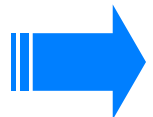


Lecture 21:
The Parity Operator

Phy851 Fall 2009

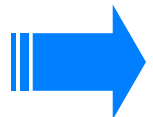


Parity inversion

- Symmetry under parity inversion is known as mirror symmetry

$$P: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$$

- Formally, we say that $f(x)$ is symmetric under parity inversion if $f(-x) = f(x)$
- We would say that $f(x)$ is antisymmetric under parity inversion if $f(-x) = -f(x)$
- The universe is not symmetric under parity inversion (beta decay)
 - Unless there is mirror matter (and mirror photons)
 - Would interact only weakly with matter via gravity



Parity Operator

- Let us define the parity operator via:

$$\Pi|x\rangle = |-x\rangle$$

- Parity operator is Hermitian:

$$\langle x|\Pi|x'\rangle = \langle x|-x'\rangle = \delta(x+x')$$

$$\langle x'|\Pi|x\rangle^* = \langle x'|-x\rangle^* = \delta(x+x')$$

$$\Pi^\dagger = \Pi$$

- Parity operator is its own inverse

$$\Pi\Pi|x\rangle = \Pi|-x\rangle = |x\rangle$$

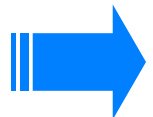
$$\Pi^2 = 1$$

- Thus it must be Unitary as well

$$\Pi^\dagger = \Pi$$

$$\Pi = \Pi^{-1}$$

$$\Pi^\dagger = \Pi^{-1}$$



Properties of the Parity operator

- Parity acting to the left:

$$\langle x | \Pi^\dagger = (\Pi | x \rangle)^\dagger = | -x \rangle^\dagger = \langle -x |$$

$$\langle x | \Pi = \langle -x |$$

-
- What is the action of the parity operator on a generic quantum state?

- Let: $|\psi'\rangle = \Pi|\psi\rangle$

$$\langle x | \psi' \rangle = \langle x | \Pi | \psi \rangle$$

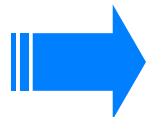
$$\langle x | \psi' \rangle = \langle -x | \psi \rangle$$

$$\psi'(x) = \psi(-x)$$

$$\psi'(-x) = \psi(x)$$

- Under parity inversion, we would say:

$$\psi'(x') = \psi(x)$$



Eigenstates of Parity Operator

- What are the eigenstates of parity?
 - What states have well-defined parity?
 - Answer: even/odd states

- Proof:

- Let: $\Pi|\pi\rangle = \pi|\pi\rangle$

- It follows that:

$$\Pi^2|\pi\rangle = \pi^2|\pi\rangle$$

- But $\Pi^2=1$, which gives:

$$|\pi\rangle = \pi^2|\pi\rangle$$

$$\pi^2 = 1$$

$$\pi = \pm 1$$

$$\pi = +1$$

$$\langle x|\Pi|+\rangle = \langle x|+\rangle$$

$$\langle -x|+\rangle = \langle x|+\rangle$$

Any Even function!

$$\pi = -1$$

$$\langle x|\Pi|-\rangle = -\langle x|-\rangle$$

$$\langle -x|-\rangle = -\langle x|-\rangle$$

Any Odd function!



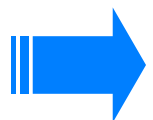
Parity acting on Momentum states

$$\begin{aligned}\Pi|p\rangle &= \int dx \Pi|x\rangle\langle x|p\rangle \\ &= \int dx | -x\rangle\langle x|p\rangle \\ &= \int dx |x\rangle\langle -x|p\rangle\end{aligned}$$

$$\langle -x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}px} = \langle x|-p\rangle$$

$$\Pi|p\rangle = \int dx |x\rangle\langle x|-p\rangle$$

$$\Pi|p\rangle = |-p\rangle$$



Commutator of X with Π

- First we can compute $\Pi X \Pi$:

$$\begin{aligned}\langle x | \Pi X \Pi | \psi \rangle &= \langle -x | X \Pi | \psi \rangle \\ &= -x \langle -x | \Pi | \psi \rangle \\ &= -x \langle x | \psi \rangle \\ &= -\langle x | X | \psi \rangle\end{aligned}$$

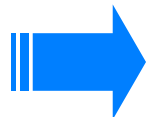
$$\Pi X \Pi = -X$$

$$\Pi X \Pi^2 = -X \Pi$$

$$\Pi X = -X \Pi$$

$$\Pi X - X \Pi = -2X \Pi$$

- So Π and X do not commute



Commutator of X with Π

- Next we can compute $[x^2, \Pi]$:

$$\begin{aligned}\langle x | \Pi X^2 \Pi | \psi \rangle &= \langle -x | X^2 \Pi | \psi \rangle \\ &= x^2 \langle -x | \Pi | \psi \rangle \\ &= x^2 \langle x | \psi \rangle \\ &= \langle x | X^2 | \psi \rangle\end{aligned}$$

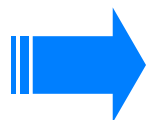
$$\Pi X^2 \Pi = X^2$$

$$\Pi X^2 \Pi^2 = X^2 \Pi$$

$$\Pi X^2 = X^2 \Pi$$

$$\Pi X^2 - X^2 \Pi = 0$$

- So Π and X^2 do commute!



Commutator with Hamiltonian

- Same results must apply for P and P^2 , as the relation between Π and P is the same as between Π and X .

- Thus
$$\left[\Pi, \frac{P^2}{2M} \right] = 0$$

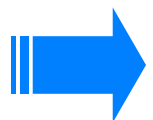
- If Π commutes with X^2 , then Π commutes with any even function of X

$$\left[\Pi, V_{\text{even}}(X^2) \right] = 0$$

- Let
$$H = \frac{P^2}{2M} + V_{\text{even}}(X)$$

- Then
$$\left[\Pi, H \right] = 0$$

- This means that simultaneous eigenstates of H and P exist



Consequences for a free particle

- The Hamiltonian of a free particle is:

$$H = \frac{P^2}{2M}$$

- Energy eigenstates are doubly-degenerate:

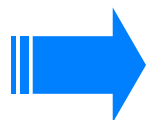
$$H|k\rangle = \frac{\hbar^2 k^2}{2M} \quad H|-k\rangle = \frac{\hbar^2 k^2}{2M}$$

$$|E,1\rangle := |k\rangle \Big|_{k=\frac{\sqrt{2ME}}{\hbar}} \quad |E,2\rangle := |k\rangle \Big|_{k=-\frac{\sqrt{2ME}}{\hbar}}$$

- Note that plane waves, $|k\rangle$, are eigenstates of momentum and energy, but NOT parity
- But $[H,\Pi]=0$, so eigenstates of energy and parity must exist

$$|E,+\rangle := \frac{1}{\sqrt{2}} (|E,1\rangle + |E,2\rangle)$$

$$|E,-\rangle := \frac{1}{\sqrt{2}} (|E,1\rangle - |E,2\rangle)$$



Consequences for the SHO

- For the SHO we have:

$$H = \frac{P^2}{2M} + \frac{1}{2}M\omega^2 X^2$$

- Therefore $[H, \Pi] = 0$, so simultaneous eigenstates of Energy and Parity must exist
- The energy levels are not-degenerate, so there is no freedom to mix and match states
- Thus the only possibility is that each energy level must have definite parity
- The Hermite Polynomials have definite parity: $H_n(-x) = (-1)^n H_n(x)$
- Thus we have:

$$\Pi|n\rangle = (-1)^n |n\rangle$$

