

Lecture 20

Bipolar Junction Transistors (BJT): Part 4

Small Signal BJT Model

Reading:

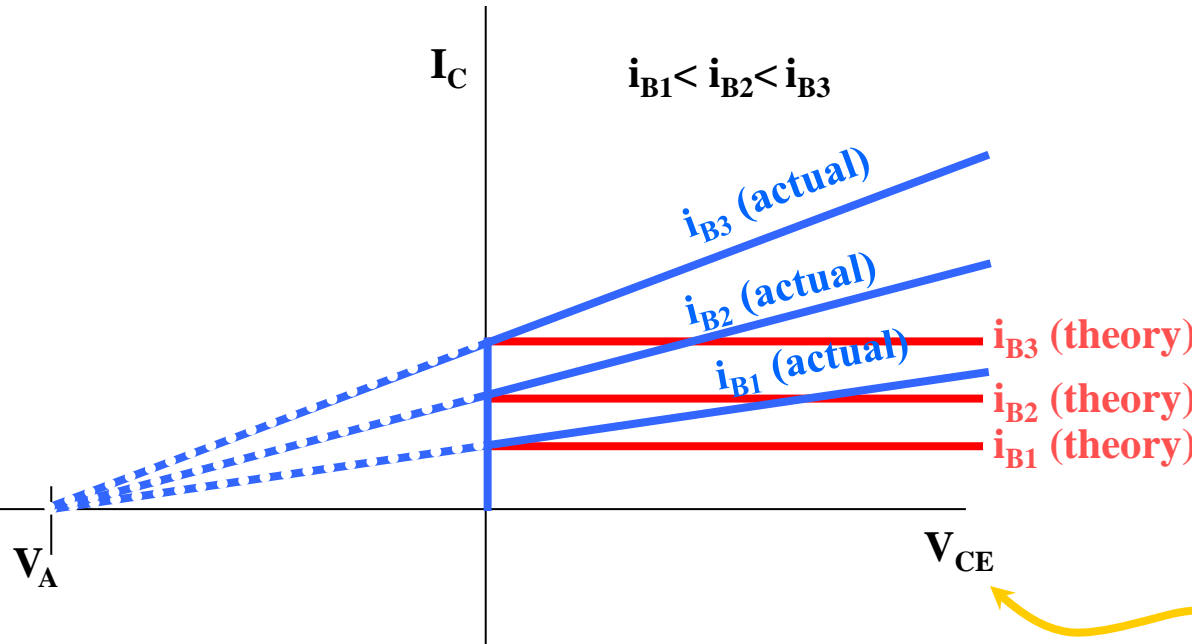
Jaeger 13.5-13.6, Notes

Further Model Simplifications

(useful for circuit analysis)

$$\begin{array}{ccc} \text{Ebers-Moll} & \text{Forward Active Mode} & \text{Neglect Small Terms} \\ \underbrace{I_C = \alpha_F I_{F0} \left(e^{V_{EB}/V_T} - 1 \right) - I_{R0} \left(e^{V_{CB}/V_T} - 1 \right)} & \Rightarrow \underbrace{I_C = \alpha_F I_{F0} \left(e^{V_{EB}/V_T} - 1 \right) + I_{R0}} & \Rightarrow \underbrace{I_C = I_S e^{V_{EB}/V_T}} \end{array}$$

Modeling the “Early Effect” (non-zero slopes in IV curves)



- Base width changes due to changes in the base-collector depletion width with changes in V_{CB} .
- This changes α_T , which changes I_C , α_{DC} and B_F

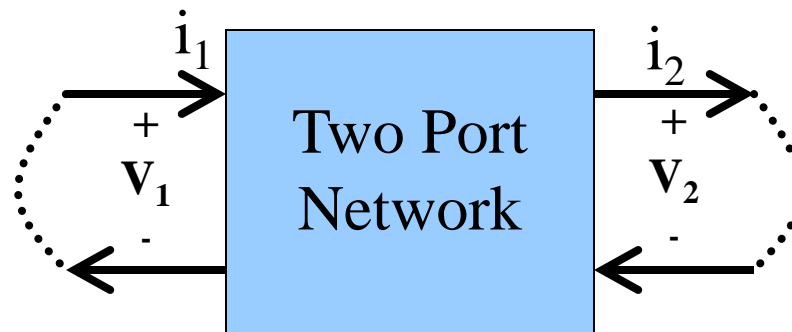
Major BJT Circuit Relationships

$$i_C = I_S e^{V_{EB}/V_T} \Rightarrow i_C = I_S e^{V_{EB}/V_T} \left[1 + \frac{V_{CE}}{V_A} \right] \quad \beta_F = \beta_{FO} \left[1 + \frac{V_{CE}}{V_A} \right] \quad i_B = \frac{i_C}{\beta_F} = \frac{I_S}{\beta_{FO}} e^{V_{EB}/V_T}$$

Small Signal Model of a BJT

- Just as we did with a p-n diode, we can break the BJT up into a large signal analysis and a small signal analysis and “linearize” the non-linear behavior of the Ebers-Moll model.
- Small signal Models are only useful for Forward active mode and thus, are derived under this condition. (Saturation and cutoff are used for switches which involve very large voltage/current swings from the on to off states.)
- Small signal models are used to determine amplifier characteristics (Example: “Gain” = Increase in the magnitude of a signal at the output of a circuit relative to its magnitude at the input of the circuit).
- Warning: Just like when a diode voltage exceeds a certain value, the non-linear behavior of the diode leads to distortion of the current/voltage curves (see previous lecture), if the inputs/outputs exceed certain limits, the full Ebers-Moll model must be used.

Consider the BJT as a two-port Network



General “y-parameter” Network

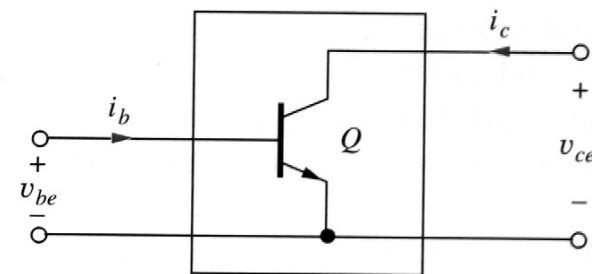
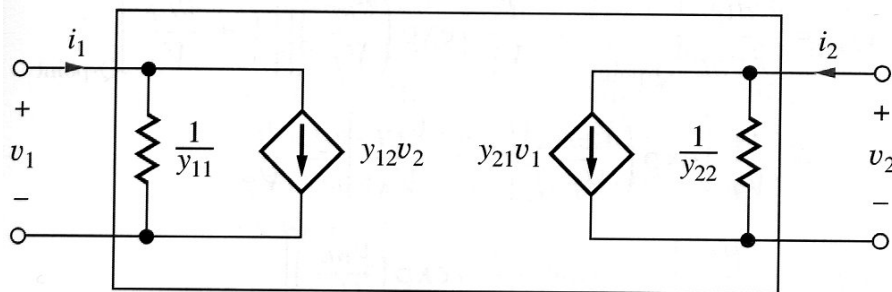
$$i_1 = y_{11}v_1 + y_{12}v_2$$

$$i_2 = y_{21}v_1 + y_{22}v_2$$

BJT “y-parameter” Network

$$i_b = y_{11}v_{be} + y_{12}v_{ce}$$

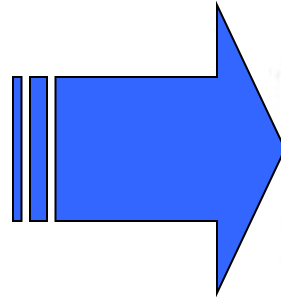
$$i_c = y_{21}v_{be} + y_{22}v_{ce}$$



Consider the BJT as a two-port Network

$$\mathbf{i}_b = y_{11}v_{be} + y_{12}v_{ce}$$

$$\mathbf{i}_c = y_{21}v_{be} + y_{22}v_{ce}$$



$$y_{11} = \left. \frac{\mathbf{i}_b}{v_{be}} \right|_{v_{ce}=0} = \left. \frac{\partial i_B}{\partial v_{BE}} \right|_{\text{Q-point}}$$

$$y_{12} = \left. \frac{\mathbf{i}_b}{v_{ce}} \right|_{v_{be}=0} = \left. \frac{\partial i_B}{\partial v_{CE}} \right|_{\text{Q-point}}$$

$$y_{21} = \left. \frac{\mathbf{i}_c}{v_{be}} \right|_{v_{ce}=0} = \left. \frac{\partial i_C}{\partial v_{BE}} \right|_{\text{Q-point}}$$

$$y_{22} = \left. \frac{\mathbf{i}_c}{v_{ce}} \right|_{v_{be}=0} = \left. \frac{\partial i_C}{\partial v_{CE}} \right|_{\text{Q-point}}$$

Consider the BJT as a two-port Network

$$y_{11} = \left. \frac{i_b}{v_{be}} \right|_{v_{ce}=0} = \left. \frac{\partial i_B}{\partial v_{BE}} \right|_{Q\text{-point}}$$

$$y_{12} = \left. \frac{i_b}{v_{ce}} \right|_{v_{be}=0} = \left. \frac{\partial i_B}{\partial v_{CE}} \right|_{Q\text{-point}}$$

$$y_{21} = \left. \frac{i_c}{v_{be}} \right|_{v_{ce}=0} = \left. \frac{\partial i_C}{\partial v_{BE}} \right|_{Q\text{-point}}$$

$$y_{22} = \left. \frac{i_c}{v_{ce}} \right|_{v_{be}=0} = \left. \frac{\partial i_C}{\partial v_{CE}} \right|_{Q\text{-point}}$$

$$y_{12} = \left. \frac{\partial i_B}{\partial v_{CE}} \right|_{Q\text{-point}} = 0$$

$$y_{21} = \left. \frac{\partial i_C}{\partial v_{BE}} \right|_{Q\text{-point}} = \frac{I_S}{V_T} \left[\exp\left(\frac{v_{BE}}{V_T}\right) \right] \left[1 + \frac{v_{CE}}{V_A} \right]_{Q\text{-point}}$$

$$y_{21} = \frac{I_S}{V_T} \left[\exp\left(\frac{V_{BE}}{V_T}\right) \right] \left[1 + \frac{V_{CE}}{V_A} \right] = \frac{I_C}{V_T}$$

$$y_{22} = \left. \frac{\partial i_C}{\partial v_{CE}} \right|_{Q\text{-point}} = \frac{I_S}{V_A} \left[\exp\left(\frac{v_{BE}}{V_T}\right) \right]_{Q\text{-point}}$$

$$y_{22} = \frac{I_S}{V_A} \left[\exp\left(\frac{V_{BE}}{V_T}\right) \right] = \frac{I_C}{V_A + V_{CE}}$$

$$y_{11} = \left. \frac{\partial i_B}{\partial v_{BE}} \right|_{Q\text{-point}} = \left[\frac{1}{\beta_F} \frac{\partial i_C}{\partial v_{BE}} - \frac{i_C}{\beta_F^2} \frac{\partial \beta_F}{\partial v_{BE}} \right]_{Q\text{-point}}$$

$$y_{11} = \frac{1}{\beta_F} \frac{\partial i_C}{\partial v_{BE}} \left[1 - \frac{i_C}{\beta_F} \frac{\partial \beta_F}{\partial i_C} \right]_{Q\text{-point}} = \frac{I_C}{\beta_F V_T} \left[1 - \left(\frac{i_C}{\beta_F} \frac{\partial \beta_F}{\partial i_C} \right)_{Q\text{-point}} \right]$$

$$y_{11} = \left[\frac{1}{\beta_F} \frac{\partial i_C}{\partial v_{BE}} - \frac{i_C}{\beta_F^2} \frac{\partial \beta_F}{\partial i_C} \frac{\partial i_C}{\partial v_{BE}} \right]_{Q\text{-point}}$$

$$y_{11} = \frac{I_C}{\beta_o V_T}$$

$$\text{where } \beta_o = \frac{\beta_F}{\left[1 - I_C \left(\frac{1}{\beta_F} \frac{\partial \beta_F}{\partial i_C} \right)_{Q\text{-point}} \right]}$$

β_o is most often taken as a constant, β_F

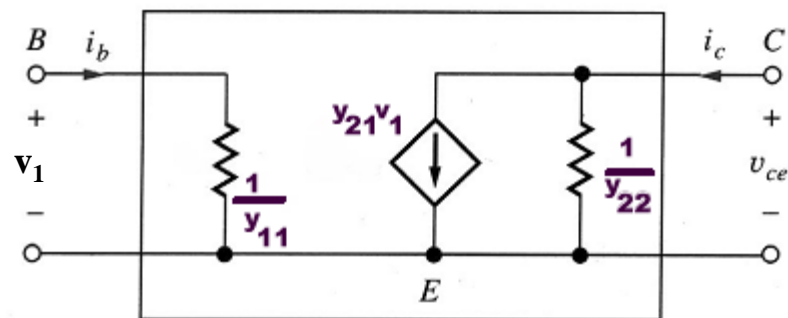
Alternative Representations

Transconductance $g_m = y_{21} = \frac{I_C}{V_T} \approx 40I_C$

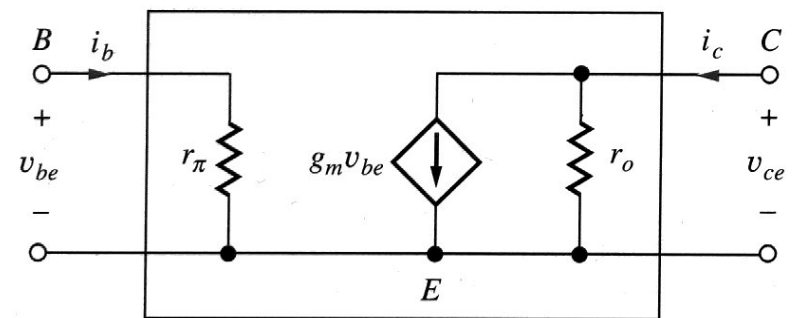
Input Resistance $r_\pi = \frac{1}{y_{11}} = \frac{\beta_o V_T}{I_C} = \frac{\beta_o}{g_m}$

Output Resistance $r_o = \frac{1}{y_{22}} = \frac{V_A + V_{CE}}{I_C}$

Y-parameter Model



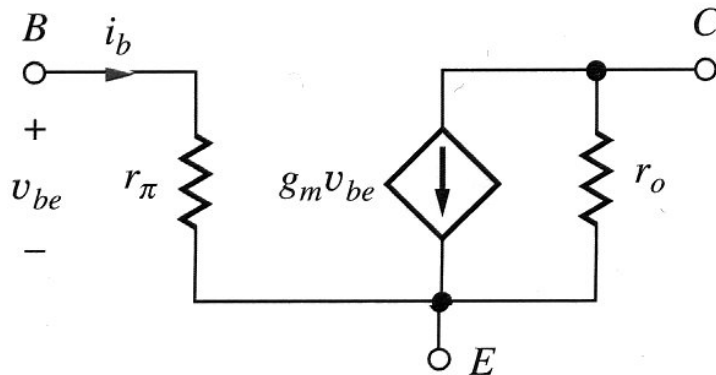
Hybrid-pi Model



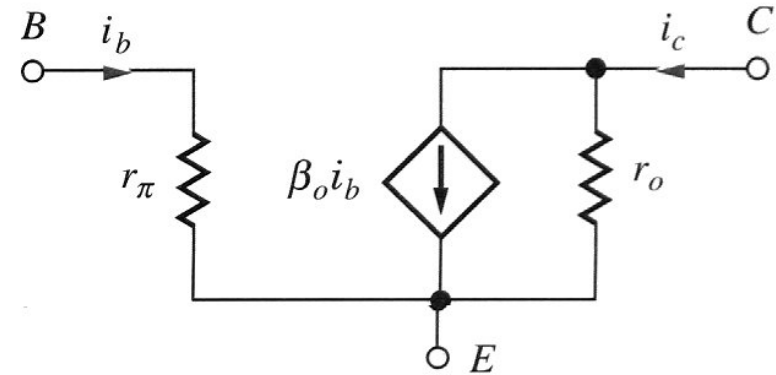
Alternative Representations

$$g_m v_{be} = g_m r_\pi i_b = \beta_o i_b$$

Voltage Controlled Current source version of Hybrid-pi Model



Current Controlled Current source version of Hybrid-pi Model



Single Transistor Amplifier Analysis: Summary of Procedure

Important!

Steps to Analyze a Transistor Amplifier

1.) Determine DC operating point and calculate small signal parameters (see next page)

2.) Convert to the AC only model.

- DC Voltage sources are shorts to ground
- DC Current sources are open circuits
- Large capacitors are short circuits
- Large inductors are open circuits

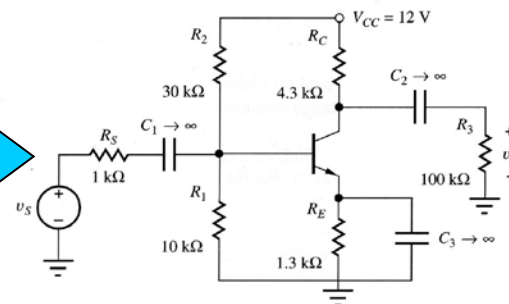
3.) Use a Thevenin circuit (sometimes a Norton) where necessary. Ideally the base should be a single resistor + a single source. **Do not confuse this with the DC Thevenin you did in step 1.**

4.) Replace transistor with small signal model

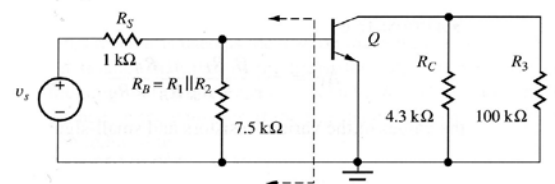
5.) Simplify the circuit as much as necessary.

6.) Calculate the small signal parameters (r_{π} , g_m , r_o etc...) and then gains etc...

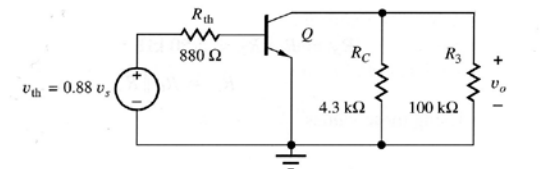
Step 1



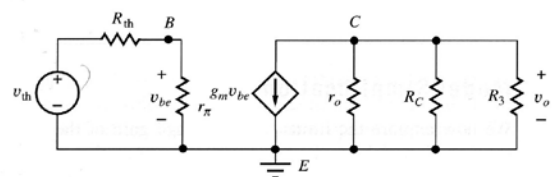
Step 2



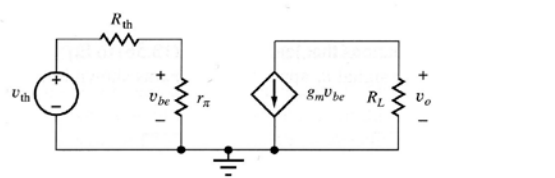
Step 3



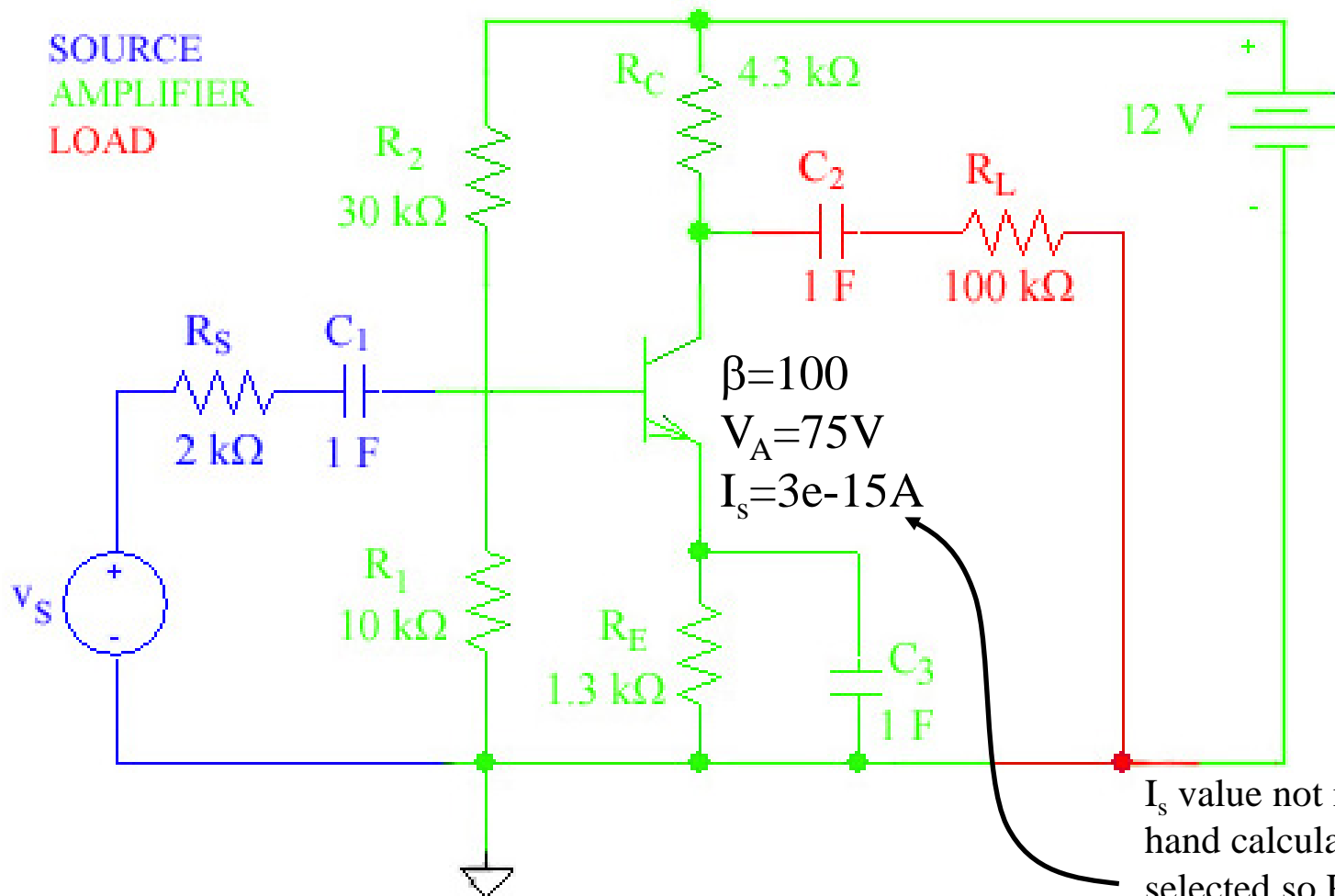
Step 4



Step 5



Detailed Example: Single Transistor Amplifier Analysis



I_s value not needed for hand calculations but is selected so PSPICE results in $V_{BE}=0.7V$ @ $I_C=1.66mA$

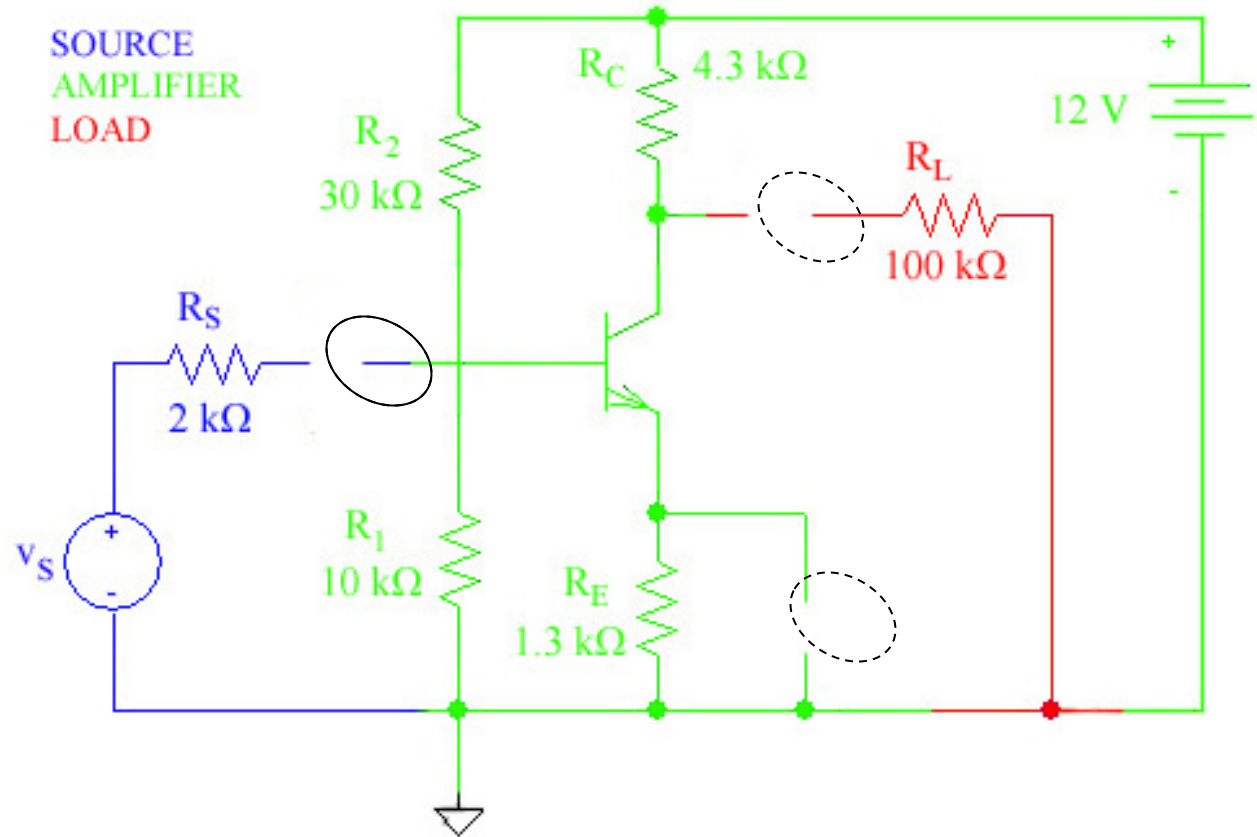
Step 1: Determine DC Operating Point

Remove the Capacitors



Because the impedance of a capacitor is $Z = 1/(j\omega C)$, capacitors have infinite impedance or are open circuits in DC ($\omega = 0$).

Inductors (not present in this circuit) have an impedance $Z = j\omega L$, and are shorts in DC.

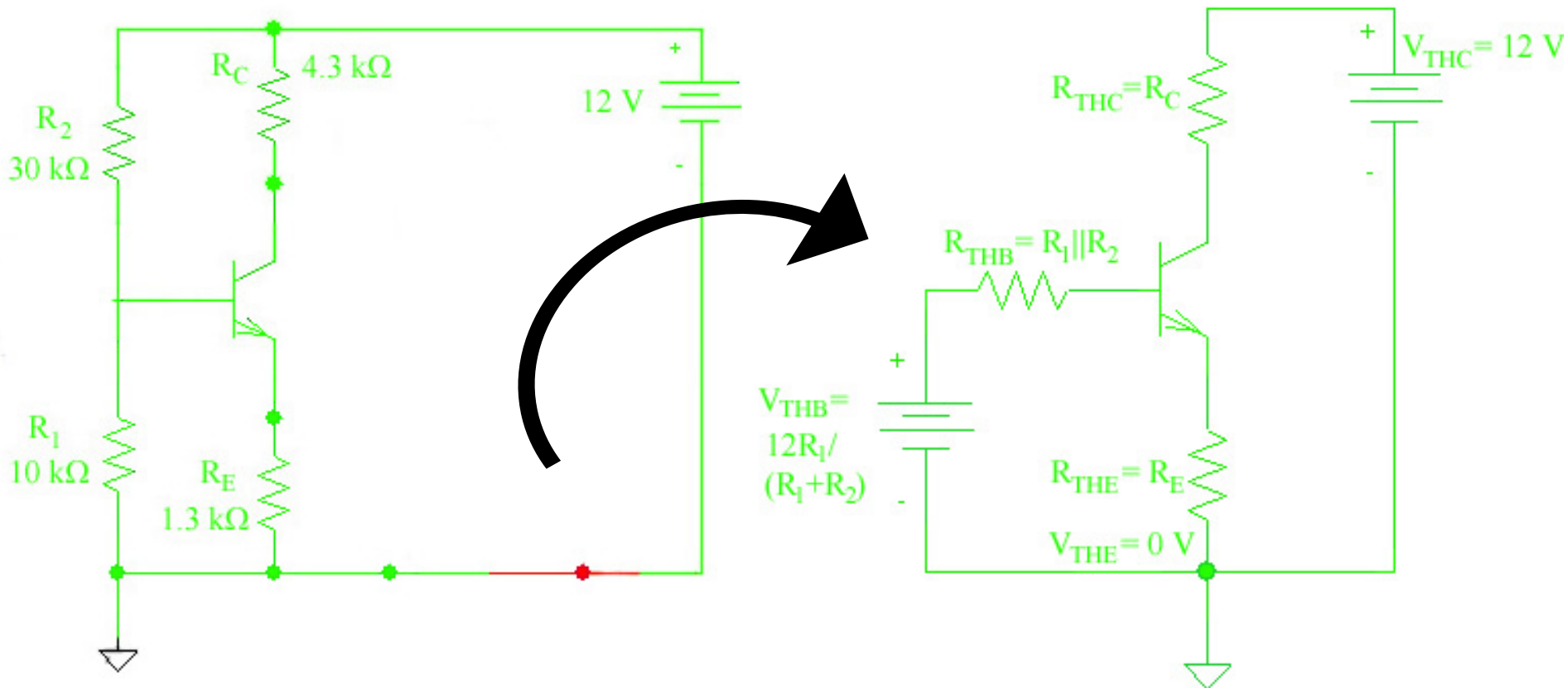


Step 1: Determine DC Operating Point

Determine the DC Thevenin Equivalent



Replace all connections to the transistor with their Thevenin equivalents.



Step 1: Determine DC Operating Point

Calculate Small Signal Parameters



Identify the type of transistor (npn in this example) and draw the **base**, **collector**, and **emitter** currents in their proper direction and their corresponding voltage polarities.

Applying KVL to the controlling loop (loop 1):

$$V_{THB} - I_B R_{THB} - V_{BE} - I_E R_E = 0$$

Applying KCL to the transistor:

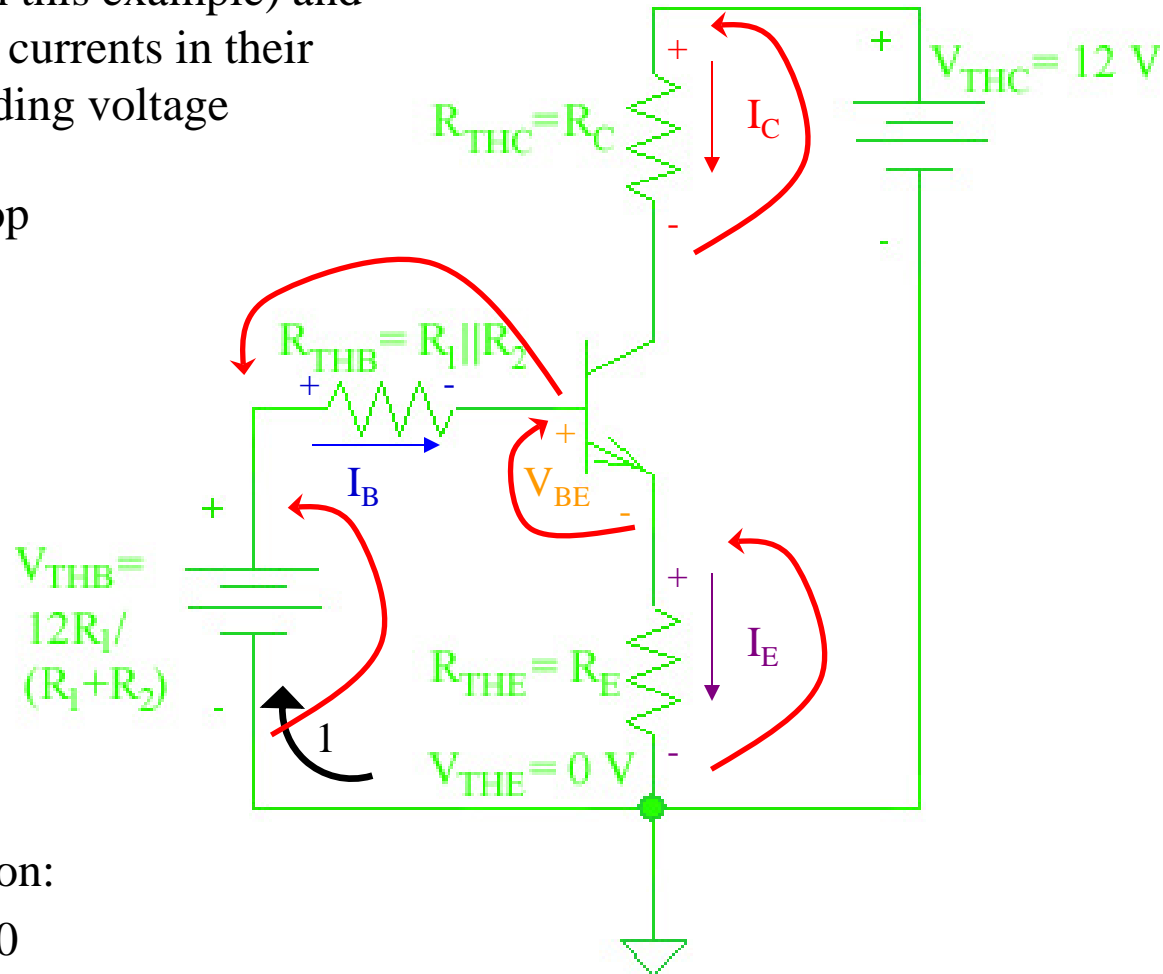
$$I_E = I_B + I_C$$

Because $I_C = \beta I_B$,

$$I_E = I_B + I_C = I_B + \beta I_B = I_B(1 + \beta)$$

Substituting for I_E in the loop equation:

$$V_{THB} - I_B R_{THB} - V_{BE} - I_B(1 + \beta)R_E = 0$$



Step 1: Determine DC Operating Point

Plug in the Numbers



$$V_{THB} - I_B R_{THB} - V_{BE} - I_B(1+\beta)R_E = 0$$

$$V_{THB} - V_{BE} - I_B(R_{THB} + (1+\beta)R_E) = 0$$

$$V_{THB} = 12R_1/(R_1+R_2) = 3 \text{ V}$$

$$R_{THB} = R_1 \parallel R_2 = 7.5 \text{ k}\Omega$$

Assume $V_{BE} = 0.7 \text{ V}$

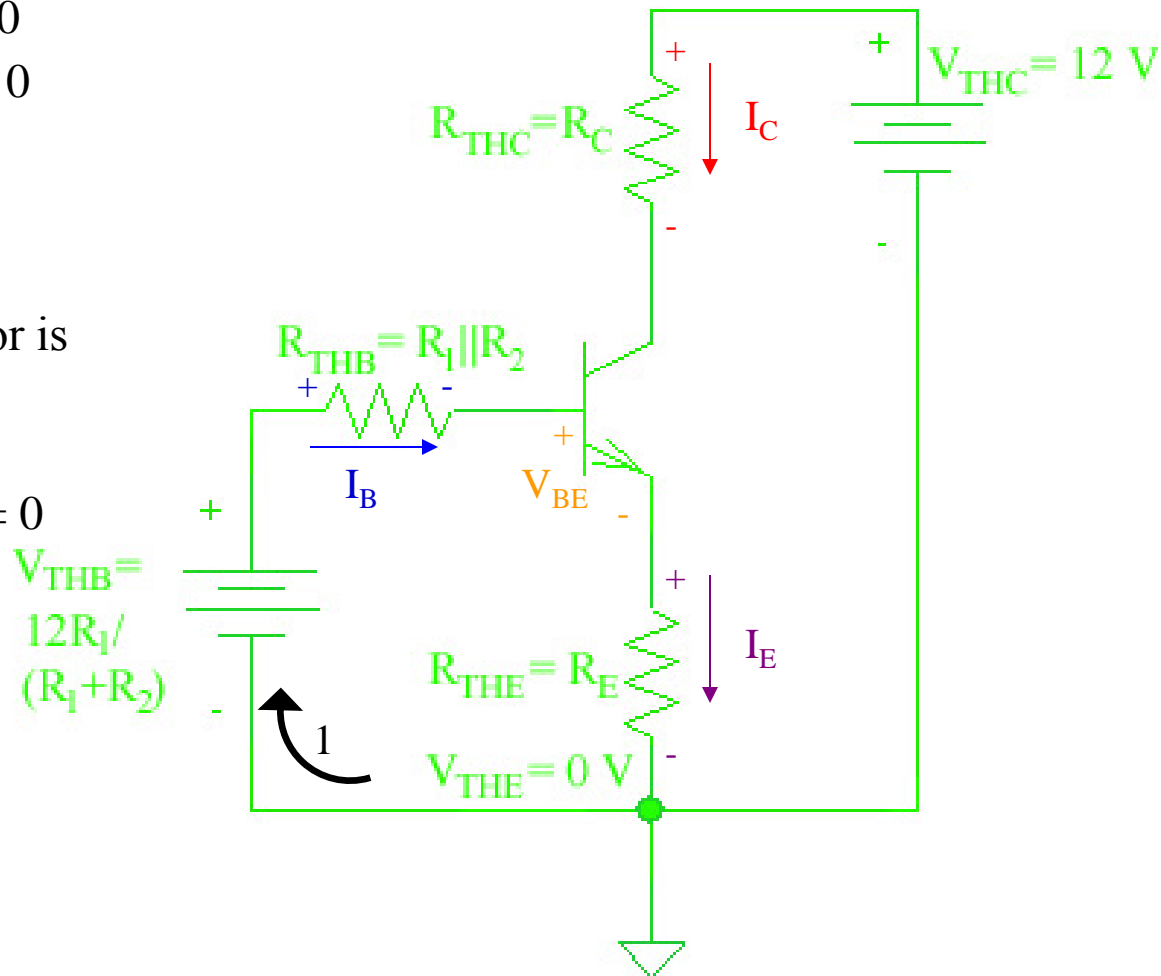
Assume β for this particular transistor is given to be 100.

$$3 - 0.7 - I_B(7500 + (1+100)*1300) = 0$$

$$I_B = 16.6 \mu\text{A}$$

$$I_C = \beta I_B = 1.66 \text{ mA}$$

$$I_E = I_B + I_C = 1.676 \text{ mA}$$



Step 1: Determine DC Operating Point

Check Assumptions: Forward Active?



$$V_C = 12 - I_C R_C = 12 - (1.66 \text{ mA})(4300) = 4.86 \text{ V}$$

$$V_E = I_E R_E = (1.67 \text{ mA})(1300) = 2.18 \text{ V}$$

$$V_B = V_{THB} - I_B R_{THB} = 3 - (16.6 \mu\text{A})(7500) = 2.88 \text{ V}$$

Check:

For an npn transistor in forward active:

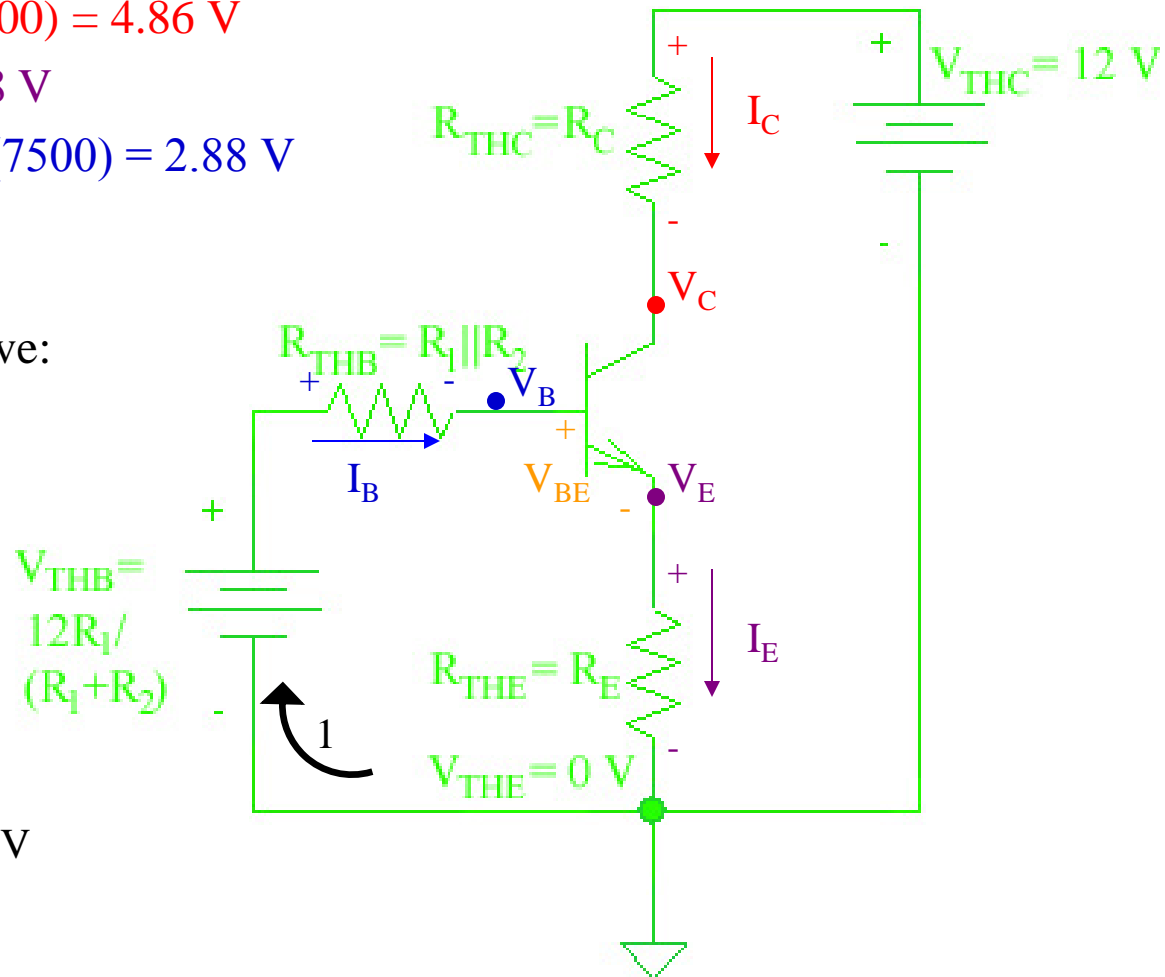
$$V_C > V_B$$

$$4.86 \text{ V} > 2.88 \text{ V}$$

$$V_B - V_E = V_{BE} = 0.7 \text{ V}$$

$$2.88 \text{ V} - 2.18 \text{ V} = 0.7 \text{ V}$$

$$V_{CE} = V_C - V_E = 4.86 \text{ V} - 2.18 \text{ V} = 2.68 \text{ V}$$

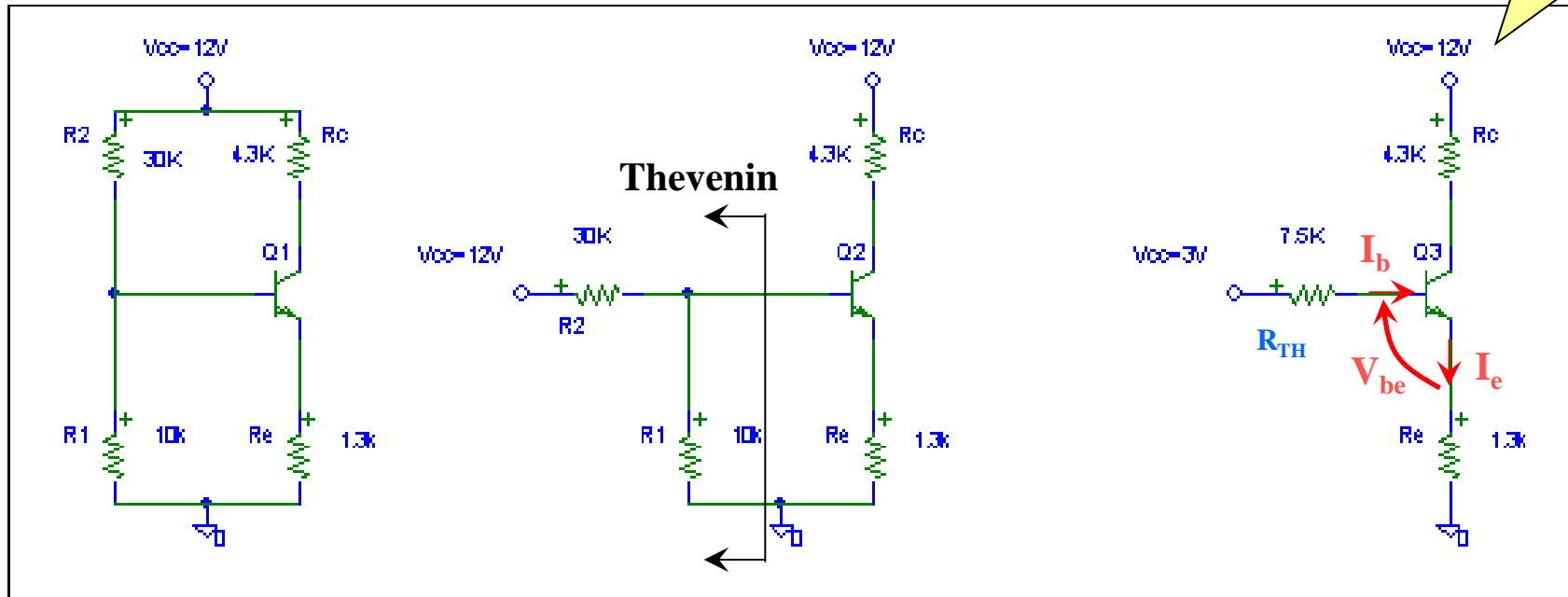


Single Transistor Amplifier Analysis

Step 1 detail

DC Bias Point (Alternative Drawing View)

Important!



$$3V = I_E R_E + V_{be} + I_B R_{TH}$$

$$3V = I_B (100 + 1) 1300 + 0.7 + I_B 7500$$

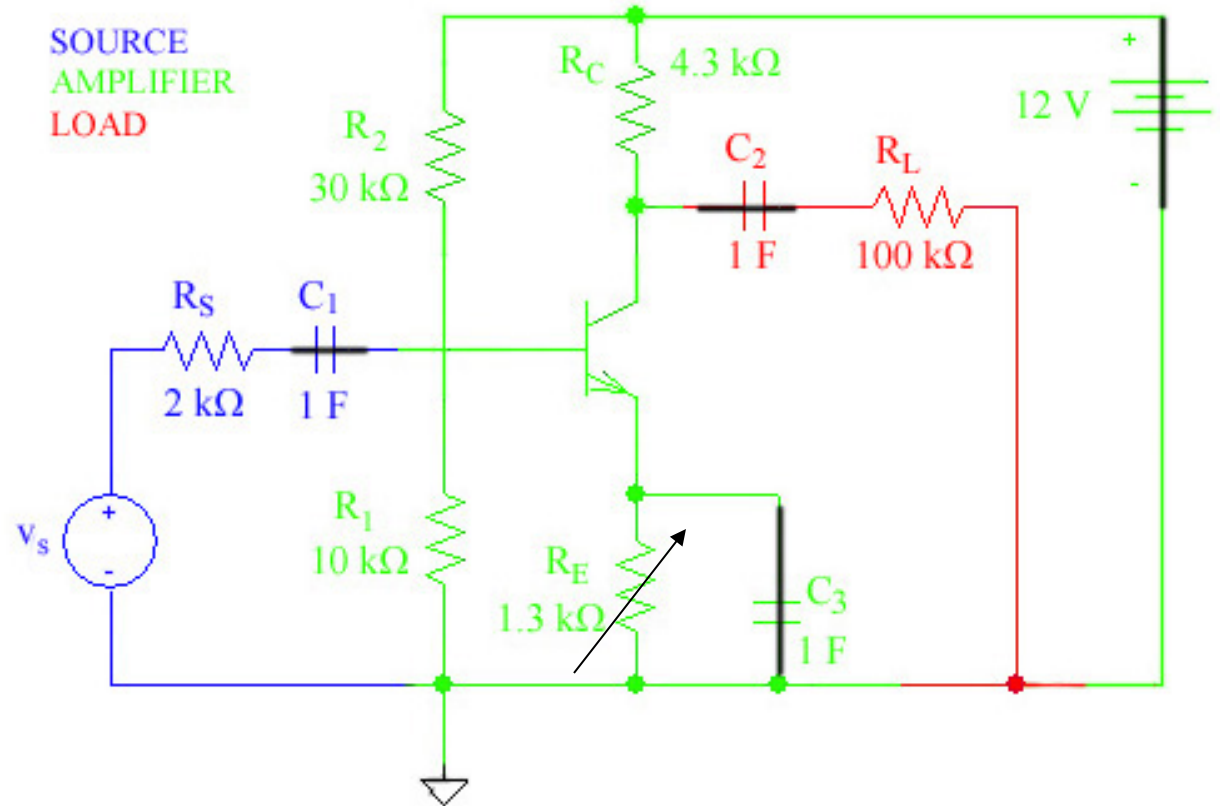
$$I_B = 16.6 \mu\text{A}, I_C = I_B \beta_o = 1.66 \text{ mA}, I_E = (\beta_o + 1) I_C / \beta_o = 1.67 \text{ mA}$$

Step 2: Convert to AC-Only Model

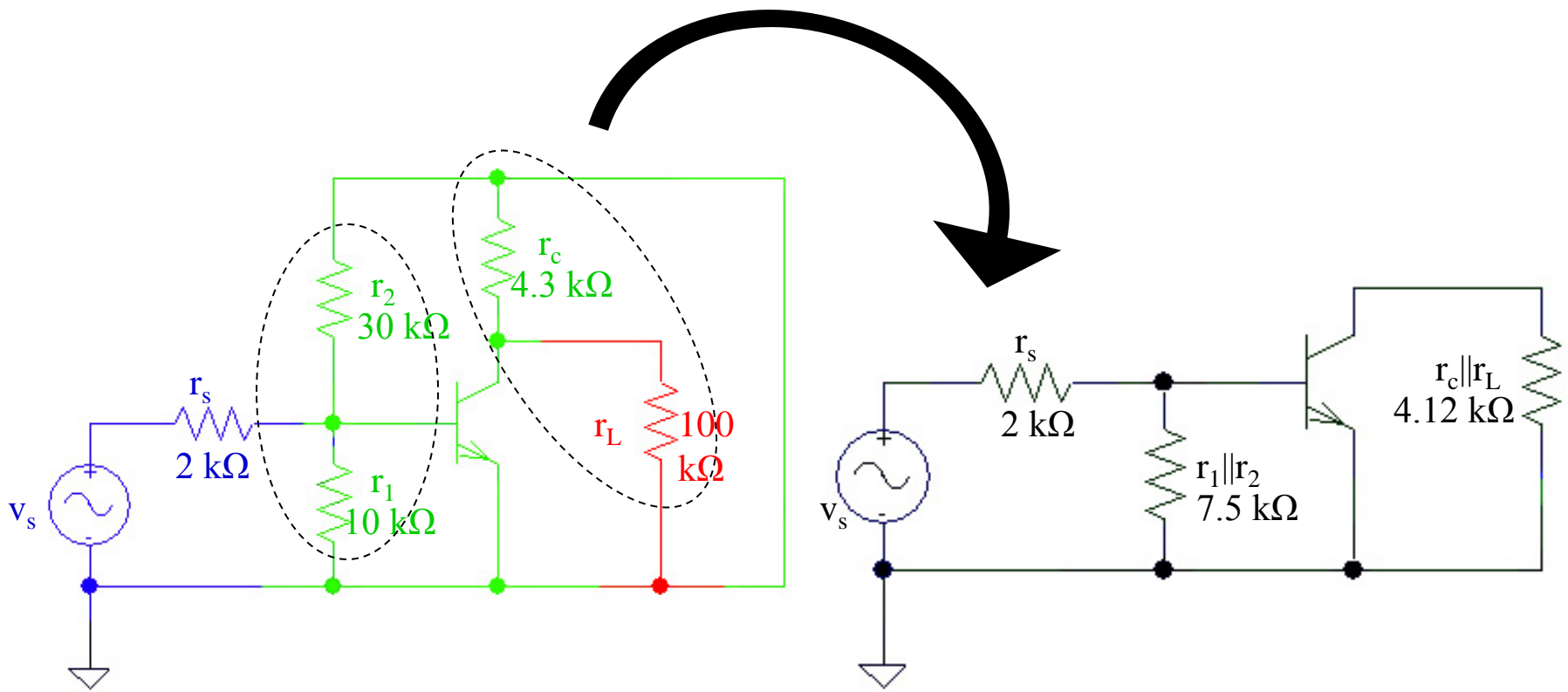
Short the Capacitors and DC Current Sources



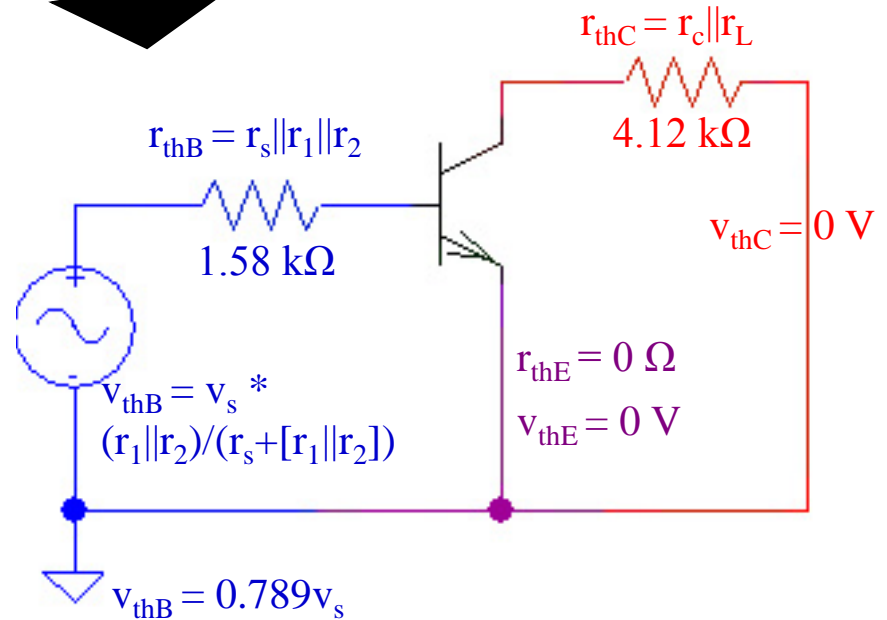
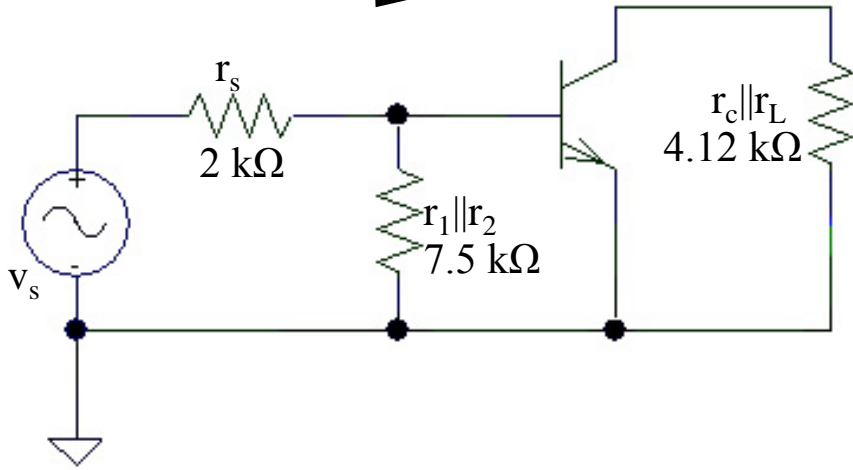
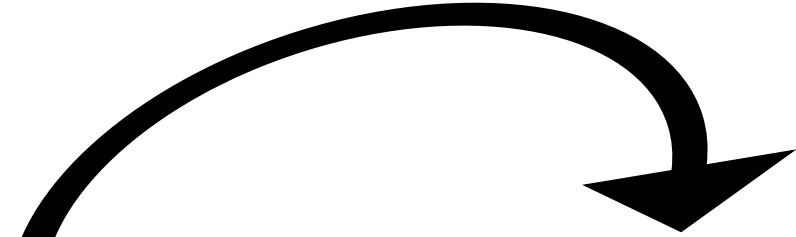
- DC voltage sources are shorts (no voltage drop/gain through a short circuit).
- DC current sources are open (no current flow through an open circuit).
- Large capacitors are shorts (if C is large, $1/j\omega C$ is small).
- Large inductors are open (if L is large, $j\omega L$ is large).



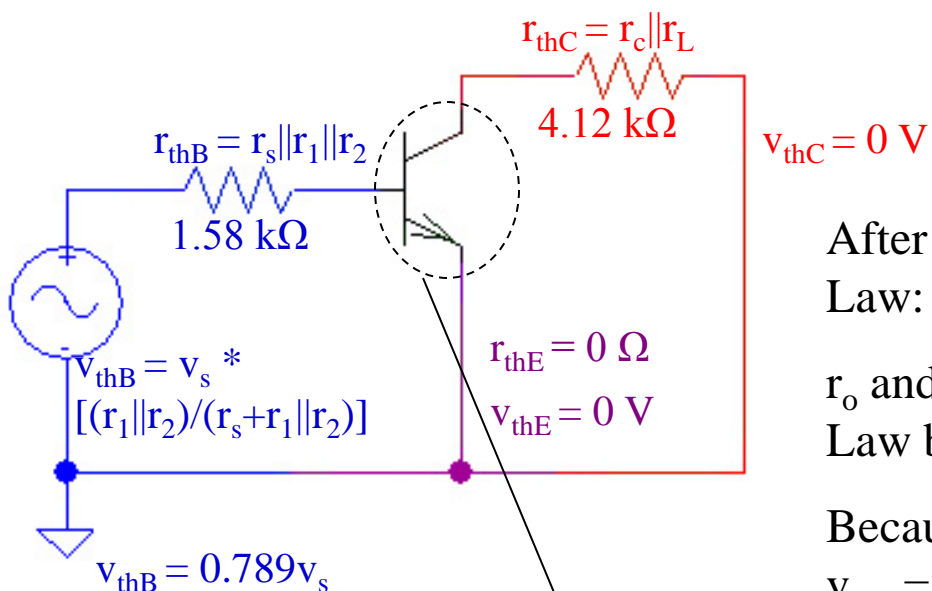
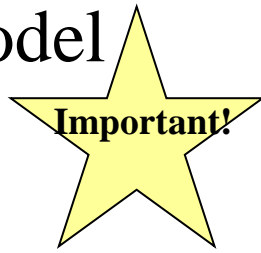
Step 2: Convert to AC-Only Model (Optional) Simplify Before Thevenizing



Step 3: Thevenize the AC-Only Model



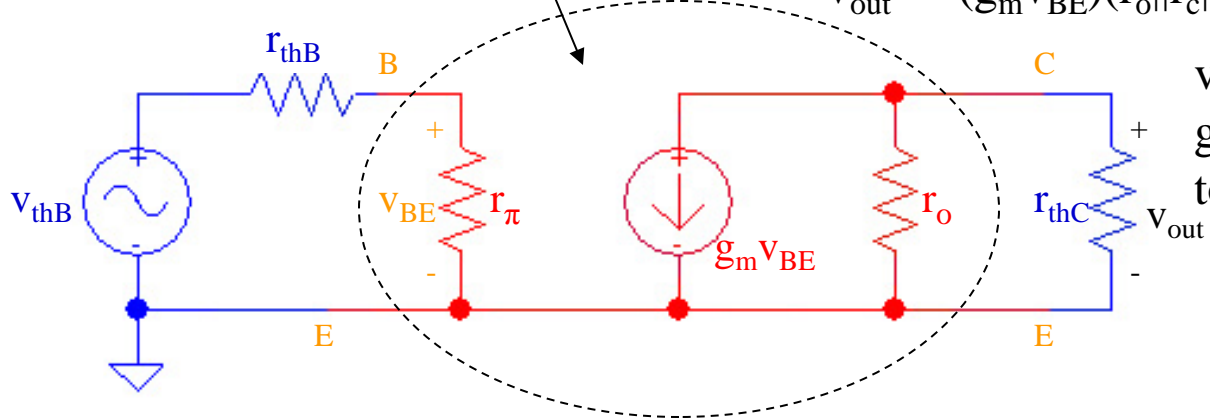
Step 4: Replace Transistor With Small Signal Model



After replacing the transistor, apply Ohm's Law: $V = IR$ to find v_{out} .

r_o and r_{thC} are in parallel, so that Ohm's Law becomes: $v_{out} = -IR = -(g_m v_{BE})(r_o || r_{thC})$

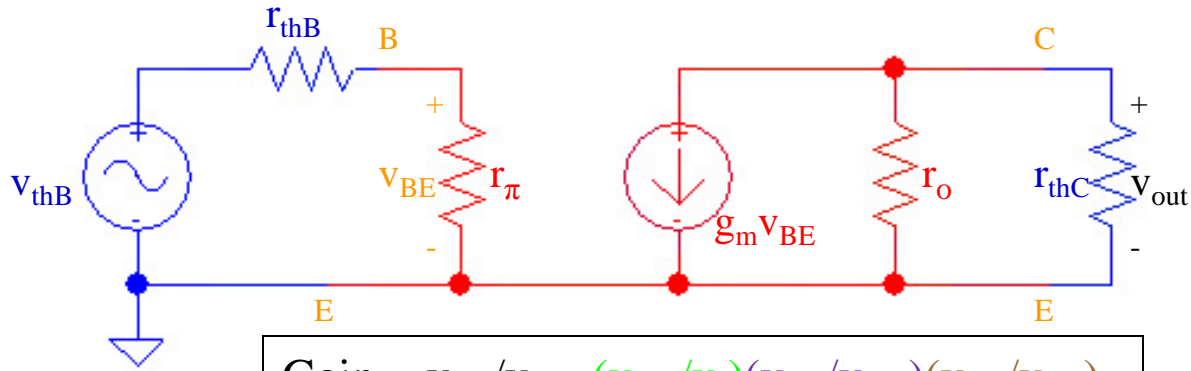
Because $r_{thC} = r_c || r_L$
 $v_{out} = -(g_m v_{BE})(r_o || r_c || r_L)$



$v_{out}/v_{BE} = -g_m(r_o || r_c || r_L)$ is the gain from transistor input (v_{BE}) to transistor/circuit output v_{out}

TRANSISTOR
EXTERNAL

Steps 5 and 6: Calculate Gain and Small Signal Parameters



TRANSISTOR
EXTERNAL

$$\text{Gain} = v_{\text{out}}/v_s = (v_{\text{thB}}/v_s)(v_{\text{BE}}/v_{\text{thB}})(v_{\text{out}}/v_{\text{BE}})$$

As previously determined:

$$v_{\text{thB}}/v_s = (r_1 || r_2) / ([r_1 || r_2] + r_s)$$

Applying a voltage divider:

$$v_{\text{BE}}/v_{\text{thB}} = r_{\pi} / (r_{\pi} + r_{\text{thB}})$$

Gain factor:

$$v_{\text{out}}/v_{\text{BE}} = -g_m(r_o || r_c || r_L)$$

Because calculating the DC operating point was done *first*, we have equations for g_m , r_{π} , and r_o in terms of previously calculated DC currents and voltages.

Transconductance	$g_m = y_{21} = \frac{I_C}{V_T} \approx 40I_C$
Input Resistance	$r_{\pi} = \frac{1}{y_{11}} = \frac{\beta_o V_T}{I_C} = \frac{\beta_o}{g_m}$
Output Resistance	$r_o = \frac{1}{y_{22}} = \frac{V_A + V_{CE}}{I_C}$

Plugging in the numbers:

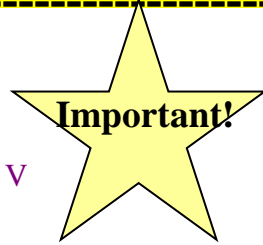
$$\text{Gain} = v_{\text{out}}/v_s = -94.8 \text{ v/v}$$

Single Transistor Amplifier Analysis

Step 6 detail

Calculate small signal parameters

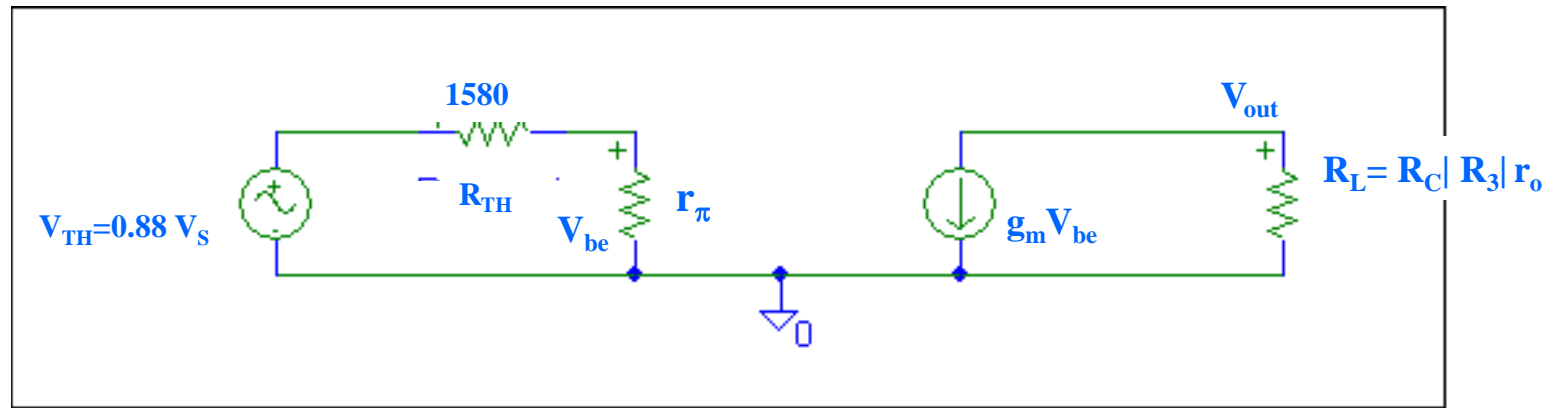
$$V_{CE} = V_C - V_E = 4.86 \text{ V} - 2.18 \text{ V} = 2.68 \text{ V}$$



Transconductance $g_m = y_{21} = \frac{I_C}{V_T} = 0.0641 \text{ S}$

Input Resistance $r_\pi = \frac{1}{y_{11}} = \frac{\beta_o V_T}{I_C} = \frac{\beta_o}{g_m} = 1560 \ \Omega$

Output Resistance $r_o = \frac{1}{y_{22}} = \frac{V_A + V_{CE}}{I_C} \approx \frac{V_A}{I_C} = \frac{75 \text{ V}}{16.6 \text{ mA}} = 45.2 \text{ K } \Omega \text{ (or } 46.8 \text{ K } \Omega)$



$$v_{out} = -g_m v_{be} R_L \quad \text{and} \quad v_{be} = v_{Th} \frac{r_\pi}{R_{Th} + r_\pi} \quad \text{and} \quad v_{Th} = 0.789 v_S$$

⇓

$$A_v \equiv \text{Voltage Gain} = \frac{v_{out}}{v_S} = \left(\frac{v_{out}}{v_{be}} \right) \left(\frac{v_{be}}{v_{th}} \right) \left(\frac{v_{th}}{v_S} \right) = (-g_m R_L) \left(\frac{r_\pi}{R_{Th} + r_\pi} \right) (0.789)$$

$$A_v = -(0.0641)(45,200 \parallel 4300 \parallel 100,000) \left(\frac{1560}{1580 + 1560} \right) (0.789)$$

$$A_v = -94.81 \text{ v/v (or } -95.2 \text{ v/v)}$$

For Extra Examples see:
Jaeger section 13.6, and
pages 627-630 (top of 630)

Interpretation/Analysis of Results



$$\text{Gain} = v_{\text{out}}/v_s = (v_{\text{thB}}/v_s)(v_{\text{BE}}/v_{\text{thB}})(v_{\text{out}}/v_{\text{BE}}) = -94.8 \text{ v/v}$$

Both terms are *loss factors*, i.e. they can never be greater than 1 in magnitude and thus cause the gain to decrease.

This term is the *gain factor* and is responsible for amplifying the signal.

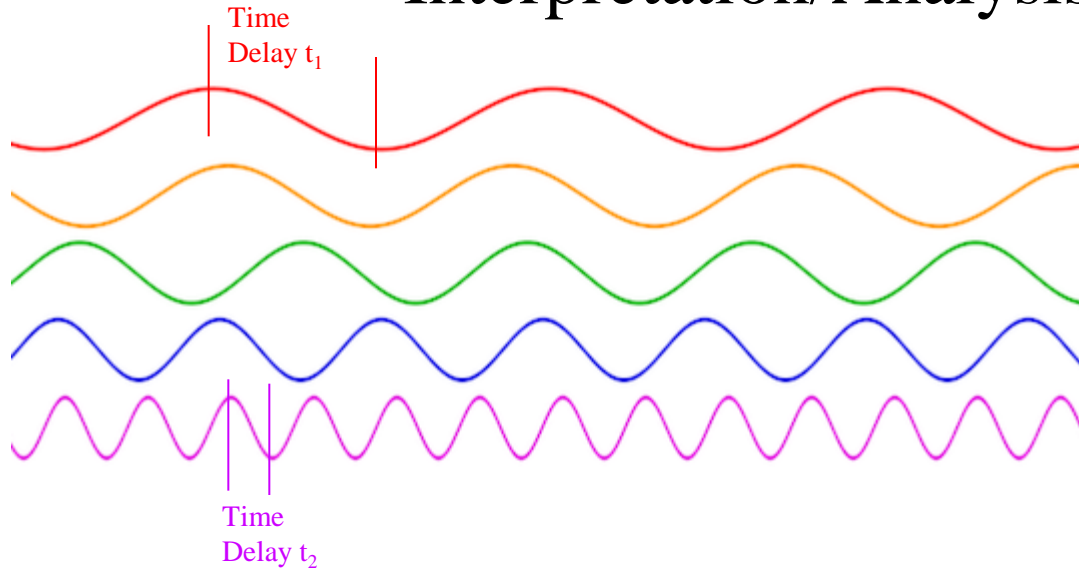
$$v_{\text{thB}}/v_s = (r_1 || r_2) / ([r_1 || r_2] + r_s)$$

$$v_{\text{BE}}/v_{\text{thB}} = r_{\pi} / (r_{\pi} + r_{\text{thB}})$$

$$v_{\text{out}}/v_{\text{BE}} = -g_m (r_o || r_c || r_L)$$

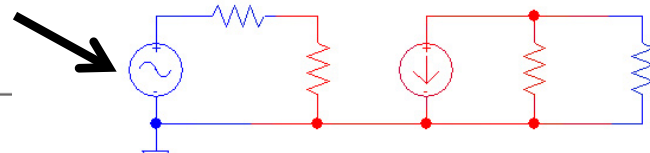
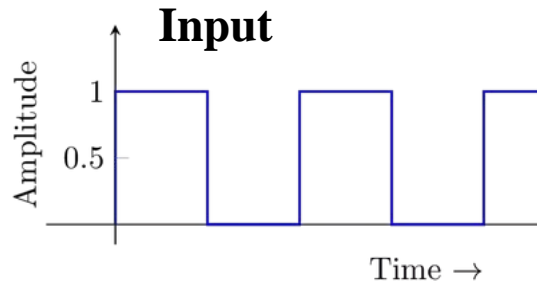
The AC input signal has been amplified ~95 times in magnitude. The negative sign indicates there has been a phase shift of 180°. A phase shift implies a time delay.

Interpretation/Analysis of Results

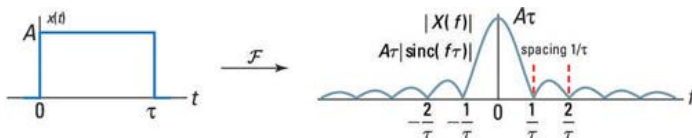
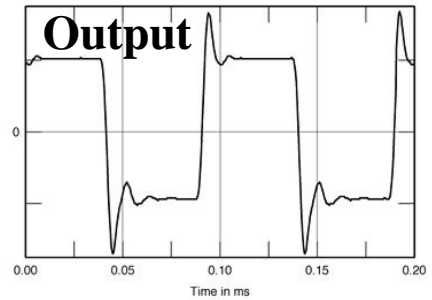


A phase shift implies a time delay. But signals that have multiple Fourier components have their Fourier components delayed by different amounts resulting in time domain distortion.

Square wave



Use Signal Processing "Transfer Function" to describe the behavior of an amplifier



Optional

Completing the Small Signal Model of the BJT Base Charging Capacitance (Diffusion Capacitance)

In active mode when the emitter-base is forward biased, the capacitance of the emitter-base junction is dominated by the diffusion capacitance (not depletion capacitance).

Recall for a diode we started out by saying:

Sum up all the minority carrier charges on either side of the junction

$$C_{Diffusion} = \frac{dQ_D}{dv'_D} = \frac{dQ_D}{dt} \frac{dt}{dv'_D}$$

Neglect charge injected from the base into the emitter due to p+ emitter in pnp

$$Q_D = qA \int_0^\infty p_{no} \left(e^{v'_D/V_T} - 1 \right) e^{-x/L_p} dx + qA \int_0^\infty n_{po} \left(e^{v'_D/V_T} - 1 \right) e^{-x/L_n} dx$$

Excess charge stored is due almost entirely to the charge injected from the emitter.

Optional

Completing the Small Signal Model of the BJT Base Charging Capacitance (Diffusion Capacitance)

- The BJT acts like a very efficient “siphon”: As majority carriers from the emitter are injected into the base and become “excess minority carriers”, the Collector “siphons them” out of the base.
- We can view the collector current as the amount of excess charge in the base collected by the collector per unit time.
- Thus, we can express the charge due to the excess hole concentration in the base as:

$$Q_B = i_C \tau_F$$

or the excess charge in the base depends on the magnitude of current flowing and the “forward” base transport time, τ_F , the average time the carriers spend in the base.

- It can be shown (see Pierret section 12.2.2) that:

$$\tau_F = \frac{W^2}{2D_B} \quad \text{where,}$$

$W \equiv$ Base Quasi – neutral region width

$D_B \equiv$ Minority carrier diffusion coefficient

Completing the Small Signal Model of the BJT

Base Charging Capacitance (Diffusion Capacitance)

Thus, the diffusion capacitance is,

$$C_B = \left. \frac{\partial Q_B}{\partial v_{BE}} \right|_{Q-point} = \left(\frac{W^2}{2D_B} \right) \left. \frac{\partial i_C}{\partial v_{BE}} \right|_{Q-point}$$

$$C_B = \tau_F \frac{I_C}{V_T} = \tau_F g_m$$

The upper operational frequency of the transistor is limited by the forward base transport time:

$$f \leq \frac{1}{2\pi\tau_F}$$

Note the similarity to the Diode Diffusion capacitance we found previously:

$$C_{Diffusion} = g_d \tau_t \quad \text{where } \tau_t = \frac{[p_{no}L_p + n_{po}L_n]qA}{I_S} \text{ is the transit time}$$

Optional

Completing the Small Signal Model of the BJT

Base Charging Capacitance (Total Capacitance)

In active mode for small forward biases the depletion capacitance of the base-emitter junction can contribute to the total capacitance

$$C_{jE} = \frac{C_{jEo}}{\sqrt{1 + \frac{V_{EB}}{V_{bi \text{ for emitter-base}}}}}$$

where,

$C_{jEo} \equiv$ zero bias depletion capacitance

$V_{bi \text{ for emitter-base}} \equiv$ built in voltage for the E – B junction

Thus, the total emitter-base capacitance is:

$$C_{\pi} = C_B + C_{jE}$$

Optional

Completing the Small Signal Model of the BJT Base Charging Capacitance (Depletion Capacitance)

In active mode when the collector-base is reverse biased, the capacitance of the collector-base junction is dominated by the depletion capacitance (not diffusion capacitance).

$$C_{\mu} = \frac{C_{\mu 0}}{\sqrt{1 + \frac{V_{CB}}{V_{bi \text{ for collector-base}}}}}$$

where,

$C_{\mu 0} \equiv$ zero bias depletion capacitance

$V_{bi \text{ for collector-base}} \equiv$ built in voltage for the B – C junction

Optional

Completing the Small Signal Model of the BJT

Collector to Substrate Capacitance (Depletion Capacitance)

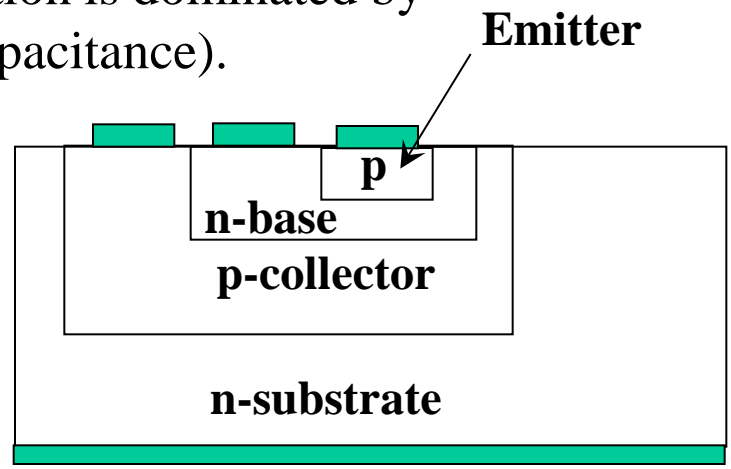
In some integrated circuit BJTs (lateral BJTs in particular) the device has a capacitance to the substrate wafer it is fabricated in. This results from a “buried” reverse biased junction. Thus, the collector-substrate junction is reverse biased and the capacitance of the collector-substrate junction is dominated by the depletion capacitance (not diffusion capacitance).

$$C_{CS} = \frac{C_{CS}}{\sqrt{1 + \frac{V_{CS}}{V_{bi \text{ for collector-substrate}}}}}$$

where,

$C_{CS} \equiv$ zero bias depletion capacitance

$V_{bi \text{ for collector-substrate}} \equiv$ built in voltage for the C – substrate junction



Optional

Completing the Small Signal Model of the BJT Parasitic Resistances

- r_b = base resistance between metal interconnect and B- E junction
- r_c = parasitic collector resistance
- r_{ex} = emitter resistance due to polysilicon contact
- These resistance's can be included in SPICE simulations, but are usually ignored in hand calculations.

Completing the Small Signal Model of the BJT

Complete Small Signal Model

