Lecture 07: Multirate Digital Signal Processing

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1st Semester 2009/ 2552

Lecture Contents

Outline

What is a Multirate Digital Signal Processing?

- A **digital signal processing system** that uses signals with **different sampling frequencies** is probably performing

  multirate digital signal processing.

- **Multirate digital signal processing** often uses **sample rate conversion** to convert from

  one sampling frequency to another *sampling frequency*.

- **Sample rate conversion** uses

  decimation to **decrease** the sampling rate,

  interpolation to **increase** the sampling rate.

Sample Rate Conversion

Changing the **sampling frequency** in the **analog domain** requires:

- **digital to analog** conversion then
- **analog to digital** conversion at a **different sampling frequency**.

Both

- **Digital to analog conversion**
- **Analog to digital conversion**

introduce **errors** and **noise** into the signal.
Therefore **sample rate conversion** is done in **digital domain** and uses a combination of:

- **Decimation,**
- and **Interpolation.**

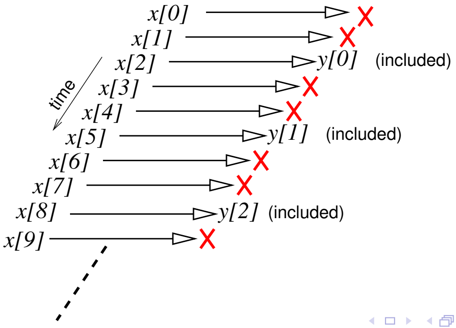
Outline

Decimation (for Downsampling)

- **Decimation** removes samples from a signal.
- **Decimation** can therefore only **downsample** the signal by an **integer factor**:

$$\frac{f_{\mathrm{s}}}{f_{\mathrm{s}}^{\mathrm{new}}} = D > 1 \qquad \text{so that } f_{\mathrm{s}} > f_{\mathrm{s}}^{\mathrm{new}}$$

where D is an integer, f_{s} is the old sampling rate (number of samples per second) and $f_{\mathrm{s}}^{\mathrm{new}}$ is the new sampling rate.



Anti-aliasing for Decimation

Decimation decreases the sampling rate.

- The **sampling theorem** states that the highest frequency in a signal should be less than **half the sampling frequency**.
- A digital **anti-aliasing filter** has to be applied to **remove frequencies** higher than:

$$f_{\mathrm{cf}} = \frac{f_{\mathrm{s}}^{\mathrm{new}}}{2}$$

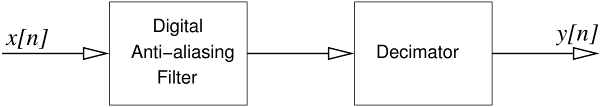
- So in digital frequency the cut-off frequency is:

$$\Omega_{\mathrm{cf}} = \frac{\Omega_{\mathrm{s}}^{\mathrm{new}}}{2} = \frac{2\pi \frac{f_{\mathrm{s}}^{\mathrm{new}}}{f_{\mathrm{s}}}}{2} = \pi \frac{f_{\mathrm{s}}^{\mathrm{new}}}{f_{\mathrm{s}}} < \pi$$

as $f_{\mathrm{s}}^{\mathrm{new}} < f_{\mathrm{s}}$.

Anti-aliasing for Decimation

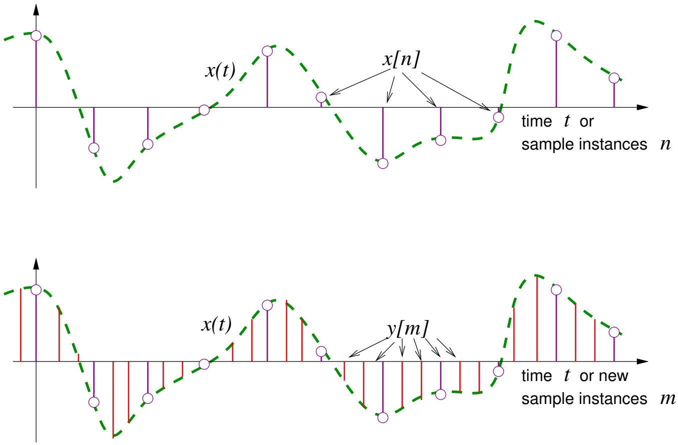
This means that the signal has to be **filtered** in the **digital domain** before **decimation**:



Outline

Interpolation (for Upsampling)

- **Interpolation** increases the sampling frequency by estimating the value of the signal between samples.



Interpolation

- The **new sampling frequency** is **greater** than the **old sampling frequency**:

$$f_s^{\text{new}} > f_s$$

where f_s is the old sampling frequency and f_s^{new} the new sampling frequency.

- Also, the new sampling frequency has to be **an integer multiple** of the original sampling frequency:

$$\frac{f_s^{\text{new}}}{f_s} = D > 1$$

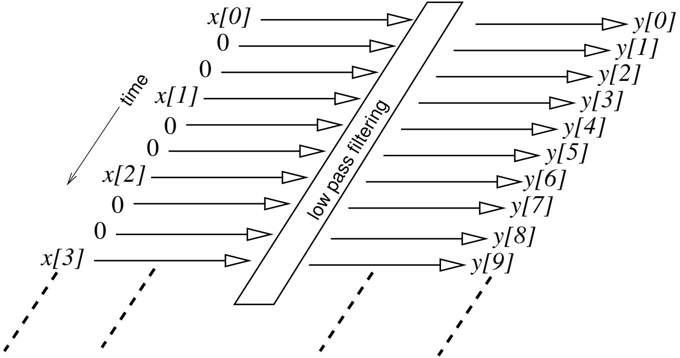
where D is an **integer**.

Zero Filling Based Interpolation

A common interpolation approach is **zero filling based interpolation**.

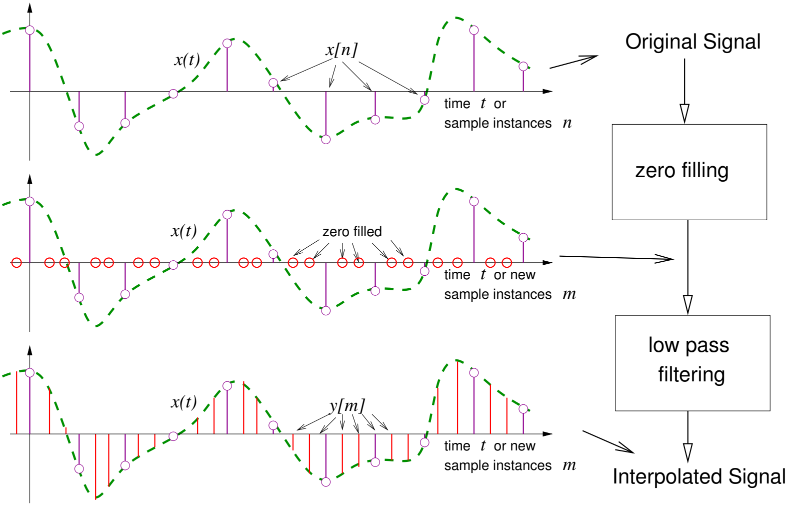
There are two stages:

1. **zero filling**
2. **low pass filtering**



Zero Filling Based Interpolation

Example: Interpolating by $\times$ 3 (two zero samples are inserted between each original sample).



Outline

Non-Integer Sample Rate Conversion

Both:

- **Decimation** *(for downsampling)*:

$$\frac{f_{\mathrm{s}}}{f_{\mathrm{s}}^{\mathrm{new}}} = D$$

- and **Interpolation** *(for upsampling)*:

$$\frac{f_{\mathrm{s}}^{\mathrm{new}}}{f_{\mathrm{s}}} = D$$

where D is an integer, can only **change** the **sampling frequency** to an **integer** of the **original frequency**.

Non-Integer Sample Rate Conversion

Example:

- A CD player stores music at 44.1kHz.
- A professional music recording device processes audio at 48kHz.
- Transfer of the music to or from the CD player and the professional audio device using:
  - **decimation only** or
  - **interpolation only**

  are **not possible** because:

$$\frac{48e3}{44.1e3} = 1.0884$$

  which is not an integer.

Non-Integer Sample Rate Conversion

Solution!

Combine decimation and interpolation to get **non-integer** *sample rate conversion*.

Similar to finding a **common denominator** in fractions...

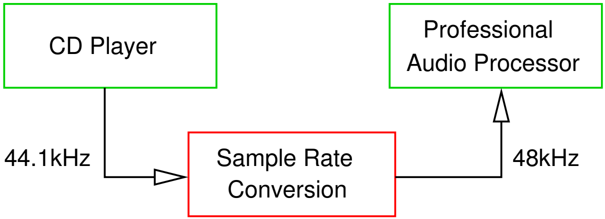
1. Find **common (integer) factor** of the two sample rates, L
2. **Interpolate (upsample)** by this **common factor** L
3. **Decimate (downsample)** to the **new sample rate** f_s^{new} by downsampling by an integer factor M.

The sample rate converison is then:

$$\frac{L}{M} = \frac{f_s^{\text{new}}}{f_s}$$

Non-Integer Sample Rate Conversion *Example*

Example:
Get audio from 44.1kHz sampled source (CD player) and transfer to professional audio processor requiring 48kHz sample rate.
A. This process requires upsampling to 48kHz from 44.1kHz.



Non-Integer Sample Rate Conversion *Example cont'd.*

1. **Worst case common factor**: $L = 48$kHz to give $f_s \times 48$kHz $= 2116.8$MHz.
   Better alternative is $L = 160$ to give $L \times 44.1$kHz $= 7056$kHz
2. So **interpolate** by factor L by **inserting 159 zeros** for each sample in 44.1kHz CD player signal then **low pass filtering.**
3. Then **decimate** to 48kHz by **removing 146 samples** in every 147 ($= L \times 44.1$kHz$/48$kHz) from the **upsampled signal** (*after applying anti-aliasing low pass filter*).

The resulting sample rate conversion is:

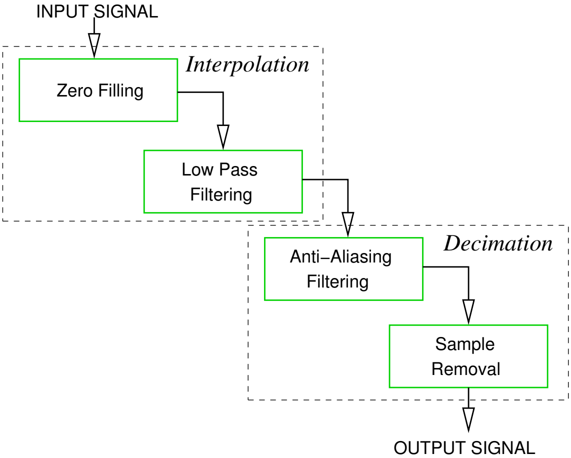
$$\frac{L}{M} = \frac{160}{147} = 1.088$$

which is the same as

$$\frac{f_s^{\text{new}}}{f_s} = \frac{48kHz}{44.1kHz} = 1.088.$$

Optimising Non-Integer Sample Rate Conversion

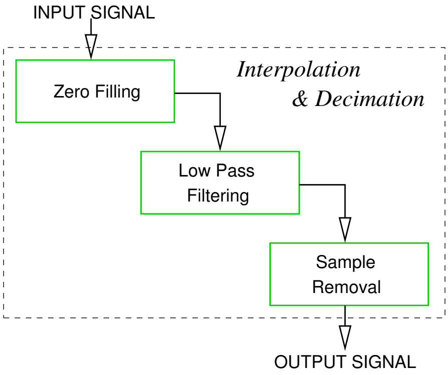
There are $\times 2$ **low pass filters** (*low pass filtering and anti-aliasing filtering*) for **non-integer sample rate conversion**:



Optimising Non-Integer Sample Rate Conversion

The **interpolation low pass filter** and the **anti-aliasing filter for the decimation** stage can be **combined**

with a cut-off frequency equal to the lower of the two filters' cut-off frequencies.



Outline

Multistage Sample Rate Conversion

Problem!

In real world applications sample rate conversion converts a **sampling frequency** to **another sampling frequency** that is:

- Very much greater ($f_s^{new} \gg f_s$) or
- Very much smaller ($f_s^{new} \ll f_s$)

than the **original signal sampling frequency**.

But what is wrong with this?

This is best explained by an example.

Multistage Sample Rate Conversion: Problem

Q. A signal $x[n]$, sampled at 4.096kHz has to be decimated to 128Hz. There should be an antialiasing filter:

- that rejects frequencies above 64Hz,
- with a stopband ripple, $\delta_s \approx 0.001$,
- and a passband ripple of $\delta_p \approx 0.001$.
- The transition width should be f_{tw} =4Hz,
- so that frequencies below 60Hz are kept.

A. A Blackman window can achieve a stop band ripple 75dB and passband ripple of 0.0014dB.
This can be compared with the requirements of this antialiasing filter of $\delta_s \approx 0.001$, which is $-20\log(0.001) = 60dB$ and a passband ripple $\delta_p \approx 0.001$, or $20\log(1+0.001) = 0.0087dB$.
However, according to the low pass FIR filter design guidelines in [Van de Vegte, 2002][1], the number of filter coefficients for a Blackman window will then be:

$$N = 5.98 \times \frac{f_{\text{s}}}{f_{\text{tw}}} = 5.98 \times 4096/4 = 6123.5$$

So the number of filter coefficients is very high.

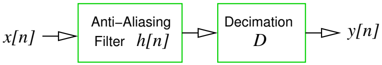
[1] Van de Vegte, "*Fundamentals of Digital Signal Processing*" Prentice Hall, 2002.

Multistage Sample Rate Conversion

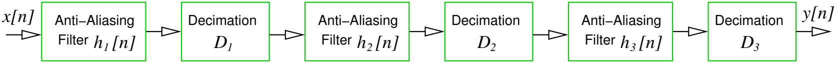
Multiple stages for **decimation** (*or interpolation*) can reduce the number of filter coefficients in the filter specifications.
The signal can be decimated more than once, using

a gradual change in sampling frequency.

Conventional decimation:



Decimation in mutliple stages (multistage):



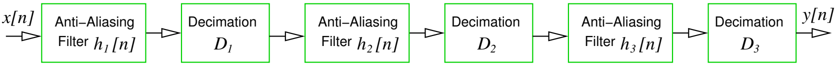
Multistage Sample Rate Conversion

The multi-stage sample rate conversion decimation values, D_i:

$$\frac{f_s}{f_s^{\text{new}}} = D = D_1 \times D_2 \times \ldots \times D_k = \prod_{i=1}^{k} D_i$$

where all D_i are integers. So for three stages, $k = 3$ and

$$D = D_1 \times D_2 \times D_3.$$



Multistage Sample Rate Conversion *Problem 2*

Q. The earlier problem can now be implemented using 2 decimation stages. Find out how many filter coefficients are necessary for a 2 stage decimation process.

A. The original sampling frequency $f_s = 4.096$kHz and the new (decimated signal) should have a sampling frequency of $f_s^{new} = 128$Hz. Multistage decimation with 2 stages requires that:

$$\frac{f_s}{f_s^{new}} = D = \frac{4096}{128} = 32 = D_1 \times D_2.$$

The multistage decimation values can therefore be $D_1 = 8$ and $D_2 = 4$, creating an intermediate signal with sampling frequency: $f_s^{(1)} = f_s/8 = 512$Hz.
The transition width can be longer with this higher sampling rate.
We can keep the same passband frequency (60Hz).
The transition width can go up to half the sampling rate:

$$f_{tw}^{(1)} \leq \frac{512Hz}{2} - 60Hz = 196Hz$$

The number of Blackman filter coefficients for this stage is:

$$N_1 = 5.98 \times \frac{f_s^{(1)}}{f_{tw}^{(1)}} = 5.98 \times \frac{512}{196} = 16, \text{ (rounded up to integer value).}$$

Multistage Sample Rate Conversion *Problem 2, cont'd.*

So $N_1 = 8$ filter coefficients are required for the first decimation stage.
The intermediate signal sampled at 512Hz is to be decimated by a factor of 4 to 128Hz for the second stage:

$$f_{\mathrm{s}}^{\mathrm{new}} = f_{\mathrm{s}}^{(2)} = \frac{512Hz}{4} = 128Hz.$$

The transition width for this (final) stage can then be:

$$f_{\mathrm{tw}}^{(2)} = \frac{128Hz}{2} - 60Hz = 4Hz$$

So that the number of Blackman filter coefficients for this stage is:

$$N_2 = 5.98 \times \frac{f_{\mathrm{s}}^{(2)}}{f_{\mathrm{tw}}^{(2)}} = 5.98 \times \frac{128}{4} = 192.$$

192 filter coefficients are required for this final stage. The combined filter coefficients for the two stages is:

$$N_1 + N_2 = 16 + 192 = 208,$$

which is considerably less than the original non-multistage decimation antialiasing filter requiring $N = 6124$ coefficients.

Outline

Lecture Summary

This lecture has covered

- **Decimation**,
- **Interpolation**,
- **Non-integer sample rate conversion**,
- **Multistage sample rate conversion**.

There are *many more* to topics and techniques in *multirate digital signal processing* including:

- **Implementation techniques, e.g. polyphase filters**
- **and Applications**.