LINEAR EQUATIONS

An **equation** is a statement that two quantities or algebraic expressions are equal. The two quantities are written with an equal sign (=) between them. An equation maybe true, false, or the truth-value cannot be determined.
For example,

$5 + 2 = 3$ *is false*
$2 + 5 = 9 - 2$ *is true*
$2 + y = 5$ *is neither true nor false—its validity depends on the value of y is which not known at this time.*

A **linear equation** in one variable is an equation of the form **ax + b** = c where **a, b,** and **c** are real numbers, c ≠ 0. A **linear equation** in one variable is the one that the **exponent on the variable** is 1. For this reason, a linear equation is often called **a first-degree equation.**

Any number used to replace a variable in an equation making the equation true is called a **solution** of the equation. That is, a solution of an equation is the number or numbers that make the equation a true statement. The collection of all solutions of an equation is called its **solution** set. The process of finding all solutions is called **solving the equation.**

TYPES OF LINEAR EQUATIONS

- **An inconsistent equation**

The equation "$x = x + 3$" has no solutions. No matter the value you choose for x, it will always be a false statement. This equation is called a **contradiction or inconsistent.**

- **An identity equation**

The equation "$x + 2 = 2 + x$" has many solutions. Every number you choose as a solution makes the equation a true statement. Is true for all real numbers and has an infinite number of solutions. This equation is called an **identity**, and has the entire set of real numbers for its solution set.

- **A conditional equation**

The equation "$x + 3 = 7$" has 4 as a solution because when x is replaced by 4, 4+3 = 7 is a true statement. It is called conditional equation because the equation is true only for some replacement of the variable and false for others. In a linear equation situation, a conditional equation has a single value as its solution.

- **Equivalent equation**

The equations "$x + 3 = 9$" and "$x = 6$" both have 6 as a solution. When two equations have exactly the same solutions, they are called **equivalent equations.**

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Before you can fully understand the process involve in solving equations, it is important to understand the difference between a term and a factor.

A **term** in mathematics is a number (say 3), a variable (say x), a product of numbers and one or more variables (say $4x$ *or* $2xy$), a quotient of numbers and variables (say $\frac{2}{y}$), or a combination of products and quotients (say $\frac{2x}{3y}$). A **plus sign (+)** and a **minus sign (-)** separate one term from another.

When a *term* is written as a ***product*** of numbers and/or variables, then any one of the ***variables or numbers entering into the product*** is called a **factor** of that term. For examples, $15 = 3 \bullet 5$, **3** and **5** are factors of the15. Similarly, $2xy = 2 \bullet x \bullet y$, **2**, $\boldsymbol{x}$ and $\boldsymbol{y}$ are factors of the $2xy$.

TECHNIQUES FOR SOLVING LINEAR EQUATIONS

1. Eliminate fractions by multiplying both sides by the least common denominator. The least common denominator of a group of numbers is the smallest number that is divisible without a remainder by each of the numbers.

2. Remove all grouping symbols (such as parenthesis, brackets, braces). If an equation involves parentheses or other grouping symbols, remove them using the *distributive property.*

3. Combine like terms on each side of the equal sign.

4. ***Check whether the unknown variable is on one side or both sides of the equation.***

> We can move one term from one side of the equality to the other by changing the sign of each term that is moved in the process. The procedure is called ***transposing.***

5. ***Transpose the terms with the unknown variable to one side of the equality and the terms without the unknown variable to the other side of the equality*** using the ***addition-subtraction rule*** (maybe repeatedly) to get the equation into the form $ax = b$.

6. ***To isolate the unknown variable factor,*** apply the ***multiplication-division rule*** to both sides of the equation to obtain a variable factor with coefficient of 1.

7. Check your solution in the original equation by substitution.

8. If an identity results, the original equation has every real number as a solution. If a contradiction results, then there is no solution.

Every equation has two members, the left hand side (LHS) and the right hand side (RHS). In the linear equation $ax + b = c$, the LHS = $ax + b$ and the RHS = c. Also note that the LHS has two terms, ax and b, but the term " ax" is a product with two factors, a and x.

$$\underbrace{\overbrace{\underbrace{a}_{factor}\;\underbrace{x}_{factor}}^{term} + \overbrace{b}^{term}}_{LHS} = \underbrace{\overbrace{c}^{term}}_{RHS}$$

- **~~Add~~ition-Subtraction rule**

If **a = b**, then **a + c = b + c** or **a – c = b – c** for any real numbers a, b, and c.

This rule allows us to add or subtract the same number or quantity to or from both sides of an equation. Note that **a = b** and **a + c = b + c** are **equivalent equations.** Adding the same number to both sides of an equation or subtracting the same number from both sides of an equation yields an equivalent equation.

Example 1

Solve. $x - 5 = 7.$

Solution:

Use the addition rule to add 5 to both sides of the equation.

$$\begin{array}{rcr} x-5 & & 7 \\ \underline{+5} & = & \underline{+5} \\ x+0 & & 12 \end{array}$$

$$x = 12$$

Problem 1

Solve.

a) x – 3 = 11

b) y – 2 = 7

c) r – 13 = 11

d) q – 8 = -7

e) a – 9 = 7

f) z – 11 = 3

g) a – 2 = -5

h) k – 7 = 6

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Example 2

Solve. $9 + x = 4$

Solution:

Use the subtraction rule to subtract 9 from both sides of the equation.

$$\begin{array}{rcr} 9+x & & 4 \\ \underline{-9} & = & \underline{-9} \\ 0+x & & -5 \end{array}$$

$$x = -5$$

Problem 2

Solve.

a) x + 3 = 11

d) 1 = c + 13

b) 2 + a = 7

e) 1 + p = 3

c) 7 = 4 + x

f) 2 + x = - 9

- **Multiplication-division rule**

If **a** = **b**, then **a • c** = **b • c** or $\frac{a}{c} = \frac{b}{c}$ for any real numbers a, b, and c, c ≠ 0.

This rule allows us to multiply or divide the same non-zero number or quantity to both sides of an equation. Note that **a** = **b** and **a • c** = **b • c** are **equivalent equations**. Multiplying the same number to both sides of an equation or dividing the same non-zero number from both sides of an equation yields an equivalent equation.

To solve any equation, you must transpose all the *terms* without the unknown variable <u>before</u> you remove the *factors* with the unknown variable. This means you must apply the addition-subtraction rule before you apply the multiplication-division rule.

Example 3

Solve.

a) $2x = 12$

Solution: To isolate the variable x, divide both sides by the factor 2

$$\frac{2x}{2} = \frac{12}{2} = \frac{\overset{6}{12 \div 2}}{\underset{1}{2 \div 2}}$$

$x = 6.$

b) $-3z = -15$

Solution: The SIDE containing the unknown variable z is the LHS.
On the LHS, there is no other term without the variable z.
To isolate the variable z, divide both sides by the factor -3

We have, $\frac{-3z}{-3} = \frac{-15}{-3} = \frac{\overset{5}{-15 \div -3}}{\underset{1}{-3 \div -3}}$ so that, $z = 5$

c) $-3x = 12$

Solution: Divide both sides by -3 to obtain an equivalent equation.

since $-3x = 12$ we have, $\frac{-3x}{-3} = \frac{12}{-3}$ and $x = -4$.

Problem 3

Solve.

a) $4x = -20$

b) $-22a = 44$

c) $45 = 9x$

d) $-7y = -84$

Example 4

Solve the equation $\frac{3}{2}x = 15$.

Solution: Multiply both sides by the reciprocal of the coefficient of x, $\frac{2}{3}$, to obtain an equivalent equation.

$$\frac{2}{3} * \frac{3}{2}x = 15 * \frac{2}{3}$$
$$x = 10$$

Problem 4

Solve.

a) $\frac{2}{7}x = 4$.

b) $\frac{2}{3}\text{x} = \frac{1}{5}$

- **Combination of the rules**

Example 5

Solve for x in the following equation $2x - 4 = 8$.

Solution: The SIDE containing the unknown variable x is the LHS.
On the LHS, the term without the variable x is -4.
Transposing -4 to the RHS, it becomes $+4$.

We have,

$$\begin{array}{rcr} 2x-4 & & 8 \\ \underline{\quad +4} & = & \underline{+4} \\ 2x+0 & & 12 \end{array}$$

$$2x = 12$$

To isolate the variable x, divide both sides by the factor 2

$$\frac{2x}{2} = \frac{12}{2} = \frac{\overset{6}{12 \div 2}}{\underset{1}{2 \div 2}}$$

$x = 6$.

Problem 5

a) $2\text{x} + 4 = 10$

b) $6 - 3\text{x} = 18$

c) $16\text{x} + 5 = -14$

d) $-8 = -\text{x} + 4$

Example 6

Solve. $3(x - 4) = 5x - 8$

Solution: $3x - 12 = 5x - 8$

$$\begin{array}{r} 3x - 12 \\ \underline{-5x + 12} \\ -2x + 0 \end{array} = \begin{array}{r} 5x - 8 \\ \underline{-5x + 12} \\ 0 + 4 \end{array} \qquad \textbf{or} \qquad 3x - 5x = -8 + 12$$

$$-2x = 4$$

$$\frac{-2x}{-2} = \frac{4}{-2}$$

$$x = -2$$

Problem 6

Solve.

a) $9(2x - 1) = 3x + 6$

b) $5(z + 1) - 4z = z - 5$

c) $2(1 - a) = 5(2a + 1) - 3(1 + a)$

d) $3(2y + 3) - 4(-y + 1) = 7y - 10$

Example 7

Solve. $\frac{2}{3}x + \frac{1}{5} = 1$

Solution: Clear the fractions. The LCD = 15

$$15\left(\frac{2}{3}x + \frac{1}{5}\right) = 15(1)$$

$$5(2)x + 3(1) = 15$$

To apply the LCD

- If a term is a **fraction**, divide LCD by denominator and multiply result by the numerator.
- If a term is **not a fraction**, just multiply the term by the LCD.

$$10x + 3 = 15$$

$$10x = 15 - 3$$

$$10x = 12$$

$$x = \frac{12}{10} = \frac{6}{5}$$

Problem 6

Solve.

a) $\frac{2}{3}x + \frac{1}{2} = 2$

b) $\frac{1}{2}x + \frac{2}{5}x = 3$

c) $\frac{2}{7} - \frac{3}{5}x = -1$

Example 7

Solve.

$$\frac{x-5}{4} - \frac{x+2}{5} = -2$$

Solution: Clear the fractions. The LCD = 20.

> To apply the LCD in an equation
> - If a term is a **fraction**, divide LCD by denominator and multiply result by the numerator.
> - If a term is **not a fraction**, just multiply the term by the LCD.

$$20\left(\frac{x-5}{4}\right) - 20\left(\frac{x+2}{5}\right) = 20(-2)$$

$$5(x - 5) - 4(x + 2) = -40$$

$$5x - 25 - 4x - 8 = -40$$

$$x - 33 = -40$$

$$x = -40 + 33$$

$$x = -7$$

Problem 7

Solve.

a) $\frac{x-5}{4} - \frac{x+5}{2} = -4$

b) $\frac{z}{5} + \frac{z+1}{2} = 4$

c) $\frac{2x-1}{6} - \frac{3x+2}{3} = \frac{1}{3}$

Additional Questions

1. $x+9=13$
2. $\frac{2}{3}+y=\frac{4}{9}$
3. $2x+3=x-3-4x$
4. $x-\frac{1}{3}=-8x$
5. $20-3(x+5)=0$
6. $x-(x+3)=x-6$
7. $3(x-4)+5=2(x+1)-3$
8. $x+\frac{3}{4}=\frac{5}{4}$
9. $-3(z-4)-2(3z+1)=-8$
10. $5(x+1)-4x=x-5$
11. $-4(-x+1)+3(2x+3)-7x=-10$
12. $\frac{x+5}{2}+\frac{x-5}{4}=\frac{1}{4}$
13. $12y=-84$, $-64=-z$
14. $-22x=-88$
15. $3-7x=17$,
16. $x+2x=8-3x+10$
17. $3z-z+10=14-5z+3$
18. $6x-4x+1=12-2x-11$
19. $4x-2(x+1)=-1$
20. $2+x=3-2[1-2(x+1)]$
21. $4(2y-1)-7(y+2)=-9$
22. $\frac{3}{4}y=-9$
23. $-2x+1=15$
24. $\frac{2x-1}{6}-\frac{3x+2}{3}=\frac{1}{3}$
25. $4x-2(x+1)=-1$
26. $6x-(2x-5)=0$

27. $3(2-4x)=13-(x+1)$

28. $2.1x + 4 = 12.4$

29. $\frac{2x}{3}=-4$

30. $2x = -1.2$

SOLVING LINEAR EQUATIONS IN ONE VARIABLE

Recall, an **algebraic equation** is a mathematical statement that two expressions have equal value.

$$expression = expression$$

The process of finding the **solution** of an equation (the value of the variable that gives a true statement) is called **solving** the equation for the variable.

LINEAR EQUATION IN ONE VARIABLE

A **linear equation in one variable** can be written in the form $ax + b = c$ where a, b, and c are real numbers and $a \neq 0$.

Example: $3x - 2 = -4$

Our goal is to simplify our equation to an equivalent equation of the form

$$\boldsymbol{x} = \textbf{number} \quad \text{or} \quad \textbf{number} = \boldsymbol{x}$$

where the **number** is the solution of the original equation. When an equation is in this form we say the variable has been **isolated** or alone. It makes no difference which side of the equation the variable is on. For example, $x = 3$ is equivalent to $3 = x$.

ADDITION PROPERTY OF EQUALITY

If a, b, and c are real numbers, then

$$a = b \quad \text{and} \quad a + c = b + c$$

are equivalent equations.

In other words, you may add (or subtract) any number to both sides of an equation and the result is an equivalent equation.

When solving equations, we try to add or subtract a number on both sides so that the equivalent equation is a simpler one to solve. Our eventual goal is to isolate the variable.

Think of an equation as a balanced scale. If you add any weight to one side, you must add the same amount of weight to the other side to keep the scale in balance. Likewise, if you remove any weight from one side, you must remove the same amount of weight to the other side to keep the scale in balance. "Whatever you do to one side of an equation, you must do to the other side."

STEPS TO SOLVING LINEAR EQUATION IN ONE VARIABLE

1. Use the distributive property to remove all grouping symbols such as parentheses.
2. Combine any like terms on each side of the equation
3. Use the addition property of equality to rewrite the equation as an equivalent equation with variable terms on one side and numbers (constants) on the other side.
4. Check the proposed solution in the original equation.

Example: $x-2=4$

$x-2+2=4+2$ add 2 to both sides in order to get x by itself.

$x=6$

Example: $3x-5=4$

$3x-5+5=4+5$ add 2 to both sides in order to get the variable by itself.

$3x=9$ divide both sides by the coefficient of x.

$$\frac{3x}{3}=\frac{9}{3}$$

$x=3$