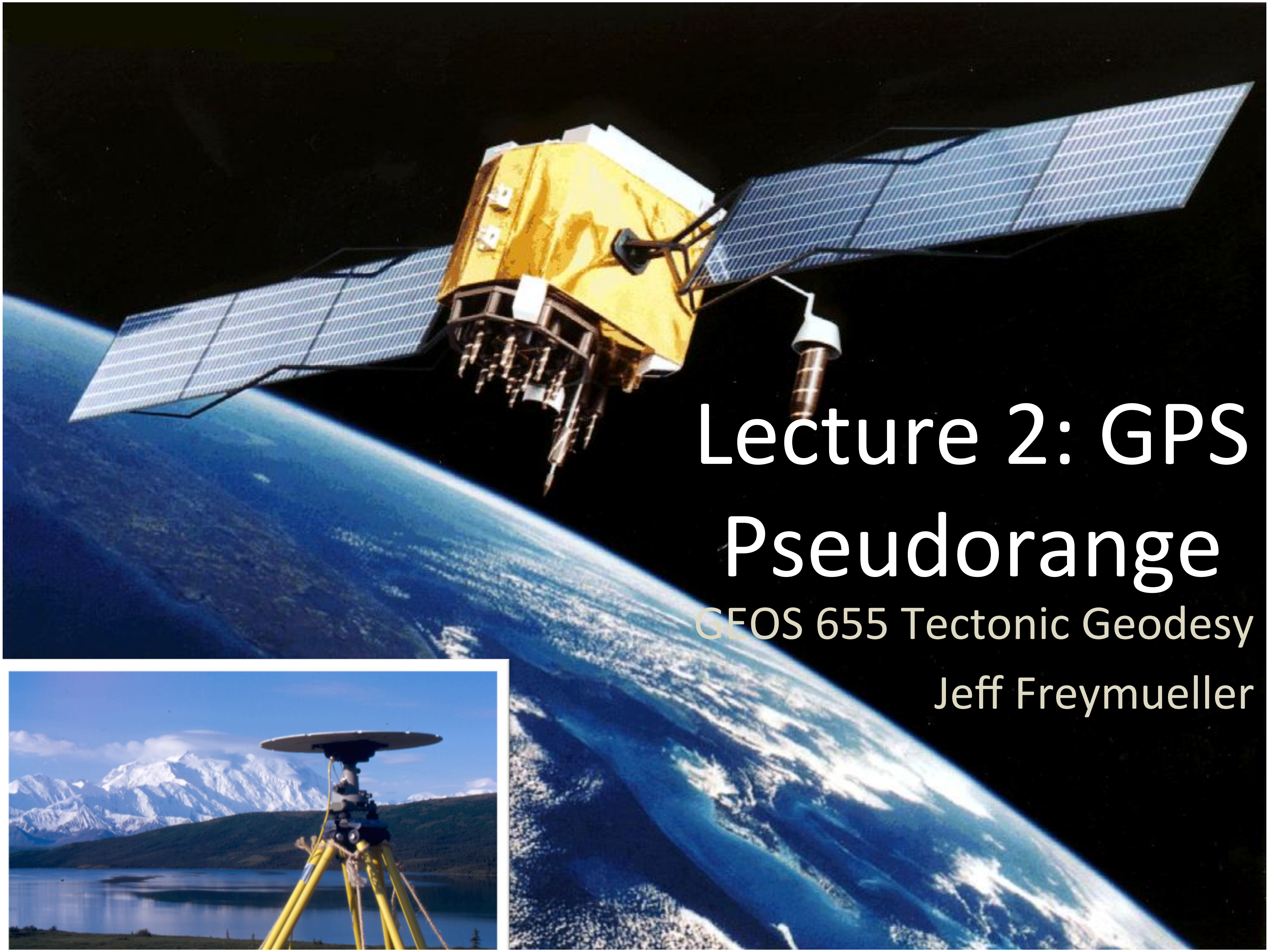
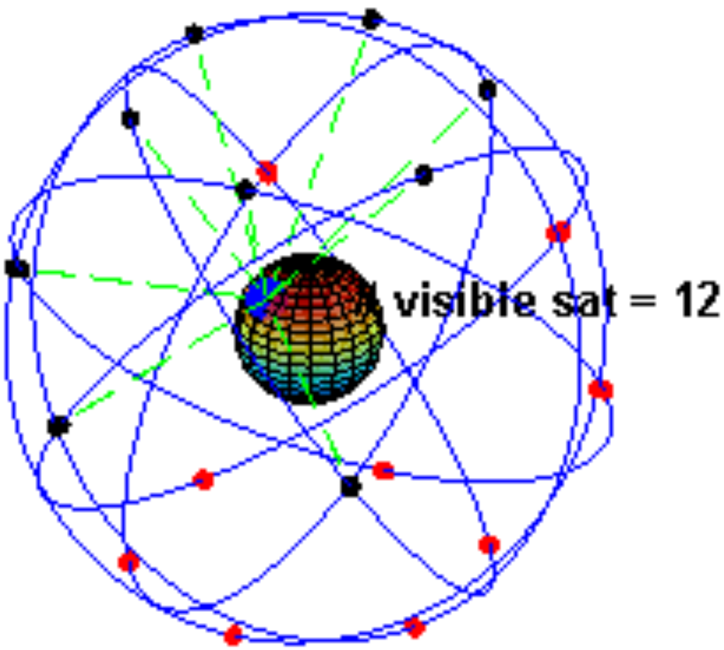


Lecture 2: GPS Pseudorange

GEOS 655 Tectonic Geodesy

Jeff Freymueller

GPS Design Timeline



- NAVSTAR = NAVigation System with Timing and Ranging
- Always-on, instant global positioning
- Development began in 1973
- First satellites launched 1978
- User equip tests 1980
- 1983: Korean Air 007 shot down by Soviet Union
  - Plane had strayed a considerable distance into Soviet airspace
  - Led US Pres. Reagan to mandate future civilian use of GPS

GPS User Hardware – Old!

GPS User Hardware – Modern!

Different Modes of Use

Navigation

- Instantaneous
- Single station
- Original intended use
- Accuracy
  - Few meters
  - Sub-meter w/differential corrections

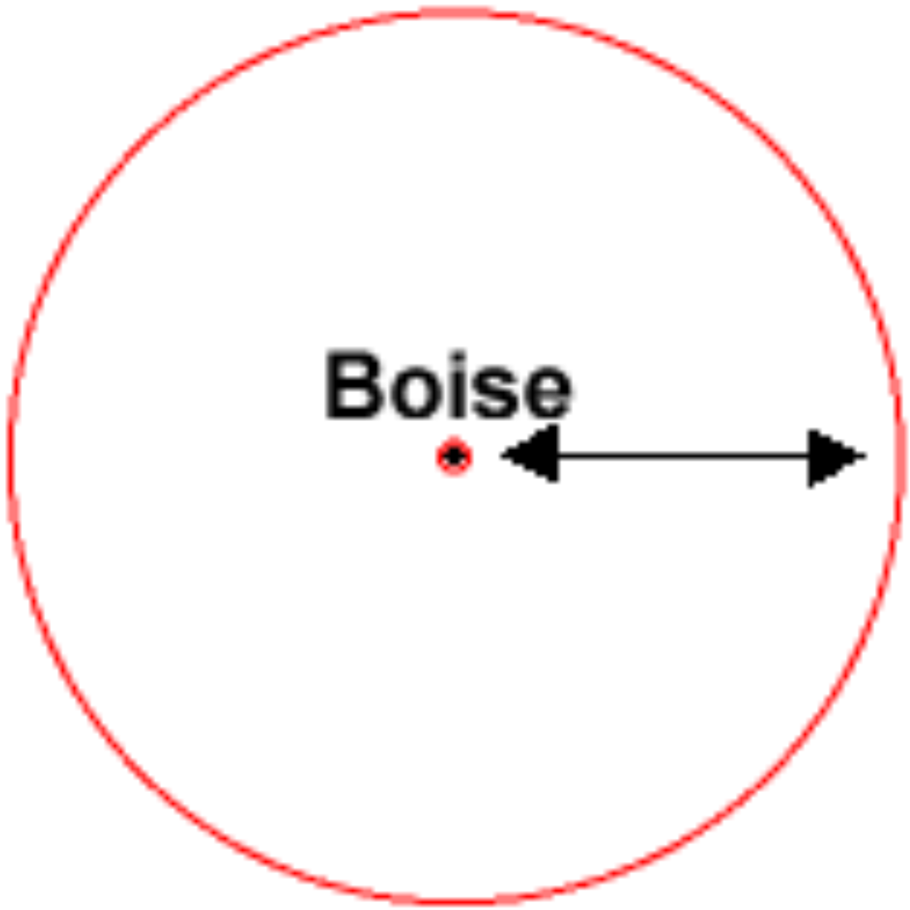
Surveying

- Usually post-process
- Usually multi-station
- Science or survey
- Accuracy
  - 1-2 cm at worst
  - 1-2 mm at best
- Also "seismology"

Basic Principles: Surveying

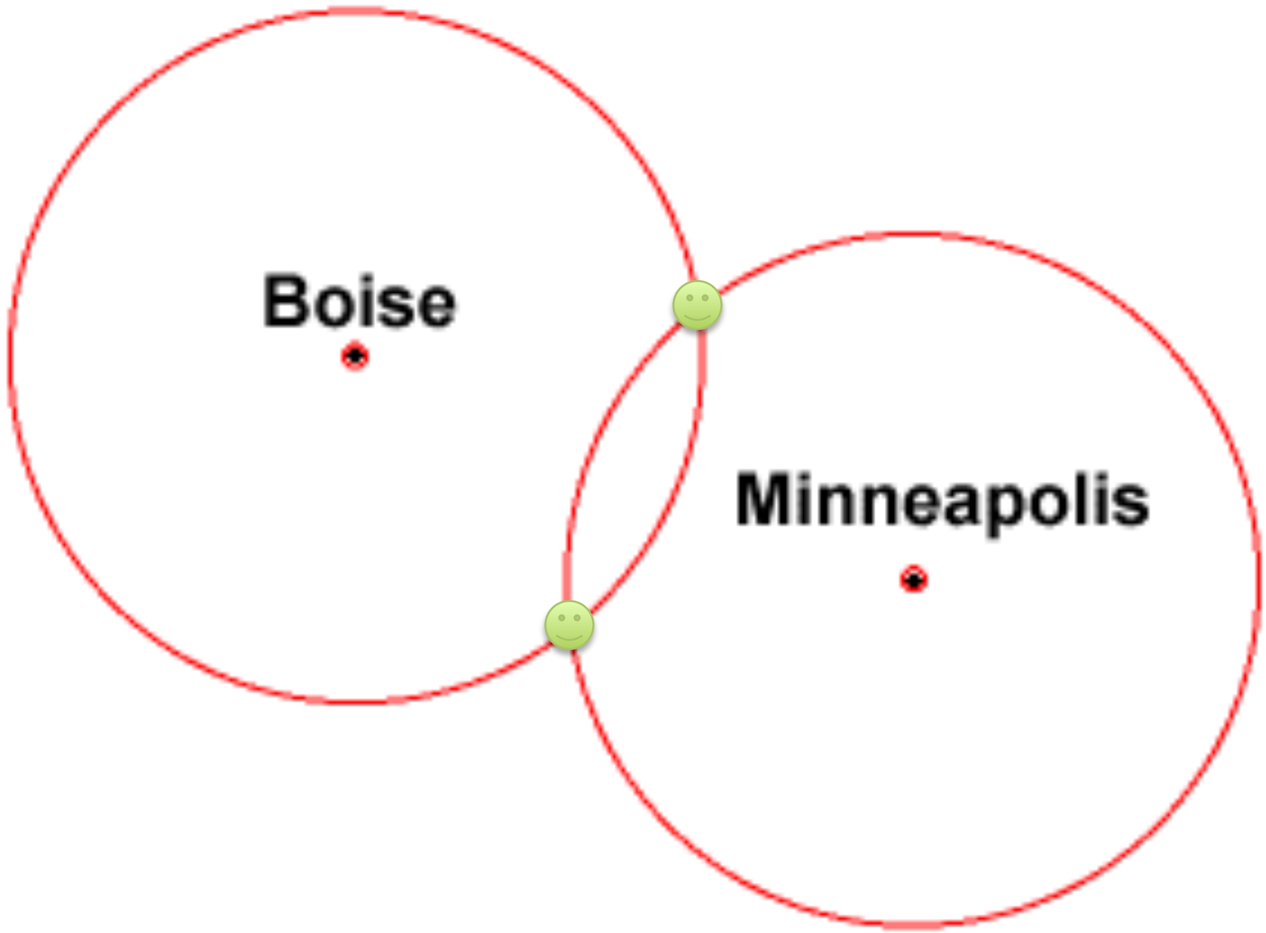
- Requires data from $n \geq 4$ satellites, $m \geq 2$ receivers
  - Point positioning approaches work with 1 receiver
- Requires continuous tracking over time
- Post-processed but real-time being developed
- Use pseudorange and carrier phase measurements from each satellite to receiver
- Orbits of satellites fixed or estimated
- Clock error on satellites estimated or differenced out
- Estimate receiver position (X,Y,Z) and clock error
- Model a wide variety of path delays and other effects

Positioning By Ranging 1



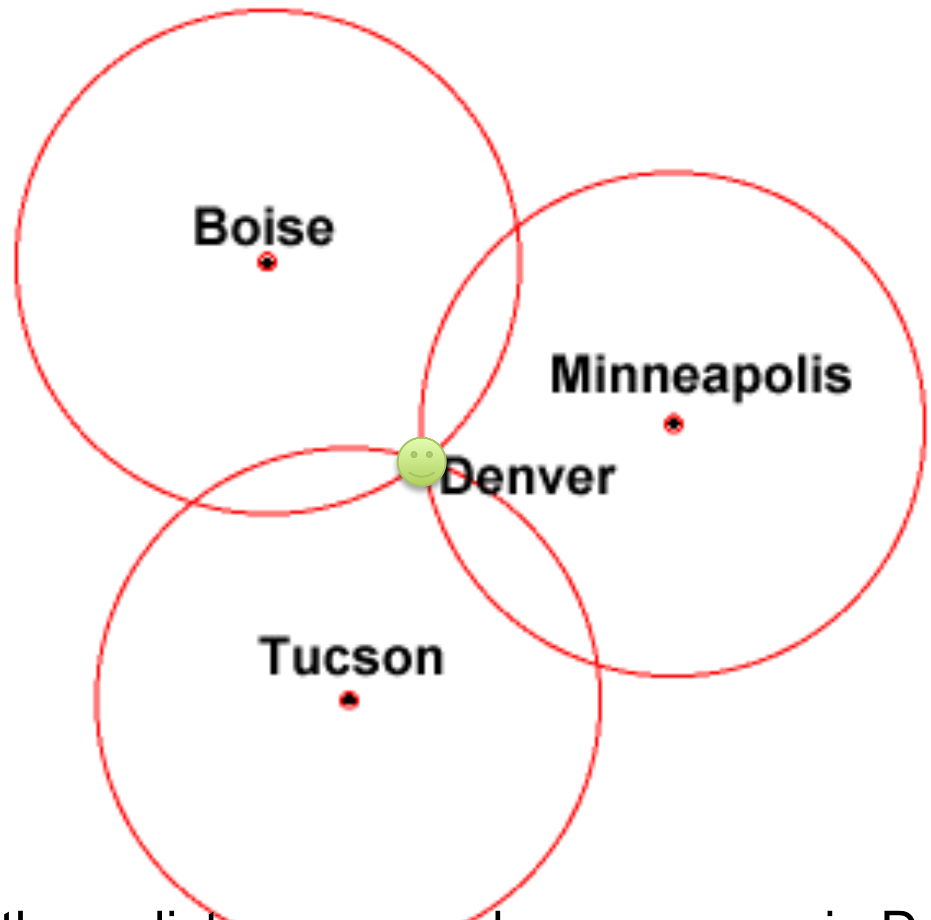
A 2D example: If you know you are a certain distance from Boise, your position could be anywhere on the circle.

Positioning By Ranging 2



With two distances, you know you are at one of two points.

Positioning By Ranging 3



With three distances, you know you are in Denver.
In 3D the circles become spheres, and with three distances you still have two possible locations – one on the surface and one out in space.

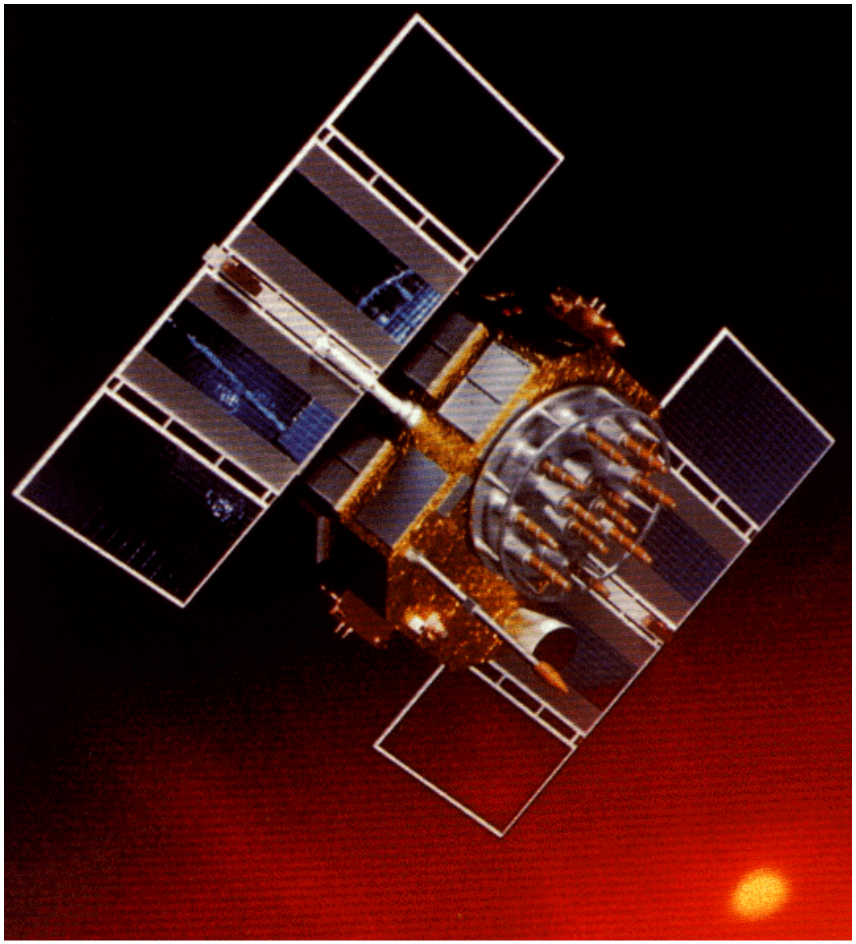
GPS Positioning

- Measure position by measuring ranges to satellites
  - A few satellites can serve an unlimited number of users on the ground, anywhere in the world
- How do we know where satellites are?
  - They broadcast their positions (orbits) in a ***navigation message***
  - (or) someone gives us precise orbits
- Measured ranges are called ***pseudoranges***
- High-precision GPS uses the phase of the GPS carrier signal to measure changes in range

Why call it a "pseudorange"?

- Range is the distance from satellite to receiver, plus path delays.
- Pseudorange is distance plus effects of clock errors
- The terminology has old roots in navigation.
  - VLBI and GPS are pseudoranging; SLR is ranging
- Geometric range ρ is true distance.
- $P = \rho + c^*(\text{clock errors}) + c^*(\text{path delays})$

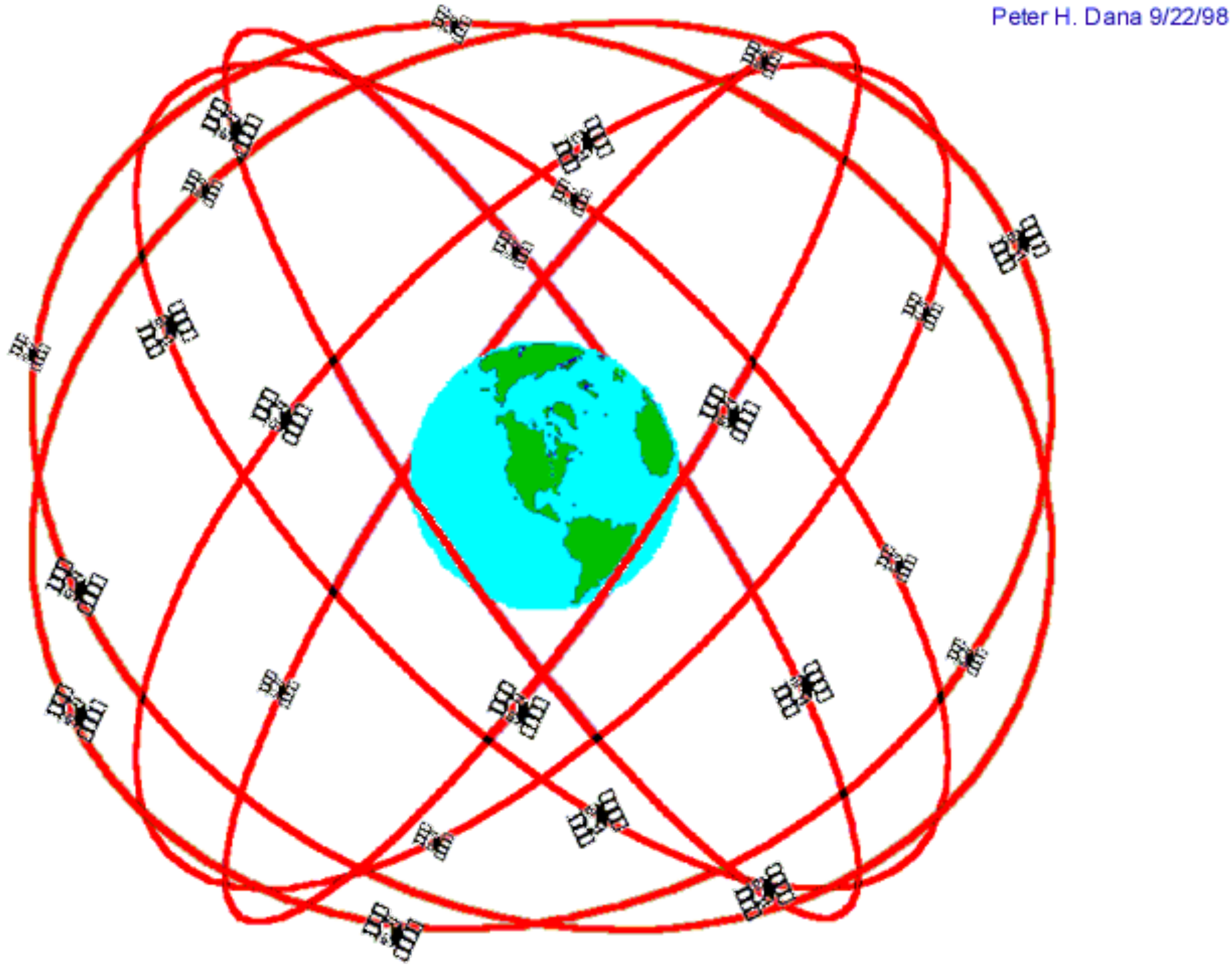
Evolving Satellites



1970s-1980s: smaller, relatively simple

2000s: big, more complex

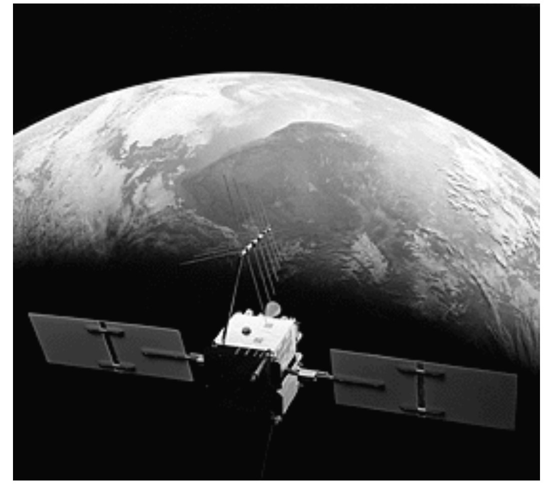
Satellite Constellation



Satellite Constellation Facts

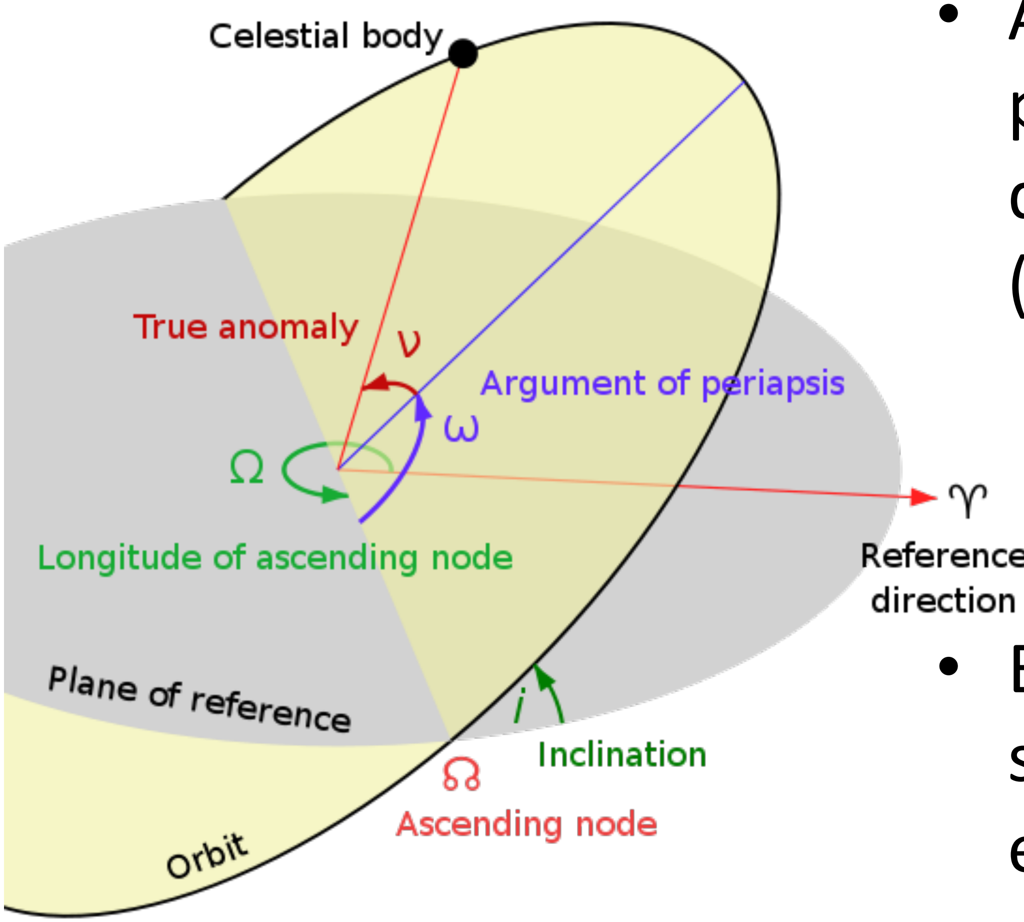
- Nominally 4 satellites (SVs) in each of 6 equally spaced orbital planes (now 5 in each plane).
- Orbital planes inclined 55° from equator.
- Nearly circular orbits $R = 26{,}600$ km ~ $4R_E$
- Orbital period is 11h 58m, two orbits per sidereal day
- Sidereal day is length of day defined by when stars appear in same place in sky
  – Differs from rotational day because of motion of earth around the sun.

Orbits



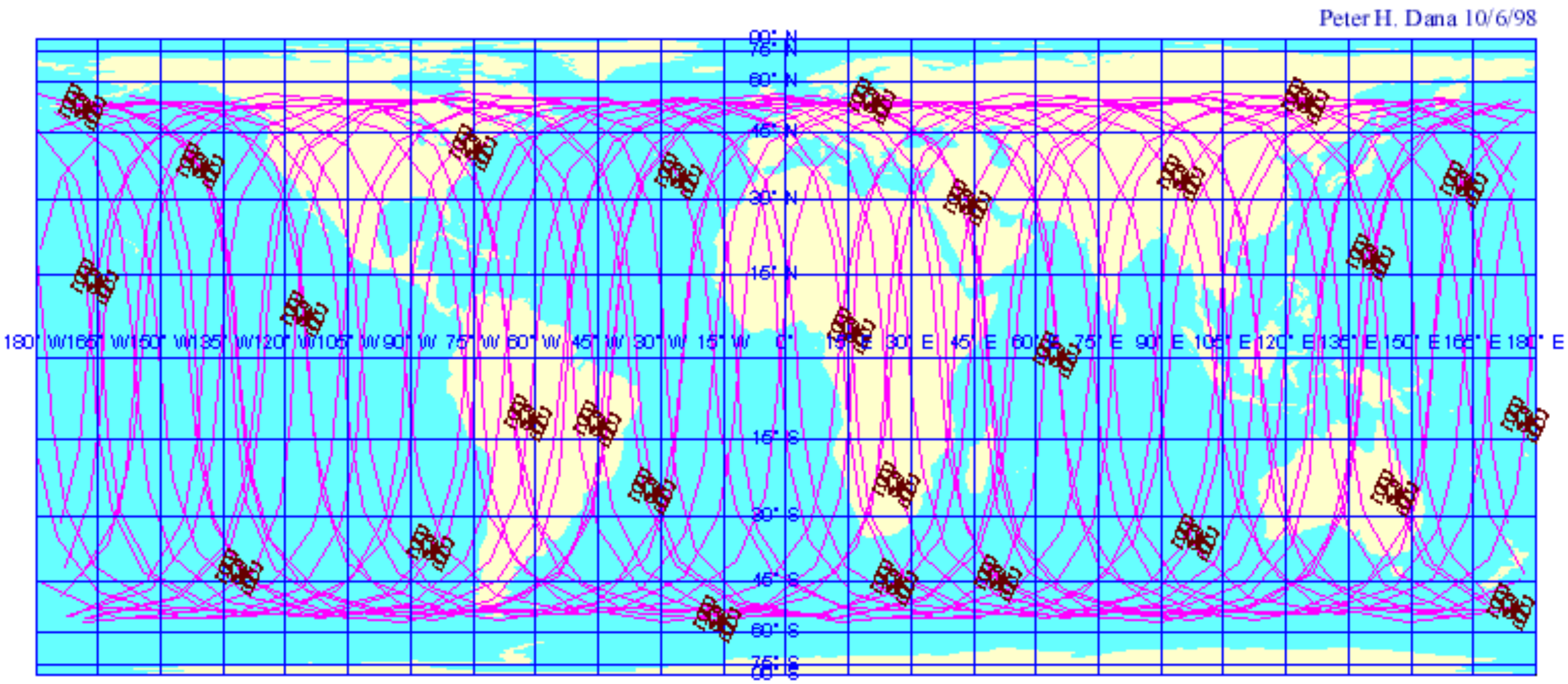
- Can estimate orbits or fix orbits to pre-determined values
- Representation of orbit
  - Broadcast: Keplerian elements + time-dependent corrections
  - Tabular file of XYZ satellite positions
  - Trajectory: initial conditions + integrate equations of motion (needed to estimate orbits)
- In practice, highly precise orbits are available from the IGS
  - Ultra-Rapid: includes predict-ahead for real time use
  - Rapid: Available next day
  - Final: Available in <2 weeks

Keplerian Elements



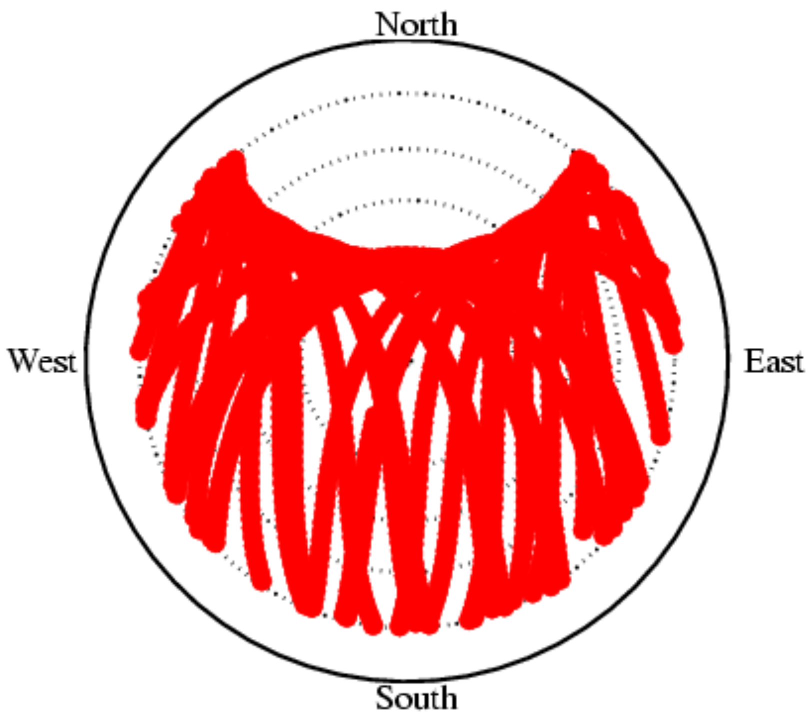
- An elliptical orbit and the position of a body can be described by 6 parameters (Keplerian elements)
  - Semi-major axis
  - Eccentricity
  - Four angles shown at left
- Earth is not a point mass, so satellite orbits are not exactly elliptical.
  - Other forces also perturb orbit

Satellite Ground Tracks

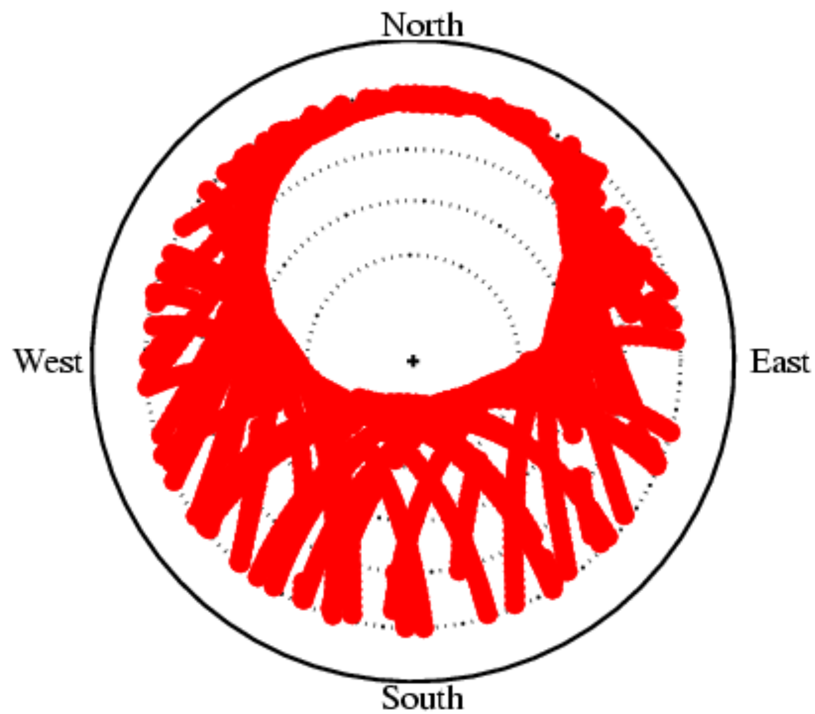


Global Positioning System Satellites and Orbits
for 27 Operational Satellites on September 29, 1998
Satellite Positions at 00:00:00 9/29/98 with 24 hours (2 orbits) of Ground Tracks to 00:00:00 9/30/98

24 hours of GPS Data skytracks



Southern California



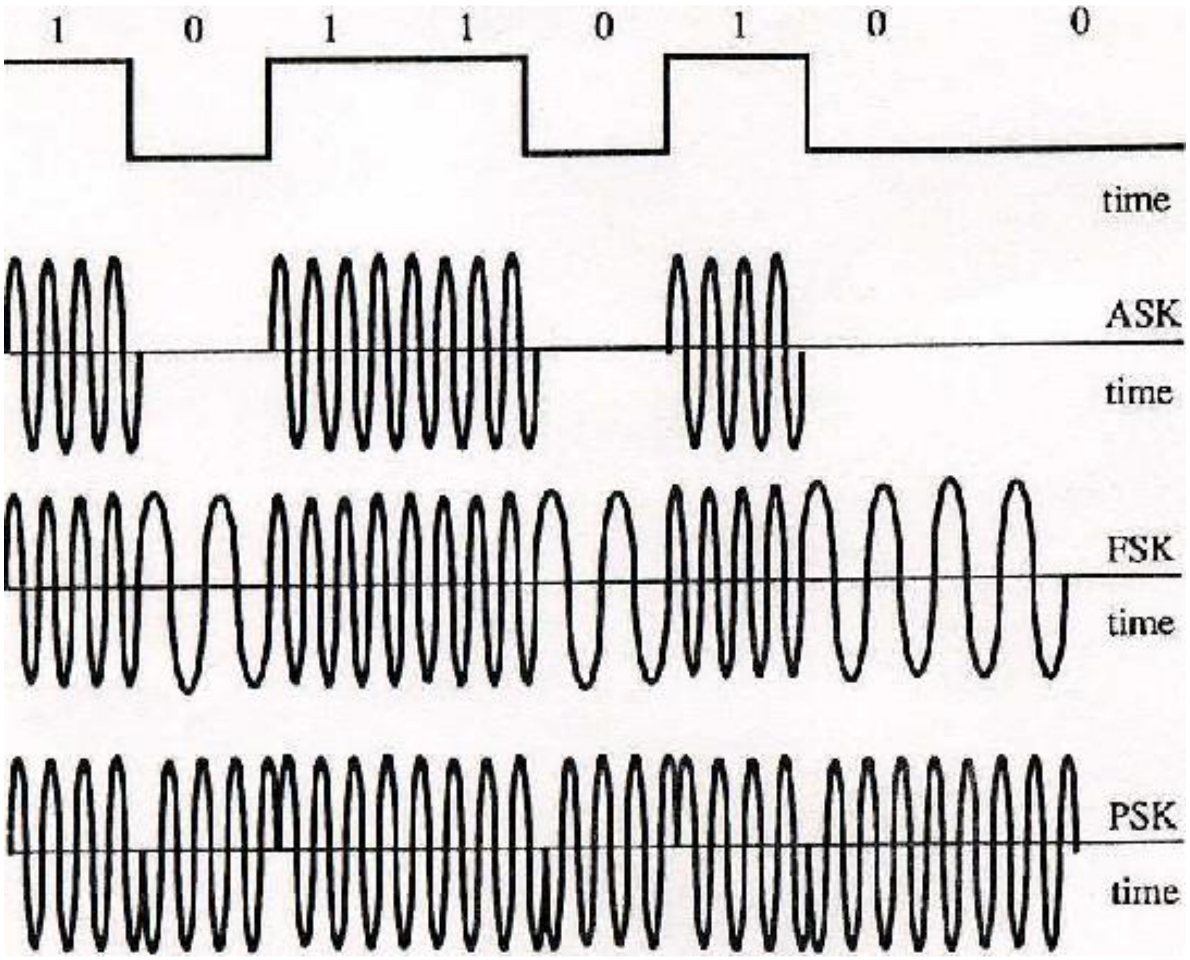
Fairbanks

These are the paths you would see in the sky if you could see the satellites

GPS Signal Structure

- Three frequencies at L-band, L1, L2, and L5
  - L1 at 154*10.23 MHz (~19 cm)
  - L2 at 120*10.23 MHz (~24 cm)
  - L5 at 115*10.23 MHz (~25 cm)
- Codes Modulated (phase modulation) onto each carrier
  - P-code at 10.23 MHz on L1 + L2
  - C/A (Coarse Acquisition) code at 1.023 MHz on L1 + L2 (new L2C)
  - Navigation message at 50 bits per second
- P and C/A codes are types of ***pseudo-random noise (PRN)*** codes

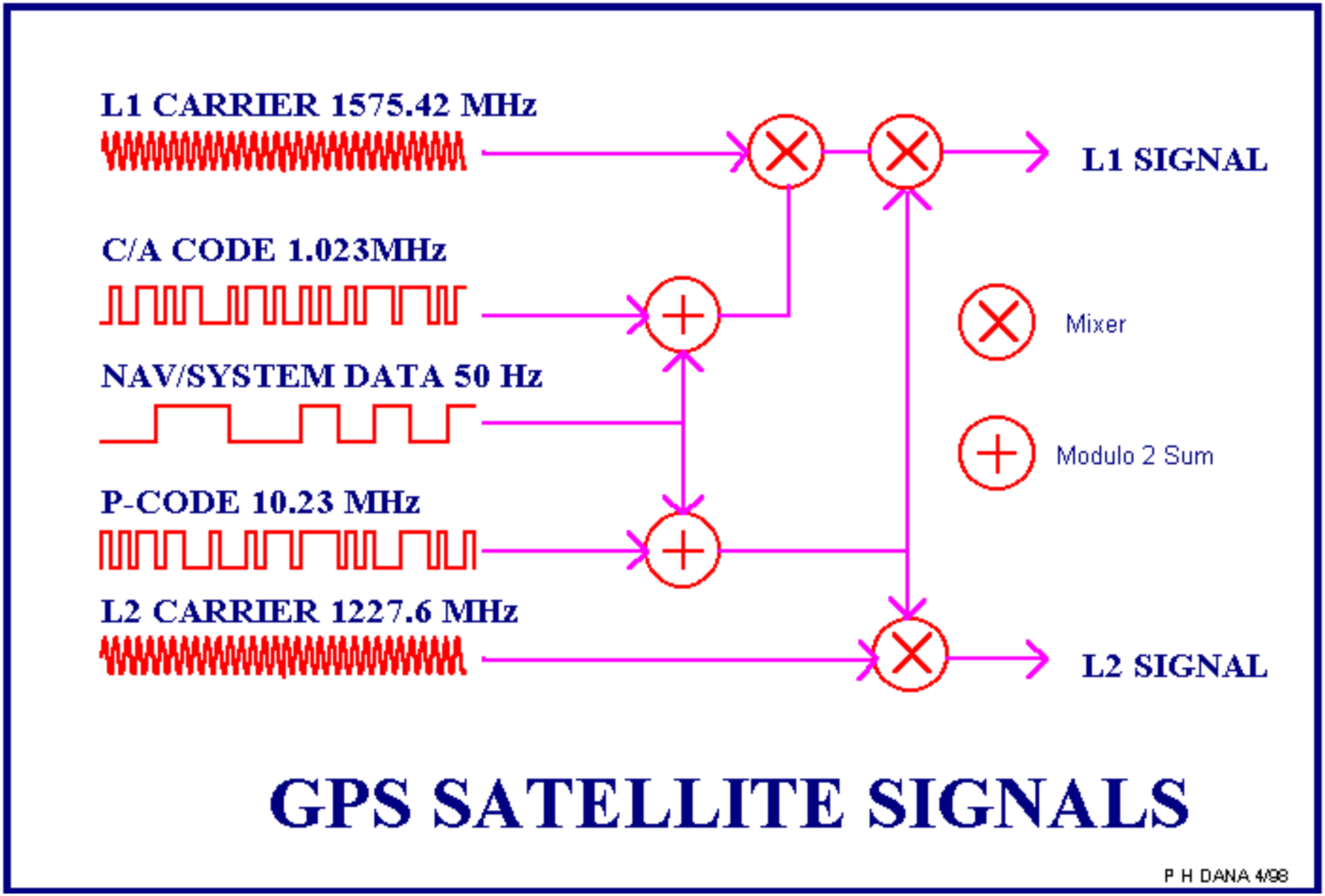
Types of signal modulation



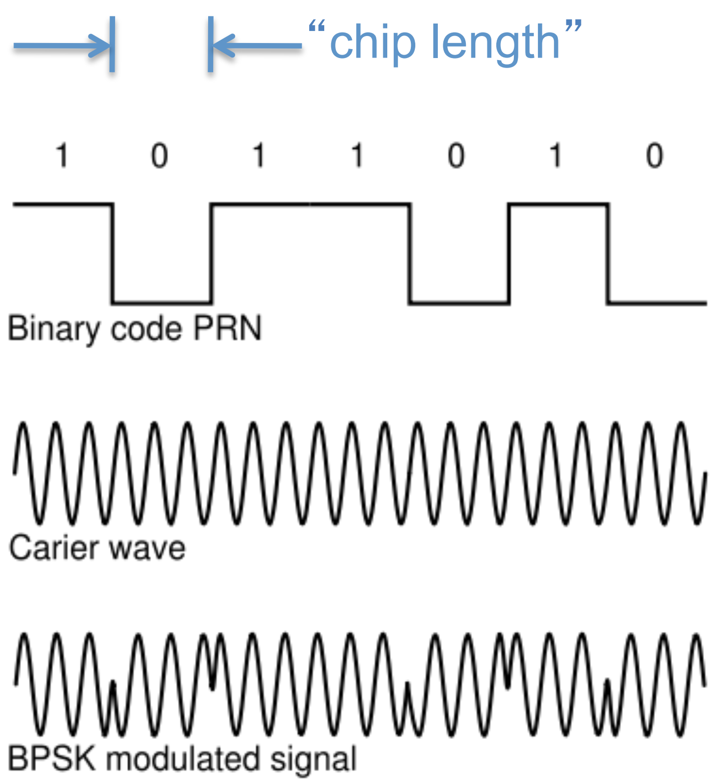
Amplitude Modulation (AM)

Frequency Modulation (FM)

Phase Modulation (PM)


L1 CARRIER 1575.42 MHz
C/A CODE 1.023MHz
NAV/SYSTEM DATA 50 Hz
P-CODE 10.23 MHz
L2 CARRIER 1227.6 MHz
L1 SIGNAL
L2 SIGNAL
Mixer
Modulo 2 Sum
GPS SATELLITE SIGNALS
P H DANA 4/98

Precision of Observations



- Code "chip length" is the distance associated with each bit of the code.
  - C/A: 293 m
    - Repeats every ~300 km
  - P: 29.3 m
- Carrier wavelength is analogous to chip length == 2-3 orders of magnitude more precise

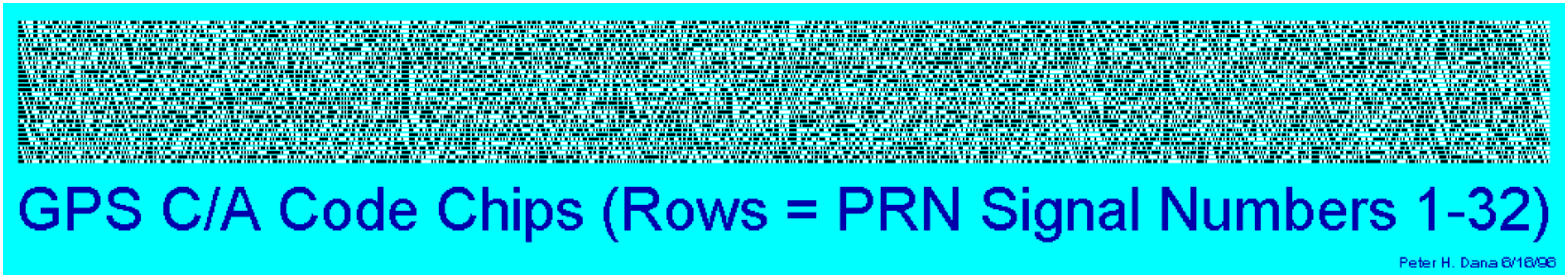
Denial of Accuracy

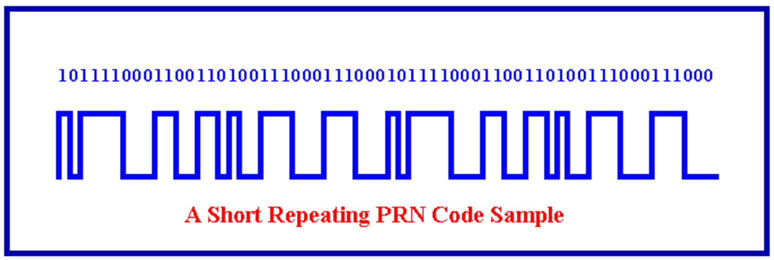
- US DoD can reduce accuracy for real-time civilian users
- (S/A) Selective Availability – on from 1990s to late 1990s
  - Epsilon (introduce errors in navigation message)
  - Dither (introduce rapid variation in SV clocks)
  - Military receivers have special chips to undo this
- (A/S) Anti-Spoofing – on since 1994
  - Encryption of P-code
  - Prevents "the enemy" from imitating (spoofing) GPS signal
  - Modern receivers get around this encryption in various ways (including a near-reverse-engineering in one case).

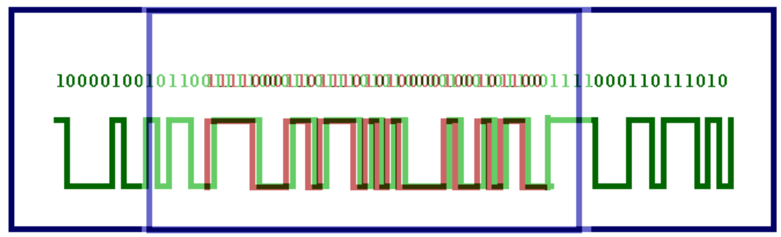
Pseudo-Random Noise

- Computers cannot generate true random numbers, but can generate a sequence of numbers with random statistical properties.
  - But the sequence can be repeated exactly
  - Begin with some starting value, then perform a series of operations
- C/A code has 1023 bits, repeats 1000 times per second
- P code has a lot of bits, repeats every 266.4 days; each SV gets a 7-day piece of code

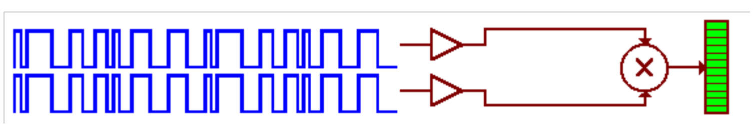
Code Correlation for Ranging

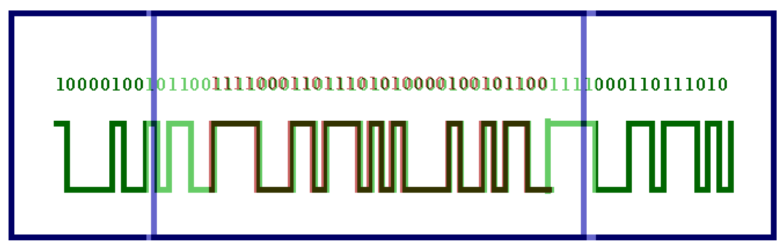






Partial Correlation of Identical Receiver and Satellite PRN Codes





Full Correlation (Code-Phase Lock) of Receiver and Satellite PRN Codes

Receiver generates a copy of the (known) code and correlates with the received code

Pseudorange Observation Model

- The correlation time shift gives an estimate of the travel time, which is the fundamental pseudorange measurement.
  - Travel time = (time of reception) – (time of transmission)
- $P^S = (T - T^S)c$
  - T = receiver clock reading at reception
  - T^S = satellite clock reading at transmission
  - c = speed of light = 299792458 m/s

Accounting for Clock Biases

- Clock bias or clock error?
  - Error == mistake
  - Error == bias, measurement error
  - Error == estimate of uncertainty in the above
- VLBI "removed" clock errors by using ultra-stable hydrogen maser clocks
  - VLBI actually models clock biases as quadratic
- GPS must estimate receiver clock bias (and satellite clock bias for high precision work)

Observation Model with Clocks

- $P^S = (T - T^S)c$
  - $T = t + \tau$ $|\tau| \leq 1$ millisecond
  - $T^S = t^S + \tau^S$ $|\tau^S|$ is small (Cesium or Rubidium clocks)
  - t, t^S are true receive, transmit times, τ are clock errors
- Substituting
  - $P^S = [(t + \tau) - (t^S + \tau^S)]c$
  - $P^S = (t - t^S)c + (\tau - \tau^S)c$
  - $P^S = \rho^S(t,t^S) + (\tau - \tau^S)c$
- $\rho^S(t,t^S)$ is range from receiver at receive time to satellite at transmit time:

$$\rho^S(t,t^S) = \sqrt{\left(x^S(t^S) - x(t)\right)^2 + \left(y^S(t^S) - y(t)\right)^2 + \left(z^S(t^S) - z(t)\right)^2}$$

Light Time Equation

- The transmission time is ~0.07 sec. To evaluate the geometric range we need to map the satellite position back to the transmission time. But we start out knowing only the receive time. We can solve this problem iteratively:

$$t^S_{(0)} = t = (T - \tau)$$

First guess: Transmit time = receive time

$$t^S_{(1)} = t - \frac{\rho^S(t, t^S_{(0)})}{c}$$

$$t^S_{(2)} = t - \frac{\rho^S(t, t^S_{(1)})}{c}$$

$$\vdots$$

Next iteration: Correct for satellite position based on the transmit time estimated from the previous iteration.

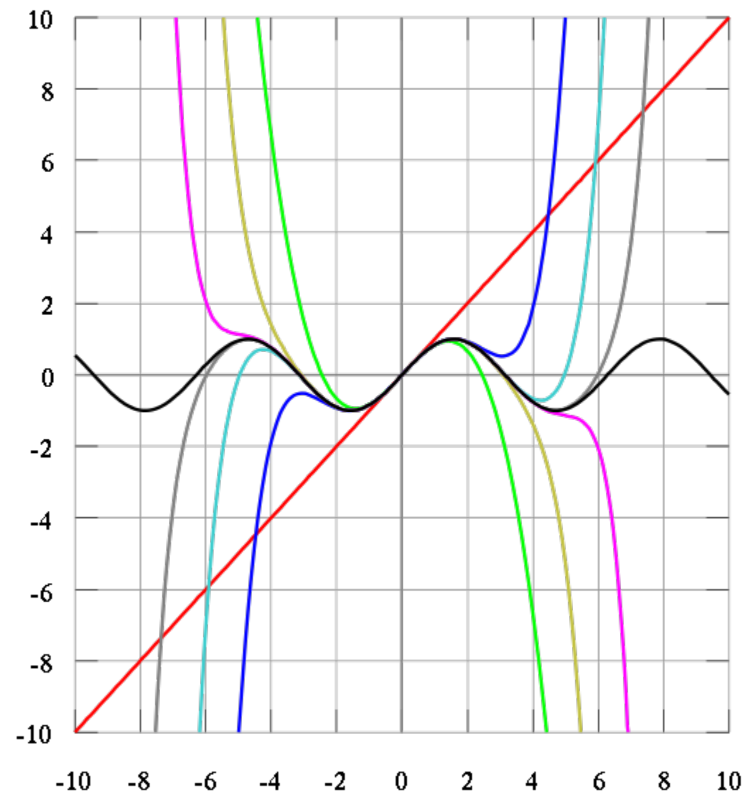
Set of Simplified Observ. Equations

- Now, generalize to multiple satellites. We use a superscript to identify each satellite (don't confuse with an exponent). Later we will have to use a subscript to keep track of multiple receivers:
  - $P^{(1)} = [\ (x^{(1)} - x)^2 + (y^{(1)} - y)^2 + (z^{(1)} - z)^2\ ]^{1/2} + c\tau - c\tau^{(1)}$
  - $P^{(2)} = [\ (x^{(2)} - x)^2 + (y^{(2)} - y)^2 + (z^{(2)} - z)^2\ ]^{1/2} + c\tau - c\tau^{(2)}$
  - $P^{(3)} = [\ (x^{(3)} - x)^2 + (y^{(3)} - y)^2 + (z^{(3)} - z)^2\ ]^{1/2} + c\tau - c\tau^{(3)}$
  - $P^{(4)} = [\ (x^{(4)} - x)^2 + (y^{(3)} - y)^2 + (z^{(3)} - z)^2\ ]^{1/2} + c\tau - c\tau^{(4)}$

x, y, z, τ: receiver position and clock error

Linearizing Nonlinear Equations

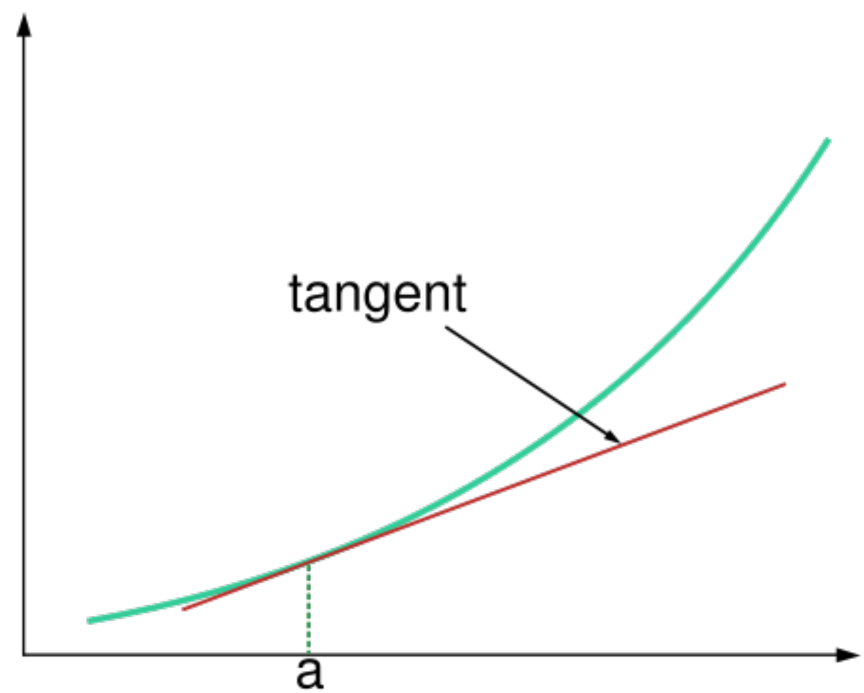
- There are simple ways to solve systems of linear equations, like matrix inversion or least squares. But we have a nonlinear problem. One approach is to linearize, or construct a linear approximation to the non-linear problem. We can do that with Taylor's theorem (Taylor Series)



$$\boxed{f(x) = f(a) + \frac{f'(a)}{1!}(x-a)} + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots,$$

Linearizing Part 2

- We approximate by taking just the linear terms of the Taylor Series.
- We linearize about approximate values (a,b)
- Partial derivatives are computed at (a,b)



$$f(x,y) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)\,.$$

Linearizing Our Equations

- We linearize our equations about approximate values (x_0, y_0, z_0, τ_0)

$$P(x,y,z,\tau) = P(x_0,y_0,z_0,\tau_0) + \frac{\partial P}{\partial x}(x - x_0) + \frac{\partial P}{\partial y}(y - y_0) + \frac{\partial P}{\partial z}(z - z_0) + \frac{\partial P}{\partial \tau}(\tau - \tau_0)$$

$$P(x,y,z,\tau) = P(x_0,y_0,z_0,\tau_0) + \frac{\partial P}{\partial x}\Delta x + \frac{\partial P}{\partial y}\Delta y + \frac{\partial P}{\partial z}\Delta z + \frac{\partial P}{\partial \tau}\Delta \tau$$

$$P(x,y,z,\tau) - P(x_0,y_0,z_0,\tau_0) = \frac{\partial P}{\partial x}\Delta x + \frac{\partial P}{\partial y}\Delta y + \frac{\partial P}{\partial z}\Delta z + \frac{\partial P}{\partial \tau}\Delta \tau$$

$$P_{observed} - P_{computed} = \frac{\partial P}{\partial x}\Delta x + \frac{\partial P}{\partial y}\Delta y + \frac{\partial P}{\partial z}\Delta z + \frac{\partial P}{\partial \tau}\Delta \tau$$

Observation Model

- To solve these equations, we need to write them in the form of an observation model.
  - Observations = Model + measurement noise
- $P_{observed} = P(x,y,z,\tau) + v$

$$P_{observed} = P_{computed} + \frac{\partial P}{\partial x}\Delta x + \frac{\partial P}{\partial y}\Delta y + \frac{\partial P}{\partial z}\Delta z + \frac{\partial P}{\partial \tau}\Delta \tau + v$$

$$P_{observed} - P_{computed} = \frac{\partial P}{\partial x}\Delta x + \frac{\partial P}{\partial y}\Delta y + \frac{\partial P}{\partial z}\Delta z + \frac{\partial P}{\partial \tau}\Delta \tau + v$$

$$\Delta P = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} & \frac{\partial P}{\partial z} & \frac{\partial P}{\partial \tau} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta \tau \end{pmatrix} + v$$

Matrix Equation

- It is easier to deal with this equation if we write it as a matrix equation:

$$\begin{pmatrix} \Delta P^{(1)} \\ \Delta P^{(2)} \\ \Delta P^{(3)} \\ \Delta P^{(4)} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial P^{(1)}}{\partial x} & \dfrac{\partial P^{(1)}}{\partial y} & \dfrac{\partial P^{(1)}}{\partial z} & \dfrac{\partial P^{(1)}}{\partial \tau} \\ \dfrac{\partial P^{(2)}}{\partial x} & \dfrac{\partial P^{(2)}}{\partial y} & \dfrac{\partial P^{(2)}}{\partial z} & \dfrac{\partial P^{(2)}}{\partial \tau} \\ \dfrac{\partial P^{(3)}}{\partial x} & \dfrac{\partial P^{(3)}}{\partial y} & \dfrac{\partial P^{(3)}}{\partial z} & \dfrac{\partial P^{(3)}}{\partial \tau} \\ \dfrac{\partial P^{(4)}}{\partial x} & \dfrac{\partial P^{(4)}}{\partial y} & \dfrac{\partial P^{(4)}}{\partial z} & \dfrac{\partial P^{(4)}}{\partial \tau} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta \tau \end{pmatrix} + \begin{pmatrix} v^{(1)} \\ v^{(2)} \\ v^{(3)} \\ v^{(4)} \end{pmatrix}$$

Evaluate the Partial Derivatives

- This is often written in matrix form like
  - **b** = **Ax** + **v**   **A** is called the "Design matrix"
  - If $\rho^{(i)} = [\,(x_0 - x^{(i)})^2 + (y_0 - y^{(i)})^2 + (z_0 - z^{(i)})^2\,]^{1/2}$

$$A = \begin{pmatrix} \boxed{\dfrac{x_0 - x^{(1)}}{\rho^{(1)}} \quad \dfrac{y_0 - y^{(1)}}{\rho^{(1)}} \quad \dfrac{z_0 - z^{(1)}}{\rho^{(1)}}} & c \\ \dfrac{x_0 - x^{(2)}}{\rho^{(2)}} \quad \dfrac{y_0 - y^{(2)}}{\rho^{(2)}} \quad \dfrac{z_0 - z^{(2)}}{\rho^{(2)}} & c \\ \dfrac{x_0 - x^{(3)}}{\rho^{(3)}} \quad \dfrac{y_0 - y^{(3)}}{\rho^{(3)}} \quad \dfrac{z_0 - z^{(3)}}{\rho^{(3)}} & c \\ \dfrac{x_0 - x^{(4)}}{\rho^{(4)}} \quad \dfrac{y_0 - y^{(4)}}{\rho^{(4)}} \quad \dfrac{z_0 - z^{(4)}}{\rho^{(4)}} & c \end{pmatrix}$$

These have the form of trig functions, and can also be written in terms of the azimuth to the satellite and the inclination of the satellite above the horizon.

Solving the Equations

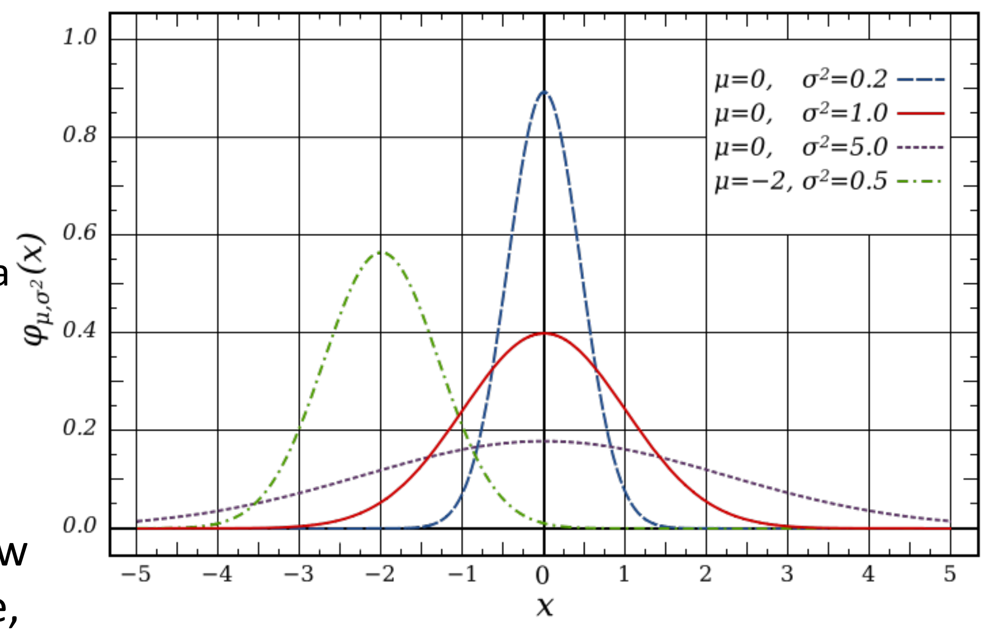
- If there are 4 observations exactly, then the system of equations can be solved exactly:
  - $b = Ax + v$ ➔ $x = A^{-1}b$ (noise $v = 0$ is assumed)
- In general, we will have more than 4 satellites observed at a time. So how do we find the "best" solution. Least squares!
  - Least squares solution minimizes the sum of squares of residuals, that is
  - Find the x that gives the minimum v^Tv.

Least Squares Solution

- The least squares solution, for equally weighted data, is
  - $x' = (A^T A)^{-1} A^T b$
- This assumes that $(A^T A)^{-1}$ exists. It will exist as long as there are 4 or more satellites located in distinct directions in the sky. Two satellites located in exactly the same place would count as one.

Biases and Errors

- Suppose we know what the measurement errors are. How would these known errors bias our estimates of position (and clock bias)?
  - $v_x = (A^TA)^{-1}A^Tv$
  - You might use this to determine whether a newly discovered error has a big impact on your estimated parameters, but in general you would simply want to correct the data!
- In reality, you don't know the measurement errors, but you may know their statistical properties. For example, you may know that the mean measurement error is 0, with uncertainty σ, and that the measurement errors follow a Gaussian or ***normal distribution***.



In this case:
Expectation: $E(v) = 0$
Covariance: $C = E(vv^T) = \sigma^2 I$

Covariance Matrix

- The covariance is given as a (symmetric) matrix. For the problem we have just solved,
  - $C_x = \sigma^2(A^TA)^{-1}$
  - σ^2 is the data noise. $(A^TA)^{-1}$ relates only to the geometry.
- Or in terms of the components:

$$C_x = \sigma^2 \begin{pmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} & \sigma_{x\tau} \\ \sigma_{yx} & \sigma_y^2 & \sigma_{yz} & \sigma_{y\tau} \\ \sigma_{zx} & \sigma_{zy} & \sigma_z^2 & \sigma_{z\tau} \\ \sigma_{\tau x} & \sigma_{\tau y} & \sigma_{\tau z} & \sigma_\tau^2 \end{pmatrix}$$

Local Coordinates

- You can also transform the XYZ coordinates to local coordinates (east, north, height). We' ll leave the equations for that for later. But if you take just the coordinates part of the covariance, you get:

$$C_L = \sigma^2 \begin{pmatrix} \sigma_e^2 & \sigma_{en} & \sigma_{eh} \\ \sigma_{ne} & \sigma_n^2 & \sigma_{nh} \\ \sigma_{he} & \sigma_{hn} & \sigma_h^2 \end{pmatrix}$$

DOPs – Dilution of Precision

- Your handheld GPS probably reports a number called "PDOP", which stands for "Position Dilution of Precision". These are other DOPs as well, which all give measures of how the satellite geometry maps into position or time precision.
  - VDOP = σ_h
  - HDOP = $(\sigma_e^2 + \sigma_n^2)^{1/2}$
  - PDOP = $(\sigma_e^2 + \sigma_n^2 + \sigma_h^2)^{1/2}$
  - GDOP = $(\sigma_e^2 + \sigma_n^2 + \sigma_h^2 + c^2\sigma_\tau^2)^{1/2}$
  - TDOP = σ_τ
- Multiply PDOP by measurement precision to get uncertainty in 3D position.

PDOP > 5 considered poor