Lecture 13
Principal Components Analysis and Factor Analysis

Prof. Dr. Svetlozar Rachev
Institute for Statistics and Mathematical Economics
University of Karlsruhe

Financial Econometrics, Summer Semester 2007

Copyright

The material is based on the text-book:

Financial Econometrics: From Basics to Advanced Modeling Techniques

(Wiley-Finance, Frank J. Fabozzi Series)

by Svetlozar T. Rachev, Stefan Mittnik, Frank Fabozzi, Sergio M. Focardi,Teo *Jašìc*.

Prof. Svetlozar (Zari) T.Rachev
Chair of Econometrics, Statistics
and Mathematical Finance
School of Economics and Business Engineering
University of Karlsruhe
Kollegium am Schloss, Bau II, 20.12, R210
Postfach 6980, D-76128, Karlsruhe, Germany
Tel. +49-721-608-7535, 0+49-721-608-2042(s)
FAX: +49-721-608-3811
http://www.statistik.uni-karlsruhe.de

Outline

- Factor models.
- Principal components analysis.
- Factor analysis.
- PCA and factor analysis compared.

Factor Models

- Factor models are statistical models that try to explain complex phenomena through a small number of basic causes or factors.
- Factor models serve two main purposes:
  1. They reduce the dimensionality of models to make estimation possible;
  2. They find the true causes that drive data.
- Factor models were introduced by Charles Spearman in 1904.

Factor Models

- The Spearman model explains intellectual abilities through one common factor, the famous "general intelligence" g factor, plus another factor s which is specific to each distinct ability.
- Louis Leon Thurstone developed the first true multifactor model of intelligence, where were identified the following seven primary mental abilities:

Verbal Comprehension	Word Fluency
Number Facility	Spatial Visualization
Associative Memory	Perceptual Speed
Reasoning.	

Factor Models

- In the early applications of factor models to psychometrics, the statistical model was essentially a conditional multivariate distribution. The objective was to explain psychometric tests as probability distributions conditional on the value of one or more factors. In this way, one can make predictions of, for example, the future success of young individuals in different activities.
- In economics, factor models are typically applied to time series. The objective is to explain the behavior of a large number of stochastic processes, typically price, returns, or rate processes, in terms of a small number of factors. These factors are themselves stochastic processes.

Factor Models

- In order to simplify both modeling and estimation, most factor models employed in financial econometrics are static models. This means that time series are assumed to be sequences of temporally independent and identically distributed (IID) random variables so that the series can be thought as independent samples extracted from one common distribution.
- In financial econometrics, factor models are needed not only to explain data but to make estimation feasible. Factor models able to explain all pairwise correlations in terms of a much smaller number of correlations between factors.

Linear Factor Models Equations

Linear factor models are regression models of the following type:

$$X_i = \alpha_i + \sum_{j=1}^{K} \beta_{ij} f_j + \varepsilon_i$$

where

X_i = a set of N random variables
f_j = a set of K common factors
ε_i = the noise terms associated with each variable X_i
β_{ij}'s are the *factor loadings* or *factor sensitivities*, which express the influence of the j-th factor on the i-th variable.

Note: In this formulation, factor models are essentially static models, but it is possible to add a dynamics to both the variables and the factors.

Factor Models

- One of the key objectives of factor models is that the covariances between the variables X_i is determined only by the covariances between factors.
- Suppose that the noise terms are mutually uncorrelated, so that

$$E(\varepsilon_i \varepsilon_j) = \begin{cases} 0 & i \neq j \\ \sigma_i^2 & i = j \end{cases}$$

  and that the noise terms are uncorrelated with the factors, that is, $E(\varepsilon_i f_j) = 0, \forall i, j$.
- Suppose also that both factors and noise terms have a zero mean, so that $E(X_i) = \alpha_i$.

Factor models that respect the above constraints are called *strict factor models*.

Factor Models

Lets compute the covariances of a strict factor model:

$$E((X_i - \alpha_i)(X_j - \alpha_j)) = E\Big(\Big(\sum_{s=1}^{K} \beta_{is} f_s + \varepsilon_i\Big)\Big(\sum_{t=1}^{K} \beta_{jt} f_t + \varepsilon_j\Big)\Big)$$

$$= E\Big(\Big(\sum_{s=1}^{K} \beta_{is} f_s\Big)\Big(\sum_{t=1}^{K} \beta_{jt} f_t\Big)\Big) + E\Big(\Big(\sum_{s=1}^{K} \beta_{is} f_s\Big)(\varepsilon_j)\Big)$$

$$+E\Big((\varepsilon_i)\sum_{t=1}^{K} \beta_{jt} f_t\Big) + E(\varepsilon_i \varepsilon_j)$$

$$= \sum_{s,t} \beta_{is} E(f_s f_t) \beta_{jt} + E(\varepsilon_i \varepsilon_j)$$

Factor Models

We can express the above compactly in matrix form. Lets write a factor model in matrix form as follows:

$$X = \alpha + \beta f + \varepsilon$$

where
$X = (X_1, \ldots, X_N)' =$ the N-vector of variables
$\alpha = (\alpha_1, \ldots, \alpha_N)' =$ the N-vector of means
$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)' =$ the N-vector of idiosyncratic noise terms
$f = (f_1, \ldots, f_K)' =$ the K-vector of factors

$$\beta = \begin{bmatrix} \beta_{11} & \cdots & \beta_{1K} \\ \vdots & \ddots & \vdots \\ \beta_{N1} & \cdots & \beta_{NK} \end{bmatrix} = \text{the } N \times K \text{ matrix of factor loadings.}$$

Factor Models

- Lets define the following:
  $\Sigma =$ the $N \times N$ variance-covariance matrix of the variables X
  $\Omega =$ the $K \times K$ variance-covariance matrix of the factors
  $\Psi = N \times N$ variance-covariance matrix of the error terms ε.
- If we assume that our model is a strict factor model, the matrix Ψ will be a diagonal matrix with the noise variances on the diagonal, that is,

$$\Psi = \begin{pmatrix} \Psi_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Psi_N^2 \end{pmatrix}$$

- We can express the variance-covariance matrix of the variables in the following way:

$$\Sigma = \beta\Omega\beta' + \Psi$$

Factor Models

- In applied work, factor models will often be approximate factor models. They allow idiosyncratic terms to be weakly correlated among themselves and with the factors.
- As many different factor models have been proposed for explaining stock returns, an important question is whether a factor model is fully determined by the observed time series.
- An estimation procedure cannot univocally determine the hidden factors and the factor loadings from the observable variables $\mathbf{X}$.

Factor Models

In fact, suppose that we multiply the factors by any nonsingular matrix $\mathbf{R}$. We obtain other factors

$$g = \mathbf{R}f$$

with a covariance matrix

$$\Omega_g = \mathbf{R}\Omega\mathbf{R}^{-1}$$

and we can write a new factor model:

$$X = \alpha + \beta f + \varepsilon = \alpha + \beta\mathbf{R}^{-1}\mathbf{R}f + \varepsilon = \alpha + \beta_g g + \varepsilon$$

Factor Models

- In order to solve this indeterminacy, we can always choose the matrix $\mathbf{R}$ so that the factors g are a set of orthonormal variables, that is, uncorrelated variables (the orthogonality condition) with unit variance (the normality condition).
- In order to make the model uniquely identifiable, we can stipulate that factors must be a set of orthonormal variables and that, in addition, the matrix of factor loadings is diagonal.
- Under this additional assumption, a strict factor model is called a *normal factor model*. The model is still undetermined under rotation, that is multiplication by any nonsingular matrix such that $\mathbf{RR}' = I$.

Factor Models

Summary:

- A set of variables has a normal factor representation if it is represented by the following factor model:

$$X = \alpha + \beta f + \varepsilon$$

  where factors are orthonormal variables and noise terms are such that the covariance matrix can be represented as follows:

$$\Sigma = \beta\beta' + \Psi$$

  where β is the diagonal matrix of factor loadings and Ψ is a diagonal matrix.
- Approximate factor models are uniquely identifiable only in the limit of an infinite number of series.

In financial econometrics, the factors used in factor models can belong to three different categories:

- Macroeconomic factors
- Fundamental factors
- Statistical factors

Macroeconomic factors are macroeconomic variables that are believed to determine asset returns (Example: GNP, the inflation rate, the unemployment rate, or the steepness of the yield curve).
Fundamental factors are variables that derive from financial analysis.
Statistical factors are factors that derive from a mathematical process.

Factor Models: Types of Factors and Their Estimation

- Macroeconomic factors are exogenous factors that must be estimated as variables exogenous to the factor model. They influence the model variables but are not influenced by them.
- A factor model is estimated as a linear regression model, means that there is indeed a linear relationship between the factors and the model variables.
- However, such a model will have no explanatory power. The variance of each variable that is not explained by common factors appears as noise.
- Adding factors might improve the explanatory power of the model but, in general, worsens the ability to estimate the model because there are more parameters to estimate. There is a trade-off between adding explanatory factors and the ability to estimate them.

Factor Models: Types of Factors and Their Estimation

- Statistical factors are obtained through a logical process of analysis of the given variables.
- Statistical factors are factors that are endogenous to the system. They are typically determined with one of two statistical processes; namely, principal component analysis or factor analysis.
- Note that factors defined through statistical analysis are linear combinations of the variables.

Principal Components Analysis

- Principal components analysis (PCA) was introduced in 1933 by Harold Hotelling as a way to determine factors with statistical learning techniques when factors are not exogenously given.
- Given a variance-covariance matrix, one can determine factors using the technique of PCA.
- The concept of PCA is the following.
  Consider a set of n stationary time series X_i.
  Consider next a linear combination of these series, that is, a portfolio of securities. Each portfolio P is identified by an n-vector of weights ω_P and is characterized by a variance σ_P^2.
  Lastly, consider a normalized portfolio, which has the largest possible variance. In this context, a normalized portfolio is a portfolio such that the squares of the weights sum to one.

Principal Components Analysis

- If we assume that returns are IID sequences, jointly normally distributed with variance-covariance matrix σ, a lengthy direct calculation demonstrates that each portfolios return will be normally distributed with variance

$$\sigma_P^2 = \omega_P^T \sigma \omega_P$$

- The normalized portfolio of maximum variance can therefore be determined in the following way: Maximize $\omega_p^T \sigma \omega_P$ subject to the normalization condition

$$\omega_P^T \omega_P = 1$$

where the product is a scalar product.

- It can be demonstrated that the solution of this problem is the eigenvector ω_2 corresponding to the largest eigenvalue λ_2 of the variance-covariance matrix σ.

Principal Components Analysis

- Consider next the set of all normalized portfolios orthogonal to ω_1, that is, portfolios completely uncorrelated with ω_1. These portfolios are identified by the following relationship:

$$\omega_1^T \omega_P = \omega_P^T \omega_1 = 0$$

  Among this set, the portfolio of maximum variance is given by the eigenvector ω_2 corresponding to the second largest eigenvalue λ_2 of the variance-covariance matrix σ.
- If there are n distinct eigenvalues, we can repeat this process n times. In this way, we determine the n portfolios P_i of maximum variance. The weights of these portfolios are the orthonormal eigenvectors of the variance- covariance matrix σ.
- These portfolios of maximum variance are all mutually uncorrelated.

Principal Components Analysis

- It can be demonstrated that we can recover all the original return time series as linear combinations of these portfolios:

$$X_j = \sum_{i=1}^{n} \alpha_{j,i} P_i$$

- Thus far we have succeeded in replacing the original n correlated time series X_j with n uncorrelated time series P_i with the additional insight that each X_j is a linear combination of the P_i.
- Suppose now that only p of the portfolios P_i have a significant variance, while the remaining n-p have very small variances. We can then implement a dimensionality reduction by choosing only those portfolios whose variance is significantly different from zero. Lets call these portfolios factors F.

- It is clear that we can approximately represent each series Xi as a linear combination of the factors plus a small uncorrelated noise. In fact we can write

$$X_j = \sum_{i=1}^{p} \alpha_{j,i} F_i + \sum_{i=p+1}^{n} \alpha_{j,i} P_i = \sum_{i=1}^{p} \alpha_{i,j} F_i + \varepsilon_j$$

  where the last term is a noise term.
- Therefore to implement PCA one computes the eigenvalues and the eigenvectors of the variance-covariance matrix and chooses the eigenvalues significantly different from zero.
- The corresponding eigenvectors are the weights of portfolios that form the factors. Criteria of choice are somewhat arbitrary.

Principal Components Analysis

- Suppose, however, that there is a *strict factor structure*, which means that returns follow a strict factor model as defined earlier in this chapter:

$$r = a + \beta f + \varepsilon$$

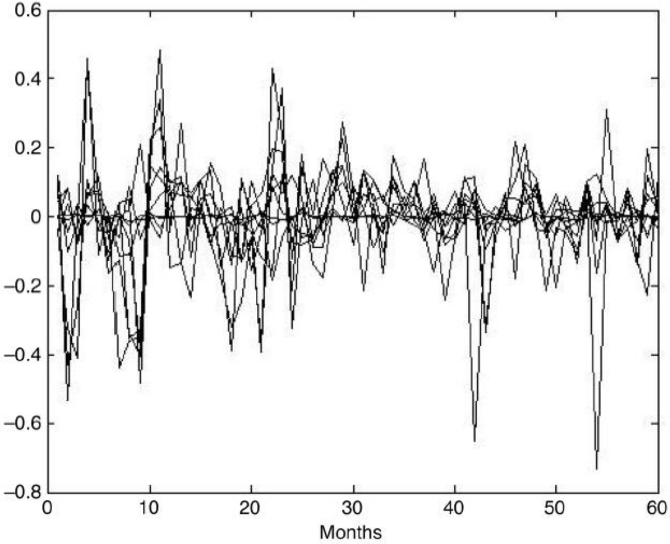
- The matrix β can be obtained diagonalizing the variance-covariance matrix.
- In general, the structure of factors will not be strict and one will try to find an approximation by choosing only the largest eigenvalues.

Principal Components Analysis

- PCA works either on the variance-covariance matrix or on the correlation matrix. The technique is the same but results are generally different.
- PCA applied to the variance-covariance matrix is sensitive to the units of measurement, which determine variances and covariances. If PCA is applied to prices and not to returns, the currency in which prices are expressed matters; one obtains different results in different currencies. In these cases, it might be preferable to work with the correlation matrix.
- PCA is a generalized dimensionality reduction technique applicable to any set of multidimensional observations. It admits a simple geometrical interpretation which can be easily visualized in the three-dimensional case.

Principal Components Analysis: Illustration

EXHIBIT 13.1 Graphics of the 10 Stock Return Processes



Principal Components Analysis: Illustration

- Performing PCA is equivalent to determining the eigenvalues and eigenvectors of the covariance matrix or of the correlation matrix. The two matrices yield different results. We perform both exercises.
- We estimate the covariance with the empirical covariance matrix. Recall that the empirical covariance σ_{ij} between variables (X_i, X_j) is defined as follows:

$$\hat{\sigma}_{ij} = \frac{1}{T}\sum_{t=1}^{T}(X_i(t) - \bar{X}_i)(X_j(t) - \bar{X}_j)$$

$$\bar{X}_i = \frac{1}{T}\sum_{t=1}^{T}X_i(t), \quad \bar{X}_j = \frac{1}{T}\sum_{t=1}^{T}X_j(t)$$

Principal Components Analysis: Illustration

EXHIBIT 13.2 The Covariance Matrix of 10 Stock Returns

	SUNW	AMZN	MERQ	GD	NOC	CPB
SUNW	0.02922	0.017373	0.020874	3.38E-05	–0.00256	–3.85E-05
AMZN	0.017373	0.032292	0.020262	5.03E-05	–0.00277	0.000304
MERQ	0.020874	0.020262	0.0355	–0.00027	–0.0035	–0.00011
GD	3.38E-05	5.03E-05	–0.00027	9.27E-05	0.000162	2.14E-05
NOC	–0.00256	–0.00277	–0.0035	0.000162	0.010826	3.04E-05
CPB	–3.85E-05	0.000304	–0.00011	2.14E-05	3.04E-05	7.15E-05

For the whole exhibit look at page 441.

EXHIBIT 13.3 The Correlation Matrix of the Same 10 Return Processes

	SUNW	AMZN	MERQ	GD	NOC	CPB
SUNW	1	0.56558	0.64812	0.020565	–0.14407	–0.02667
AMZN	0.56558	1	0.59845	0.029105	–0.14815	0.20041
MERQ	0.64812	0.59845	1	–0.14638	–0.17869	–0.06865
GD	0.020565	0.029105	–0.14638	1	0.16217	0.26307
NOC	–0.14407	–0.14815	–0.17869	0.16217	1	0.034519
CPB	–0.02667	0.20041	–0.06865	0.26307	0.034519	1

For the whole exhibit look at page 442.

EXHIBIT 13.4 Eigenvectors and Eigenvalues of the Covariance Matrix

Panel A: Eigenvectors

	1	2	3	4	5	6
1	–0.50374	0.50099	0.28903	–0.59632	–0.01824	–0.01612
2	–0.54013	–0.53792	0.51672	0.22686	–0.06092	0.25933
3	–0.59441	0.32924	–0.4559	0.52998	0.051976	0.015346
4	0.001884	–0.00255	0.018107	–0.01185	0.013384	0.01246
5	0.083882	0.10993	0.28331	0.19031	0.91542	–0.06618
6	–0.00085	–0.01196	0.016896	0.006252	–0.00157	0.01185

For the whole exhibit look at page 443.

Principal Components Analysis: Illustration

Panel B: Eigenvalues of the covariance matrix

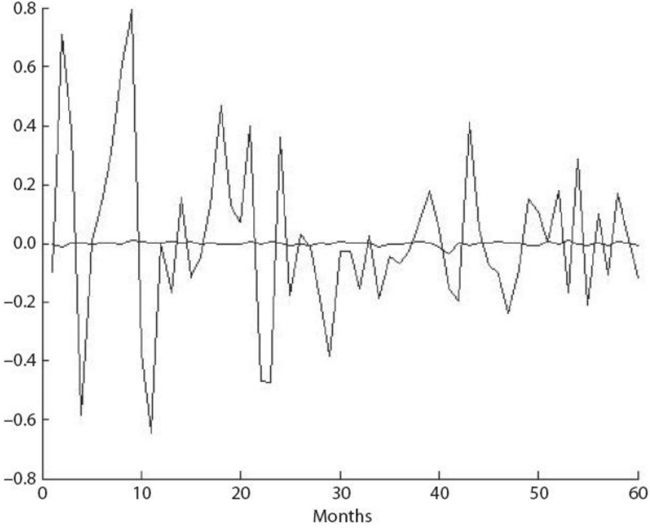
1	0.0783
2	0.0164
3	0.0136
4	0.0109
5	0.0101
6	0.0055
7	0.0039
8	0.0028
9	0.0001
10	0.0001

Principal Components Analysis: Illustration

- If we form portfolios whose weights are the eigenvectors, we can form 10 portfolios that are orthogonal (i.e., uncorrelated). These orthogonal portfolios are called *principal components.*
- The variance of each principal component will be equal to the corresponding eigenvector. Thus the first principal component will have the maximum possible variance and the last principal component will have the smallest variance.

Principal Components Analysis: Illustration

EXHIBIT 13.5 Graphic of the Portfolios of Maximum and Minimum Variance Based on the Covariance Matrix



Principal Components Analysis: Illustration

- The 10 principal components thus obtained are linear combinations of the original series, $X = (X_1, \ldots, X_N)'$ that is, they are obtained by multiplying X by the matrix of the eigenvectors.
- If the eigenvalues and the corresponding eigenvectors are all distinct we can apply the inverse transformation and recover the X as linear combinations of the principal components.
- PCA is interesting if, in using only a small number of principal components, we nevertheless obtain a good approximation. So we regress the original series X onto a small number of principal components. By choosing as factors the components with the largest variance, we can explain a large portion of the total variance of X.

EXHIBIT 13.8 Eigenvectors and Eigenvalues of the Correlation Matrix

Panel A: Eigenvectors

	1	2	3	4	5	6
1	–0.4341	0.19295	–0.26841	0.040065	–0.19761	0.29518
2	–0.45727	0.18203	0.20011	0.001184	0.013236	0.37606
3	–0.47513	–0.03803	–0.16513	0.16372	–0.01282	0.19087
4	0.06606	0.63511	0.18027	–0.16941	–0.05974	–0.24149
5	0.17481	0.33897	–0.21337	0.14797	0.84329	0.23995
6	–0.00505	0.42039	0.57434	0.40236	–0.15072	–0.05018

For the whole exhibit look at page 448.

Principal Components Analysis: Illustration

Panel B: Eigenvalues

1	3.0652
2	1.4599
3	1.1922
4	0.9920
5	0.8611
6	0.6995
7	0.6190
8	0.5709
9	0.3143
10	0.2258

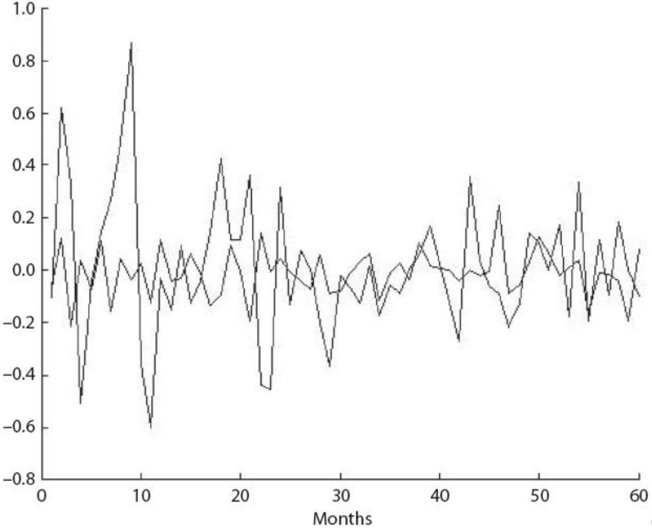
Principal Components Analysis: Illustration

EXHIBIT 13.9 Percentage of the Total Variance Explained by a Growing Number of Components Using the Correlation Matrix

Principal Component	Percentage of Total Variance Explained
1	30.6522%
2	45.2509
3	57.1734
4	67.0935
5	75.7044
6	82.6998
7	88.8901
8	94.5987
9	97.7417
10	100.0000

Principal Components Analysis: Illustration

EXHIBIT 13.10 Graphic of the Portfolios of Maximum and Minimum Variance Based on the Correlation Matrix



PCA and Factor Analysis with Stable Distributions

- PCA and factor analysis can be applied provided that all variances and covariances exist. Applying them does not require that distributions are normal, but only that they have finite variances and covariances.
- Variances and covariances are not robust but are sensitive to outliers. Robust equivalent of variances and covariances exist.
- In order to make PCA and factor analysis insensitive to outliers, one could use robust versions of variances and covariances and apply PCA and factor analysis to these robust estimates.

PCA and Factor Analysis with Stable Distributions

- In many cases, however, distributions might exhibit fat tails and infinite variances. In this case, large values cannot be trimmed but must be taken into proper consideration. However, if variances and covariances are not finite, the least squares methods used to estimate factor loadings cannot be applied. In addition, the concept of PCA and factor analysis cannot be applied.
- In fact, if distributions have infinite variances, it does not make sense to determine the portfolio of maximum variance as all portfolios will have infinite variance and it will be impossible, in general, to determine an ordering based on the size of variance.

Factor Analysis

- Here we consider an alternative technique for determining factors: *factor analysis* (FA).
- Suppose we are given T independent samples of a random vector $X = (X_1, \ldots, X_N)'$, N time series of stock returns at T moments, as in the case of PCA.
- Assuming that the data are described by a strict factor model with K factors, the objective of factor analysis (FA) consists of determining a model of the type

$$X = \alpha + \beta f + \varepsilon$$

with covariance matrix

$$\Sigma = \beta\beta' + \Psi$$

- The estimation procedure is performed in two steps:
  1. We estimate the covariance matrix and the factor loadings.
  2. We estimate factors using the covariance matrix and the factor loadings.
- Assuming that the variables are jointly normally distributed and temporally independently and identically distributed (IID), we can estimate the covariance matrix with maximum likelihood methods. Iterative methods such as the *expectation maximization* (EM) algorithm are generally used for the estimation of factor models.

Factor Analysis

- After estimating the matrices β and Ψ factors can be estimated as linear regressions.
- In fact, assuming that factors are zero means, we can write the factor model as

$$X - \alpha = \beta f + \varepsilon$$

which shows that, at any given time, factors can be estimated as the regression coefficients of the regression of $(X\alpha)$ onto β.

- Using the standard formulas of regression analysis we can now write factors, at any given time, as follows:

$$\hat{f}_t = (\hat{\beta}' \hat{\Psi}^{-1} \hat{\beta})^{-1} \hat{\beta}' \hat{\Psi}^{-1} (X_t - \hat{\alpha})$$

Factor Analysis

- Maximum likelihood estimates implies that the number of factors is known. The correct number of factors is determined basing on the heuristic procedure when estimates of q factors stabilize and cannot be rejected on the basis of p probabilities.
- The factor loadings matrix can also be estimated with *ordinary least squares* (OLS) methods.
- OLS estimates of the factor loadings are inconsistent when the number of time points goes to infinity but the number of series remains finite, unless we assume that the idiosyncratic noise terms all have the same variance.
- The OLS estimators remain consistent under the assumption when both the number of processes and the time go to infinity.

So, to perform factor analysis, we need to estimate only the factor loadings and the idiosyncratic variances of noise terms.

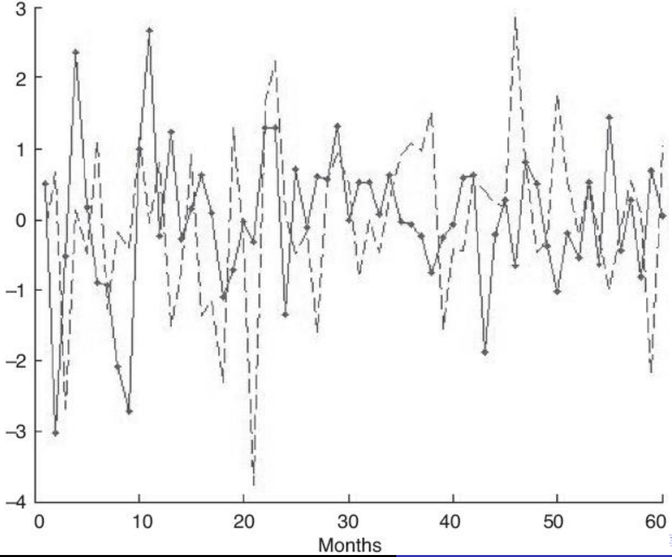
An Illustration of Factor Analysis

EXHIBIT 13.11 A Factor Loadings and Idiosyncratic Variances

	Factor Loadings			
	β_1	β_2	β_3	Variance
SUNW	0.656940	0.434420	0.27910	0.301780
AMZN	0.959860	–0.147050	–0.00293	0.057042
MERQ	0.697140	0.499410	–0.08949	0.256570
GD	0.002596	–0.237610	0.43511	0.754220
NOC	–0.174710	–0.119960	0.23013	0.902130
CPB	0.153360	–0.344400	0.13520	0.839590
KO	0.170520	0.180660	–0.46988	0.717500
MLM	0.184870	0.361180	0.28657	0.753250
HLT	0.593540	0.011929	–0.18782	0.612300
UTX	0.385970	0.144390	–0.15357	0.806590

An Illustration of Factor Analysis

EXHIBIT 13.12 Graphics of the Three Factors



Applying PCA to Bond Portfolio Management

There are two applications in bond portfolio management where PCA has been employed.

- The first application is explaining the movement or dynamics in the yield curve and then applying the resulting principal components to measure and manage yield curve risk.
- The second application of PCA is to identify risk factors beyond changes in the term structure.

For example, given historical bond returns and factors that are believed to affect bond returns, PCA can be used to obtain principal components that are linear combinations of the variables that explain the variation in returns.

Using PCA to Control Interest Rate Risk

- Using PCA, several studies have investigated the factors that have affected the historical returns on Treasury portfolios.
- Robert Litterman and Jose Scheinkman found that three factors explained historical bond returns for U.S. Treasuries zero-coupon securities:
  1. The changes in the *level of rates*;
  2. The changes in the *slope of the yield curve*;
  3. The changes in the *curvature of the yield curve*.
- After identifying the factors, Litterman and Scheinkman use regression analysis to assess the relative contribution of these three factors in explaining the returns on zero-coupon Treasury securities of different maturities.

Using PCA to Control Interest Rate Risk

- On average, the first principal component explained about 90% of the returns, the second principal component 8%, and the third principal component 2%.
- Thus, only three principal components were needed to fully explain the dynamics of the yield curve.
- There have been several studies that have examined the yield curve movement using PCA and reported similar results.

Using PCA to Control Interest Rate Risk

Once yield curve risk is described in terms of principal components, the factor loadings can be used to:

- Construct hedges that neutralize exposure to changes in the direction of interest rates.
- Construct hedges that neutralize exposure to changes in nonparallel shifts in the yield curve.
- Structure yield curve trades.

PCA of the dynamics of the yield curve have lead to the use of what is now referred to as *principal component duration*. Moreover, PCA can be used to estimate the probability associated with a given hypothetical interest rate shock so that a bond portfolio manager can better analyze the interest rate risk of a bond portfolio and traders can better understand the risk exposure of a bond trading strategy.

Factor Analysis: Bond Risk Factors

- For a bond index that includes nongovernment securities, there are risk factors other than term structure factors.
- Using PCA, Gauthier and Goodman have empirically identified the risk factors that generate nominal excess returns for the Salomon Smith Barney Broad Investment Grade Index (SSB BIG Index) for the period January 1992 to March 2003.
- The results of their PCA for the first six principal components for each bond sector of the SSB BIG Index are presented on the next slide.

An Illustration of Factor Analysis

	Component					
	1	2	3	4	5	6
Nominal Returns						
Agy. Callable	0.28	0.00	0.41	0.16	0.00	0.85
Agy. NC	0.54	0.24	−0.20	0.60	0.46	−0.22
MBS	0.30	0.00	0.75	0.00	−0.34	−0.46
Credit	0.48	−0.82	−0.28	−0.11	0.00	0.00
ABS	0.31	0.15	0.19	−0.73	0.56	0.00
Treasury	0.47	0.49	−0.34	−0.25	−0.60	0.00
Factor contribution (%)	92.7	3.1	2.3	0.9	0.5	—
Cumulative Importance (%)	92.7	95.8	98.1	99	99.5	1
Duration-Adjusted Returns						
Agy. Callable	0.18	0.28	0.76	−0.10	−0.53	−0.12
Agy. NC	0.21	0.52	0.17	0.67	0.45	
MBS	0.23	0.65	−0.23	−0.66	0.21	
Credit	0.91	−0.40	0.00		0.12	
ABS	0.23	0.26	−0.58	0.32	−0.64	−0.18
Treasury	0.00	0.00	0.00		−0.22	0.97
Factor contribution (%)	80.3	12.1	2.9	2.5	1.8	—
Cumulative importance (%)	80.3	92.4	95.3	97.8	99.6	1.0

Factor Analysis: Bond Risk Factors

- The **first** principal component explains 92.7% of the variation.
- The **second** principal component explains 3.1% of nominal excess returns. Gauthier and Goodman identify this factor as the *credit specific factor* because of the high negative factor loadings on the credit index combined with a high positive weighting on Treasuries.
- Gauthier and Goodman identify the **third** principal component as an *optionality factor*. This can be supported by noting that the factor loadings on the assets classes that have some optionality is positive, while the factor loading on the noncallable series is negative.

Factor Analysis: Bond Risk Factors

- This third principal component, which represents optionality, is consistent with studies of the movements of the yield curve.
- Gauthier and Goodman show that there is a high positive correlation between the optionality factor and the slope of the yield, but a negative relationship with 5-year cap volatility.

This suggests

1. the steeper the yield curve slope, the better a callable series should do;
2. the higher the volatility, the lower the return on the callable series.

PCA and Factor Analysis Compared

- The two illustrations of PCA and FA are relative to the same data and will help clarify the differences between the two methods.
- Lets first observe that PCA does not imply any specific restriction on the process. Given a nonsingular covariance matrix, we can always perform PCA as an exact linear transformation of the series. When we consider a smaller number of principal components, we perform an approximation which has to be empirically justified.
- Factor analysis, on the other hand, assumes that the data have a strict factor structure in the sense that the covariance matrix of the data can be represented as a function of the covariances between factors plus idiosyncratic variances.

PCA and Factor Analysis Compared

- PCA tends to be a dimensionality reduction technique that can be applied to any multivariate distribution and that yields incremental results. This means that there is a trade-off between the gain in estimation from dimensionality reduction and the percentage of variance explained.
- Factor analysis, on the other hand, tends to reveal the exact factor structure of the data. That is, FA tends to give an explanation in terms of what factors explain what processes.
- Factor rotation can be useful both in the case of PCA and FA.

An Illustration of Factor Analysis

EXHIBIT 13.16 Graphical Representation of Factor Loadings

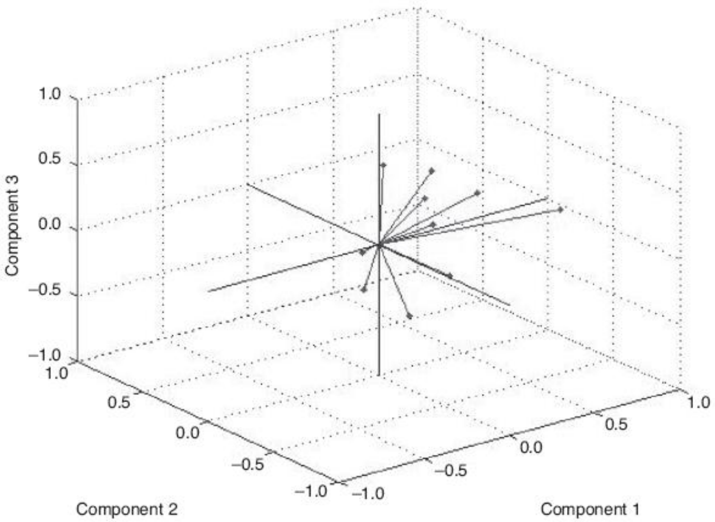
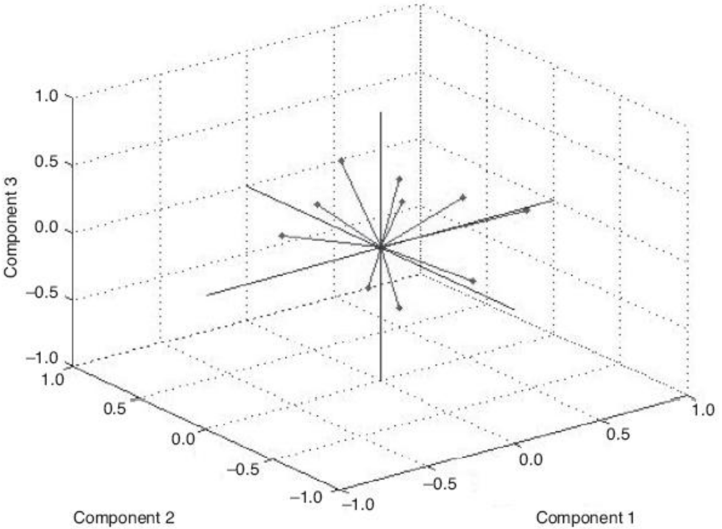


EXHIBIT 13.17 Factor Loadings after Rotation

	F1	F2	F3
SUNW	0.214020	0.750690	0.101240
AMZN	0.943680	0.127310	0.104990
MERQ	0.218340	0.578050	–0.294340
GD	0.163360	0.073269	0.544220
NOC	–0.070130	–0.003990	0.278000
CPB	0.393120	–0.178070	0.301920
KO	0.032397	–0.100020	–0.545120
MLM	–0.137130	0.561640	0.123670
HLT	0.513660	0.048842	–0.168290
UTX	0.229400	0.133510	–0.204650

An Illustration of Factor Analysis

EXHIBIT 13.18 Relationship of Times Series to the Factors after Rotation



Final Remarks

- The required textbook is "Financial Econometrics: From Basics to Advanced Modeling Techniques".
- Please read Chapter 13 for today's lecture.
- All concepts explained are listed in page 464 of the textbook.