Lecture 13

Extra Sums of Squares

STAT 512

Spring 2011

Background Reading

KNNL: 7.1-7.4

Topic Overview

- Extra Sums of Squares (Defined)
- Using and Interpreting R^2 and Partial-R^2
- Getting ESS and Partial-R^2 from SAS
- General Linear Test (Review Section 2.8)
- Testing single $\beta_k = 0$
- Testing several $\beta_k = 0$
- Other General Linear Tests

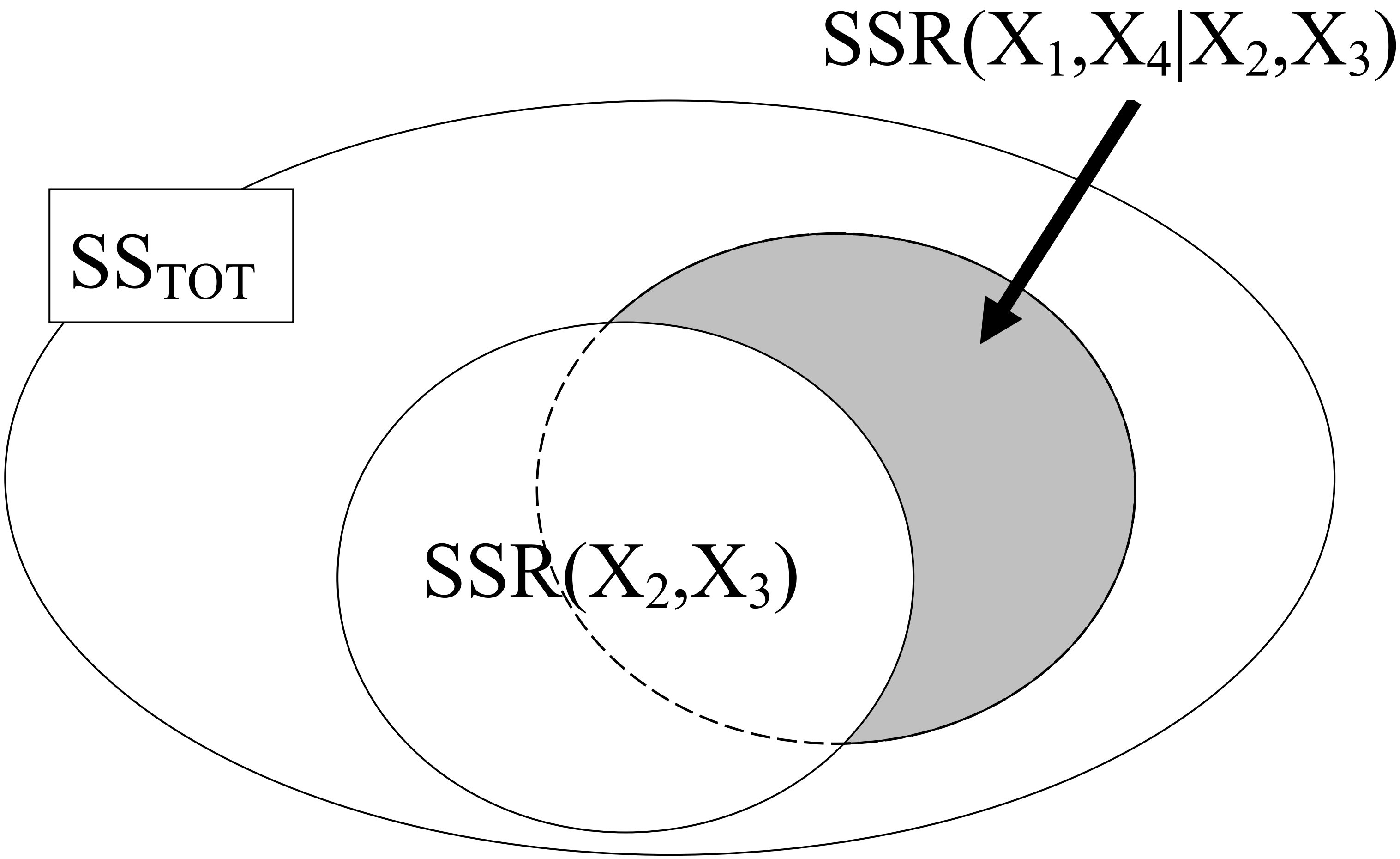
Extra Sums of Squares

- ESS measure the ***marginal*** reduction in the error sum of squares from the addition of a group of predictor variables to the model.
- Examples
  - $SSR(X_1, X_2, X_3)$ is the total variation explained by X_1 X_2, and X_3 in a model
  - $SSR(X_1 \mid X_2)$ is the additional variation explained by X_1 when added to a model already containing X_2
  - $SSR(X_1, X_4 \mid X_2, X_3)$ is the additional variation explained by X_1 and X_4 when added to a model already containing X_2 and X_3

Extra Sums of Squares (2)

- Can also view in terms of SSE's
- ESS represents the part of the SSE that is explained by an added group of variables that was not previously explained by the rest.
- Examples
  - $SSR(X_1 \mid X_2) = SSE(X_2) - SSE(X_1, X_2)$
  - $SSR(X_1, X_4 \mid X_2, X_3) = SSE(X_2, X_3)$ $- SSE(X_1, X_2, X_3, X_4)$

Extra Sums of Squares (3)



Decomposition of SSR (TYPE I)

- Regression SS can be partitioned into pieces (in any order):

$$\begin{aligned} SSR(X_1, X_2, X_3, X_4) = SSR(X_1) \\ +SSR(X_2 \mid X_1) \\ +SSR(X_3 \mid X_1, X_2) \\ +SSR(X_4 \mid X_1, X_2, X_3) \end{aligned}$$

- This particular breakdown is called TYPE I sums of squares (variables added in order).

Extended ANOVA Table

- Row for "Model" or "Regression" becomes $p-1$ rows, in terms of Type I SS and MS.

SOURCE	DF	Sum of Sq	Mean Square
X1	1	SSR(X1)	MSR(X1)
X2	1	SSR(X2|X1)	MSR(X2|X1)
X3	1	SSR(X3|X1,X2)	MSR(X3|X1,X2)
ERROR	n-4	SSE(X1,X2,X3)	MSE(X1,X2,X3)
Total	n-1	SST	

- Decomposition can be obtained in SAS

Type III SS

- Type III sums of squares refers to variables added last. These do NOT add to the SSR.

$$SSR\left(X_1 \mid X_2, X_3, X_4\right)$$

$$SSR\left(X_2 \mid X_1, X_3, X_4\right)$$

$$SSR\left(X_3 \mid X_1, X_2, X_4\right)$$

$$SSR\left(X_4 \mid X_1, X_2, X_3\right)$$

- Also can be obtained from SAS; Type III SS leads to variable-added-last tests.

Getting ESS from SAS

- New Procedure: GLM (stands for general linear model)
- GLM is quite similar to REG, but can handle ANOVA when we get there
- Computer Science Example

```
proc glm data=cs;
  model gpa = hsm hss hse satm satv
                    /clparm alpha=0.05;
```

- Note: Output gives way more decimals than needed. OK to cut to reasonable.

GLM Output

Source	DF	SS	MS	F Value	Pr > F
Model	5	28.64	5.73	11.69	<.0001
Error	218	106.82	0.49		
Total	223	135.46			

R-Square	Coeff Var	Root MSE	gpa Mean
0.2114	26.56	0.7000	2.635

- Standard output that we are used to. F-test is for the overall model – answers question of whether any important variables are involved.

GLM Output (2)

Source	DF	Type I SS	MS	F Value	Pr > F
hsm	1	25.810	25.810	52.67	<.0001
hss	1	1.237	1.237	2.52	0.1135
hse	1	0.665	0.665	1.36	0.2452
satm	1	0.699	0.699	1.43	0.2337
satv	1	0.233	0.233	0.47	0.4915

- Type I – Variables Added In Order; SS add to SSR on previous slide.
- F-tests are testing each variable *given* previous variables already in model

GLM Output (3)

Source	DF	Type III SS	MS	F Value	Pr > F
hsm	1	6.772	6.772	13.82	0.0003
hss	1	0.442	0.442	0.90	0.3432
hse	1	0.957	0.957	1.95	0.1637
satm	1	0.928	0.928	1.89	0.1702
satv	1	0.233	0.233	0.47	0.4915

- Type III – Variables Added Last
- F-tests are testing variables *given* that all of the other variables already in model

Coefficients of Partial Determination

- Recall: R^2 is the coefficient of determination, and may be interpreted as the percentage of the total variation that has been explained by the model.
- Example: $R^2 = 0.87$ means 87% of the Total SS has been explained by the regression model (of however many variables)
- Can also consider the amount of *remaining variation* explained by a variable *given* other variables already in the model – this is called *partial determination*.

Coef. of Partial Determination (2)

- Notation: $R^2_{Y1|23}$ represents the percentage of the leftover variation in Y (after regressing on X_2 and X_3) that is explained by X_1.
- Mathematically,

$$R^2_{Y1|23} = \frac{SSE(X_2, X_3) - SSE(X_1, X_2, X_3)}{SSE(X_2, X_3)}$$

$$= \frac{SSR(X_1 \mid X_2, X_3)}{SSE(X_2, X_3)}$$

- Subscripts after bar (|) represent variables already in model.

Example

- Suppose total sums of squares is 100, and X_1 explains 60.
- Of the remaining 40, X_2 then explains 20, and of the remaining 20, X_3 explains 5.
- Then

$$R^2_{Y2|1} = \frac{20}{40} = 0.50$$

$$R^2_{Y3|12} = \frac{5}{20} = 0.25$$

Coefficient of Partial Correlation

- Square Root of the coefficient of partial determination
- Given plus/minus sign according to the corresponding regression coefficient
- Can be useful in model selection (Chapter 9); but no clear interpretation like R^2.
- Notation: $r_{Y1|23}$

Getting Partial R^2 from SAS

- PROC REG can produce these, along with sums of squares (in REG, the TYPE III SS are actually denoted as TYPE II – there is no difference between the two types for normal regression, but there is for ANOVA so we'll discuss this later)
- CS Example

```
proc reg data=cs;
  model gpa = hsm hss hse satm satv
                /ss1 ss2 pcorr1 pcorr2;
```

REG Output

Variable	DF	SS(I)	SS(II)	Squared Partial Corr(I)	Squared Partial Corr(II)
Intercept	1	1555	0.327	.	.
hsm	1	25.8	6.772	0.19053	0.05962
hss	1	1.2	0.442	0.01128	0.00412
hse	1	0.7	0.957	0.00614	0.00888
satm	1	0.7	0.928	0.00648	0.00861
satv	1	0.2	0.233	0.00217	0.00217

- Example: HSE explains 0.6% of remaining variation after HSM and HSS in model

REG Output (2)

- Can get any partial coefficient of determination that we want, but may have to rearrange model to do it.
- Example: If we want HSE given HSM, we would need to list variables HSM and HSE as the first and second in the model
- Can get any desired Type I SS in the same way.

REG Output (3)

Variable	DF	SS(I)	SS(II)	Corr(I)	Corr(II)
Intercept	1	1555	0.327	.	.
hsm	1	25.8	6.772	0.19053	0.05962
hse	1	1.5	0.957	0.01362	0.00412

- Interpretation: Once HSM is in the model, of the remaining variation (SSE=109) HSE explains only 1.36% of it.

General Linear Test

- Compare two models:
  - Full Model: All variables / parameters
  - Reduced Model: Apply NULL hypothesis to full model.
- Example: 4 variables, $H_0 : \beta_2 = \beta_3 = 0$

  FULL: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \varepsilon$

  REDUCED: $Y = \beta_0 + \beta_1 X_1 + \beta_4 X_4 + \varepsilon$
- F-statistic is

$$F = \frac{\left(SSE(R) - SSE(F)\right) / \left(df_R - df_F\right)}{SSE(F) / df_F}$$

General Linear Test (2)

- Numerator of F-test is the difference in SSE's, or the EXTRA SS associated to the "added" variables; divided by number of variables being "added" (d.f.)
- Denominator is MSE for full model.
- For the example, test statistic will be

$$F = \frac{SSR\left(X_2, X_3 \mid X_1, X_4\right) / 2}{MSE\left(X_1, X_2, X_3, X_4\right)}$$

- Compare to F-distribution on 2 and $n - 5$ degrees of freedom.

Alternative Hypotheses

- Alternative is simply that the null is false.
- Most of the time, the alternative will be that at least one of the variables in the null group is important.
- Often looking to *"fail to reject"* when performing a test like this – our goal is to eliminate unnecessary variables.
- This means POWER / sample size must be a consideration! If our sample size is too small, we may incorrectly remove variables.

CS Example (ess.sas)

- Test whether HSS, SATM, SATV as a group are important when added to model containing HSM and HSE.
- SSE for HSM/HSE model is 108.16 on 221 degrees of freedom
- SSE for full model is 106.82 on 218 degrees of freedom; MSE is 0.49
- F statistic is

$$F = \frac{(108.16 - 106.82)/3}{0.49} = 0.91$$

CS Example (2)

- $F < 1$ so no need to even look up a value; fail to reject.
- With 224 data points, we likely have the power required to conclude that the three variables are not useful in the model that already contains HSM and HSE.
- Can obtain this test in SAS using a test statement:

```
proc reg data=cs;
  model gpa = hsm hss hse satm satv;
  TEST1:  test hss=0, satm=0, satv=0;
```

TEST output

Test TEST1 Results

Source	DF	Mean Square	F Value	Pr > F
Numerator	3	0.44672	0.91	0.4361
Denominator	218	0.49000		

- P-value is 0.4361 (as long as its > 0.1 and the sample size is reasonably large, we can discard the additional variables)

CS Example (3)

- Can also obtain the numbers we need from TYPE I / III Sums of Squares.
- How would we test...
  - Importance of HSS in addition to rest.
  - Importance of SAT’s added to HS’s
  - Importance of HSE after HSM/HSS
  - Importance of HSE after HSM
- Can obtain the numbers you need for any partial-F test by arranging the variables correctly.

Upcoming in Lecture 14….

- Diagnostics and Remedial Measures