Lecture 21:
The Parity Operator

Phy851 Fall 2009

Parity inversion

- Symmetry under parity inversion is known as mirror symmetry

$$P: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$$

- Formally, we say that f(x) is symmetric under parity inversion if f(-x) = f(x)

- We would say that f(x) is antisymmetric under parity inversion if f(-x)=-f(x)

- The universe is not symmetric under parity inversion (beta decay)
  - Unless there is mirror matter (and mirror photons)
  - Would interact only weakly with matter via gravity

Parity Operator

- Let us define the parity operator via:

$$\Pi\left|x\right\rangle = \left|-x\right\rangle$$

- Parity operator is Hermitian:

$$\left\langle x\right|\Pi\left|x'\right\rangle = \left\langle x\right|-x'\rangle = \delta(x+x')$$

$$\left\langle x'\right|\Pi\left|x\right\rangle^* = \left\langle x'\right|-x\rangle^* = \delta(x+x')$$

$$\Pi^\dagger = \Pi$$

- Parity operator is it's own inverse

$$\Pi\Pi\left|x\right\rangle = \Pi\left|-x\right\rangle = \left|x\right\rangle$$

$$\Pi^2 = 1$$

- Thus it must be Unitary as well

$$\Pi^\dagger = \Pi$$

$$\Pi = \Pi^{-1}$$

$$\Pi^\dagger = \Pi^{-1}$$

Properties of the Parity operator

- Parity acting to the left:

$$\langle x|\Pi^\dagger = \left(\Pi|x\rangle\right)^\dagger = |-x\rangle^\dagger = \langle -x|$$

$$\langle x|\Pi = \langle -x|$$

- What is the action of the parity operator on a generic quantum state?
  - Let: $|\psi'\rangle = \Pi|\psi\rangle$

$$\langle x|\psi'\rangle = \langle x|\Pi|\psi\rangle$$

$$\langle x|\psi'\rangle = \langle -x|\psi\rangle$$

$$\psi'(x) = \psi(-x)$$

$$\psi'(-x) = \psi(x)$$

- Under parity inversion, we would say:

$$\psi'(x') = \psi(x)$$

Eigenstates of Parity Operator

- What are the eigenstates of parity?
  - What states have well-defined parity?
  - Answer: even/odd states

- Proof:
  - Let: $\Pi|\pi\rangle = \pi|\pi\rangle$
  - It follows that:
$$\Pi^2|\pi\rangle = \pi^2|\pi\rangle$$
  - But $\Pi^2 = 1$, which gives:
$$|\pi\rangle = \pi^2|\pi\rangle$$
$$\pi^2 = 1$$
$$\pi = \pm 1$$

$\pi = +1$	$\pi = -1$
$\langle x\|\Pi\|+\rangle = \langle x\|+\rangle$	$\langle x\|\Pi\|-\rangle = -\langle x\|-\rangle$
$\langle -x\|+\rangle = \langle x\|+\rangle$	$\langle -x\|-\rangle = -\langle x\|-\rangle$
Any Even function!	Any Odd function!

Parity acting on Momentum states

$$\Pi|p\rangle = \int dx\, \Pi|x\rangle\langle x|p\rangle$$

$$= \int dx\, |-x\rangle\langle x|p\rangle$$

$$= \int dx\, |x\rangle\langle -x|p\rangle$$

$$\langle -x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}px} = \langle x|-p\rangle$$

$$\Pi|p\rangle = \int dx\, |x\rangle\langle x|-p\rangle$$

$$\Pi|p\rangle = |-p\rangle$$

Commutator of X with Π

- First we can compute $\Pi X \Pi$:

$$\langle x|\Pi X \Pi|\psi\rangle = \langle -x|X\Pi|\psi\rangle$$
$$= -x\langle -x|\Pi|\psi\rangle$$
$$= -x\langle x|\psi\rangle$$
$$= -\langle x|X|\psi\rangle$$

$$\Pi X \Pi = -X$$

$$\Pi X \Pi^2 = -X\Pi$$

$$\Pi X = -X\Pi$$

$$\Pi X - X\Pi = -2X\Pi$$

- So Π and X do not commute

<u>Commutator of X with Π</u>

- Next we can compute $[x^2,\Pi]$:

$$\langle x|\Pi X^2 \Pi|\psi\rangle = \langle -x|X^2 \Pi|\psi\rangle$$
$$= x^2\langle -x|\Pi|\psi\rangle$$
$$= x^2\langle x|\psi\rangle$$
$$= \langle x|X^2|\psi\rangle$$

$$\Pi X^2 \Pi = X^2$$

$$\Pi X^2 \Pi^2 = X^2 \Pi$$

$$\boxed{\Pi X^2 = X^2 \Pi}$$

$$\Pi X^2 - X^2 \Pi = 0$$

- So Π and X^2 do commute!

Commutator with Hamiltonian

- Same results must apply for P and P^2, as the relation between Π and P is the same as between Π and X.
- Thus $$\left[\Pi, \frac{P^2}{2M}\right] = 0$$
- If Π commutes with X^2, then Π commutes with any even function of X $$\left[\Pi, V_{even}\left(X^2\right)\right] = 0$$
- Let $$H = \frac{P^2}{2M} + V_{even}(X)$$
- Then $$\left[\Pi, H\right] = 0$$
- This means that simultaneous eigenstates of H and P exist

Consequences for a free particle

- The Hamiltonian of a free particle is:

$$H = \frac{P^2}{2M}$$

- Energy eigenstates are doubly-degenerate:

$$H\left|k\right\rangle = \frac{\hbar^2 k^2}{2M} \qquad H\left|-k\right\rangle = \frac{\hbar^2 k^2}{2M}$$

$$\left|E,1\right\rangle := \left|k\right\rangle\Big|_{k=\frac{\sqrt{2ME}}{\hbar}} \qquad \left|E,2\right\rangle := \left|k\right\rangle\Big|_{k=-\frac{\sqrt{2ME}}{\hbar}}$$

- Note that plane waves, $|k\rangle$, are eigenstates of momentum and energy, but NOT parity

- But [H,Π]=0, so eigenstates of energy and parity must exist

$$\left|E,+\right\rangle := \frac{1}{\sqrt{2}}\left(\left|E,1\right\rangle + \left|E,2\right\rangle\right)$$

$$\left|E,-\right\rangle := \frac{1}{\sqrt{2}}\left(\left|E,1\right\rangle - \left|E,2\right\rangle\right)$$

Consequences for the SHO

- For the SHO we have:

$$H = \frac{P^2}{2M} + \frac{1}{2} M\omega^2 X^2$$

- Therefore [H,Π]=0, so simultaneous eigenstates of Energy and Parity must exist

- The energy levels are not-degenerate, so there is no freedom to mix and match states

- Thus the only possibility is that each energy level must have definite parity

- The Hermite Polynomials have definite parity: $H_n(-x)=(-1)^n H_n(x)$

- Thus we have:

$$\Pi|n\rangle = (-1)^n|n\rangle$$