LEAST-SQUARES FINITE ELEMENT MODELS

- General idea of the least-squares formulation applied to an abstract boundary-value problem
- Works of our group
- Application to Poisson's equation
- Application to flows of viscous incompressible fluids
- Numerical Examples

Least-Squares Variational Formulation: *Abstract Nonlinear Formulation*

- Abstract nonlinear boundary value problem

$$\mathcal{L}(\mathbf{u}) = \mathbf{f} \quad \text{in } \Omega$$
$$g(\mathbf{u}) = \mathbf{h} \quad \text{on } \Gamma$$

Ω: domain of boundary value problem
Γ: boundary of Ω
$\mathcal{L}$: first-order nonlinear partial differential operator
g: linear boundary condition operator
$\mathbf{f}$, $\mathbf{h}$: data

Least-Squares Variational Formulation: *Abstract Nonlinear Formulation*

- Abstract least-squares variational principle

$$\mathcal{J}(\mathbf{u};\mathbf{f},\mathbf{h}) = \frac{1}{2}\left(\left\|\mathcal{L}(\mathbf{u})-\mathbf{f}\right\|_{\Omega,0}^{2} + \left\|g(\mathbf{u})-\mathbf{h}\right\|_{\Gamma,0}^{2}\right)$$

  – Find $\mathbf{u} \in \mathcal{V}$ such that $\mathcal{J}(\mathbf{u}; \mathbf{f}, \mathbf{h}) \leq \mathcal{J}(\tilde{\mathbf{u}}; \mathbf{f}, \mathbf{h})$ for all $\tilde{\mathbf{u}} \in \mathcal{V}$, where $\mathcal{V}$ is an appropriate vector space, such as $\mathbf{H}^1(\Omega)$

- Necessary condition for minimization:

$$\mathcal{G}(\tilde{\mathbf{u}},\mathbf{u}) = \left(\nabla\mathcal{L}(\mathbf{u})\cdot\tilde{\mathbf{u}}, \mathcal{L}(\mathbf{u})-\mathbf{f}\right)_{\Omega,0} + \left(g(\tilde{\mathbf{u}}), g(\mathbf{u})-\mathbf{h}\right)_{\Gamma,0}$$

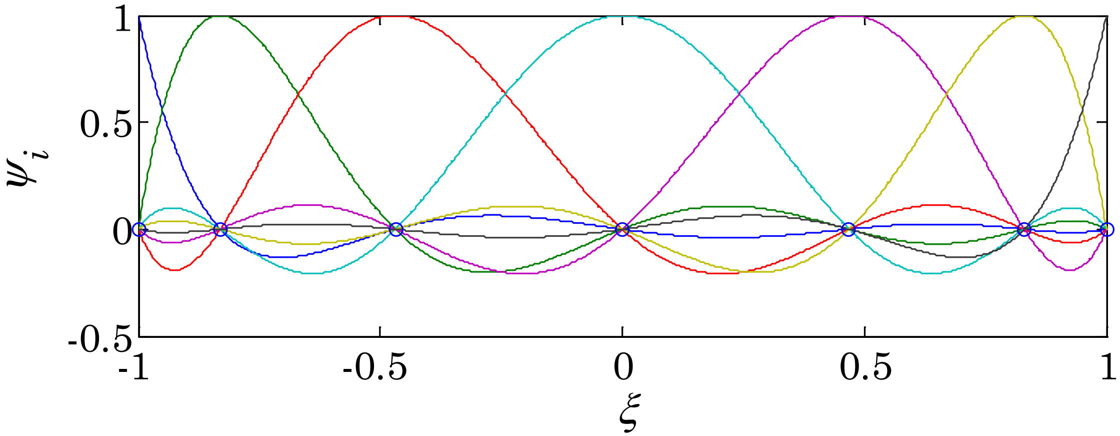
Least-Squares Finite Element Models: *Formulations for Nonlinear Problems*

- Two approaches may be adopted when formulating least-squares finite element models of nonlinear problems
  - (1) Linearize PDE prior to construction and minimization of least-squares functional $\mathcal{J}$
    - Element matrices will always be symmetric
    - Simplest possible form of the element matrices
  - (2) Linearize finite element equations following construction and minimization of least-squares functional $\mathcal{J}$
    - Approach is consistent with variational setting
    - Finite element matrices are more complicated
    - Resulting coefficient matrix may not be symmetric

Finite Element Implementation
Spectral/hp Finite Elements

- One-dimensional high-order Lagrange interpolation functions

$$\psi_i(\xi) = \frac{(\xi-1)(\xi+1)L_p'(\xi)}{p(p+1)L_p(\xi_i)(\xi-\xi_i)}$$



- Multi-dimensional interpolation functions constructed from tensor products of the one-dimensional functions
- We employ full Gauss Legendre quadrature rules in evaluation of the integrals

APPLICATIONS OF LSFEM TO DATE by JNReddy and his coauthors

- Fluid Dynamics (2-D)
  - Viscous incompressible fluids
  - Viscous compressible fluids (with shocks)
  - Non-Newtonian (polymer and power-law) fluids
  - Coupled fluid flow and heat transfer
  - Fluid-solid interaction

- Solid Mechanics (static and free vibration analysis)
  - Beams
  - Plates
  - Shells
  - Fracture mechanics
  - Helmholtz equation

Finite Element Formulations Of the Poisson Equation

(Primal) Problem :

$$-\nabla^2 u = f \quad \text{in} \quad \Omega$$

$$(-\nabla^2 = -\nabla \cdot \nabla)$$

$$u = \hat{u} \quad \text{on} \quad \Gamma_u$$

$$\frac{\partial u}{\partial n} = \hat{g} \quad \text{on} \quad \Gamma_g$$

(Mixed) Problem :

$$\mathbf{v} - \nabla u = 0 \quad \text{in} \quad \Omega$$

$$-\nabla \cdot \mathbf{v} = f \quad \text{in} \quad \Omega$$

$$u = \hat{u} \quad \text{on} \quad \Gamma_u$$

$$\hat{\mathbf{n}} \cdot \mathbf{v} = \hat{g} \quad \text{on} \quad \Gamma_g$$

Least-Squares Formulation - Primal

1. $I_1(u) = \| -\nabla^2 u - f \|_{0,\Omega}^2 + \| \frac{\partial u}{\partial n} - \hat{g} \|_{0,\Gamma_g}^2$

2. **Minimize** $I_1(u)$

$$B_1(u,v) = l_1(v)$$

$$B_1(u,v) = \left(-\nabla^2 v, -\nabla^2 u\right)_{0,\Omega} + \left(\frac{\partial v}{\partial n}, \frac{\partial u}{\partial n}\right)_{0,\Gamma_g}$$

$$l_1(v) = \left(\nabla^2 v, f\right)_{0,\Omega} + \left(\frac{\partial v}{\partial n}, g\right)_{0,\Gamma_g}$$

Least Squares Formulation - Mixed

$$I_m(u) = \| \mathbf{v} - \nabla u \|_{0,\Omega}^2 + \| -\nabla \cdot \mathbf{v} - f \|_{0,\Omega}^2 + \| \hat{\mathbf{n}} \cdot \mathbf{v} - \hat{g} \|_{0,\Gamma_g}^2$$

Minimize I_m **:** $\delta I_m = 0 \;\; gives$

$$B_m((u,\mathbf{v}),(\delta u,\delta \mathbf{v})) = l_m((\delta u,\delta \mathbf{v}))$$

Least-squares Mixed Fe Model

Finite element approximation

$$u(\mathbf{x}) \approx u_h(\mathbf{x}) = \sum_{j=1}^{m} u_j \psi_j(\mathbf{x}), \;\; \mathbf{v}(\mathbf{x}) \approx \mathbf{v}_h(\mathbf{x}) = \sum_{j=1}^{n} \mathbf{v}_j \varphi_j(\mathbf{x})$$

Finite element model
$$\begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} \\ \left(\mathbf{K}^{12}\right)^{\mathrm{T}} & \mathbf{K}^{22} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{v} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^{1} \\ \mathbf{F}^{2} \end{Bmatrix}$$

$$K_{ij}^{11} = \int_{\Omega} \nabla\psi_i \cdot \nabla\psi_j \; d\mathbf{x}, \;\; K_{ij}^{12} = -\int_{\Omega} \nabla\psi_i \cdot \varphi_j \; d\mathbf{x} = K_{ji}^{21}$$

$$K_{ij}^{22} = \int_{\Omega} \left(\varphi_i\varphi_j + \nabla\varphi_i \cdot \nabla\varphi_j\right) \; d\mathbf{x} + \oint_{\Gamma} \left(\hat{\mathbf{n}} \cdot \varphi_i\right)\left(\hat{\mathbf{n}} \cdot \varphi_j\right) ds$$

$$F_i^1 = 0, \;\; F_i^2 = -\int_{\Omega} f \, \nabla \cdot \varphi_i \, d\mathbf{x} + \oint_{\Gamma} \hat{\mathbf{n}} \cdot \varphi_i \hat{g} \; ds$$

Example (Using LSFEM Mixed Model):

Differential Equation

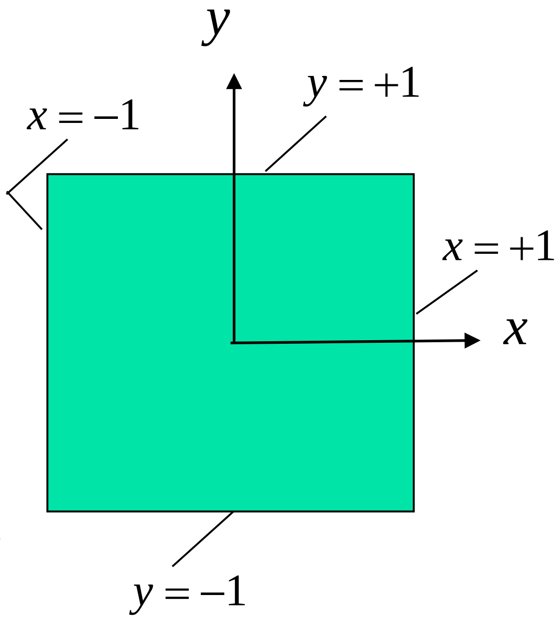
$$-\nabla^2 u = f \ \textbf{in -1} \leq x, y \leq \textbf{1}$$

Boundary Conditions

$$\frac{\partial u}{\partial y} \equiv v = 0 \text{ on } y = \pm 1$$

$$\frac{\partial u}{\partial x} \equiv w = q^*(y) = 0 \text{ on } x = -1$$

$$u = u^*(y) = 8 \cos \pi y \text{ on } x = 1$$

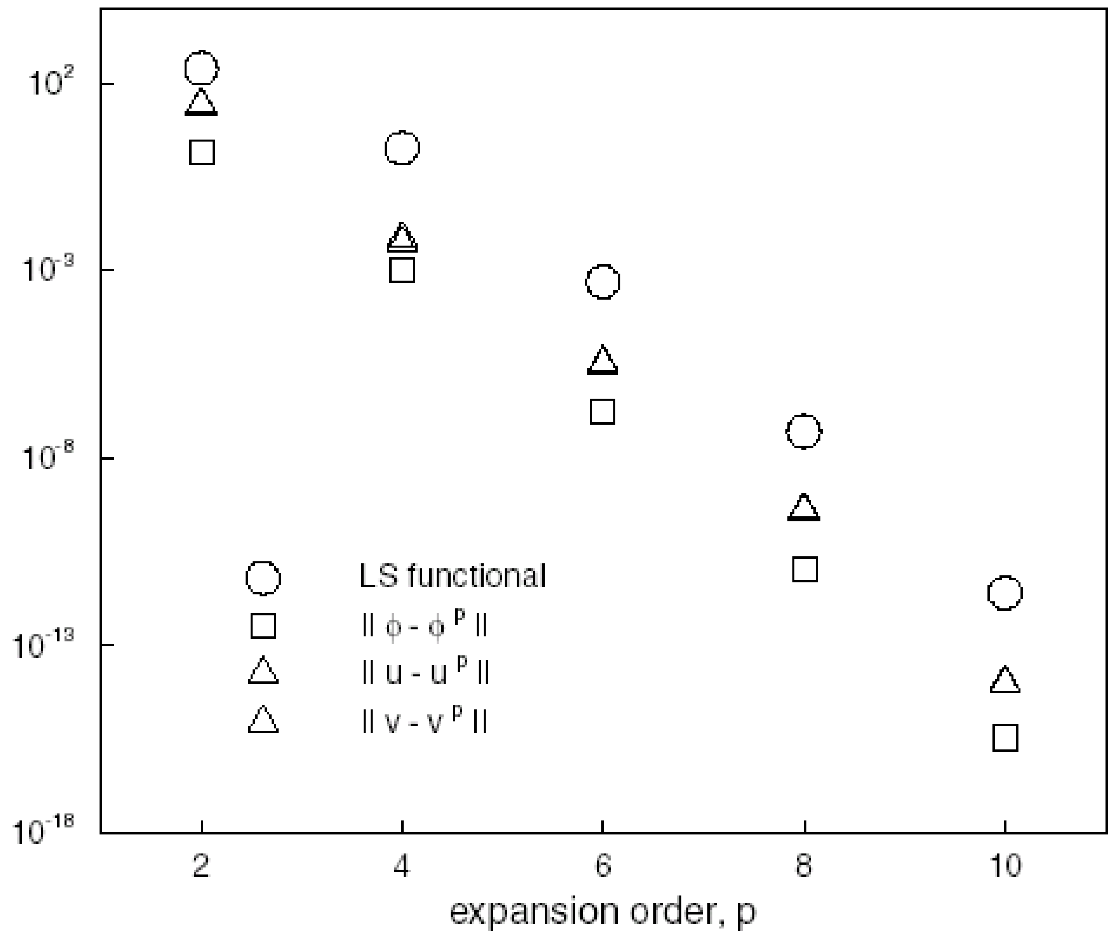


Analytical solution:

$$u(x, y) = (7x + x^7) \cos \pi y$$

Plots of the L_2-Error norms as a function of p

L_2 Norm of Errors



Polynomial order, p

LEAST-SQUARES FORMULATION OF VISCOUS INCOMPRESSIBLE FLUIDS

Governing equations (Navier-Stokes equations)

$$(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p-\frac{1}{\mathrm{Re}}\nabla\cdot[(\nabla\mathbf{u})+(\nabla\mathbf{u})^{T}]=\mathbf{f}\quad\text{in}\quad\Omega$$

$$\nabla\cdot\mathbf{u}=0\quad\text{in}\quad\Omega$$

$$\mathbf{u}=\hat{\mathbf{u}}\quad\text{on}\quad\Gamma_{\mathrm{u}}$$

$$\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}=\hat{\mathbf{t}}\quad\text{on}\quad\Gamma_{\sigma}$$

VELOCITY-PRESSURE-VORTICITY FORMULATION OF N-S EQUATIONS FOR VISCOUS INCOMPRESSIBLE FLUIDS

$$(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p-\frac{1}{Re}\nabla\times\boldsymbol{\omega}=\mathbf{f}$$

$$\boldsymbol{\omega}-\nabla\times\mathbf{u}=\mathbf{0}$$

$$\nabla\cdot\mathbf{u}=0$$

$$\nabla\cdot\boldsymbol{\omega}=0 \quad in \;\; \Omega$$

$$\mathbf{u}=\hat{\mathbf{u}} \quad on \;\; \Gamma_u$$

$$\boldsymbol{\omega}=\hat{\boldsymbol{\omega}} \quad on \;\; \Gamma_\omega$$

$$(\mathbf{u}\cdot\mathbf{U})^T+\nabla p-\frac{1}{Re}(\nabla\cdot\mathbf{U})^T=\mathbf{f}$$

$$\mathbf{U}-(\nabla\mathbf{u})^T=\mathbf{0}$$

$$\nabla\cdot\mathbf{u}=0$$

$$\nabla\times\mathbf{U}=\mathbf{0}$$

$$\nabla(tr\,\mathbf{U})=\mathbf{0}$$

$$\mathbf{u}=\hat{\mathbf{u}} \quad on \;\; \Gamma_u$$

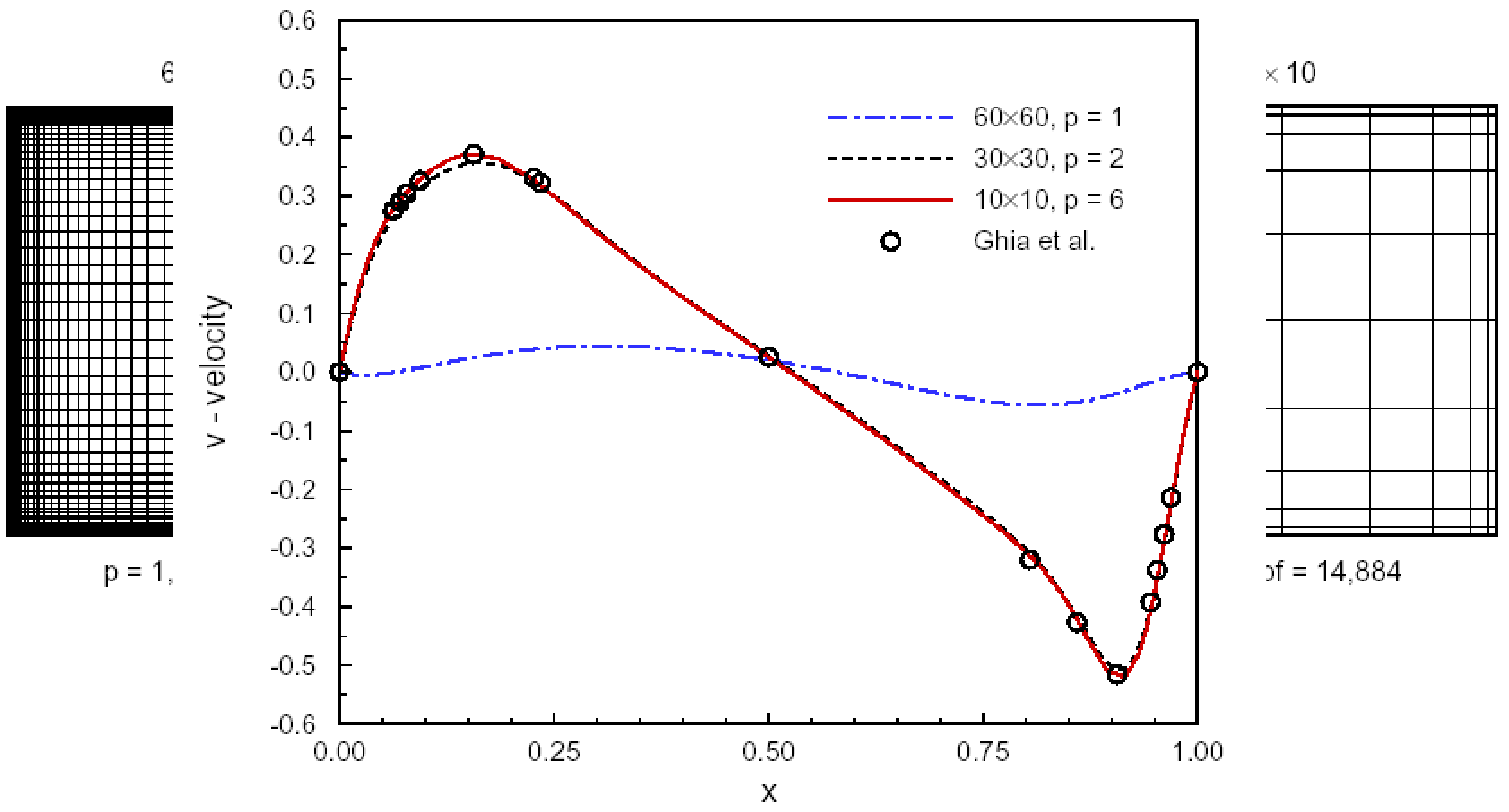
$$\mathbf{U}=\hat{\mathbf{U}} \quad on \;\; \Gamma_\omega$$

$$\mathcal{J}(\mathbf{u},p,\omega;\mathrm{f})=\frac{1}{2}\Big(\|(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p+\frac{1}{\mathrm{Re}}\nabla\times\omega-\mathrm{f}\|_0^2+\|\omega-\nabla\times\mathbf{u}\|_0^2$$

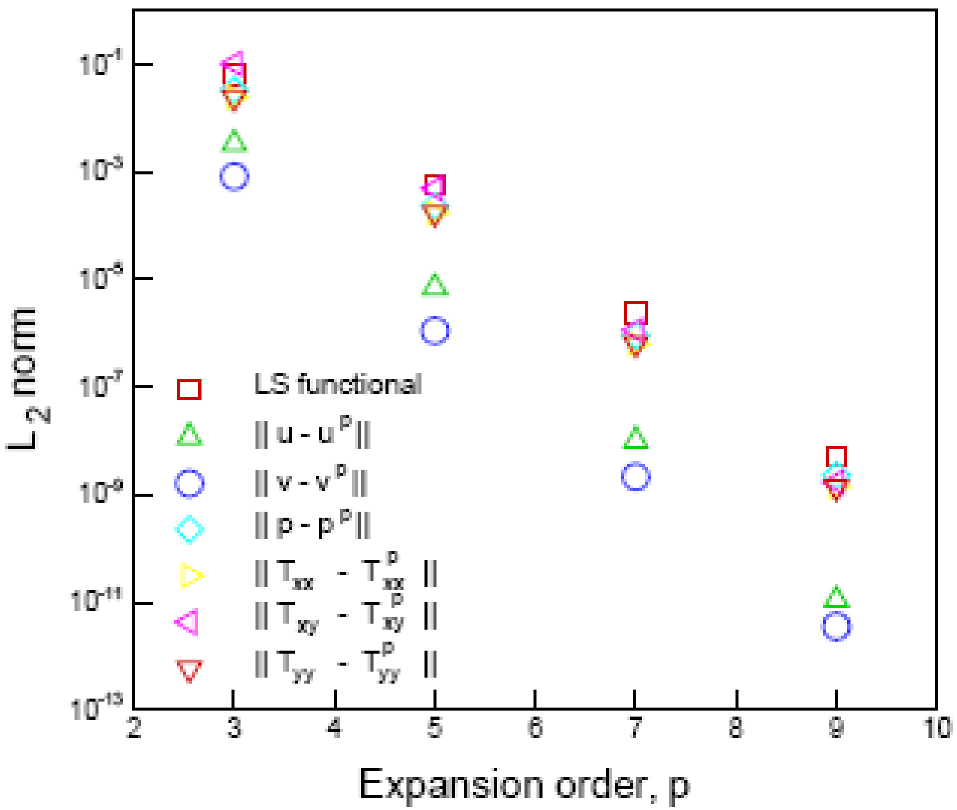
$$+\|\nabla\cdot\mathbf{u}\|_0^2+\|\nabla\cdot\omega\|_0^2\Big)$$

NUMERICAL EXAMPLES

Lid-driven Cavity-1

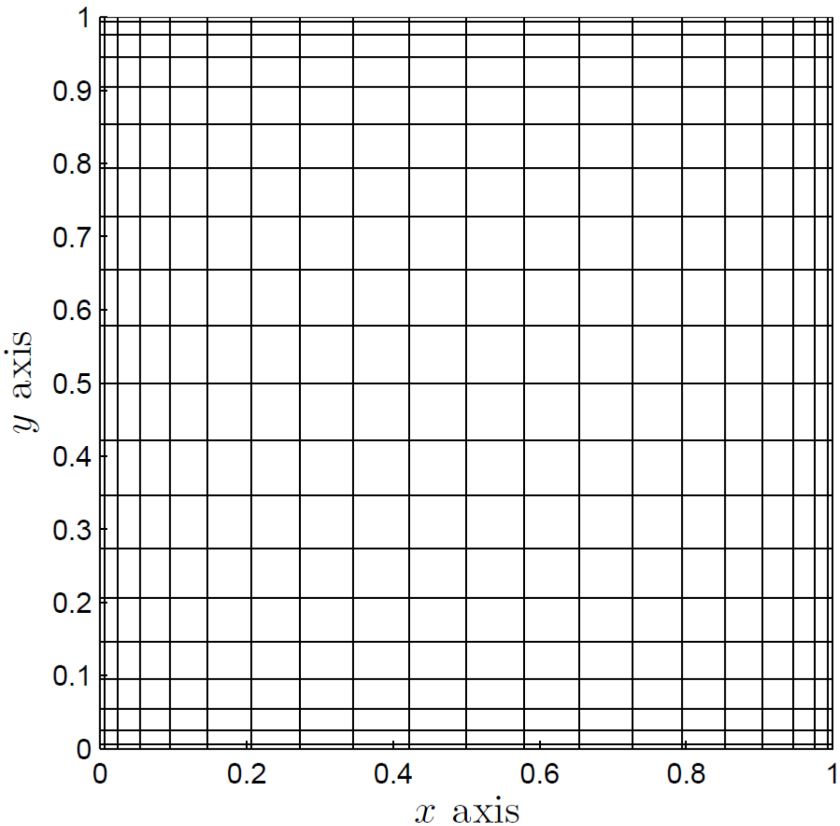


Lid-driven Cavity-2

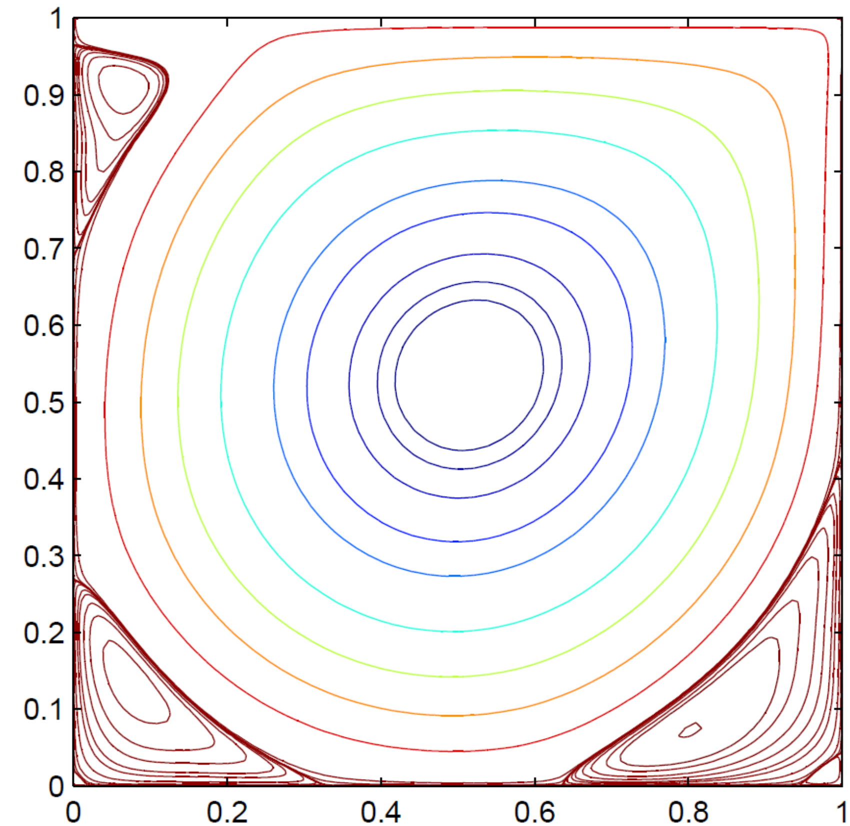


Lid-driven Cavity-3

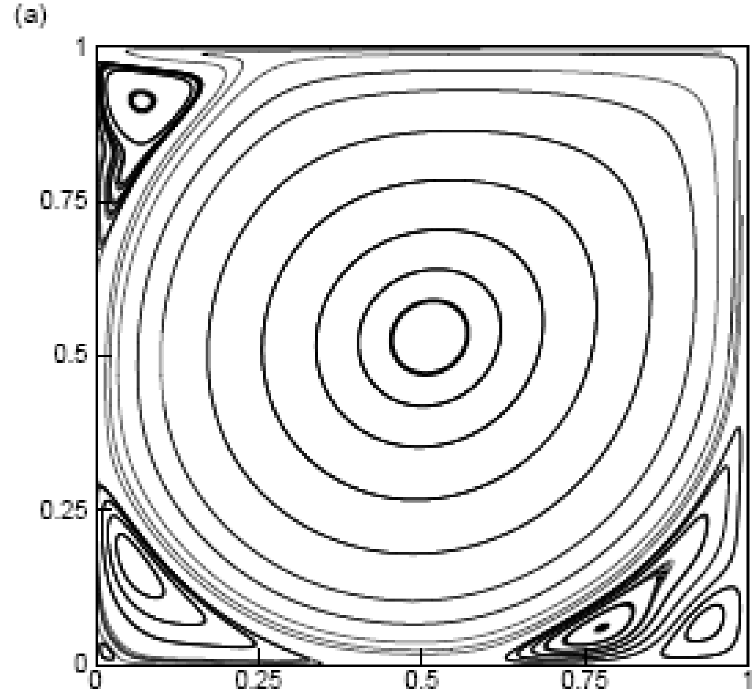
Finite element mesh (20x20)



Stream function (Re=5,000,

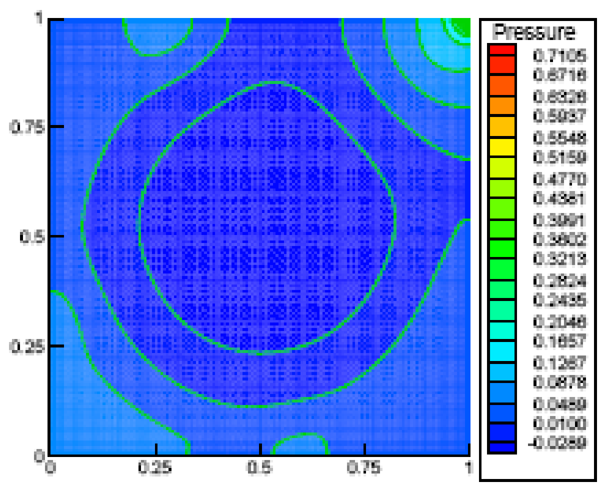


$$\mathrm{Re} = 10^4$$

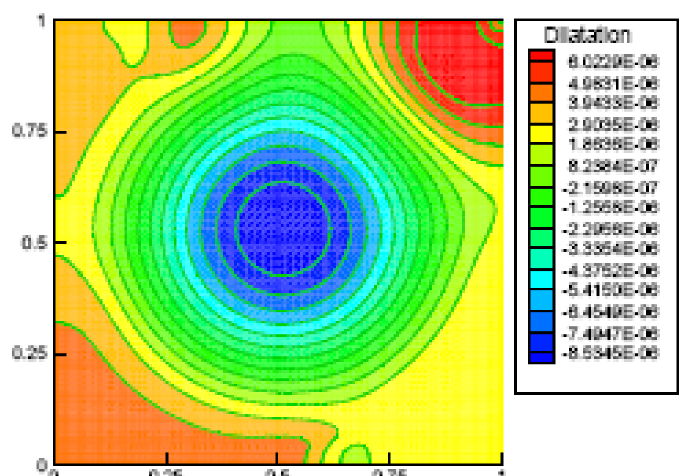


Streamlines

Pressure contours



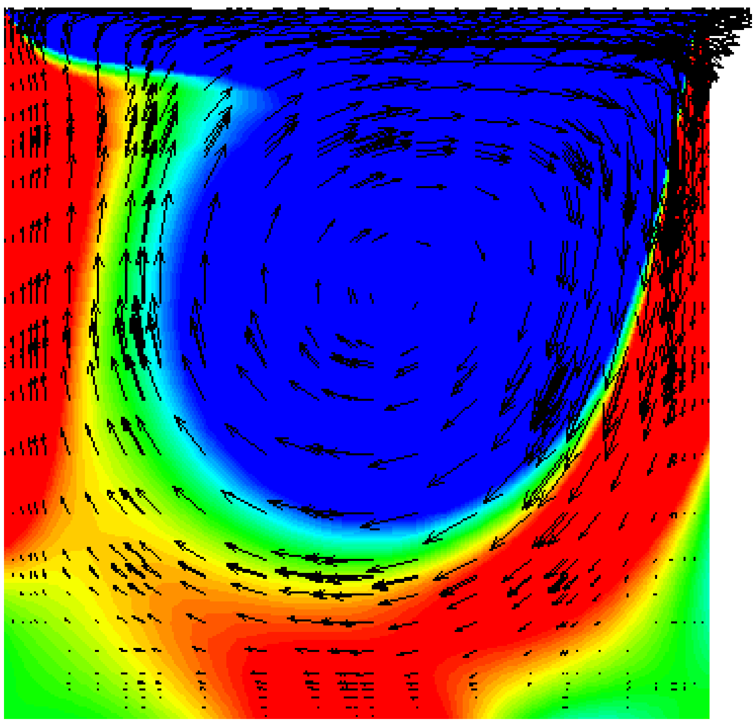
Dilatation contours



Oscillatory flow of a viscous incompressible fluid in a lid-driven cavity

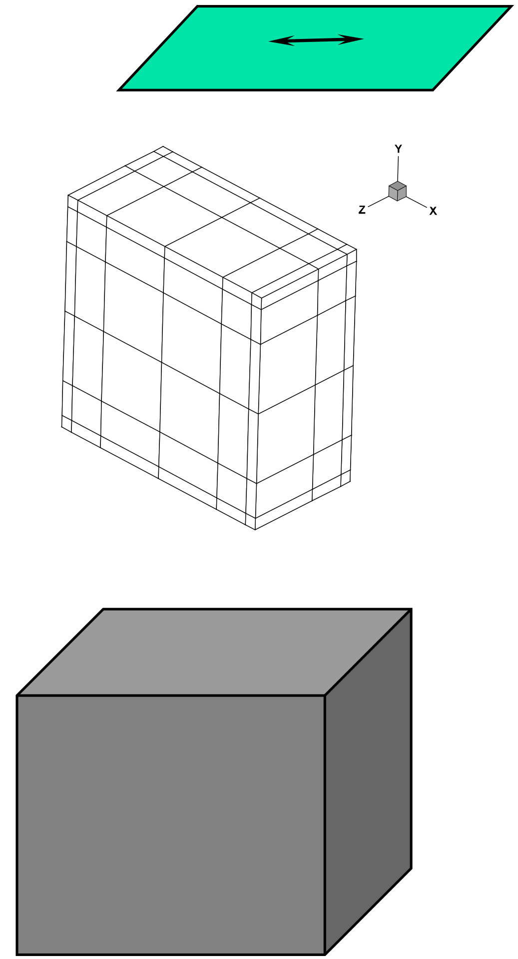
Oscillatory Lid-Driven Cavity Flow

Velocity Vector Field and Vorticity Contours

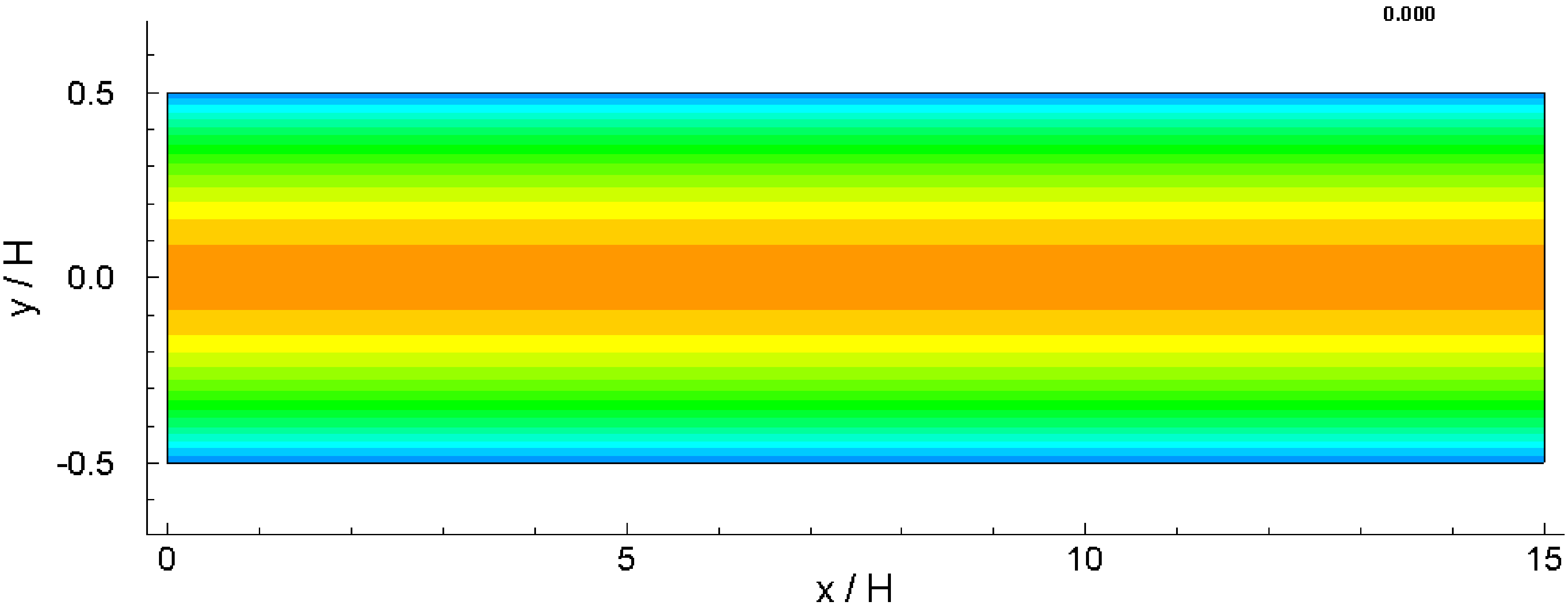


Non-stationary incompressible N-S equations, Re = 400
Least-Squares space / time decoupled formulation
6 × 6 mesh with p = 5

J.P. Pon
TAMU



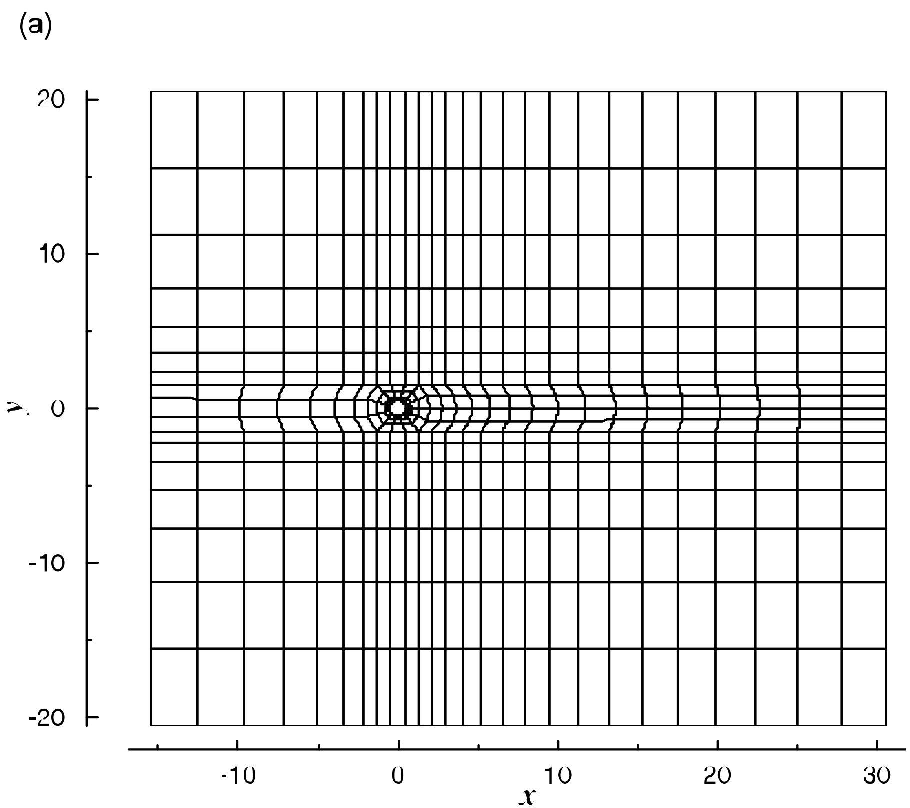
Flow of a viscous fluid in a narrow channel (backward facing step)

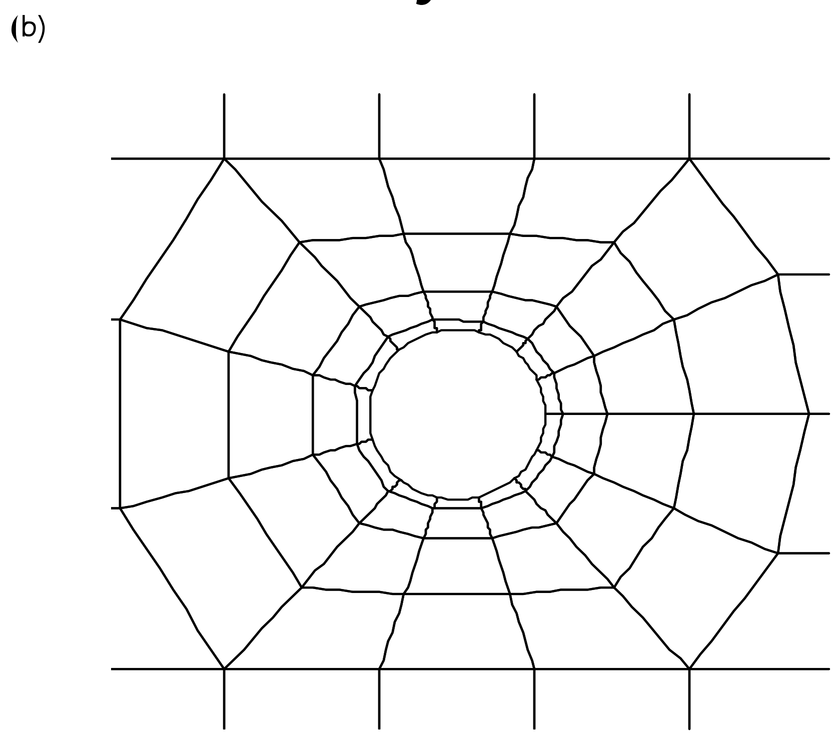


Flow of a Viscous Incompressible Fluid around a Cylinder-1

Mesh (501 elements; p=4)

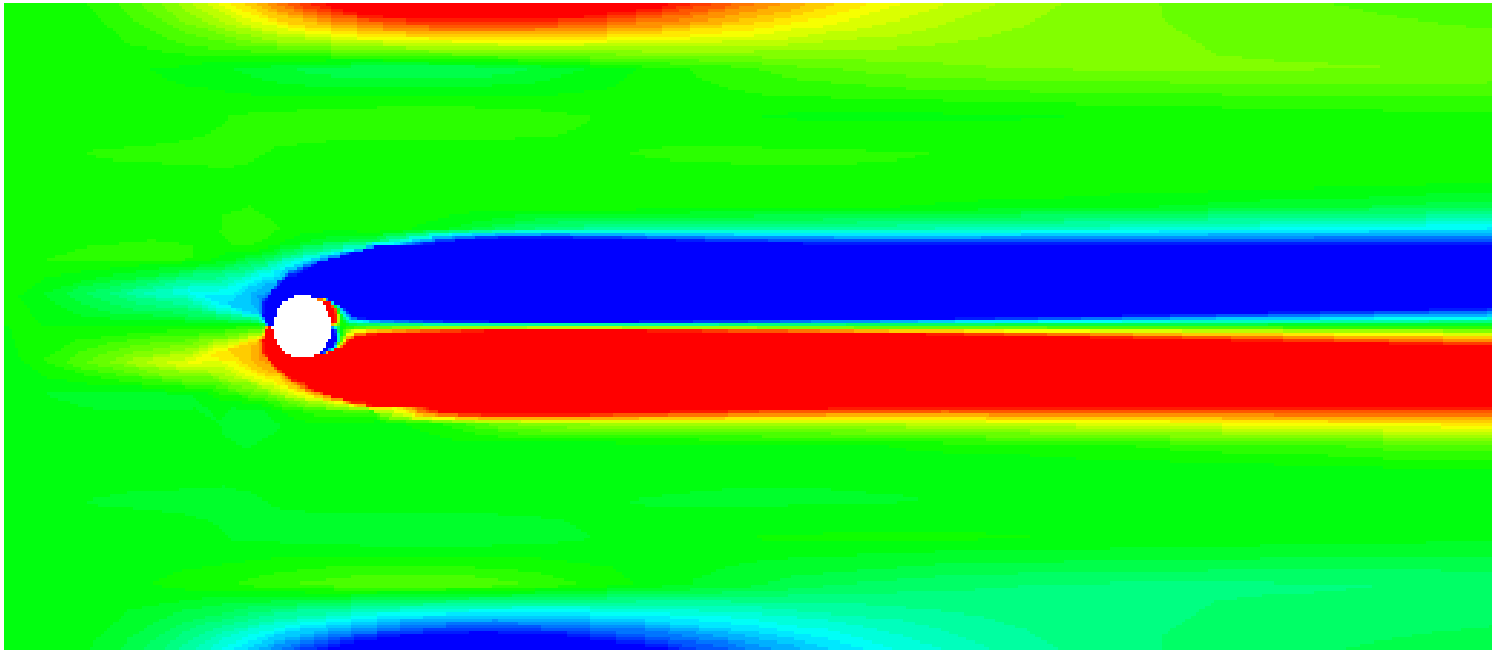
Close-up of mesh around the cylinder





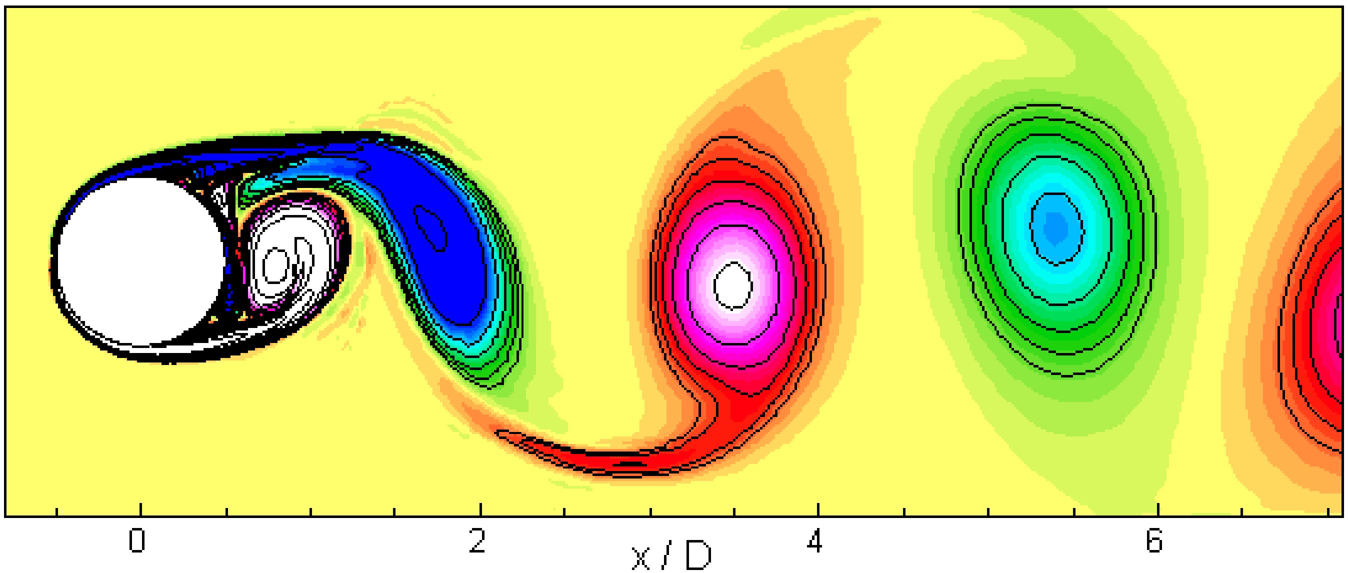
Circular Cylinder in Crossflow

Vorticity Contours



Non-stationary incompressible N-S equations, Re = 100
Least-Squares time / space decoupled formulation
1200 elements with p = 2

2D Flows Past a Circular Cylinder-2

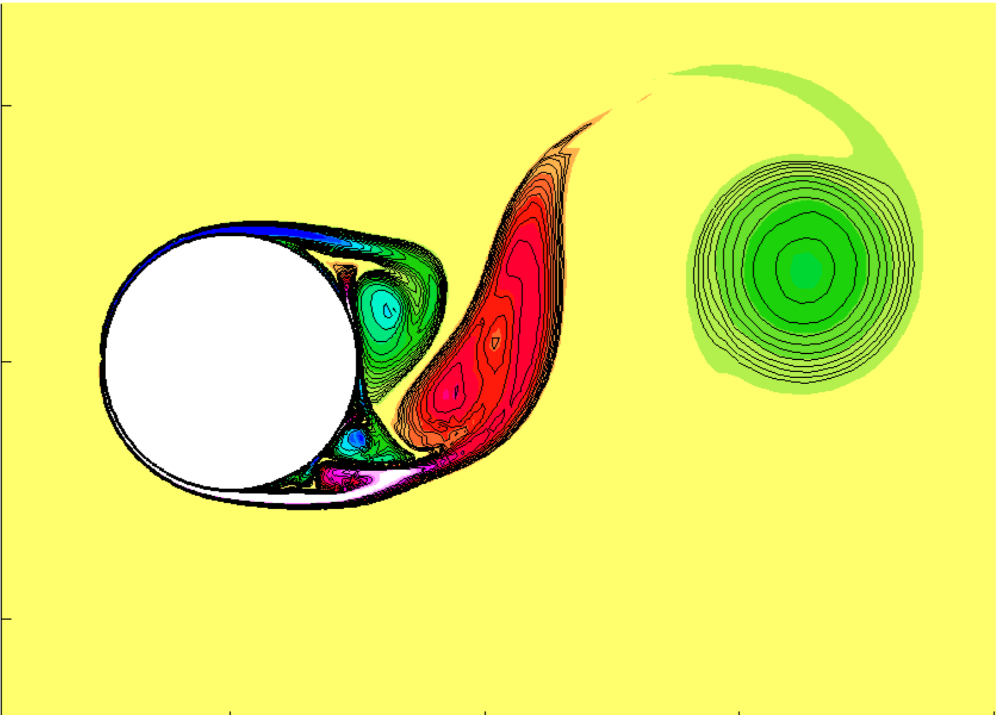


- Robust at moderately high Reynolds numbers: Re = 100 – 10^4
- High *p*-level solution: $p = 4, 6, 8, 10$
- No filters or stabilization are needed

Flow of a viscous fluid past a circular cylinder-3

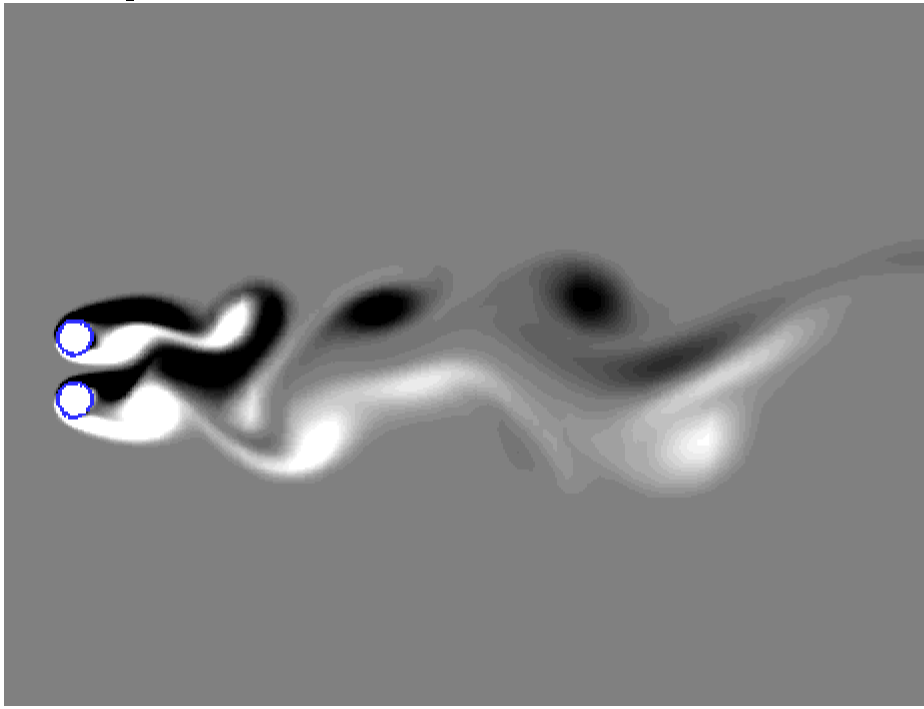
vorticity contours

25.000



Incompressible flow past two circular cylinders in a side-by-side arrangement
surface-to-surface gap, S/D = 0.85 , Re = 100
vorticity, ω_z , contours

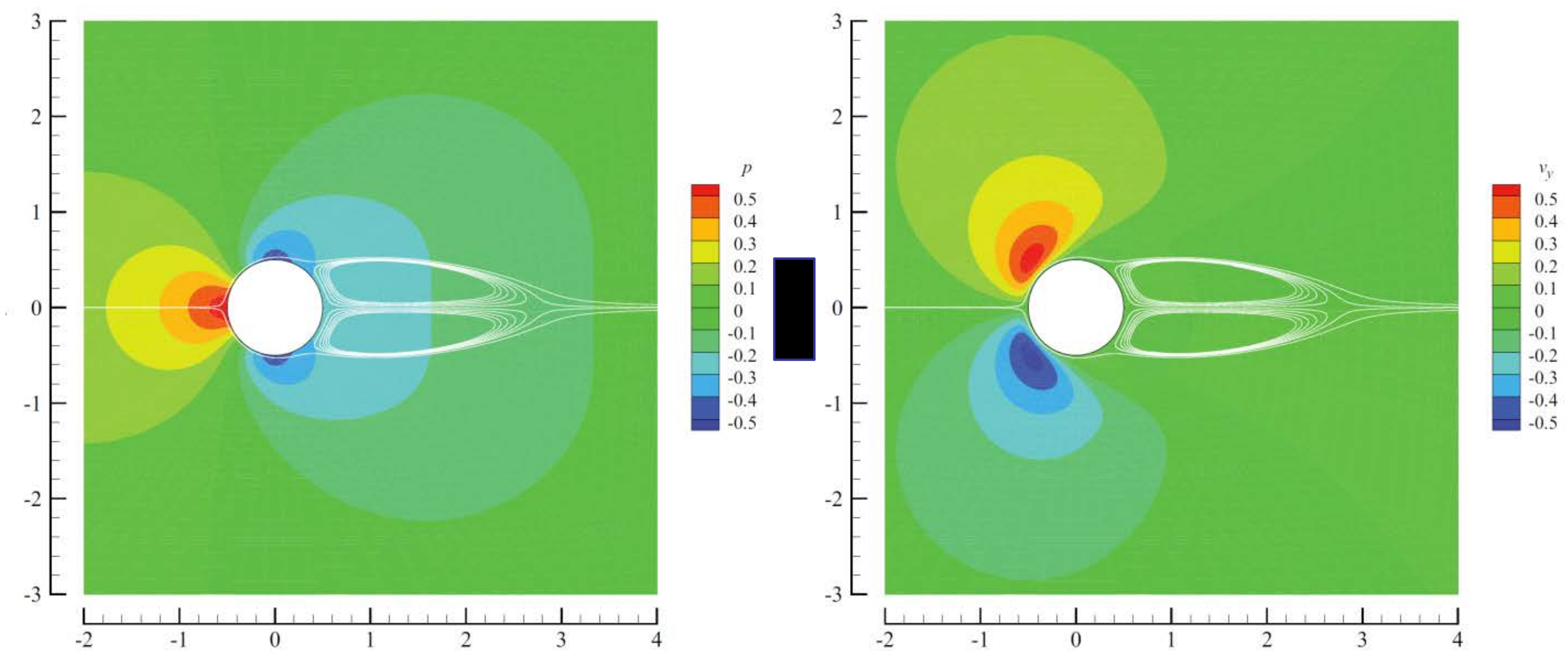
300.000



Least-squares finite element formulation
p-levels of 4/4/2 in space-time

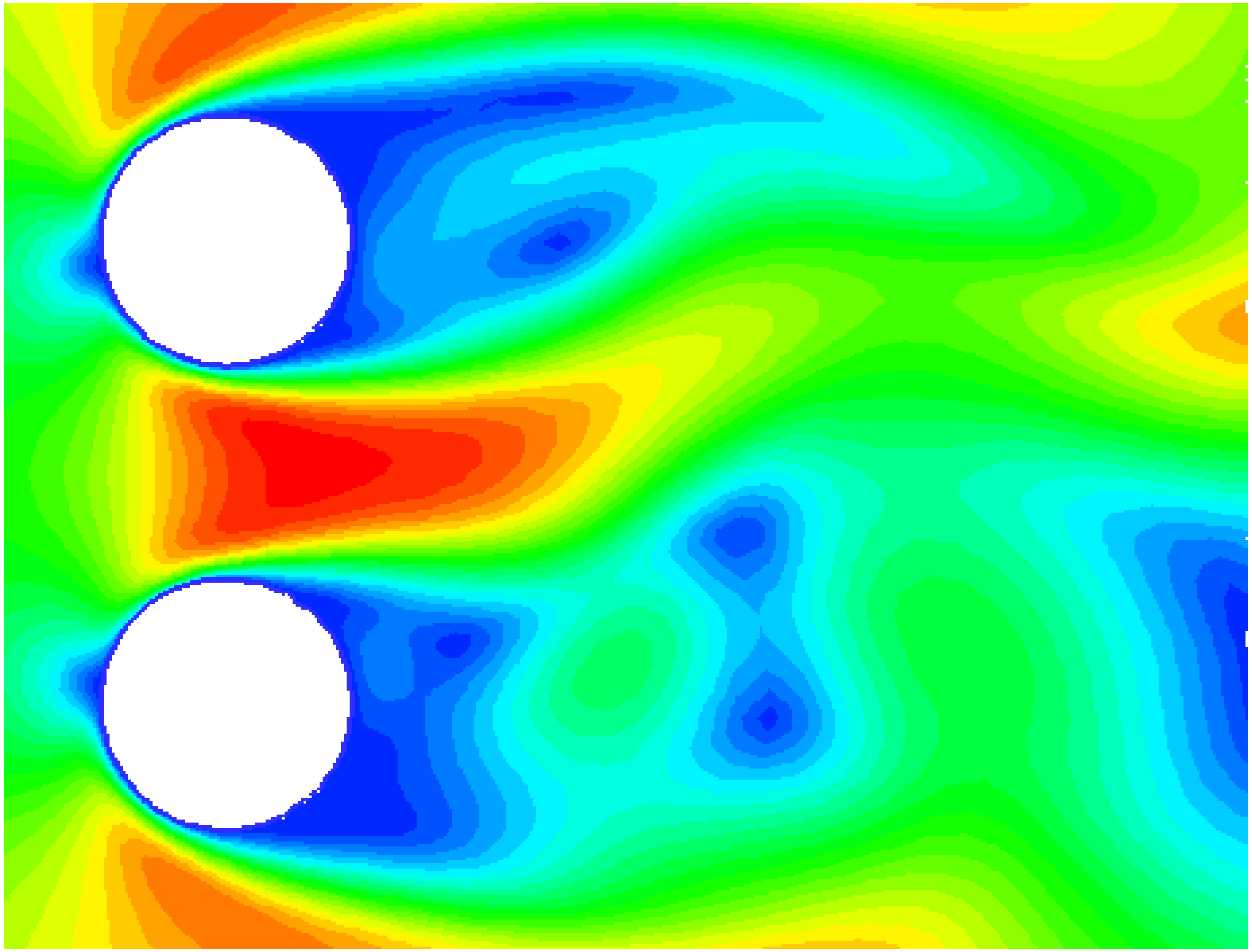
J.P. Pontaza, 20

Steady Flow Past a Circular Cylinder-4



Flow Past Two Circular Cylinders

Incompressible flow past two circular cylinders in a side-by-side arrangement
surface-to-surface gap, S/D=0.85 , Re=100
velocity magnitude contours showing the "bistable gap jet" **300.000**

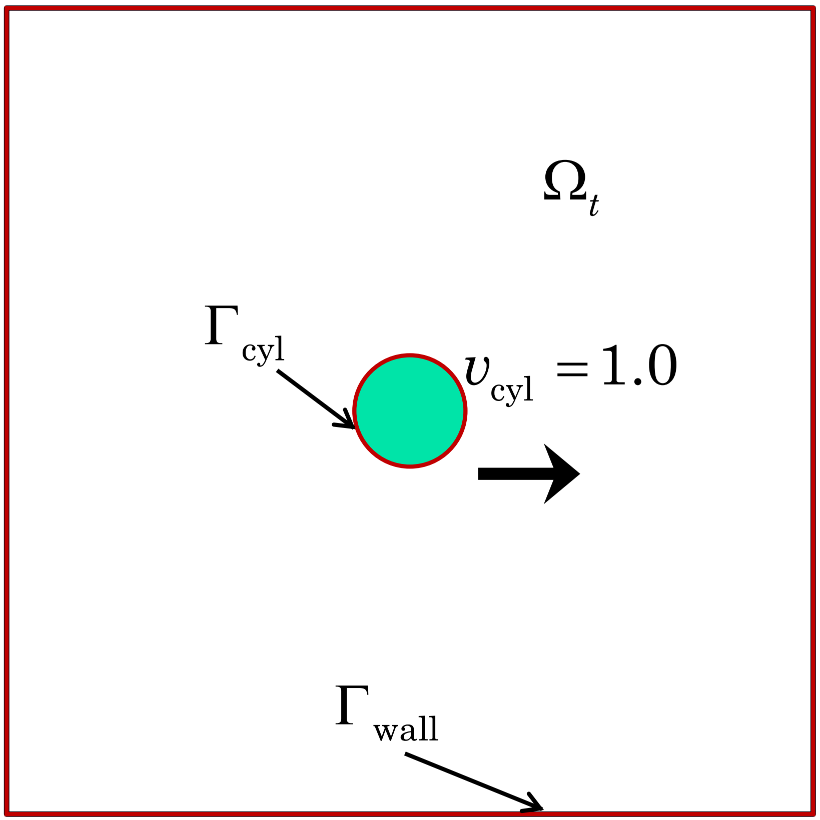


Least-squares finite element formulation
p-levels of 4/4/2 in space-time

J.P. Pontaza, 2004

Motion of a Cylinder in a Square Cavity

Initial boundary value problem



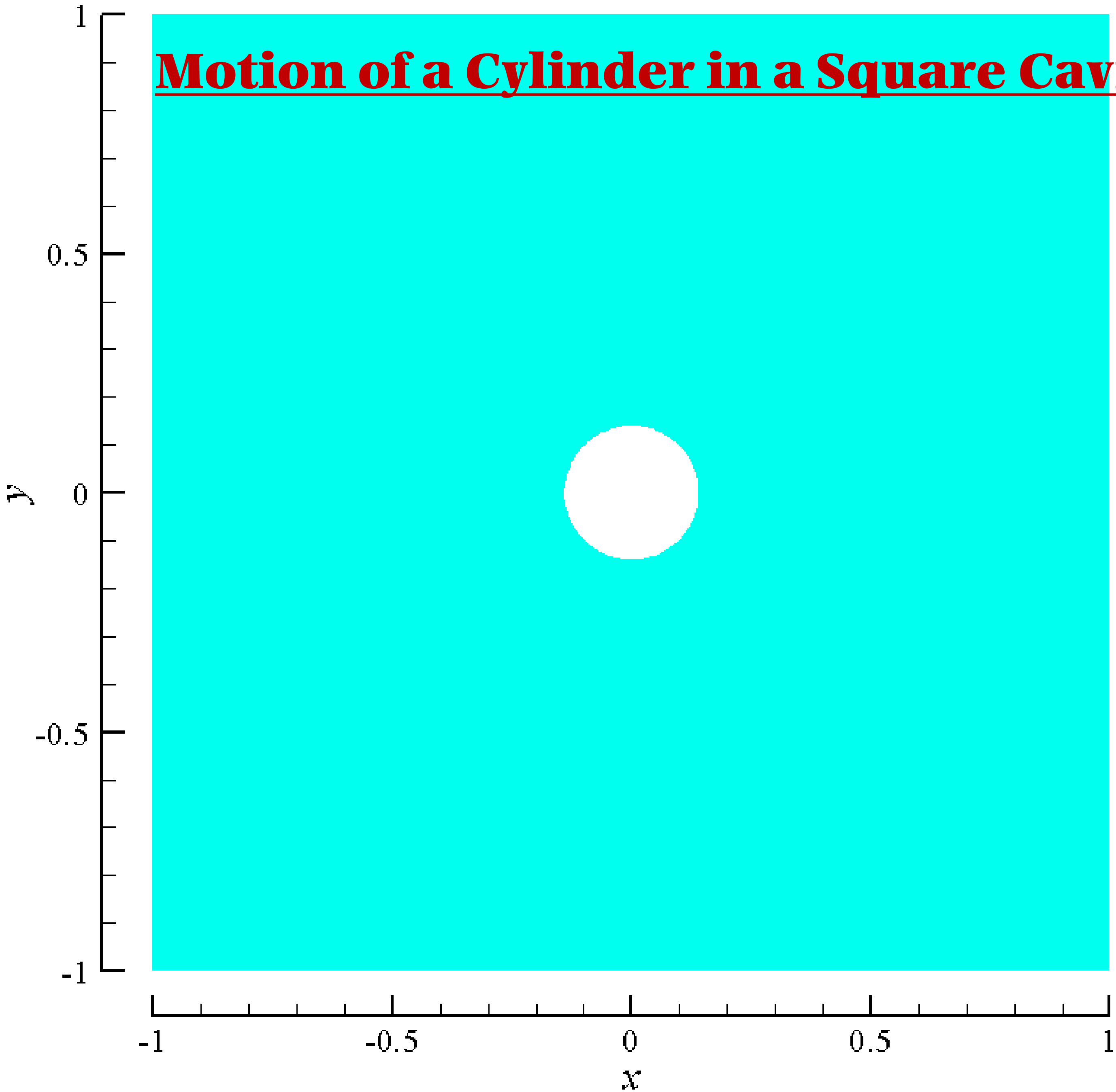
Problem parameters

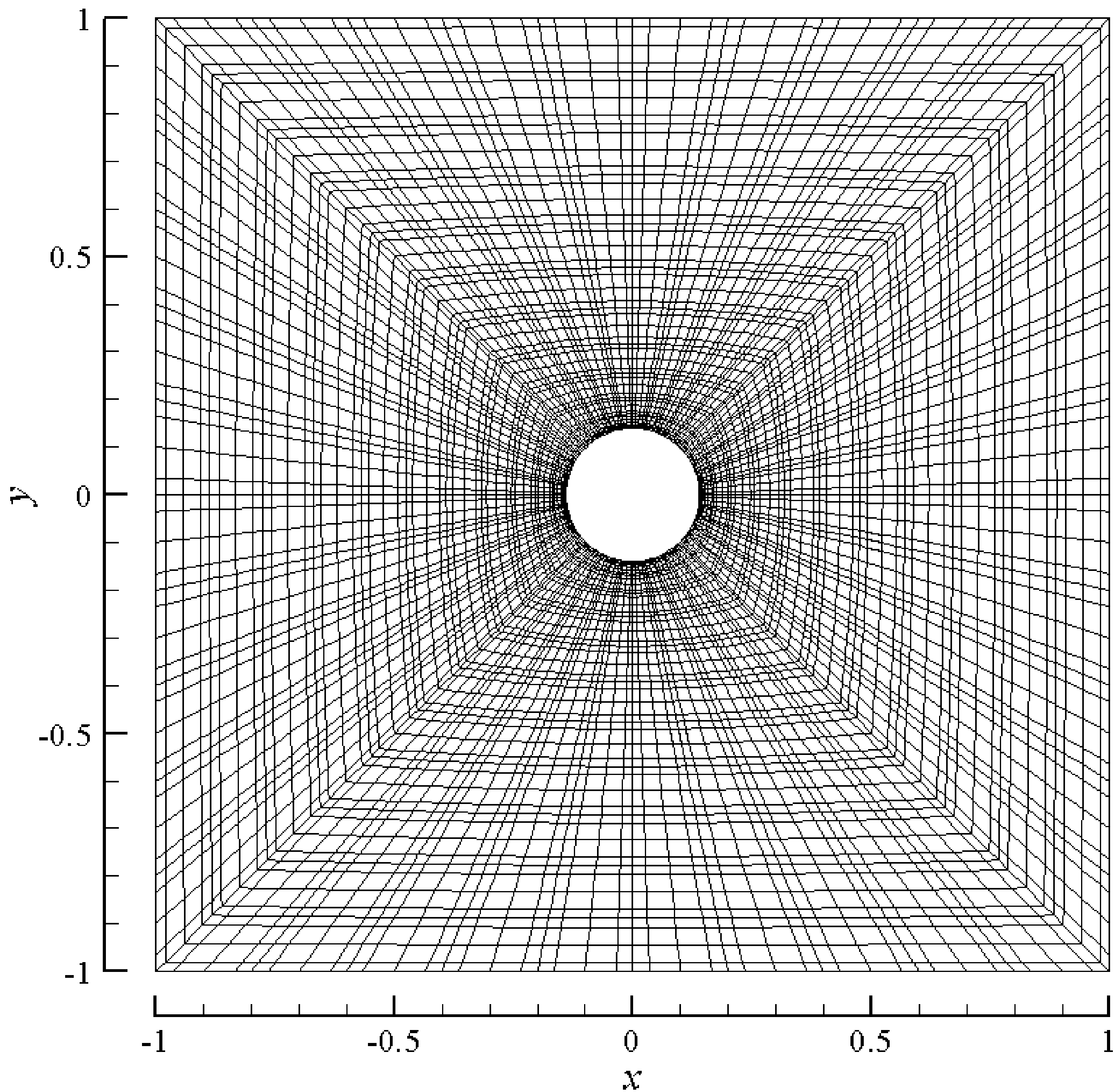
- $\rho = 1$, Re = 100
- $\mathbf{v}_{walls} = 0$ $\quad v_{cyl} = 1.0$
- $t \in (0, 0.70]$

Finite element discretization

- NE = 400, p-level = 4
- 31,360 degrees of freedom
- Time step: Δt=0.005
- $\alpha = 0.5$ (α-family)
- $\varepsilon = 10^{-6}$ (nonlinear iteration)

Motion of a Cylinder in a Square Cavity





1
0.5
0
-0.5
-1
y
-1
-0.5
0
0.5
1
x

Fluid-Solid Interaction
(movement of a rigid solid circular cylinder in a viscous fluid)