# Lecture 11: Homogenization II.

Jack Xin*

**Abstract**

Classnotes on homogenization, references and practice.

## 1 Examples of Linear Transport Equations

Consider:

$$\begin{aligned} u_t &+ \mathbf{a}(\frac{x}{\epsilon}) \cdot \nabla_x u = 0, \\ u|_{t=0} &= u^0(x, \frac{x}{\epsilon}), \end{aligned} \tag{1.1}$$

where $\epsilon > 0$, $t \in [0,T]$, $\forall T > 0$, $u^0(x,y) \in C^1_0(R^2 \times T^2)$, $\mathbf{a} : T^2 \to R^2$ smooth, $div\mathbf{a} = 0$, $\mathbf{a} \neq 0$. Here $T^2$ denotes the 2-D unit torus, and $\mathbf{a}$ is the transport vector field for $u$. We are interested in the behavior of $u^\epsilon$ as $\epsilon \downarrow 0$ and the transport vector field oscillates faster and faster.

The leading order equation from two-scale expansion gives:

$$\mathbf{a}(y) \cdot \nabla_y U = 0 \quad y \in T^2 , \tag{1.2}$$

also known as the cell-problem. In general, (1.2) does not have a unique solution (unique up to a multiplicative function of (x,t)). Its uniqueness depends on the ergodicity of the flows defined by:

$$\frac{dy}{dt} = \mathbf{a}(y) \quad y \in T^2 . \tag{1.3}$$

The ergodicity is equivalent to saying that $U =$ constant (in $y$) is the only solution of (1.2) on $T^2$. In this case, two-scale expansion gives the homogenized equation:

$$\bar{u}_t + < \mathbf{a} > \cdot \nabla \bar{u} = 0. \tag{1.4}$$

*Department of Mathematics and TICAM, University of Texas at Austin, Austin, TX 78712. Email: jxin@math.utexas.edu.

## 2 A Two-scale Convergence Framework

We first introduce some notations.

$C_p(R^n)$: the space of continuous unit periodic functions on $R^n$.

$L_{K_0}(R^n)$: the space of all functions in $L^2(R^n)$ having compact support in $K_0$, where $K_0$ is a fixed compact set in $R^n$.

$L^2_p(R^n)$: the space of unit periodic functions in $L^2_{loc}(R^n)$ equipped with norm: $\|w\|_{L^2} = (\int_Y w^2 dy)^{1/2}$, where $Y$ is the unit lattice in $R^n$.

$L^2(R^n; L^2_p)$: the space of measurable functions $u(x,y)$ on $R^n \times R^n$, such that for almost all $x$, the function $y \to u(x,y)$ is in $L^2_p(R^n)$ and

$$\int_{R^n \times Y} |u|^2(x,y) dx dy \; < \; +\infty \; .$$

We define the norm for this space:

$$\|u\|_{L^2(R^n \times Y)} = (\int_{R^n \times Y} |u(x,y)|^2 dx dy)^{1/2} \; .$$

$K(R^n; C_p)$: space of continuous functions on $R^n$ with values in $C_p(R^n)$ and compact supports.

The following convergence theorem is due to Nguetseng [1] :

**Theorem 2.1** *Let $u^\epsilon(x) \in L^2_{K_0}(R^n)$, where $K_0$ is independent of $\epsilon$, $\epsilon \; > \; 0$. Suppose that $\|u^\epsilon\|_{L^2} \; \leq \; C$, for some constant $C \; > \; 0$ independent of $\epsilon$. Then there exist a subsequence from $\epsilon$, still denoted by $\epsilon$, and a function $U \in L^2(R^n; L^2_p)$ such that as $\epsilon \downarrow 0$:*

$$\int_{R^n} u^\epsilon \psi(x, \epsilon^{-1} x) dx \to \int_{R^n \times Y} U(x,y) \psi(x,y) dx dy$$

*for all $\psi \in K(R^n; C_p)$.*

We will use this theorem to derive the homogenized equations in two sample cases.

From equation (1.1), we see that due to the finite speed of propagation and $u^0$ being compactly supported, $u^\epsilon$ is supported in some compact set $K_0$ in $R^2 \times [0,T]$, for any $T \; > \; 0$. Multiplying $u$ and integration by parts give:

$$1/2 \partial_t \int_{R^2} u^2 dx + 1/2 \int_{R^2} \mathbf{a} \cdot \nabla_x u^2 dx = 0 \; , \tag{2.5}$$

and

$$1/2 \partial_t \int_{R^2} u^2 dx - 1/2 \int_{R^2} (div \mathbf{a}) u^2 dx = 1/2 \partial_t \int_{R^2} u^2 dx = 0 \; . \tag{2.6}$$

Thus

$$\int_{R^2} u^2 dx = \int_{R^2} (u^0)^2 dx < \ C < \ +\infty \ ,$$

$$\int_0^T \int_{R^2} u^2 dxdt \ \leq CT \ .$$

Therefore $u^\epsilon(x,t)$ satisfies the conditions in the above theorem, and there exists $U = U(x,t,y,\tau)$ such that as $\epsilon \downarrow 0$:

$$\begin{aligned} \int_{R^2\times[0,T]} \quad & u^\epsilon(x,t)\psi(x,t,\epsilon^{-1}x,\frac{t}{\epsilon})dxdt \to \\ & \int_{R^2\times[0,T]\times T^2\times[0,1]} U(x,t,y,\tau)\psi(x,t,y,\tau)dxdtdyd\tau \end{aligned} \tag{2.7}$$

for any $\psi \in K(R^2 \times [0,T]; C_p)$. If $\psi = \psi(x,t,\epsilon^{-1}x)$, then

$$\int_{R^2\times[0,T]} u^\epsilon(x,t)\psi dxdt \to \int_{R^2\times[0,T]\times T^2} \overline{U}(x,t,y)\psi(x,t,y)dxdtdy \tag{2.8}$$

where $\overline{U}(x,t,y) = \int_0^1 U(x,t,y,\tau)d\tau$.

In what follows, we will neglect the overbar and repeatedly use (2.8). First let $\varphi(x,t) \in C_0^\infty(R^2 \times [0,T))$, and $\varphi_0 = \varphi(x,0)$. We denote $\mathbf{a}(\epsilon^{-1}x)$ by $\mathbf{a}^\epsilon$, and $\mathbf{a}(y)$ by $\mathbf{a}$. Multiplying $\varphi$ to (1.1) and integrating by parts give:

$$\int_{R^2\times[0,T]} \{\varphi u_t^\epsilon + \varphi\nabla_x \cdot (\mathbf{a}^\epsilon u^\epsilon)\}dxdt = 0 \ , \tag{2.9}$$

and

$$\int_{supp\varphi} \varphi_t u^\epsilon dxdt + \int_{supp\varphi} (\nabla_x\varphi \cdot \mathbf{a}^\epsilon)u^\epsilon dxdt + \int_{supp\varphi_0} \varphi_0 u^0(x,\epsilon^{-1}x)dx = 0 \ . \tag{2.10}$$

Letting $\epsilon$ go to zero, and using (2.8), we have:

$$\begin{aligned} \int_{supp\varphi} \varphi_t (\int_{T^2} U(x,t,y)dy)dxdt \ & + \ \int_{supp\varphi} (\int_{T^2} (\nabla_x\varphi \cdot \mathbf{a})U(x,t,y)dy)dxdt \\ & + \ \int_{supp\varphi_0} \varphi_0 (\int_{T^2} u^0(x,y)dy)dx = 0 \ , \end{aligned}$$

or

$$\begin{aligned} \int_{supp\varphi} \quad & \varphi_t (\int_{T^2} U(x,t,y)dy)dxdt + \int_{supp\varphi} \nabla_x\varphi \cdot (\int_{T^2} U(x,t,y)\mathbf{a}dy)dxdt \\ & + \int_{supp\varphi_0} \varphi_0 (\int_{T^2} u^0(x,y)dy)dx = 0 \ . \end{aligned} \tag{2.11}$$

It follows that

$$(\int_{T^2} U(x,t,y)dy)_t + \nabla_x \cdot (\int_{T^2} U(x,t,y)\mathbf{a}(y)dy) = 0 \tag{2.12}$$

for any $t > 0$ in the weak $L^2$ sense, and

$$\int_{T^2} U(x,0,y)dy = \int_{T^2} u^0(x,y)dy \ , \tag{2.13}$$

where $U(x,0,y)$ is $\lim_{t\downarrow 0^+} U(x,t,y)$, and it is always understood as such in what follows.

As we see, equation (2.12) itself is not enough to determine the weak limit of $u^\epsilon$. This is due to the fact that the test fuction we used in obtaining (2.12) does not contain any information of the oscillatory component of $u^\epsilon$. To identify $U(x,t,y)$, we propose to use a complete set of oscillatory test functions $\varphi e^{2\pi i k\cdot\epsilon^{-1}x}$, where $\varphi = \varphi(x,t)$ belongs to $C_0^\infty(R^2 \times [0,T))$. It follows from (2.10) with the above chosen test function replacing $\varphi$:

$$\begin{aligned}\int_{R^2\times[0,T]} (\varphi_t e^{2\pi i k\cdot\epsilon^{-1}x} u^\epsilon \quad &+ \quad (\nabla_x\varphi\cdot\mathbf{a}^\epsilon) e^{2\pi i k\cdot\epsilon^{-1}x} u^\epsilon + \frac{2\pi i(k\cdot\mathbf{a}^\epsilon)}{\epsilon}\varphi e^{2\pi i k\cdot\epsilon^{-1}x} u^\epsilon)dxdt \\ &+ \quad \int_{R^2} \varphi_0 e^{2\pi i k\cdot\epsilon^{-1}x} u^0(x,\epsilon^{-1}x)dx = 0 \ .\end{aligned} \tag{2.14}$$

Multiplying $\epsilon$ and letting $\epsilon \downarrow 0$, we get:

$$\int_{R^2\times[0,T]\times T^2} (k\cdot\mathbf{a})\varphi(x,t)e^{2\pi i k\cdot y} U(x,t,y)dxdtdy = 0 \ , \tag{2.15}$$

thus for any $(x,t)$, $t > 0$:

$$\int_{T^2} (k\cdot\mathbf{a})e^{2\pi i k\cdot y} U(x,t,y)dy = 0 \ . \tag{2.16}$$

We recognize from (2.16) that $U$ is a weak solution of

$$div_y(U\mathbf{a}(y)) = 0.$$

In case that $\mathbf{a}(y)$ generates an ergodic flow [2], $U = U(x,t)$ is the only solution of the above equation. Let

$$\overline{u} = \int_{T^2} U(x,t,y)dy,$$

the weak limit of $u^\epsilon$, then by (2.12), $\overline{u}$ satisfies:

$$\overline{u}_t + \overline{\mathbf{a}}\cdot\nabla_x\overline{u} = 0, \tag{2.17}$$

where $\overline{\mathbf{a}} = \int_{T^2} \mathbf{a}(y)dy$, and $\overline{u}|_{t=0} = \int_{T^2} u^0(x,y)dy$.

Let us consider the case of $\mathbf{a} = \mathbf{a}(y)$ so that we can solve for $U(x,t,y)$ from (2.14) and (2.16) , but not from (2.16) alone. Assume that:

$$\mathbf{a}(y) = (\mathbf{a}_1, \mathbf{a}_2) = (m\rho(y), n\rho(y)) \tag{2.18}$$

where $\rho = g(-ny_1 + my_2)$, $g$ is a smooth positive 1-periodic function, $(m,n) \in Z^2$, $m^2+n^2 \neq 0$. It is easy to check that all the conditions on $\mathbf{a}(y)$ are satisfied. Define:

$$L = L_{(m,n)} = \{k \in Z^2 \mid k \cdot (m,n) = 0\} \,. \tag{2.19}$$

Then, if $k \notin L$, (2.16) shows:

$$\int_{T^2} \rho(y) e^{2\pi i k\cdot y} U(x,t,y)dy = 0 \quad t > 0 \,,$$

which implies:

$$\begin{aligned} \rho(y)U(x,t,y) &= \sum_{l\in L} e^{2\pi i l\cdot y} b_l(x,t) \,, \\ U(x,t,y) &= \sum_{l\in L} e^{2\pi i l\cdot y} \rho^{-1}(y) b_l(x,t) \quad t > 0 \,. \end{aligned} \tag{2.20}$$

On the other hand, if $k \in L$, then (2.14) shows that

$$\begin{aligned} \int_{R^2\times[0,T]} (\varphi_t e^{2\pi i k\cdot\epsilon^{-1}x} u^\epsilon &+ (\nabla_x\varphi \cdot \mathbf{a}^\epsilon) e^{2\pi i k\cdot y} u^\epsilon) dxdt \\ &+ \int_{R^2} \varphi_0 u^0(x, \epsilon^{-1}x) e^{2\pi i k\cdot\epsilon^{-1}x} dx = 0 \,. \end{aligned} \tag{2.21}$$

Letting $\epsilon \downarrow 0$ in (2.21) yields:

$$\begin{aligned} \int_{R^2\times[0,T]\times T^2} & (\varphi_t e^{2\pi i k\cdot y} U(x,t,y) + (\nabla_x\varphi \cdot \mathbf{a}) e^{2\pi i k\cdot y} U(x,t,y)) dxdtdy \\ & + \int_{R^2\times T^2} \varphi_0 u^0(x,y) e^{2\pi i k\cdot y} dxdy = 0 \,. \end{aligned} \tag{2.22}$$

It follows that

$$(\int_{T^2} e^{2\pi i k\cdot y} U(x,t,y)dy)_t + \nabla_x \cdot (\int_{T^2} e^{2\pi i k\cdot y} \mathbf{a}(y) U(x,t,y) dy) = 0 \,, \tag{2.23}$$

for any $t > 0$, and

$$\int_{T^2} U(x,0,y) e^{2\pi i k\cdot y} dy = \int_{T^2} u^0(x,y) e^{2\pi i k\cdot y} dy \,, \tag{2.24}$$

for any $k \in L$. Substituting (2.20) into (2.23), we get

$$(\int_{T^2} \sum_{l\in L} b_l \frac{e^{2\pi i (l+k)\cdot y}}{\rho(y)} dy)_t + \nabla_x \cdot (\int_{T^2} \sum_{l\in L} b_l e^{2\pi i (l+k)\cdot y} \frac{\mathbf{a}}{\rho(y)} dy) = 0 \,, \tag{2.25}$$

or

$$(\sum_{l\in L} b_l < \rho^{-1} >_{l+k})_t + m(b_{-k})_{x_1} + n(b_{-k})_{x_2} = 0 \;, \tag{2.26}$$

where $< \rho^{-1} >_{l+k} = \int_{T^2} e^{2\pi i(l+k)\cdot y}\rho^{-1} dy$ and $k \in L$. Substituting (2.20) into (2.24), we get

$$\sum_{l\in L} \int_{T^2} \rho^{-1}(y) e^{2\pi i(l+k)\cdot y} b_l(x,0) dy = \int_{T^2} u^0(x,y) e^{2\pi i k\cdot y} dy \;, \tag{2.27}$$

for $k \in L$. That is:

$$\sum_{l\in L} < \rho^{-1} >_{l+k} b_l(x,0) = < u^0 >_k \;, \tag{2.28}$$

which is the same as

$$\sum_{l\in L} < \rho^{-1} >_{l-k} b_l(x,0) = < u^0 >_{-k} \;, \tag{2.29}$$

for all $k \in L$.

Let $C = (c_{lk}) = (< \rho^{-1} >_{l-k})$ and $b = (b_l)^T$, where $k$ and $l$ are in $L$. It is easy to check that due to $\rho^{-1}$ being positive, $C$ is a bounded positive Hermitian infinite matrix from $l^2(C) \to l^2(C)$. That is, there exists a positive constant $c_0$, such that for any $x \in l^2(C)$

$$c_0 \|x\|_{l^2} \le x^T C \overline{x} \le c_0^{-1} \|x\|_{l^2}^2 \;.$$

Equation (2.26) with $k$ in the place of $-k$ then becomes:

$$C b_t + m b_{x_1} + n b_{x_2} = 0 \;. \tag{2.30}$$

Similarly equation (2.29) can be written as:

$$C b|_{t=0} = (< u^0(x,y) >_{-k})^T_{k\in L} \;, \tag{2.31}$$

which implies:

$$b|_{t=0} = C^{-1} (< u^0 >_{-k})^T_{k\in L} \;. \tag{2.32}$$

Equations (2.30) and (2.32) form an initial value problem of an infinite symmetric hyperbolic system. In fact, (2.30) and (2.32) can be solved by Fourier transform. Therefore,

$$u^\epsilon \rightharpoonup \overline{u}(x,t) = \sum_{l\in L} < \rho^{-1} >_l b_l = \int_{T^2} U(x,t,y) dy$$

where $b_l$'s are given by (2.30) and (2.32).

The reason that we find infinitely many equations indexed by $k \in L$ is that the underlying flow on $T^2$ has infinitely many isolated open channels (or that many ergodic components). In general, if the flow on $T^2$ has islands (area inside closed streamlines) and open channels, one gets zero homogenized speeds for closed islands and a number of nonzero speed wave equations for the open channels (same as the number of independent channels in the flow), see [4].

In summary, strong convergence (or $U = U(t,x)$) corrersponds to having one ergodic channel, weak convergence (or $U = U(t,x,y)$) corresponds to having more than one ergodic components in the flow on torus $T^2$.

*Practice Problem:* Derive the homogenized elliptic equation in Lecture 10 using the above framework.

# References


[1] G. Nguetseng, *A General Convergence Result for a Functional Related to the Theory of Homogenization.* SIAM J. Math. Anal., Vol. 20, No. 3, pp 608-623, May, 1989.

[2] J. G. Sinai, *Introduction to Ergodic Theory.* Princeton University Press: Princeton. 1976.

[3] T. Hou, J. Xin, *Homogenization of Linear Transport Equations with Oscillatory Vector Fields*, SIAM Journal Applied Math, Vol. 52, No. 1, pp 34-45, 1992.

[4] W. E, *Homogenization of Linear and Nonlinear Transport Equations*, Comm Pure Appl Math, XLV, pp 301-326, 1992.