Last Topics #1: Reference frame theory/induction machines

There are some additional topics that are appropriate to include in a course on power systems dynamics, and I do so here because they are highly relevant to what might be called "current events." In these notes, we provide a broader view of reference frame theory and then apply it to provide induction machine models appropriate for modeling load and wind turbines in power system stability studies.

1.0 Introduction

You recall that in developing a model for the synchronous machine that we found the differential equations included time varying coefficients that resulted from the dependence of inductances (stator-stator and stator-rotor) on rotor position.

To address this, so that we could obtain differential equations with constant coefficients, we performed a transformation on the a-b-c voltages and currents. We called that transformation "Park's transformation."

It is worthwhile to read what Paul Krause says in his very good text on electric machinery [1] (references within quotes are not included here). The below quotes are from his chapter 3, titled "Reference-frame theory." You would do well to have this book on your bookshelf.

"In the late 1920's, R. H. Park [1] introduced a new approach to electric machine analysis. He formulated a change of variables which, in effect, replaced the variables (voltages, currents, and flux linkages) associated with the stator windings of a synchronous machine with variables associated with fictitious windings rotating with the rotor. In other words, he transformed, or referred, the stator variables to a frame of reference fixed in the rotor. Park's transformation, which revolutionized electric machine analysis, has the unique property of eliminating all time-varying inductances from the voltage equations of the synchronous machine which occur due to (1) electric circuits in relative motion and (2) electric circuits with varying magnetic reluctance.

In the late 1930's, H. C. Stanley [2] employed a change of variables in the analysis of induction machines. He showed that the time-varying inductances in the voltage equations of an induction machine due to electric circuits in relative motion could be eliminated by transforming the variables associated with the rotor windings (rotor variables) to variables associated with fictitious stationary windings. In this case the rotor variables are transformed to a frame of reference fixed in the stator.

G. Kron [3] introduced a change of variables which eliminated the time-varying inductances of a symmetrical induction machine by transforming both the stator variables and the rotor variables to a reference frame rotating in synchronism with the rotating magnetic field. This reference frame is commonly referred to as the synchronously rotating reference frame.

D. S. Brereton et al. [4] employed a change of variables which also eliminated the time-varying inductances of a symmetrical induction machine by transforming the stator variables to a reference frame fixed in the rotor. This is essentially Park's transformation applied to induction machines.

Park, Stanley, Kron, and Brereton et al. developed changes of variables each of which appeared to be uniquely suited for a particular application. Consequentially, each transformation was derived and treated separately in the literature until it was noted in 1965 [5] that all known real transformations used in induction machine analysis are contained in one general transformation which eliminates all time-varying inductances by referring the stator and rotor variables to a frame of reference which may rotate at any angular velocity or remain stationary. All known real transformations may then be obtained by simply assigning the appropriate speed of rotation to this so-called arbitrary reference frame. Later, it was noted that the stator variables of a synchronous machine could also be referred to the arbitrary reference frame [6]. However, we will find that the time-varying inductances of a synchronous machine are eliminated only if the reference frame is fixed in the rotor (Park's transformation); consequently the arbitrary reference frame does not offer the advantage in the analysis of synchronous machines that it does in the case of induction machines."

Krause also indicates later in the same chapter

"…The transformation of stationary circuits to a stationary reference frame was developed by E. Clarke [7] who used the notation f_α, f_β …"

There are three main points to be made from the above.

1. Other transformations: There have been a number of proposed transformations besides Park's.
2. Powerful tool: The ability to make transformations is apparently a powerful tool for analysis of electric machines.
3. General theory: There is a higher-level general theory for transformations under which all of them fall.

We have two motivations for being interested in induction machines (and reference frame theory). First, we know when we perform a stability analysis we need to model the load in some way. It was common at first to model the load with just constant impedance load. Then the ZIP model was introduced that improved the constant impedance model. But the zip model was recognized early-on to not very well capture the main portion of the load, which is dynamical (not static) in the form of induction motors. To capture induction motor load dynamics, we need an electromechanical model. And an electromechanical model of induction motors needs reference frame theory, because induction motors, although similar to synchronous machines, are different from synchronous machines. The upshot here is that *load modeling is our first motivation for being interested in induction machines and reference frame theory*.

Our second motivation for being interested in induction machines and reference frame theory is the growth of wind turbines. Indeed:

- wind turbines have primarily used induction machines;
- the doubly-fed induction generator (DFIG) has been particularly attractive;
- control of the DFIGs requires back-to-back voltage-source-inverters (VSI);
- design of VSI is most effectively done using space-vector-modulation (SVM);

- SVM requires one kind of transformation for the grid-side converter and one kind of transformation for the rotor-side converter.

Figure 1 [2] illustrates a DFIG, and Fig. 2 [2] provides an expanded illustration of the inverter.

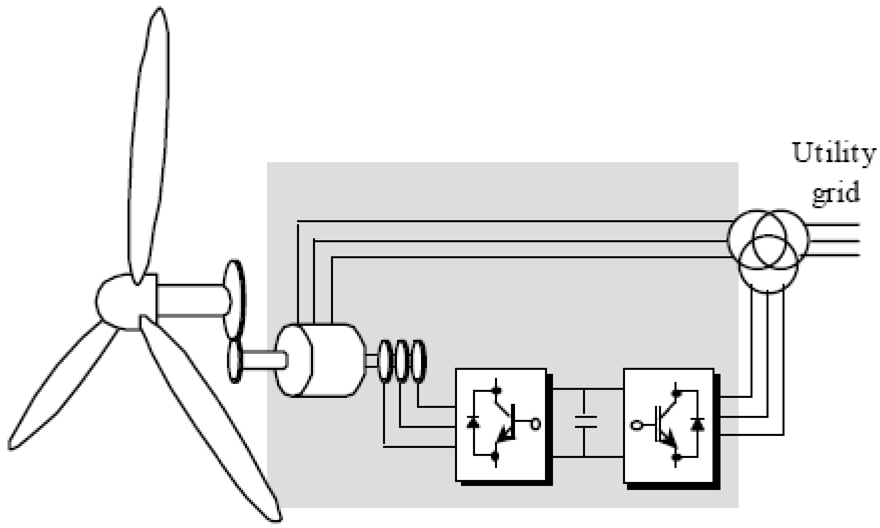


Fig. 1 [2]

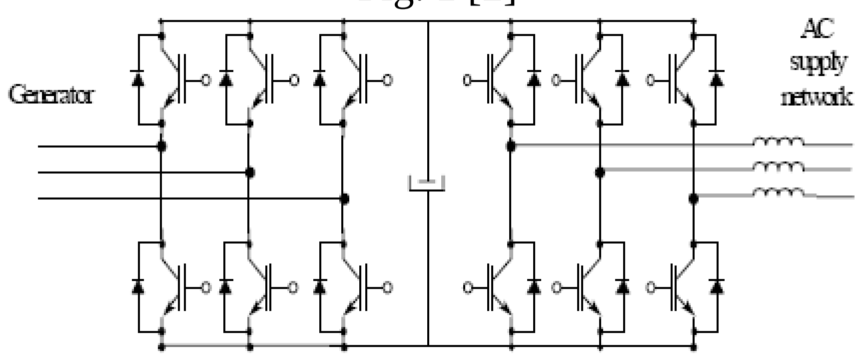


Fig. 2 [2]

Figure 2 represents a back-to-back four-quadrant VSI. Control of this converter is accomplished through space-vector modulation as illustrated in Fig. 3 [2].

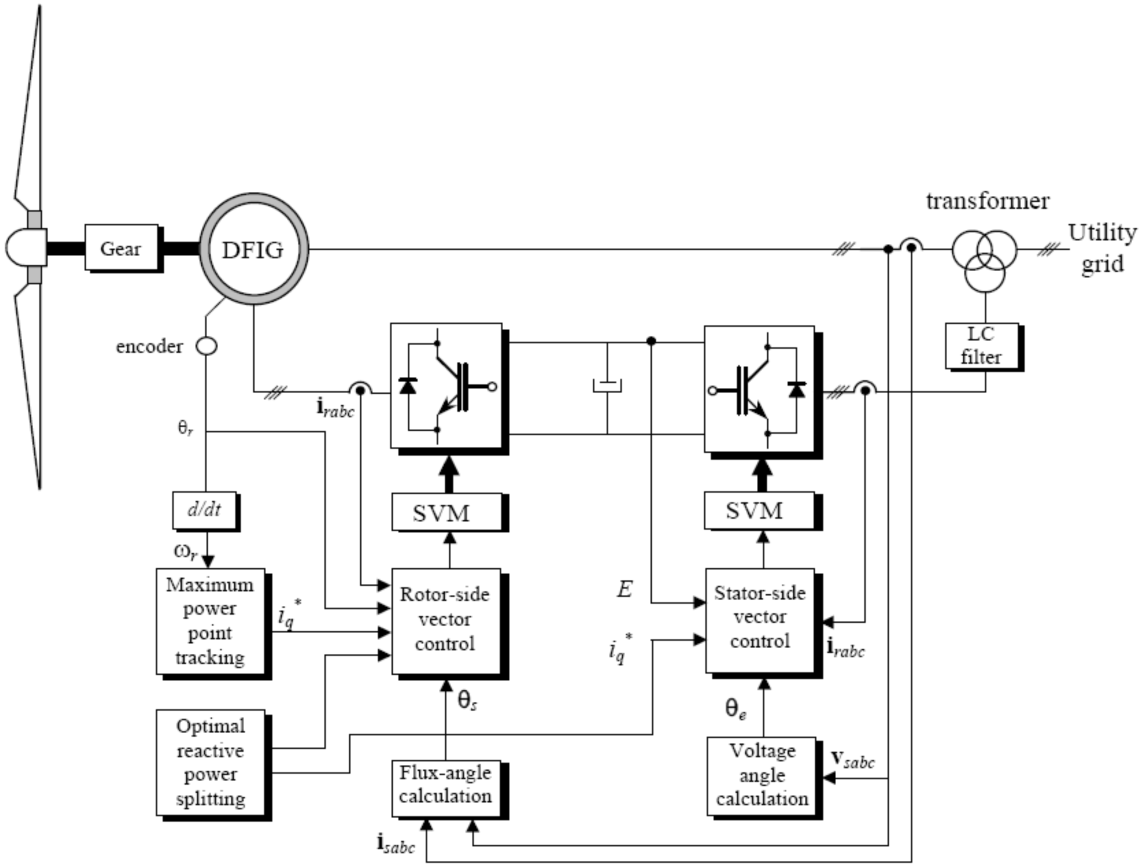


Fig. 3 [2]

In these notes, I want to provide you with enough material to appreciate the main points associated with Dr. Krause's text (other transformations, powerful tool, and general theory) as listed above, together with an appreciation of the relevance of reference frame theory to the analysis of DFIG.

2.0 A&F vs. Krause

We recall from our work earlier in the semester that we used Park's transformation as

$$\underline{P} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \cos\theta & \cos(\theta - 120) & \cos(\theta + 120) \\ \sin\theta & \sin(\theta - 120) & \sin(\theta + 120) \end{bmatrix} \tag{1}$$

This enabled us to transform any a-b-c quantities into a rotating reference frame fixed on the rotor, according to

$$\underline{f}_{0dq} = \underline{P}\,\underline{f}_{abc} \tag{2}$$

Krause uses a slightly different transformation, calling it the "transformation to the arbitrary reference frame," given by

$$\underline{K}_S = \frac{2}{3}\begin{bmatrix} \cos\theta & \cos(\theta-120) & \cos(\theta+120) \\ \sin\theta & \sin(\theta-120) & \sin(\theta+120) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{3}$$

where the angle θ is given by

$$\theta = \int_0^t \omega(\tau)d\tau + \theta(0) \tag{4}$$

It enables us to transform any a-b-c- quantities into a reference frame according to

$$\underline{f}_{qd0} = \underline{K}_S\,\underline{f}_{abc} \tag{5}$$

Krause explains that

> "The connotation of arbitrary stems from the fact that the angular velocity of the transformation is unspecified and can be selected arbitrarily to expedite the solution of the system equations or to satisfy the system constraints."

In this sense, A&F's transformation P is also a "transformation to the arbitrary reference frame." Thus, the angular velocity may be specified to be

- rotor angular velocity (Park), ω_r;
- a stationary reference frame, (Stanley, Clarke), $\omega=0$;
- a synchronously rotating reference frame (Kron), ω_{Re};
- rotor angular velocity, including rotor windings (Brereton), ω_r;

Other significant differences between Krause's transformations and A&F's include:

- The q-axis quantities of Krause are obtained through the row that starts with cosθ, whereas the d-axis quantities of A&F are obtained through the row that starts with cosθ. This is because

Krause has the q-axis 90° ahead of the d-axis, whereas A&F have the d-axis 90° ahead of the q-axis.

- The constants applied to each row differ. This causes Krause's transformation to not be power invariant or orthogonal.

3.0 Krause's generalization of transformations

Krause also provides a table that is useful in seeing the additional transformations that one can use, adapted here as Table 1 below.

Table 1

Reference frame speed	Interpretation
ω (unspecified)	Stationary circuit variables referred to the arbitrary reference frame.
0	Stationary circuit variables referred to the stationary reference frame (Stanley, Clarke).
ω_r or ω_m	Stationary circuit variables referred to a reference frame fixed in the rotor (Park, Brereton).
ω_{Re}	Stationary circuit variables referred to the synchronously rotating reference frame (Kron).

Krause goes on to develop a generalized approach to transforming quantities from any one reference frame to another. If $\underline{K}_S^x$ transforms to reference frame x and $\underline{K}_S^y$ transforms to reference frame y, then ${}^x\underline{K}_S^y$ transforms variables in reference frame y to variables in reference frame x according to

$${}^x\underline{K}_S^y = \underline{K}_S^y \left(\underline{K}_S^x\right)^{-1} \tag{6}$$

4.0 Transformations for induction motors

In this section, we apply Park's transformation to the three-phase induction machine, which was expressed in Chapter 4 as

$$\underline{P} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \cos\theta & \cos(\theta-120) & \cos(\theta+120) \\ \sin\theta & \sin(\theta-120) & \sin(\theta+120) \end{bmatrix}$$

$$\theta = \int_0^t \omega(\gamma)d\gamma + \theta(0)$$

and the inverse transformation was given as

$$\underline{P}^{-1} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \cos\theta & \sin\theta \\ \frac{1}{\sqrt{2}} & \cos(\theta-120) & \sin(\theta-120) \\ \frac{1}{\sqrt{2}} & \cos(\theta+120) & \sin(\theta+120) \end{bmatrix}$$

The angle θ used in Park's transformation is the angle between the rotating d-q reference frame and the a-axis, where the a-axis is fixed on the stator frame and is defined by the location of the phase-a winding. We expressed this angle analytically using

$$\theta = \int_0^t \omega(\gamma)d\gamma + \theta(0)$$

where ω is the rotational speed of the d-q coordinate axes, i.e., the synchronous speed. This transformation will allow us to transform the stator circuit voltage equation to the q-d-0 coordinates.

But for the induction machine, however, we need to apply our transformation to the rotor a-b-c windings in order to obtain the rotor circuit voltage equation in q-d-0 coordinates. In doing so, we observe that whereas the stator phase-a winding (and thus its a-axis) is fixed, the rotor phase-a winding (and thus it's a-axis) rotates. If we apply the $\underline{P}$ transformation to the rotor, we will not account for its rotation, i.e., we will be treating it as if it were fixed. To understand how to handle this issue, consider the Figure 1 below where we show

- θ, the angle between the stator a-axis and the q-axis of the synchronously rotating reference frame;
- θ_m, the angle between the stator a-axis and the rotor a-axis and
- β, the angle between the rotor a-axis and the q-axis of the synchronously rotating reference frame.

Here, the stator a-axis is stationary, the q-d axis rotates at ω, and the rotor a-axis rotates at ω_m. Observe that $\beta=\theta-\theta_m$.

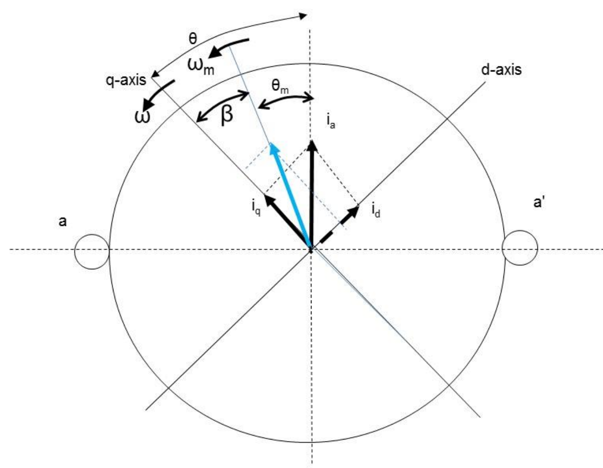


Fig. 1

In Fig. 1, the stator a-axis is stationary, the q-d axis rotates at ω, and the rotor a-axis rotates at ω_m. Consider the i_{ar} vector, in blue, which is coincident with the rotor a-axis. Observe that we may decompose it in the q-d reference frame, but we need to use β instead of θ. Thus, we conclude that the transformation we need for the rotor is exactly like the transformation we used for the stator, except we substitute β for θ, and account for the fact that, to the rotor windings, the q-d coordinate system appears to be moving at $\omega-\omega_m$. Thus, Park's transformation for the rotor winding will be

$$\underline{P}_r = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \cos\beta & \cos(\beta-120) & \cos(\beta+120) \\ \sin\beta & \sin(\beta-120) & \sin(\beta+120) \end{bmatrix}$$

$$\beta = \int_0^t \underbrace{\omega(\gamma)-\omega_m(\gamma)}_{\omega_r} d\gamma + \underbrace{\theta(0)-\theta_m(0)}_{\beta(0)}$$

and the inverse transformation is given as

$$\underline{P}_r^{-1} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \cos\beta & \sin\beta \\ \frac{1}{\sqrt{2}} & \cos(\beta-120) & \sin(\beta-120) \\ \frac{1}{\sqrt{2}} & \cos(\beta+120) & \sin(\beta+120) \end{bmatrix}$$

5.0 Transformed voltage equations for induction machines

We now take the following approach:
Step 1: Write voltage eqts in terms of stator/rotor abc quantities.
Step 2: Multiple through by transformation matrices P and $\underline{P}_r$.
Step 3: Manipulate resulting eqts to write them in terms of stator and rotor qd0 quantities, eliminating stator and rotor abc quantities.
Steps 4a,4b: Express the flux linkages in terms of currents.

Step 1: Our voltage equations are as follows:

$$\begin{bmatrix} v_{as} \\ v_{bs} \\ v_{cs} \\ v_{ar} \\ v_{br} \\ v_{cr} \end{bmatrix} = \begin{bmatrix} r_s & 0 & 0 & 0 & 0 & 0 \\ 0 & r_s & 0 & 0 & 0 & 0 \\ 0 & 0 & r_s & 0 & 0 & 0 \\ 0 & 0 & 0 & r_r & 0 & 0 \\ 0 & 0 & 0 & 0 & r_r & 0 \\ 0 & 0 & 0 & 0 & 0 & r_r \end{bmatrix} \begin{bmatrix} i_{as} \\ i_{bs} \\ i_{cs} \\ i_{ar} \\ i_{br} \\ i_{cr} \end{bmatrix} + \begin{bmatrix} \dot{\lambda}_{as} \\ \dot{\lambda}_{bs} \\ \dot{\lambda}_{cs} \\ \dot{\lambda}_{ar} \\ \dot{\lambda}_{br} \\ \dot{\lambda}_{cr} \end{bmatrix}$$

where

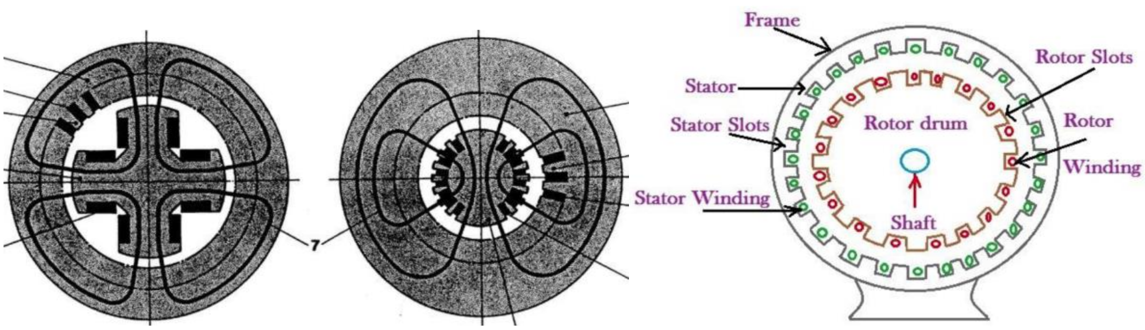
$$\begin{bmatrix} \lambda_{sa} \\ \lambda_{sb} \\ \lambda_{sc} \\ \lambda_{ra} \\ \lambda_{rb} \\ \lambda_{rc} \end{bmatrix} = \begin{bmatrix} \underline{L}_s & \underline{L}_{sr} \\ \underline{L}_{rs} & \underline{L}_r \end{bmatrix} \begin{bmatrix} i_{sa} \\ i_{sb} \\ i_{sc} \\ i_{ra} \\ i_{rb} \\ i_{rc} \end{bmatrix}$$

The individual sub-matrices in the inductance matrix can be developed using considerations similar to those used in developing the inductance matrix for the synchronous machine (see A&F section 4.3 and notes called "WindingsAxes"). An important difference between the synchronous and induction machine is that,

- whereas in the synchronous machine, the movement of the rotor changes the permeance of the path seen by flux linking any of the stator windings so that related inductances (stator-stator self and mutual terms) are a function of rotor position and thus of time;

- in the induction machine, the movement of the rotor does not change the permeance of the path seen by fluxes linking stator windings alone (stator-stator self and mutual terms) or rotor windings alone (rotor-rotor self and mutual terms).

This difference is because the rotor of an induction machine is round with three symmetric windings, in contrast to the synchronous machine which is salient (or round) with a main field winding construction developed to direct flux along the polar axis. Synchronous machine and induction machine structures are illustrated below.



Salient pole synch machine; Round rotor synch machine; Induction machine (wound)

As a result, in the induction machine, the rotor-rotor self and mutual terms, and the stator-stator self and mutual terms, are all constants (independent of rotor position and therefore independent of time). The rotor-rotor and stator-stator matrices are therefore expressed as

$$\underline{L}_r = \begin{bmatrix} l_r + L_{mr} & -\frac{1}{2}L_{mr} & -\frac{1}{2}L_{mr} \\ -\frac{1}{2}L_{mr} & l_r + L_{mr} & -\frac{1}{2}L_{mr} \\ -\frac{1}{2}L_{mr} & -\frac{1}{2}L_{mr} & l_r + L_{mr} \end{bmatrix} \quad \underline{L}_s = \begin{bmatrix} l_s + L_{ms} & -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & l_s + L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} & l_s + L_{ms} \end{bmatrix} \qquad (7a)$$

where l_r and L_{mr} are rotor-side leakage and magnetizing inductances, respectively, and l_s and L_{ms} are stator-side leakage and magnetizing inductances, respectively, so that

- diagonals express self inductance (leakage plus magnetizing);
- off-diagonals express mutual inductance (magnetizing only).

It is different for the rotor-stator mutuals, however, because the positions (as expressed by relative angles) of the rotor-to-stator windings change with rotor movement. Thus, the mutual inductance between any pair of rotor-stator windings is a function of rotor position, as expressed by (7b) below.

$$\underline{L}_{rs} = L_{sr} \begin{bmatrix} \cos\theta_m & \cos(\theta_m - 120) & \cos(\theta_m + 120) \\ \cos(\theta_m + 120) & \cos\theta_m & \cos(\theta_m - 120) \\ \cos(\theta_m - 120) & \cos(\theta_m + 120) & \cos\theta_m \end{bmatrix} = \underline{L}_{sr}^T \qquad (7b)$$

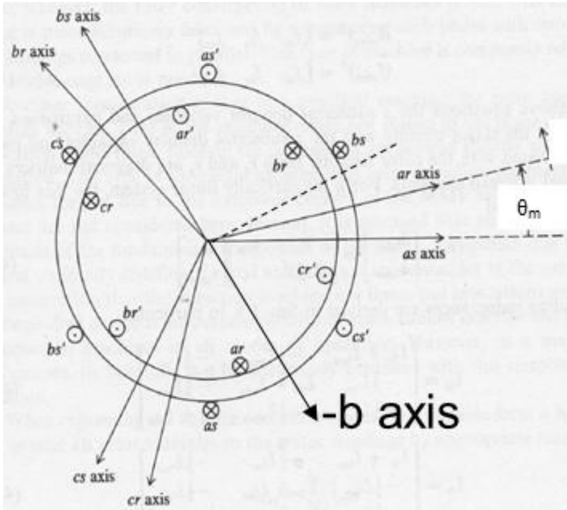
And the matrix of stator-rotor mutuals will just be the transpose:

$$\underline{L}_{sr} = L_{sr} \begin{bmatrix} \cos\theta_m & \cos(\theta_m + 120) & \cos(\theta_m - 120) \\ \cos(\theta_m - 120) & \cos\theta_m & \cos(\theta_m + 120) \\ \cos(\theta_m + 120) & \cos(\theta_m - 120) & \cos\theta_m \end{bmatrix} \qquad (7c)$$

Writing the voltage equations in compact notation:

$$\begin{bmatrix} \underline{v}_{abcs} \\ \underline{v}_{abcr} \end{bmatrix} = \begin{bmatrix} \underline{r}_s & \underline{0} \\ \underline{0} & \underline{r}_r \end{bmatrix} \begin{bmatrix} \underline{i}_{abcs} \\ \underline{i}_{abcr} \end{bmatrix} + \begin{bmatrix} \dot{\underline{\lambda}}_{abcs} \\ \dot{\underline{\lambda}}_{abcr} \end{bmatrix}$$

Some additional comments are provided on the inductance matrices:
- Diagonal elements are the self-inductance of each winding and include leakage plus mutual.
- Off-diagonal elements are mutual inductances between windings and are negative because 120° axis offset between any pair of windings results in flux contributed by one winding to have negative component along the main axis of another winding – see below fig.
- The “1/2” in off-diagonals results from definition of self and mutual inductances, e.g., see. Eqs. 1.5-16 and 1.5-21 and in Krause (pp. 52-53).
- On this page, and in these notes, I have maintained distinct notation for the mutuals between stator windings, L_{ms}, the mutuals between rotor windings L_{mr}, and the mutuals between stator and rotor windings L_{sr}. However, Krause in his book indicates they are equal, i.e., that $L_{ms}=L_{mr}=L_{sr}$ as a result of referring all quantities to the stator (see pp. 144-145 of his 2002 edition). I adopt this latter approach (that they are all equal) on slide 37 and also in my notes on (slide 8 of) torque and power.



Step 2: Now let's apply our d-q transformations:

$$\underbrace{\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{P}_r \end{bmatrix}\begin{bmatrix} \underline{v}_{abcs} \\ \underline{v}_{abcr} \end{bmatrix}}_{Term1} = \underbrace{\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{P}_r \end{bmatrix}\begin{bmatrix} \underline{r}_s & \underline{0} \\ \underline{0} & \underline{r}_r \end{bmatrix}\begin{bmatrix} \underline{i}_{abcs} \\ \underline{i}_{abcr} \end{bmatrix}}_{Term2} + \underbrace{\begin{bmatrix} \underline{P} & \underline{0} \\ \underline{0} & \underline{P}_r \end{bmatrix}\begin{bmatrix} \underline{\dot{\lambda}}_{abcs} \\ \underline{\dot{\lambda}}_{abcr} \end{bmatrix}}_{Term3}$$

Performing the multiplications for terms 1, 2, and 3, we obtain

$$\underbrace{\begin{bmatrix} \underline{v}_{0dqs} \\ \underline{v}_{0dqr} \end{bmatrix}}_{Term1} = \underbrace{\begin{bmatrix} \underline{P}\underline{r}_s & \underline{0} \\ \underline{0} & \underline{P}_r\underline{r}_r \end{bmatrix}\begin{bmatrix} \underline{i}_{abcs} \\ \underline{i}_{abcr} \end{bmatrix}}_{Term2} + \underbrace{\begin{bmatrix} \underline{P}\underline{\dot{\lambda}}_{abcs} \\ \underline{P}_r\underline{\dot{\lambda}}_{abcr} \end{bmatrix}}_{Term3} \tag{8}$$

Step 3: Using the inverse Park's transformation, we can write

$$\begin{bmatrix} \underline{i}_{abcs} \\ \underline{i}_{abcr} \end{bmatrix} = \begin{bmatrix} \underline{P}^{-1} & \underline{0} \\ \underline{0} & \underline{P}_r^{-1} \end{bmatrix}\begin{bmatrix} \underline{i}_{0dqs} \\ \underline{i}_{0dqr} \end{bmatrix}$$

Now substitute into (8) to obtain

$$\underbrace{\begin{bmatrix} \underline{v}_{0dqs} \\ \underline{v}_{0dqr} \end{bmatrix}}_{Term1} = \underbrace{\begin{bmatrix} \underline{P}\underline{r}_s & \underline{0} \\ \underline{0} & \underline{P}_r\underline{r}_r \end{bmatrix}\begin{bmatrix} \underline{P}^{-1} & \underline{0} \\ \underline{0} & \underline{P}_r^{-1} \end{bmatrix}\begin{bmatrix} \underline{i}_{0dqs} \\ \underline{i}_{0dqr} \end{bmatrix}}_{Term2} + \underbrace{\begin{bmatrix} \underline{P}\underline{\dot{\lambda}}_{abcs} \\ \underline{P}_r\underline{\dot{\lambda}}_{abcr} \end{bmatrix}}_{Term3}$$

Performing the matrix multiplication for term 2, we obtain:

$$\underbrace{\begin{bmatrix} \underline{v}_{0dqs} \\ \underline{v}_{0dqr} \end{bmatrix}}_{Term1} = \underbrace{\begin{bmatrix} \underline{P}\underline{r}_s\underline{P}^{-1} & \underline{0} \\ \underline{0} & \underline{P}_r\underline{r}_r\underline{P}_r^{-1} \end{bmatrix}\begin{bmatrix} \underline{i}_{0dqs} \\ \underline{i}_{0dqr} \end{bmatrix}}_{Term2} + \underbrace{\begin{bmatrix} \underline{P}\underline{\dot{\lambda}}_{abcs} \\ \underline{P}_r\underline{\dot{\lambda}}_{abcr} \end{bmatrix}}_{Term3} \tag{9}$$

The term 2 matrix elements collapse to $\underline{r}_s$ and $\underline{r}_r$, respectively, because $\underline{KRK}^{-1}=\underline{R}$ if $\underline{R}$ is diagonal having equal diagonal elements.

This follows from $\underline{KRK}^{-1}=\underline{K}r\underline{UK}^{-1}=r\underline{KUK}^{-1}=r\underline{KK}^{-1}=r\underline{U}=\underline{R}$, where r is the scalar value of each element in $\underline{R}$, and $\underline{U}$ is the identity matrix. Therefore (9) becomes:

$$\underbrace{\begin{bmatrix} \underline{v}_{0dqs} \\ \underline{v}_{0dqr} \end{bmatrix}}_{Term1} = \underbrace{\begin{bmatrix} \underline{r}_s & \underline{0} \\ \underline{0} & \underline{r}_r \end{bmatrix}\begin{bmatrix} \underline{i}_{0dqs} \\ \underline{i}_{0dqr} \end{bmatrix}}_{Term2} + \underbrace{\begin{bmatrix} \underline{P}\dot{\underline{\lambda}}_{abcs} \\ \underline{P}_r\dot{\underline{\lambda}}_{abcr} \end{bmatrix}}_{Term3} \tag{10}$$

Focusing for the moment on just the top element in the term 3 vector, consider the transformation of stator flux linkages according to

$$\underline{\lambda}_{0dqs} = \underline{P}\underline{\lambda}_{abcs}$$

Use the chain rule to differentiate both sides:

$$\dot{\underline{\lambda}}_{0dqs} = \underline{P}\dot{\underline{\lambda}}_{abcs} + \dot{\underline{P}}\underline{\lambda}_{abcs}$$

Now solve for the first term on the right-hand-side:

$$\underline{P}\dot{\underline{\lambda}}_{abcs} = \dot{\underline{\lambda}}_{0dqs} - \dot{\underline{P}}\underline{\lambda}_{abcs}$$

Use $\underline{\lambda}_{abcs} = \underline{P}^{-1}\underline{\lambda}_{qd0s}$ on the right-hand-side:

$$\underline{P}\dot{\underline{\lambda}}_{abcs} = \dot{\underline{\lambda}}_{0dqs} - \dot{\underline{P}}\underline{P}^{-1}\underline{\lambda}_{0dqs} \tag{11a}$$

A similar process for the rotor quantities results in

$$\underline{P}_r\dot{\underline{\lambda}}_{abcr} = \dot{\underline{\lambda}}_{0dqr} - \dot{\underline{P}}_r\underline{P}_r^{-1}\underline{\lambda}_{0dqr} \tag{11b}$$

Substituting these (11a) and (11b) into (10) yields:

$$\underbrace{\begin{bmatrix} \underline{v}_{0dqs} \\ \underline{v}_{0dqr} \end{bmatrix}}_{Term1} = \underbrace{\begin{bmatrix} \underline{r}_s & \underline{0} \\ \underline{0} & \underline{r}_r \end{bmatrix}\begin{bmatrix} \underline{i}_{0dqs} \\ \underline{i}_{0dqr} \end{bmatrix}}_{Term2} + \underbrace{\begin{bmatrix} \dot{\underline{\lambda}}_{0dqs} \\ \dot{\underline{\lambda}}_{0dqr} \end{bmatrix} - \begin{bmatrix} \dot{\underline{P}}\underline{P}^{-1}\underline{\lambda}_{0dqs} \\ \dot{\underline{P}}_r\underline{P}_r^{-1}\underline{\lambda}_{0dqr} \end{bmatrix}}_{Term3} \tag{12}$$

This provides that both stator and rotor equations are expressed in terms of qd0 quantities instead of abc quantities.

Step 4a: One thing remains, in that we have both currents and flux linkages as state variables. Let’s express the flux linkages in terms of currents, resulting in a state-space equation in current variables.

To do this, we express flux linkage in terms of current according to:

$$\begin{bmatrix}\underline{\lambda}_{abcs}\\ \underline{\lambda}_{abcr}\end{bmatrix}=\begin{bmatrix}\underline{L}_{s} & \underline{L}_{sr}\\ \underline{L}_{rs} & \underline{L}_{r}\end{bmatrix}\begin{bmatrix}\underline{i}_{abcs}\\ \underline{i}_{abcr}\end{bmatrix} \tag{13}$$

Now use Park's transformation to replace the abc currents in (13) by their qd0 values:

$$\begin{bmatrix}\underline{\lambda}_{abcs}\\ \underline{\lambda}_{abcr}\end{bmatrix}=\begin{bmatrix}\underline{L}_{s} & \underline{L}_{sr}\\ \underline{L}_{rs} & \underline{L}_{r}\end{bmatrix}\begin{bmatrix}\underline{P}^{-1} & 0\\ 0 & \underline{P}_r^{-1}\end{bmatrix}\begin{bmatrix}\underline{i}_{0dqs}\\ \underline{i}_{0dqr}\end{bmatrix} \tag{14}$$

Then (14) can be used in the transformation relation for flux linkages, i.e.,

$$\underbrace{\begin{bmatrix}\underline{\lambda}_{0dqs}\\ \underline{\lambda}_{0dqr}\end{bmatrix}=\begin{bmatrix}\underline{P} & 0\\ 0 & \underline{P}_r\end{bmatrix}\begin{bmatrix}\underline{\lambda}_{abcs}\\ \underline{\lambda}_{abcr}\end{bmatrix}}_{\text{Transformation Relation for flux linkages}}=\begin{bmatrix}\underline{P} & 0\\ 0 & \underline{P}_r\end{bmatrix}\begin{bmatrix}\underline{L}_{s} & \underline{L}_{sr}\\ \underline{L}_{rs} & \underline{L}_{r}\end{bmatrix}\begin{bmatrix}\underline{P}^{-1} & 0\\ 0 & \underline{P}_r^{-1}\end{bmatrix}\begin{bmatrix}\underline{i}_{0dqs}\\ \underline{i}_{0dqr}\end{bmatrix} \tag{15}$$

Performing the matrix multiplication on the right-hand-side:

$$\begin{bmatrix}\underline{\lambda}_{0dqs}\\ \underline{\lambda}_{0dqr}\end{bmatrix}=\begin{bmatrix}\underline{P}\,\underline{L}_{s}\,\underline{P}^{-1} & \underline{P}\,\underline{L}_{sr}\,\underline{P}_r^{-1}\\ \underline{P}_r\,\underline{L}_{rs}\,\underline{P}^{-1} & \underline{P}_r\,\underline{L}_{r}\,\underline{P}_r^{-1}\end{bmatrix}\begin{bmatrix}\underline{i}_{0dqs}\\ \underline{i}_{0dqr}\end{bmatrix} \tag{16}$$

Recall the inductance matrices given in (7a), (7b), and (7c), repeated here for convenience:

$$\underline{L}_r=\begin{bmatrix} l_r+L_{mr} & -\frac{1}{2}L_{mr} & -\frac{1}{2}L_{mr}\\ -\frac{1}{2}L_{mr} & l_r+L_{mr} & -\frac{1}{2}L_{mr}\\ -\frac{1}{2}L_{mr} & -\frac{1}{2}L_{mr} & l_r+L_{mr}\end{bmatrix}\quad \underline{L}_s=\begin{bmatrix} l_s+L_{ms} & -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms}\\ -\frac{1}{2}L_{ms} & l_s+L_{ms} & -\frac{1}{2}L_{ms}\\ -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} & l_s+L_{ms}\end{bmatrix} \tag{7a}$$

$$\underline{L}_{rs}=L_{sr}\begin{bmatrix}\cos\theta_m & \cos(\theta_m-120) & \cos(\theta_m+120)\\ \cos(\theta_m+120) & \cos\theta_m & \cos(\theta_m-120)\\ \cos(\theta_m-120) & \cos(\theta_m+120) & \cos\theta_m\end{bmatrix}=\underline{L}_{sr}^{T} \tag{7b}$$

$$\underline{L}_{sr}=L_{sr}\begin{bmatrix}\cos\theta_m & \cos(\theta_m+120) & \cos(\theta_m-120)\\ \cos(\theta_m-120) & \cos\theta_m & \cos(\theta_m+120)\\ \cos(\theta_m+120) & \cos(\theta_m-120) & \cos\theta_m\end{bmatrix} \tag{7c}$$

We will not take time to evaluate each of the terms in (16), but we will rather just provide the results, which are as follows.

$$\underline{P}\,\underline{L}_s\,\underline{P}^{-1} = \begin{bmatrix} l_s + M & 0 & 0 \\ 0 & l_s + M & 0 \\ 0 & 0 & l_s \end{bmatrix} \equiv \underline{L}_{s0dq}$$

$$\underline{P}_s\,\underline{L}_{sr}\,\underline{P}_r^{-1} = \underline{P}_r\,\underline{L}_{rs}\,\underline{P}^{-1} = \begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv \underline{L}_{m0dq} \qquad (17)$$

$$\underline{P}_r\,\underline{L}_r\,\underline{P}_r^{-1} = \begin{bmatrix} l_r + M & 0 & 0 \\ 0 & l_r + M & 0 \\ 0 & 0 & l_r \end{bmatrix} \equiv \underline{L}_{r0dq}$$

One comment here: Krause indicates that $L_M=3/2(L_{ms})=3/2(L_{mr})$, ch. 4.5, pp. 150 of his 2002 edition. I have renamed L_M as "M". Also, see last bullet in comments on p. 12 of these notes.

Substituting the expressions of (17), as denoted by the right-hand-side nomenclature, into (16), we have

$$\begin{bmatrix} \underline{\lambda}_{0dqs} \\ \underline{\lambda}_{0dqr} \end{bmatrix} = \begin{bmatrix} \underline{P}\,\underline{L}_s\,\underline{P}^{-1} & \underline{P}\,\underline{L}_{sr}\,\underline{P}_r^{-1} \\ \underline{P}_r\,\underline{L}_{rs}\,\underline{P}^{-1} & \underline{P}_r\,\underline{L}_r\,\underline{P}_r^{-1} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dqs} \\ \underline{i}_{0dqr} \end{bmatrix} \qquad (16)$$

$$\begin{bmatrix} \underline{\lambda}_{0dqs} \\ \underline{\lambda}_{0dqr} \end{bmatrix} = \begin{bmatrix} \underline{L}_{s0dq} & \underline{L}_{m0dq} \\ \underline{L}_{m0dq} & \underline{L}_{r0dq} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dqs} \\ \underline{i}_{0dqr} \end{bmatrix} \qquad (18)$$

Since from (17) our inductance matrix is constant, we can easily differentiate (18) to obtain

$$\begin{bmatrix} \dot{\underline{\lambda}}_{0dqs} \\ \dot{\underline{\lambda}}_{0dqr} \end{bmatrix} = \begin{bmatrix} \underline{L}_{s0dq} & \underline{L}_{m0dq} \\ \underline{L}_{m0dq} & \underline{L}_{r0dq} \end{bmatrix} \begin{bmatrix} \dot{\underline{i}}_{0dqs} \\ \dot{\underline{i}}_{0dqr} \end{bmatrix} \qquad (19)$$

Now we can substitute the above expression (19) for flux linkage derivatives into our voltage equation (12):

$$\underbrace{\begin{bmatrix} \underline{v}_{0dqs} \\ \underline{v}_{0dqr} \end{bmatrix}}_{Term1} = \underbrace{\begin{bmatrix} \underline{r}_s & \underline{0} \\ \underline{0} & \underline{r}_r \end{bmatrix} \begin{bmatrix} \underline{i}_{0dqs} \\ \underline{i}_{0dqr} \end{bmatrix}}_{Term2} + \underbrace{\begin{bmatrix} \dot{\underline{\lambda}}_{0dqs} \\ \dot{\underline{\lambda}}_{0dqr} \end{bmatrix} - \begin{bmatrix} \dot{\underline{P}}\,\underline{P}^{-1}\,\underline{\lambda}_{0dqs} \\ \dot{\underline{P}}_r\,\underline{P}_r^{-1}\,\underline{\lambda}_{0dqr} \end{bmatrix}}_{Term3} \qquad (12)$$

resulting in

$$\begin{bmatrix} \underline{v}_{0dqs} \\ \underline{v}_{0dqr} \end{bmatrix} = \begin{bmatrix} \underline{r}_s & \underline{0} \\ \underline{0} & \underline{r}_r \end{bmatrix} \begin{bmatrix} \underline{i}_{0dqs} \\ \underline{i}_{0dqr} \end{bmatrix} + \begin{bmatrix} \underline{L}_{s0dq} & \underline{L}_{m0dq} \\ \underline{L}_{m0dq} & \underline{L}_{r0dq} \end{bmatrix} \begin{bmatrix} \dot{\underline{i}}_{0dqs} \\ \dot{\underline{i}}_{0dqr} \end{bmatrix} - \begin{bmatrix} \dot{\underline{P}}\,\underline{P}^{-1}\,\underline{\lambda}_{0dqs} \\ \dot{\underline{P}}_r\,\underline{P}_r^{-1}\,\underline{\lambda}_{0dqr} \end{bmatrix} \qquad (20)$$

Step 4b: This still leaves us with flux linkages in the last term of (20). Addressing this is step 4b. We will take this step with two sub-steps.

A. Obtain $\dot{\underline{P}}\underline{P}^{-1}$ *and* $\dot{\underline{P}}_r\underline{P}_r^{-1}$

B. Express individual qd0 elements of $\underline{\lambda}_{qd0s}$ *and* $\underline{\lambda}_{qd0r}$ *in terms of qd0 currents.*

A. Step 4b-A - Obtain $\dot{\underline{P}}\underline{P}^{-1}$ *and* $\dot{\underline{P}}_r\underline{P}_r^{-1}$*:*

To obtain the first product, we express $\underline{P}$ and $\underline{P}^{-1}$:

$$\underline{P}=\sqrt{\frac{2}{3}}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \cos\theta & \cos(\theta-120) & \cos(\theta+120)\\ \sin\theta & \sin(\theta-120) & \sin(\theta+120)\end{bmatrix}$$

$$\underline{P}^{-1}=\sqrt{\frac{2}{3}}\begin{bmatrix}\frac{1}{\sqrt{2}} & \cos\theta & \sin\theta\\ \frac{1}{\sqrt{2}} & \cos(\theta-120) & \sin(\theta-120)\\ \frac{1}{\sqrt{2}} & \cos(\theta+120) & \sin(\theta+120)\end{bmatrix}$$

To obtain $\dot{\underline{P}}$, we need the derivative of θ. We recall the definition of θ, which is

$$\theta=\int_0^t\omega(\gamma)d\gamma+\theta(0)$$

By the fundamental theorem of calculus, we can write that

$$\dot{\theta}(t)=\omega$$

Therefore

$$\dot{\underline{P}}=\sqrt{\frac{2}{3}}\omega\begin{bmatrix}0 & 0 & 0\\ -\sin\theta & -\sin(\theta-120) & -\sin(\theta+120)\\ \cos\theta & \cos(\theta-120) & \cos(\theta+120)\end{bmatrix} \tag{21}$$

Likewise, to obtain the second product, we express $\underline{P}_r$ and $\underline{P}_r^{-1}$:

$$\underline{P}_r=\sqrt{\frac{2}{3}}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \cos\beta & \cos(\beta-120) & \cos(\beta+120)\\ \sin\beta & \sin(\beta-120) & \sin(\beta+120)\end{bmatrix}$$

$$\underline{P}_r^{-1} = \sqrt{\frac{2}{3}}\begin{bmatrix} \frac{1}{\sqrt{2}} & \cos\beta & \sin\beta \\ \frac{1}{\sqrt{2}} & \cos(\beta-120) & \sin(\beta-120) \\ \frac{1}{\sqrt{2}} & \cos(\beta+120) & \sin(\beta+120) \end{bmatrix}$$

we recall that

$$\beta = \int_0^t \underbrace{\omega(\gamma) - \omega_m(\gamma)}_{\omega_r} d\gamma + \underbrace{\theta(0) - \theta_m(0)}_{\beta(0)}$$

By the fundamental theorem of calculus, we can write that

$$\dot{\beta}(t) = \omega - \omega_m$$

Therefore

$$\underline{\dot{P}}_r = \sqrt{\frac{2}{3}}(\omega-\omega_m)\begin{bmatrix} 0 & 0 & 0 \\ -\sin\beta & -\sin(\beta-120) & -\sin(\beta+120) \\ \cos\beta & \cos(\beta-120) & \cos(\beta+120) \end{bmatrix}$$

Now we can obtain $\underline{\dot{P}}\underline{P}^{-1}$ and $\underline{\dot{P}}_r\underline{P}_r^{-1}$

$$\underline{\dot{P}}\underline{P}^{-1} = \sqrt{\frac{2}{3}}\omega\begin{bmatrix} 0 & 0 & 0 \\ -\sin\theta & -\sin(\theta-120) & -\sin(\theta+120) \\ \cos\theta & \cos(\theta-120) & \cos(\theta+120) \end{bmatrix}\sqrt{\frac{2}{3}}\begin{bmatrix} \frac{1}{\sqrt{2}} & \cos\theta & \sin\theta \\ \frac{1}{\sqrt{2}} & \cos(\theta-120) & \sin(\theta-120) \\ \frac{1}{\sqrt{2}} & \cos(\theta+120) & \sin(\theta+120) \end{bmatrix} \quad (22)$$

$$= \frac{2}{3}\omega\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{2} \\ 0 & \frac{3}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix}$$

$$\underline{\dot{P}}_r\underline{P}_r^{-1} = \sqrt{\frac{2}{3}}(\omega-\omega_m)\begin{bmatrix} 0 & 0 & 0 \\ -\sin\beta & -\sin(\beta-120) & -\sin(\beta+120) \\ \cos\beta & \cos(\beta-120) & \cos(\beta+120) \end{bmatrix}\sqrt{\frac{2}{3}}\begin{bmatrix} \frac{1}{\sqrt{2}} & \cos\beta & \sin\beta \\ \frac{1}{\sqrt{2}} & \cos(\beta-120) & \sin(\beta-120) \\ \frac{1}{\sqrt{2}} & \cos(\beta+120) & \sin(\beta+120) \end{bmatrix} \quad (23)$$

$$= \frac{2}{3}(\omega-\omega_m)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{2} \\ 0 & \frac{3}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -(\omega-\omega_m) \\ 0 & (\omega-\omega_m) & 0 \end{bmatrix}$$

Now substitute (22) and (23) into the last term of the voltage equations (20), repeated here for convenience:

$$\begin{bmatrix} \underline{v}_{0dqs} \\ \underline{v}_{0dqr} \end{bmatrix} = \begin{bmatrix} \underline{r}_s & \underline{0} \\ \underline{0} & \underline{r}_r \end{bmatrix} \begin{bmatrix} \underline{i}_{0dqs} \\ \underline{i}_{0dqr} \end{bmatrix} + \begin{bmatrix} \underline{L}_{s0dq} & \underline{L}_{m0dq} \\ \underline{L}_{m0dq} & \underline{L}_{r0dq} \end{bmatrix} \begin{bmatrix} \dot{\underline{i}}_{0dqs} \\ \dot{\underline{i}}_{0dqr} \end{bmatrix} - \underbrace{\begin{bmatrix} \dot{\underline{P}}\underline{P}^{-1}\underline{\lambda}_{0dqs} \\ \dot{\underline{P}}_r\underline{P}_r^{-1}\underline{\lambda}_{0dqr} \end{bmatrix}}_{InsertHere} \tag{20}$$

Fully expanding (20), we obtain (note that here I have re-ordered the voltage and current vectors – I need to change this to make it consistent with what we did for synchronous machines):

$$\begin{bmatrix} v_{0s} \\ v_{ds} \\ v_{qs} \\ v_{0r} \\ v_{dr} \\ v_{qr} \end{bmatrix} = \begin{bmatrix} r_s & 0 & 0 & 0 & 0 & 0 \\ 0 & r_s & 0 & 0 & 0 & 0 \\ 0 & 0 & r_s & 0 & 0 & 0 \\ 0 & 0 & 0 & r_r & 0 & 0 \\ 0 & 0 & 0 & 0 & r_r & 0 \\ 0 & 0 & 0 & 0 & 0 & r_r \end{bmatrix} \begin{bmatrix} i_{0s} \\ i_{ds} \\ i_{qs} \\ i_{0r} \\ i_{dr} \\ i_{qr} \end{bmatrix} + \begin{bmatrix} l_s & 0 & 0 & 0 & 0 & 0 \\ 0 & l_s+M & 0 & 0 & M & 0 \\ 0 & 0 & l_s+M & 0 & 0 & M \\ 0 & 0 & 0 & l_r & 0 & 0 \\ 0 & M & 0 & 0 & l_r+M & 0 \\ 0 & 0 & M & 0 & 0 & l_r+M \end{bmatrix} \begin{bmatrix} \dot{i}_{0s} \\ \dot{i}_{ds} \\ \dot{i}_{qs} \\ \dot{i}_{0r} \\ \dot{i}_{dr} \\ \dot{i}_{qr} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 & 0 & 0 \\ 0 & -\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega-\omega_m \\ 0 & 0 & 0 & 0 & -(\omega-\omega_m) & 0 \end{bmatrix} \begin{bmatrix} \lambda_{0s} \\ \lambda_{ds} \\ \lambda_{qs} \\ \lambda_{0r} \\ \lambda_{dr} \\ \lambda_{qr} \end{bmatrix} \tag{24}$$

Note the “Speed voltages” in the

- second equation, $\omega\lambda_{qs}$
- third equation, $-\omega\lambda_{ds}$
- fifth equation, $(\omega- \omega_m)\lambda_{qr}$, and
- sixth equation, $-(\omega- \omega_m) \lambda_{ds}$.

Comments on speed voltages: $\omega\lambda_{qs}$, $-\omega\lambda_{ds}$, $(\omega- \omega_m)\lambda_{qr}$, $-(\omega- \omega_m) \lambda_{ds}$:

- These *speed voltages* represent the fact that a rotating flux wave will create voltages in windings that are stationary relative to that flux wave.
- Speed voltages are so named to contrast them from what may be called *transformer voltages*, which are induced as a result of a time varying magnetic field.
- You may have run across the concept of “speed voltages” in Physics, where you computed a voltage induced in a coil of wire as it moved through a static magnetic field, in which case, you may have used the equation Blv where B is flux density, l is conductor length, and v is the component of the velocity of the moving conductor (or moving field) that is normal with respect to the field flux direction (or conductor).
- The first speed voltage term, $\omega\lambda_{qs}$, appears in the v_{ds} equation. The second speed voltage term, $-\omega\lambda_{ds}$, appears in the v_{qs} equation. Thus, we see that the d-axis flux causes a speed voltage in the q-axis winding, and the q-axis flux causes a speed voltage in the d-axis winding. A similar thing is true for the rotor winding.

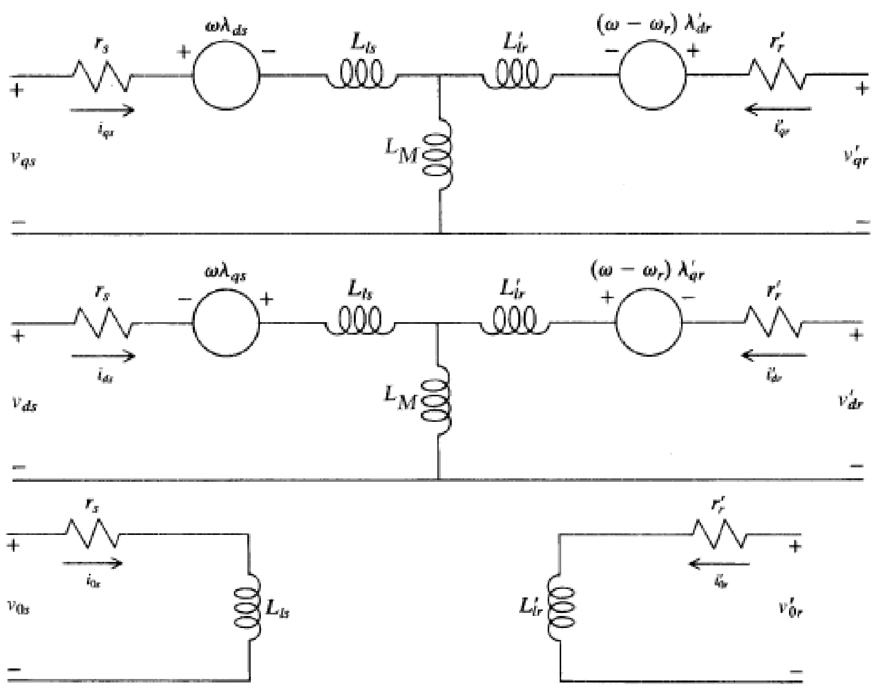
The voltage equation as we now have it is:

$$\begin{bmatrix} v_{0s}\\ v_{ds}\\ v_{qs}\\ v_{0r}\\ v_{dr}\\ v_{qr}\end{bmatrix} = \begin{bmatrix} r_s&0&0&0&0&0\\ 0&r_s&0&0&0&0\\ 0&0&r_s&0&0&0\\ 0&0&0&r_r&0&0\\ 0&0&0&0&r_r&0\\ 0&0&0&0&0&r_r\end{bmatrix}\begin{bmatrix} i_{0s}\\ i_{ds}\\ i_{qs}\\ i_{0r}\\ i_{dr}\\ i_{qr}\end{bmatrix} + \begin{bmatrix} l_s&0&0&0&0&0\\ 0&l_s+M&0&0&M&0\\ 0&0&l_s+M&0&0&M\\ 0&0&0&l_r&0&0\\ 0&M&0&0&l_r+M&0\\ 0&0&M&0&0&l_r+M\end{bmatrix}\begin{bmatrix} \dot{i}_{0s}\\ \dot{i}_{ds}\\ \dot{i}_{qs}\\ \dot{i}_{0r}\\ \dot{i}_{dr}\\ \dot{i}_{qr}\end{bmatrix} - \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&\omega&0&0&0\\ 0&-\omega&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&\omega-\omega_m\\ 0&0&0&0&-(\omega-\omega_m)&0\end{bmatrix}\begin{bmatrix} \lambda_{0s}\\ \lambda_{ds}\\ \lambda_{qs}\\ \lambda_{0r}\\ \lambda_{dr}\\ \lambda_{qr}\end{bmatrix}$$

Let's collapse the last matrix-vector product by performing the multiplication, resulting in:

$$\begin{bmatrix} v_{0s}\\ v_{ds}\\ v_{qs}\\ v_{0r}\\ v_{dr}\\ v_{qr}\end{bmatrix} = \begin{bmatrix} r_s&0&0&0&0&0\\ 0&r_s&0&0&0&0\\ 0&0&r_s&0&0&0\\ 0&0&0&r_r&0&0\\ 0&0&0&0&r_r&0\\ 0&0&0&0&0&r_r\end{bmatrix}\begin{bmatrix} i_{0s}\\ i_{ds}\\ i_{qs}\\ i_{0r}\\ i_{dr}\\ i_{qr}\end{bmatrix} + \begin{bmatrix} l_s&0&0&0&0&0\\ 0&l_s+M&0&0&M&0\\ 0&0&l_s+M&0&0&M\\ 0&0&0&l_r&0&0\\ 0&M&0&0&l_r+M&0\\ 0&0&M&0&0&l_r+M\end{bmatrix}\begin{bmatrix} \dot{i}_{0s}\\ \dot{i}_{ds}\\ \dot{i}_{qs}\\ \dot{i}_{0r}\\ \dot{i}_{dr}\\ \dot{i}_{qr}\end{bmatrix} - \begin{bmatrix} 0\\ \omega\lambda_{qs}\\ -\omega\lambda_{ds}\\ 0\\ (\omega-\omega_m)\lambda_{qr}\\ -(\omega-\omega_m)\lambda_{dr}\end{bmatrix} \qquad (25)$$

Krause, on p. 151 of his 2002 edition, provides equivalent circuits that "fit" these equations, as follows:



Here, L_M=M, and L_{ls} and L_{lr} are l_s and l_r, respectively, and the primed notation indicates quantities referred to the stator, which we have assumed in the development of these notes.

B. Express individual 0dq elements of $\underline{\lambda}_{0dqs}$ and $\underline{\lambda}_{0dqr}$ in terms of 0dq currents.

Equation (18) is repeated here for convenience:

$$\begin{bmatrix} \underline{\lambda}_{0dqs} \\ \underline{\lambda}_{0dqr} \end{bmatrix} = \begin{bmatrix} \underline{L}_{s0dq} & \underline{L}_{m0dq} \\ \underline{L}_{m0dq} & \underline{L}_{r0dq} \end{bmatrix} \begin{bmatrix} \underline{i}_{0dqs} \\ \underline{i}_{0dqr} \end{bmatrix} \tag{18}$$

which can be expanded according to

$$\begin{bmatrix} \lambda_{0s} \\ \lambda_{ds} \\ \lambda_{qs} \\ \lambda_{0r} \\ \lambda_{dr} \\ \lambda_{qr} \end{bmatrix} = \begin{bmatrix} l_s & 0 & 0 & 0 & 0 & 0 \\ 0 & l_s+M & 0 & 0 & M & 0 \\ 0 & 0 & l_s+M & 0 & 0 & M \\ 0 & 0 & 0 & l_r & 0 & 0 \\ 0 & M & 0 & 0 & l_r+M & 0 \\ 0 & 0 & M & 0 & 0 & l_r+M \end{bmatrix} \begin{bmatrix} i_{0s} \\ i_{ds} \\ i_{qs} \\ i_{0r} \\ i_{dr} \\ i_{qr} \end{bmatrix}$$

From the above, we observe[1][2]

$$\lambda_{qs} = \left(l_s+M\right)i_{qs} + Mi_{qr} \qquad \lambda_{qr} = Mi_{qs} + \left(l_r+M\right)i_{qr}$$

$$\lambda_{ds} = \left(l_s+M\right)i_{ds} + Mi_{dr} \qquad \lambda_{dr} = Mi_{ds} + \left(l_r+M\right)i_{dr}$$

Now substitute these terms into (25), repeated here for convenience:

$$\begin{bmatrix} v_{0s} \\ v_{ds} \\ v_{qs} \\ v_{0r} \\ v_{dr} \\ v_{qr} \end{bmatrix} = \begin{bmatrix} r_s & 0 & 0 & 0 & 0 & 0 \\ 0 & r_s & 0 & 0 & 0 & 0 \\ 0 & 0 & r_s & 0 & 0 & 0 \\ 0 & 0 & 0 & r_r & 0 & 0 \\ 0 & 0 & 0 & 0 & r_r & 0 \\ 0 & 0 & 0 & 0 & 0 & r_r \end{bmatrix} \begin{bmatrix} i_{0s} \\ i_{ds} \\ i_{qs} \\ i_{0r} \\ i_{dr} \\ i_{qr} \end{bmatrix} + \begin{bmatrix} l_s & 0 & 0 & 0 & 0 & 0 \\ 0 & l_s+M & 0 & 0 & M & 0 \\ 0 & 0 & l_s+M & 0 & 0 & M \\ 0 & 0 & 0 & l_r & 0 & 0 \\ 0 & M & 0 & 0 & l_r+M & 0 \\ 0 & 0 & M & 0 & 0 & l_r+M \end{bmatrix} \begin{bmatrix} \dot{i}_{0s} \\ \dot{i}_{ds} \\ \dot{i}_{qs} \\ \dot{i}_{0r} \\ \dot{i}_{dr} \\ \dot{i}_{qr} \end{bmatrix} - \begin{bmatrix} 0 \\ \omega\lambda_{qs} \\ -\omega\lambda_{ds} \\ 0 \\ (\omega-\omega_m)\lambda_{qr} \\ -(\omega-\omega_m)\lambda_{dr} \end{bmatrix} \tag{25}$$

and we obtain:

$$\begin{bmatrix} v_{0s} \\ v_{ds} \\ v_{qs} \\ v_{0r} \\ v_{dr} \\ v_{qr} \end{bmatrix} = \begin{bmatrix} r_s & 0 & 0 & 0 & 0 & 0 \\ 0 & r_s & 0 & 0 & 0 & 0 \\ 0 & 0 & r_s & 0 & 0 & 0 \\ 0 & 0 & 0 & r_r & 0 & 0 \\ 0 & 0 & 0 & 0 & r_r & 0 \\ 0 & 0 & 0 & 0 & 0 & r_r \end{bmatrix} \begin{bmatrix} i_{0s} \\ i_{ds} \\ i_{qs} \\ i_{0r} \\ i_{dr} \\ i_{qr} \end{bmatrix} + \begin{bmatrix} l_s & 0 & 0 & 0 & 0 & 0 \\ 0 & l_s+M & 0 & 0 & M & 0 \\ 0 & 0 & l_s+M & 0 & 0 & M \\ 0 & 0 & 0 & l_r & 0 & 0 \\ 0 & M & 0 & 0 & l_r+M & 0 \\ 0 & 0 & M & 0 & 0 & l_r+M \end{bmatrix} \begin{bmatrix} \dot{i}_{0s} \\ \dot{i}_{ds} \\ \dot{i}_{qs} \\ \dot{i}_{0r} \\ \dot{i}_{dr} \\ \dot{i}_{qr} \end{bmatrix} - \begin{bmatrix} 0 \\ \omega\left[\left(l_s+M\right)i_{qs} + Mi_{qr}\right] \\ -\omega\left[\left(l_s+M\right)i_{ds} + Mi_{dr}\right] \\ 0 \\ (\omega-\omega_m)\left[Mi_{qs} + \left(l_r+M\right)i_{qr}\right] \\ -(\omega-\omega_m)\left[Mi_{ds} + \left(l_r+M\right)i_{dr}\right] \end{bmatrix} \tag{26}$$

Now observe that the four non-zero elements in the last vector are multiplied by four currents from the current vector which multiplies the resistance matrix: i_{ds}, i_{qs}, i_{dr}, i_{qr}. I have highlighted the relevant terms in the voltage equation below.

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[1] Note that these equations are given on p. 150 of Krause's 2002 edition as eqs. 4.5-16, 4.5-17, 4.5-19, and 4.5-20, respectively.

[2] We also observe $\lambda_{0s}=l_s\, i_{0s}$ and $\lambda_{0r}=l_r\, i_{0r}$ but we will not need these.

$$\begin{bmatrix} v_{0s} \\ v_{ds} \\ v_{qs} \\ v_{0r} \\ v_{dr} \\ v_{qr} \end{bmatrix} = \begin{bmatrix} r_s & 0 & 0 & 0 & 0 & 0 \\ 0 & r_s & 0 & 0 & 0 & 0 \\ 0 & 0 & r_s & 0 & 0 & 0 \\ 0 & 0 & 0 & r_r & 0 & 0 \\ 0 & 0 & 0 & 0 & r_r & 0 \\ 0 & 0 & 0 & 0 & 0 & r_r \end{bmatrix} \begin{bmatrix} i_{0s} \\ i_{ds} \\ i_{qs} \\ i_{0r} \\ i_{dr} \\ i_{qr} \end{bmatrix} + \begin{bmatrix} l_s & 0 & 0 & 0 & 0 & 0 \\ 0 & l_s+M & 0 & 0 & M & 0 \\ 0 & 0 & l_s+M & 0 & 0 & M \\ 0 & 0 & 0 & l_r & 0 & 0 \\ 0 & M & 0 & 0 & l_r+M & 0 \\ 0 & 0 & M & 0 & 0 & l_r+M \end{bmatrix} \begin{bmatrix} \dot{i}_{0s} \\ \dot{i}_{ds} \\ \dot{i}_{qs} \\ \dot{i}_{0r} \\ \dot{i}_{dr} \\ \dot{i}_{qr} \end{bmatrix} - \begin{bmatrix} 0 \\ \omega\left[(l_s+M)i_{qs} + Mi_{qr}\right] \\ -\omega\left[(l_s+M)i_{ds} + Mi_{dr}\right] \\ 0 \\ (\omega-\omega_m)\left[Mi_{qs} + (l_r+M)i_{qr}\right] \\ -(\omega-\omega_m)\left[Mi_{ds} + (l_r+M)i_{dr}\right] \end{bmatrix}$$

So let's now expand back out the last vector so that it is a product of a matrix and a current vector.

$$\begin{bmatrix} v_{0s} \\ v_{ds} \\ v_{qs} \\ v_{0r} \\ v_{dr} \\ v_{qr} \end{bmatrix} = \begin{bmatrix} r_s & 0 & 0 & 0 & 0 & 0 \\ 0 & r_s & 0 & 0 & 0 & 0 \\ 0 & 0 & r_s & 0 & 0 & 0 \\ 0 & 0 & 0 & r_r & 0 & 0 \\ 0 & 0 & 0 & 0 & r_r & 0 \\ 0 & 0 & 0 & 0 & 0 & r_r \end{bmatrix} \begin{bmatrix} i_{0s} \\ i_{ds} \\ i_{qs} \\ i_{0r} \\ i_{dr} \\ i_{qr} \end{bmatrix} + \begin{bmatrix} l_s & 0 & 0 & 0 & 0 & 0 \\ 0 & l_s+M & 0 & 0 & M & 0 \\ 0 & 0 & l_s+M & 0 & 0 & M \\ 0 & 0 & 0 & l_r & 0 & 0 \\ 0 & M & 0 & 0 & l_r+M & 0 \\ 0 & 0 & M & 0 & 0 & l_r+M \end{bmatrix} \begin{bmatrix} \dot{i}_{0s} \\ \dot{i}_{ds} \\ \dot{i}_{qs} \\ \dot{i}_{0r} \\ \dot{i}_{dr} \\ \dot{i}_{qr} \end{bmatrix}$$

$$- \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega(l_s+M) & 0 & 0 & \omega M \\ 0 & -\omega(l_s+M) & 0 & 0 & -\omega M & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\omega-\omega_m)M & 0 & 0 & (\omega-\omega_m)(l_r+M) \\ 0 & -(\omega-\omega_m)M & 0 & 0 & -(\omega-\omega_m)(l_r+M) & 0 \end{bmatrix} \begin{bmatrix} i_{0s} \\ i_{ds} \\ i_{qs} \\ i_{0r} \\ i_{dr} \\ i_{qr} \end{bmatrix}$$

Now change the sign on the last matrix and on all of its elements…

$$\begin{bmatrix} v_{0s} \\ v_{ds} \\ v_{qs} \\ v_{0r} \\ v_{dr} \\ v_{qr} \end{bmatrix} = \begin{bmatrix} r_s & 0 & 0 & 0 & 0 & 0 \\ 0 & r_s & 0 & 0 & 0 & 0 \\ 0 & 0 & r_s & 0 & 0 & 0 \\ 0 & 0 & 0 & r_r & 0 & 0 \\ 0 & 0 & 0 & 0 & r_r & 0 \\ 0 & 0 & 0 & 0 & 0 & r_r \end{bmatrix} \begin{bmatrix} i_{0s} \\ i_{ds} \\ i_{qs} \\ i_{0r} \\ i_{dr} \\ i_{qr} \end{bmatrix} + \begin{bmatrix} l_s & 0 & 0 & 0 & 0 & 0 \\ 0 & l_s+M & 0 & 0 & M & 0 \\ 0 & 0 & l_s+M & 0 & 0 & M \\ 0 & 0 & 0 & l_r & 0 & 0 \\ 0 & M & 0 & 0 & l_r+M & 0 \\ 0 & 0 & M & 0 & 0 & l_r+M \end{bmatrix} \begin{bmatrix} \dot{i}_{0s} \\ \dot{i}_{ds} \\ \dot{i}_{qs} \\ \dot{i}_{0r} \\ \dot{i}_{dr} \\ \dot{i}_{qr} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega(l_s+M) & 0 & 0 & -\omega M \\ 0 & \omega(l_s+M) & 0 & 0 & \omega M & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\omega-\omega_m)M & 0 & 0 & -(\omega-\omega_m)(l_r+M) \\ 0 & (\omega-\omega_m)M & 0 & 0 & (\omega-\omega_m)(l_r+M) & 0 \end{bmatrix} \begin{bmatrix} i_{0s} \\ i_{ds} \\ i_{qs} \\ i_{0r} \\ i_{dr} \\ i_{qr} \end{bmatrix}$$

Notice that the resistance matrix and the last matrix multiply the same vector (the current vector); therefore, we can combine these two matrices. For example, element (2,3) in the last matrix will go into element (2,3) of the resistance matrix, as shown below:

$$\begin{bmatrix} v_{0s} \\ v_{ds} \\ v_{qs} \\ v_{0r} \\ v_{dr} \\ v_{qr} \end{bmatrix} = \begin{bmatrix} r_s & 0 & 0 & 0 & 0 & 0 \\ 0 & r_s & 0 & 0 & 0 & 0 \\ 0 & 0 & r_s & 0 & 0 & 0 \\ 0 & 0 & 0 & r_r & 0 & 0 \\ 0 & 0 & 0 & 0 & r_r & 0 \\ 0 & 0 & 0 & 0 & 0 & r_r \end{bmatrix} \begin{bmatrix} i_{0s} \\ i_{ds} \\ i_{qs} \\ i_{0r} \\ i_{dr} \\ i_{qr} \end{bmatrix} + \begin{bmatrix} l_s & 0 & 0 & 0 & 0 & 0 \\ 0 & l_s+M & 0 & 0 & M & 0 \\ 0 & 0 & l_s+M & 0 & 0 & M \\ 0 & 0 & 0 & l_r & 0 & 0 \\ 0 & M & 0 & 0 & l_r+M & 0 \\ 0 & 0 & M & 0 & 0 & l_r+M \end{bmatrix} \begin{bmatrix} \dot{i}_{0s} \\ \dot{i}_{ds} \\ \dot{i}_{qs} \\ \dot{i}_{0r} \\ \dot{i}_{dr} \\ \dot{i}_{qr} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega(l_s+M) & 0 & 0 & -\omega M \\ 0 & \omega(l_s+M) & 0 & 0 & \omega M & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\omega-\omega_m)M & 0 & 0 & -(\omega-\omega_m)(l_r+M) \\ 0 & (\omega-\omega_m)M & 0 & 0 & (\omega-\omega_m)(l_r+M) & 0 \end{bmatrix} \begin{bmatrix} i_{0s} \\ i_{ds} \\ i_{qs} \\ i_{0r} \\ i_{dr} \\ i_{qr} \end{bmatrix}$$

This results in the expression below….

$$\begin{bmatrix} v_{0s} \\ v_{ds} \\ v_{qs} \\ v_{0r} \\ v_{dr} \\ v_{qr} \end{bmatrix} = \begin{bmatrix} r_s & 0 & 0 & 0 & 0 & 0 \\ 0 & r_s & -\omega(l_s+M) & 0 & 0 & -\omega M \\ 0 & \omega(l_s+M) & r_s & 0 & \omega M & 0 \\ 0 & 0 & 0 & r_r & 0 & 0 \\ 0 & 0 & -(\omega-\omega_m)M & 0 & r_r & -(\omega-\omega_m)(l_r+M) \\ 0 & (\omega-\omega_m)M & 0 & 0 & (\omega-\omega_m)(l_r+M) & r_r \end{bmatrix} \begin{bmatrix} i_{0s} \\ i_{ds} \\ i_{qs} \\ i_{0r} \\ i_{dr} \\ i_{qr} \end{bmatrix} + \begin{bmatrix} l_s & 0 & 0 & 0 & 0 & 0 \\ 0 & l_s+M & 0 & 0 & M & 0 \\ 0 & 0 & l_s+M & 0 & 0 & M \\ 0 & 0 & 0 & l_r & 0 & 0 \\ 0 & M & 0 & 0 & l_r+M & 0 \\ 0 & 0 & M & 0 & 0 & l_r+M \end{bmatrix} \begin{bmatrix} \dot{i}_{0s} \\ \dot{i}_{ds} \\ \dot{i}_{qs} \\ \dot{i}_{0r} \\ \dot{i}_{dr} \\ \dot{i}_{qr} \end{bmatrix} \quad (27)$$

These are the complete transformed induction machine voltage equations in "current form."

6.0 Torque equation for induction machine

The electromagnetic torque of an induction machine may be evaluated according to

$$T_{em} = \frac{\partial W_c}{\partial \Theta_m}$$

where W_c is the co-energy of the coupling fields associated with the various windings, and Θ_m is the angle in mechanical degrees between the main rotor axis and fixed reference.

We are not considering saturation here, assuming the flux-current relations are linear, in which case the co-energy W_c of the coupling field equals its energy, W_f, so that:

$$T_{em} = \frac{\partial W_f}{\partial \Theta_m}$$

We use electric rad/sec by substituting $\Theta_m=\theta_m/p$ where p is the number of pole pairs.

$$T_{em} = p\frac{\partial W_f}{\partial \theta_m} \tag{28}$$

For a linear electromagnetic system with J electrical inputs (windings), the total field energy is given by [1, p. 22-24]:

$$W_f = \frac{1}{2}\sum_{p=1}^{J}\sum_{q=1}^{J} L_{pq} i_p i_q \tag{29}$$

where L_{pq} is the winding's self inductance when p=q and when p≠q, it is the mutual inductance between the two windings (the energy stored in the leakage inductances is not a part of the energy stored in the coupling field).

Consider the abc inductance matrices given by (7a), (7b), and (7c) of these notes. Applying (29), the stored energy is given by:

$$W_f = \frac{1}{2}\underline{i}_{abcs}^T(\underline{L}_s - l_s\underline{U})\underline{i}_{abcs} + \underline{i}_{abcs}^T\underline{L}_{sr}\,\underline{i}_{abcr} + \frac{1}{2}\underline{i}_{abcr}^T(\underline{L}_r - l_r\underline{U})\underline{i}_{abcr}$$

Applying (28) to (29), and observing that dependence on θ_m only occurs in the middle term, we obtain

$$\frac{\partial W_f}{\partial \theta_m} = \frac{\partial}{\partial \theta_m}\underline{i}_{abcs}^T\underline{L}_{sr}\,\underline{i}_{abcr}$$

so that

$$T_{em} = p\frac{\partial}{\partial \theta_m}\underline{i}_{abcs}^T\underline{L}_{sr}\,\underline{i}_{abcr}$$

But only $\underline{L}_{sr}$ depends on θ_m, so

$$T_{em} = p\underline{i}_{abcs}^T\frac{\partial \underline{L}_{sr}}{\partial \theta_m}\underline{i}_{abcr}$$

We may perform the above differentiation and associated matrix multiplication to show that the above evaluates to

$$T_{em} = -pL_m \left\{ \left[i_{as}\left(i_{ar} - \frac{1}{2}i_{br} - \frac{1}{2}i_{cr} \right) + i_{bs}\left(i_{br} - \frac{1}{2}i_{ar} - \frac{1}{2}i_{cr} \right) + i_{cs}\left(i_{cr} - \frac{1}{2}i_{br} - \frac{1}{2}i_{ar} \right) \right] \sin\theta_m \right.$$

$$\left. + \frac{\sqrt{3}}{2}\left[i_{as}\left(i_{br} - i_{cr} \right) + i_{bs}\left(i_{cr} - i_{ar} \right) + i_{cs}\left(i_{ar} - i_{br} \right) \right] \cos\theta_m \right\}$$

To complete our abc model we relate torque to rotor speed according to:

$$T_{em} = \frac{J}{p}\frac{d\omega_m}{dt} + T_m$$

where J is the inertia of the rotor in kg-m^2 (or joule-sec^2), the first term on the right is the inertial torque, and the second term is the mechanical torque (and has a negative value for generation).

➔Start with slide #6 in “Torque&Power” powerpoint.

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[1] P. Krause, O. Wasynczuk, and S. Sudhoff, “Analysis of Electric Machinery,” 1995, IEEE Press.
[2]J. Marques, H. Pinheiro, H. A. Gründling, J. R. Pinheiro and H. L. Hey, “A Survey On Variable-Speed Wind Turbine System,” Proc. of ????, pp. 732-738.