, we will get the upper bound.

Current Work

Let J_n be the number of intersecting points of two independent random walks start at the origin.

CONJECTURE

Let $d \geq 3$. For every $c > 0$,

$$\lim_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}(J_n \geq cn) = -\frac{1}{d}\hat{I}_d^{\kappa}(c),$$

where

$$\hat{I}_d^{\kappa}(c) = \inf_{\phi\in\Phi_d^{\kappa}(c)} [\int_{\mathbb{R}^d} |\nabla\phi|^2(x)dx]$$

with

$$\Phi_d^{\kappa}(c) = \{\phi \in H^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} \phi^2(x)dx = 1, \int_{\mathbb{R}^d} (1 - e^{-(\kappa/c)\phi^2(x)})^2 dx \geq 1\}.$$

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