LECTURE 20
MATH 242

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The goal is to prove (some of) Arnold's conjecture on fixed points of Hamiltonian symplectomorphisms. In particular we will look at the nondegenerate version. To do to this we will need Floer homology for (Hamiltonian) symplectomorphisms. First will review Morse homology.

1. Morse homology

1.1. Setup. Let X be a closed smooth finite dimensional manifold, and $f : X \to \mathbb{R}$ a smooth function. A critical point is a point $p \in X$ such that $df_p : T_pX \to \mathbb{R}$ is 0. Given a critical point, define the Hessian

$$H = D(df) : T_pX \to T_p^*X \ .$$

We could equivalently think of this as a map

$$H : T_pX \otimes T_pX \to \mathbb{R} \ .$$

In local coordinates $x_1, \ldots, x_n$ the Hessian is

$$H\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial^2 f}{\partial x_i \partial x_j}(p) \ .$$

The critical point p is nondegenerate if

$$H : T_pX \to T_P^*X$$

is invertible, or equivalently

$$H : T_pX \otimes T_pX \to \mathbb{R}$$

is nondegenerate. In this case, define the *Morse index* $\operatorname{ind}(p)$ to be the number of negative eigenvalues.

Lemma 1. *If p is a nondegenerate critical point of index i, then there exist local coordinates $x_1, \ldots, c_n$ with $(0, \ldots, 0) = p$ such that*

$$f = f(p) - x_1^2 - \cdots - x_i + x_{i+1}^2 + \cdots + x_n^2 \ .$$

Remark 1. Most of what we will say about Morse theory generalizes to infinite dimensions. Lemma 1 does not.

Example 1. Consider the height functor on a torus. Then the critical points have indices as in fig. 1. If we put the torus on the table, then we have two critical points of the function which correspond to circles in the torus, but these are degenerate so they have no index. It is a general fact that nondegenerate critical points are isolated. This sort of function is called *Morse-Bott*.

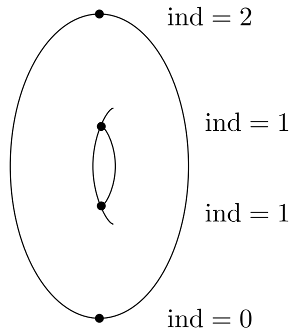

FIGURE 1. The height function for the Torus has four critical points. The bottom has index 0, the middle two have index 1, and the top point has index 2.

Definition 1. f is *Morse* if all critical points are nondegenerate.

Fact 1. *All critical points are nondegenerate for generic f.*

Remark 2. Classical Morse theory is concerned with relating critical points of f (and more) to the topology of f. E.g. what is the maximum number of critical points of a Morse function on X? As it turns out we have the following theorem:

Theorem 1. *The number of index i critical points is $\geq b_i(X)$.*

The idea is to keep track of the critical points using a chain complex. Define the *Morse complex* over $\mathbb{Z}$ (or $\mathbb{Z}/2\mathbb{Z}$ if we are lazy about orientations) by setting

$$C_i = \mathbb{Z}\{\text{index } i \text{ critical points}\} \ .$$

To define the differential $\partial : C_i \to C_{i-1}$, choose a Riemannian metric g on X to get ∇f.

Definition 2. If p_+ and p_- are critical points, a downward gradient flow line from p_+ to p_- is a smooth map $u : \mathbb{R} \to X$ such that

$$\frac{d}{ds} u(s) = \nabla f(u(s))$$

and

$$\lim_{s \to \pm\infty} u(s) = p_\pm \ .$$

The picture is as in fig. 2.

The differential will count these.

Assume $p_+ \neq p_-$. Let

$$\tilde{m}(p_+, p_-)$$

consist of the flow lines as above. Then write

$$m(p_+, p_-) = \tilde{m}(p_+, p_-)/\mathbb{R}$$

where $\mathbb{R}$ acts on $\tilde{m}$ by

$$(t \cdot u)(s) = u(s - t) \ .$$

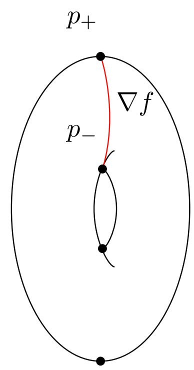

FIGURE 2. A downward flow line from p_+ to p_-.

This is a free action when $p_+ \neq p_-$. If p is a critical point, define the *descending manifold* to be

$$\mathcal{D}(p) = \{x \in X \mid \text{ flow of } \nabla f \text{ starting at } X \text{ converges to } p\}$$

and the ascending manifold to be

$$\mathcal{A}(p) = \{x \in X \mid \text{ flow of } \nabla f \text{ ending at } X \text{ converges to } p\} \ .$$

Fact 2. *$\mathcal{D}(p)$ is a smoothly embedded open ball of dimension* $\operatorname{ind}(p)$ *and $\mathcal{A}(p)$ is a smoothly embedded open ball of dimension $n - \operatorname{ind}(p)$.*

$T_p\mathcal{D}(p)$ is the negative eigenspace of

$$H(p) : T_pX \to T_p^*X \xrightarrow{g} T_pX$$

and $T_p\mathcal{A}(p)$ is the positive eigenspace.

By definition we have a bijection

$$\tilde{m}(p_+, p_-) \longrightarrow \mathcal{D}(p_+) \cap \mathcal{A}(p_-)$$

$$u \longmapsto u(0)$$

Definition 3. The pair (f, g) is *Morse-Smale* if $\mathcal{D}(p_+) \pitchfork \mathcal{A}(p_-)$ where $p_+ \neq p_-$ and f is Morse.

Fact 3. *Given a Morse function f, (f, g) is Morse-Smale for generic g.*

In this case,

$$\begin{aligned}
\dim \tilde{m}(p_+, p_-) &= \dim \mathcal{D}(p_+) + \dim \mathcal{A}(p_-) - n \\
&= \operatorname{ind}(p_+) + (n - \operatorname{ind}(p_-)) - n \\
&= \operatorname{ind}(p_+) - \operatorname{ind}(p_-)
\end{aligned}$$

and we get:

$$\boxed{\dim m(p_+, p_-) = \operatorname{ind}(p_+) - \operatorname{ind}(p_-) - 1} \ .$$

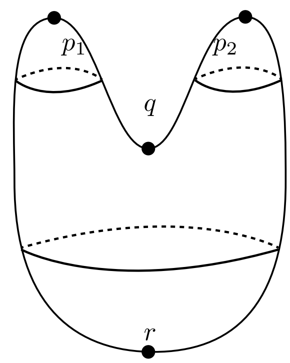

FIGURE 3. We can deform the sphere to get these four critical points.

We now define the differential. Assume (f, g) is Morse-Smale and p is a critical point of index i. Then define

$$\partial(p) = \sum_{\operatorname{ind} q = i-1} \# m(p, q)\, q$$

where # denotes some sort of signed count.

Lemma 2. $\operatorname{ind} p - \operatorname{ind} q = 1$ *implies* $m(p, q)$ *is compact, and hence finite.*

Lemma 3. $\partial^2 = 0$.

This allows us to define the Morse homology

$$H_i^{\mathrm{Morse}}(f, g) = \ker\left(\partial|_{C_i}\right) / \partial(C_{i+1})$$

Counterexample 1. Our working example T^2 doesn't actually work, because it is not Morse Smale. For example, $\mathcal{D}(q)$, which is a circle minus p, and $\mathcal{A}(p)$, which is a circle minus q, do not intersect transversely, where q is the upper critical point of index 1, and p is the lower critical point of index 1. If it was Morse Smale, then we would have that

$$\dim m(p, q) = \operatorname{ind}(p) - \operatorname{ind}(q) - 1$$

but the index of both of these points is 1, and $|m(p, q)| = 2$.

Example 2. Consider a heart-shaped sphere as in fig. 3. The critical points of the height function have the following indices:

$$\operatorname{ind}(p_i) = 2 \qquad \operatorname{ind}(q) = 1 \qquad \operatorname{ind}(r) = 0$$

and

$$|m(q, r)| = 2\ .$$

The differential sends $\partial q = 0$ and $\partial r = 0$. Then

$$H_2 = \mathbb{Z}\{[p_1 + p_2]\} \qquad H_1 = 0 \qquad H_0 = \mathbb{Z}\{[r]\}\ .$$

This of course agrees with the usual homology of the sphere.

Theorem 2. $H_i^{Morse}(f, g) = H_i(X)$.

Corollary 1. *The number of index* i *critical points is* $\geq b_i(X)$.

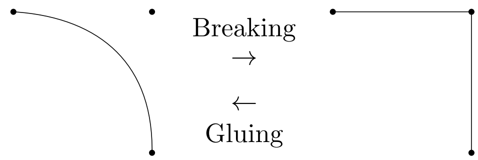

FIGURE 4. Breaking and gluing trajectories.

Why is $\partial^2 = 0$? Suppose $\operatorname{ind} p_+ - \operatorname{ind} p_- = 2$. Then $\dim m(p_+, p_-) = 1$.

Lemma 4. *$m(p_+, p_-)$ has a compactification $\overline{m(p_+, p_-)}$ which is a compact (oriented) 1-manifold with boundary*

$$\partial\overline{m(p_+, p_-)} = \coprod_{\operatorname{ind}(p_0)=\operatorname{ind}(p_+)-1} m(p_+, p_0) \times m(p_0, p_-) \ .$$

This compactification is related to breaking and gluing as in fig. 4.

Example 3. Consider the example in fig. 3. The denscending manifold of p_1 is the open half of the heart. The compactified descending manifold of p_1 is this plus the pieces

$$m(p_1, q) \times \mathcal{D}(q) \qquad m(p_1, r) \times \mathcal{D}(r) \qquad m(p_1, q) \times m(q, r) \times \mathcal{D}(r)$$

which gives us a closed disk.

Proof that $\partial^2 = 0$. Suppose $\operatorname{ind}(p_+) - \operatorname{ind}(p_-) = 2$. Then we have

$$\begin{aligned}
\left\langle \partial^2 p_+, p_- \right\rangle &= \sum_{\operatorname{ind} p_0 = ind p_+ - 1} \langle \partial p_+, p_0 \rangle \langle \partial p_0, p_- \rangle \\
&= \sum_{\operatorname{ind} p_0 = ind p_+ - 1} \# m(p_+, p_0) \cdot \# m(p_0, p_-) \\
&= \# \partial \overline{m(p_+, p_-)} \\
&= 0 \ .
\end{aligned}$$

□

Sketch of proof that Morse homology agrees with usual homology. Given a critical point p, define

$$\overline{\mathcal{D}(p)} = \coprod_{k \geq 0} \coprod_{p = p_0 \neq p_1 \neq \ldots \neq p_k} m(p_0, p_1) \times \ldots \times m(p_{k-1}, p_k) \times \mathcal{D}(p_k) \ .$$

Then there is a projection $e : \overline{\mathcal{D}(p)} \to X$ which projects onto $\mathcal{D}(p_k)$.

Fact 4 (Non-trivial)**.** *The pairs $\left(\overline{\mathcal{D}(p)}, e\right)$ give X the structure of a CW-complex with one i-cell for each index i critical point. Then the Morse differential is the CW-complex differential.*

□