Advanced Computation:
Computational Electromagnetics

Diffraction Gratings & the Plane Wave Spectrum

1

Outline

- Fourier Series
- Diffraction from gratings
- The plane wave spectrum
- Plane wave spectrum for crossed gratings
- Power flow from gratings

2

Fourier Series

Jean Baptiste Joseph Fourier

Born: March 21, 1768 in Yonne, France.

Died: May 16, 1830 in Paris, France.

Slide 3

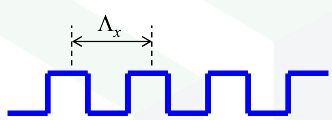
3

1D Complex Fourier Series

If a function $f(x)$ is periodic with period Λ_x, it can be expanded into a complex Fourier series.

$$f(x) = \sum_{p=-\infty}^{\infty} a_p e^{-j\frac{2\pi px}{\Lambda_x}}$$

$$a_p = \frac{1}{\Lambda_x} \int_{-\Lambda_x/2}^{\Lambda_x/2} f(x) e^{j\frac{2\pi px}{\Lambda_x}} dx$$



Typically, only a finite number of terms are retained in the expansion.

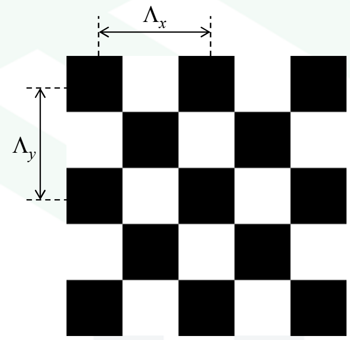
$$f(x) = \sum_{p=-P}^{P} a_p e^{-j\frac{2\pi px}{\Lambda_x}}$$

EMPossible

Slide 4

4

2D Complex Fourier Series



For 2D periodic functions, the complex Fourier series generalizes to

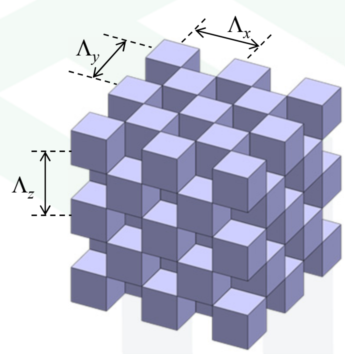
$$f(x,y)=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}a_{p,q}e^{-j\left(\frac{2\pi px}{\Lambda_x}+\frac{2\pi qy}{\Lambda_y}\right)} \qquad a_{p,q}=\frac{1}{A}\iint_A f(x,y)e^{j\left(\frac{2\pi px}{\Lambda_x}+\frac{2\pi qy}{\Lambda_y}\right)}dA$$

EMPossible

Slide 5

5

3D Complex Fourier Series



For 3D periodic functions, the complex Fourier series generalizes to

$$f(x,y,z)=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}a_{p,q,r}e^{-j\left(\frac{2\pi px}{\Lambda_x}+\frac{2\pi qy}{\Lambda_y}+\frac{2\pi rz}{\Lambda_z}\right)} \qquad a_{p,q,r}=\frac{1}{V}\iiint_V f(x,y,z)e^{j\left(\frac{2\pi px}{\Lambda_x}+\frac{2\pi qy}{\Lambda_y}+\frac{2\pi rz}{\Lambda_z}\right)}dV$$

EMPossible

Slide 6

6

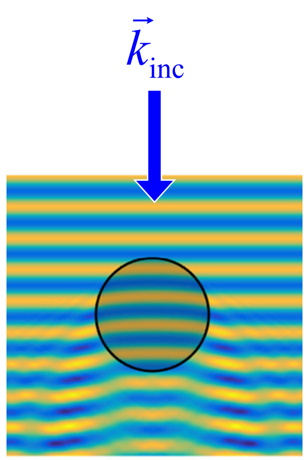
Diffraction from Gratings

EMPossible

Slide 7

7

Fields are Perturbed by Objects



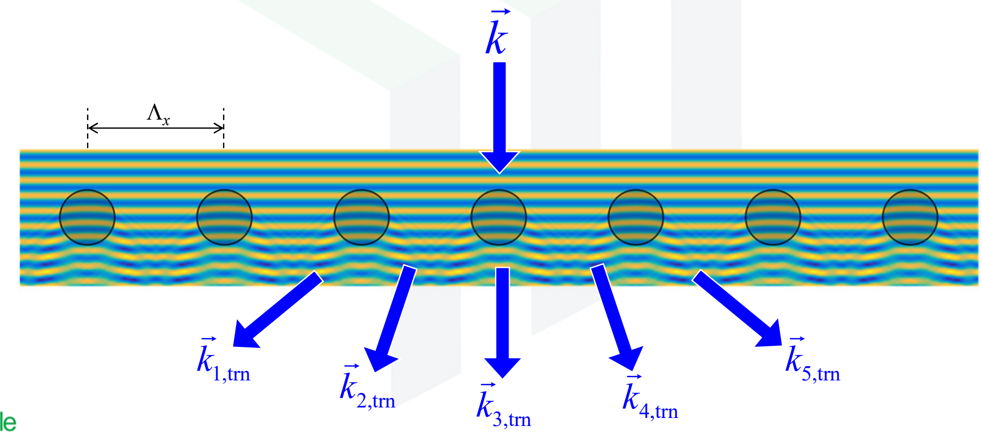
A portion of the wave front is delayed after travelling through the dielectric object.

Slide 8

8

Fields in Periodic Structures

Waves in periodic structures take on the same periodicity as their host.



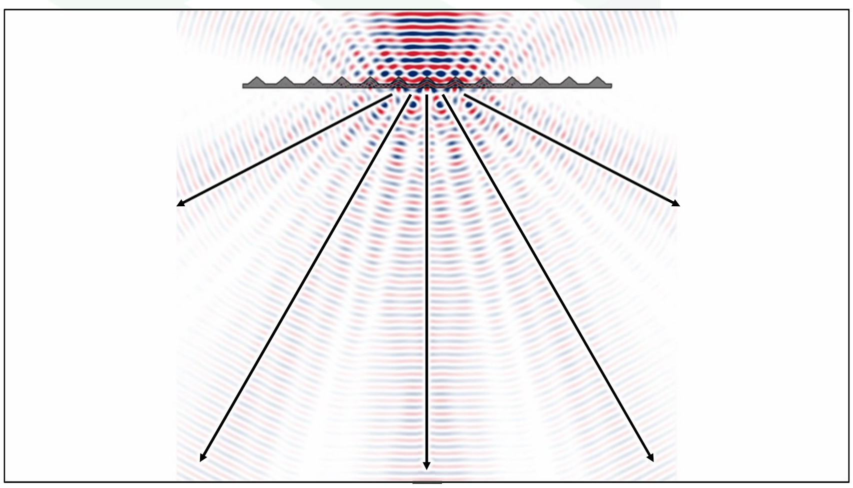
EMPossible

Slide 9

9

Diffraction from Gratings

The field is no longer a pure plane wave. The grating "chops" the wave front and sends the power into multiple discrete directions that are called *diffraction orders*.



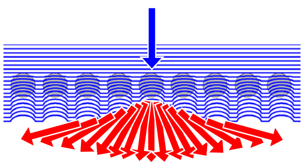
EMPossible

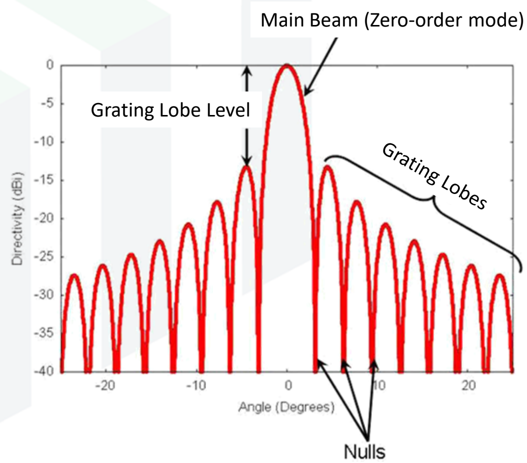
Slide 10

10

Grating Lobes

If we were to plot the power exiting a periodic structure as a function of angle, we would get the following. The power peaks are called *grating lobes* or sometimes *side lobes*. The power minimums are called *nulls*.





EMPossible

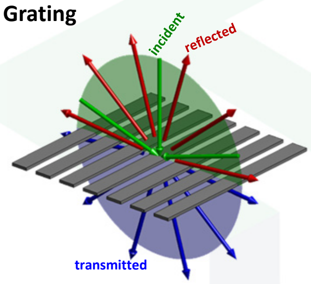
Slide 11

11

Diffraction Configurations

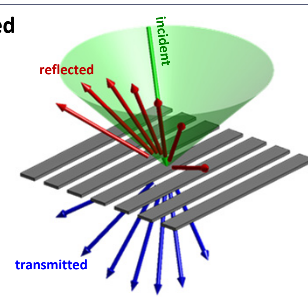
Planar Diffraction from a Ruled Grating

- Diffraction is confined within a plane
- Numerically much simpler than other cases
- E and H modes are independent



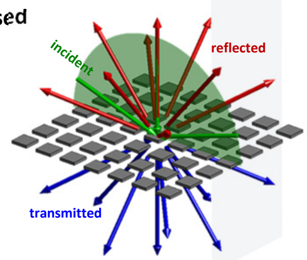
Conical Diffraction from a Ruled Grating

- Diffraction is no longer confined to a plane
- Almost same analytical complexity as crossed grating case, but simpler numerically
- E and H modes are coupled



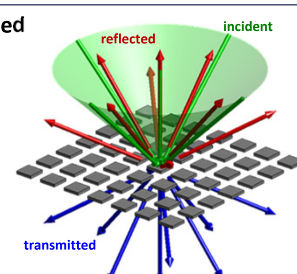
Conical Diffraction from a Crossed Grating with Planar Incidence

- Diffraction occurs in all directions
- Almost same numerical complexity as next case
- E and H modes are coupled



Conical Diffraction from a Crossed Grating

- Diffraction occurs in all directions
- Most complicated case numerically
- E and H modes are coupled
- Essentially the same as previous case



EMPossible

Slide 12

12

Grating Equation for Planar Diffraction

The angles of the diffracted modes are related to the wavelength and grating period through the grating equation.

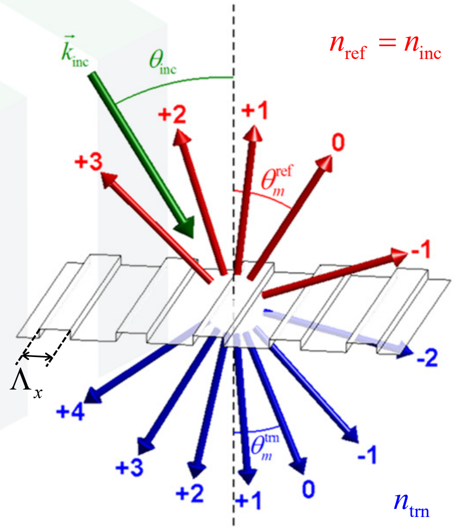
The grating equation only predicts the directions of the modes, not how much power is in them.

Reflection Region

$$n_{\text{ref}} \sin\theta_m = n_{\text{inc}} \sin\theta_{\text{inc}} - m\frac{\lambda_0}{\Lambda_x}$$

Transmission Region

$$n_{\text{trn}} \sin\theta_m = n_{\text{inc}} \sin\theta_{\text{inc}} - m\frac{\lambda_0}{\Lambda_x}$$



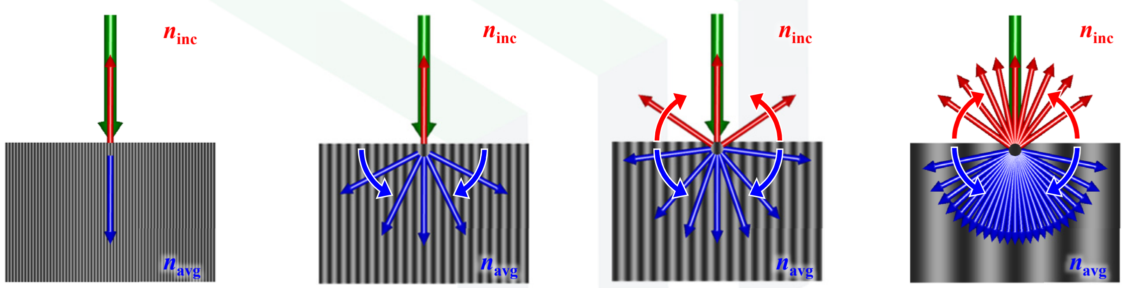
EMPossible

Slide 13

13

Effect of Grating Periodicity

Subwavelength Grating	"Subwavelength" Grating	Low Order Grating	High Order Grating



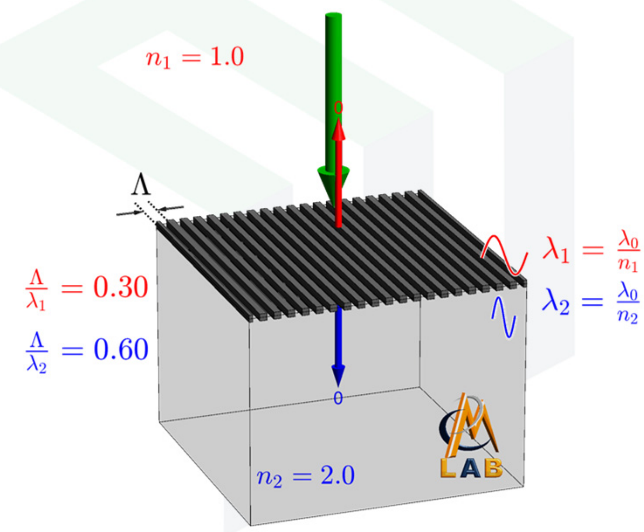
$\Lambda_x < \dfrac{\lambda_0}{n_{\text{avg}}}$	$\dfrac{\lambda_0}{n_{\text{avg}}} < \Lambda_x < \dfrac{\lambda_0}{n_{\text{inc}}}$	$\Lambda_x > \dfrac{\lambda_0}{n_{\text{inc}}}$	$\Lambda_x >> \dfrac{\lambda_0}{n_{\text{inc}}}$

EMPossible

Slide 14

14

Animation of Grating Diffraction (1 of 3)

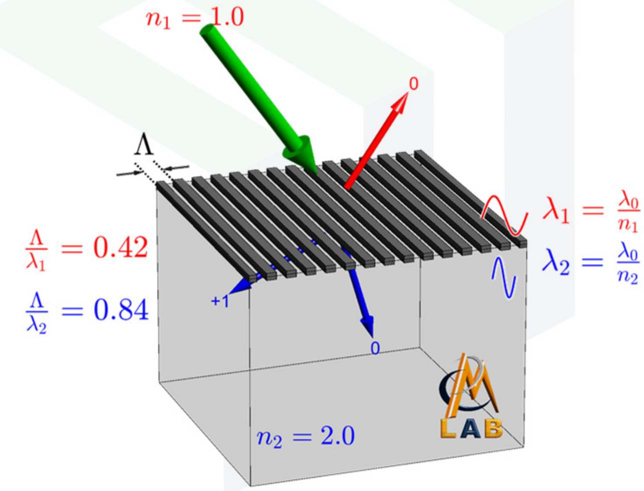


EMPossible

Slide 15

15

Animation of Grating Diffraction (2 of 3)

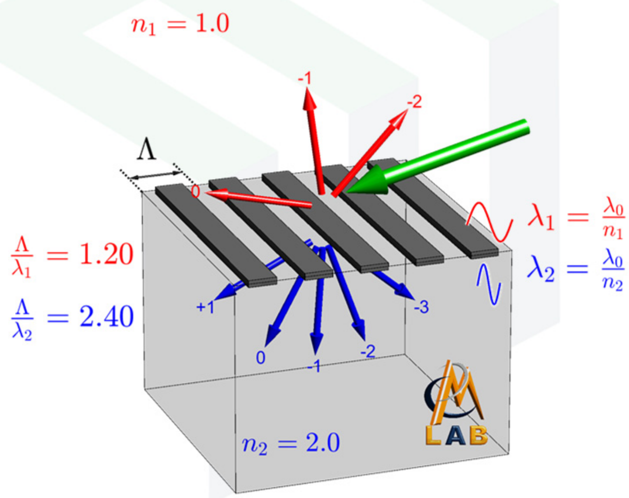


EMPossible

Slide 16

16

Animation of Grating Diffraction (3 of 3)



EMPossible

Slide 17

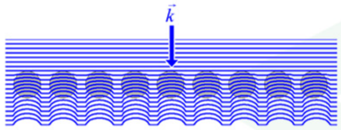
17

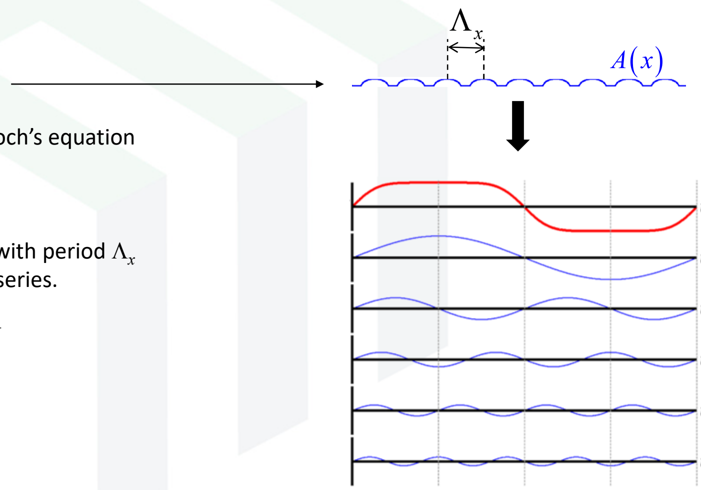
The Plane Wave Spectrum

Slide 18

18

Periodic Functions Can Be Expanded into a Fourier Series





Waves in periodic structures obey Bloch's equation

$$E(x,y)=A(x)e^{j\vec{\beta}\bullet\vec{r}}$$

The envelop $A(x)$ is periodic along x with period Λ_x so it can be expanded into a Fourier series.

$$A(x)=\sum_{m=-\infty}^{\infty}S(m)e^{-j\frac{2\pi mx}{\Lambda_x}}$$

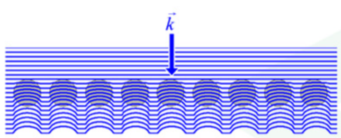
$$S(m)=\int_{\Lambda}A(x)e^{j\frac{2\pi mx}{\Lambda_x}}dx$$

EMPossible

Slide 19

19

Rearrange the Fourier Series (1 of 2)



A periodic field can be expanded into a Fourier series.

$$\begin{aligned}E(x,y)&=A(x)e^{j\vec{\beta}\bullet\vec{r}}\\&=\left[\sum_{m=-\infty}^{\infty}S(m)e^{-j\frac{2\pi mx}{\Lambda_x}}\right]e^{j\vec{\beta}\bullet\vec{r}}\\&=\sum_{m=-\infty}^{\infty}S(m)e^{j\vec{\beta}\bullet\vec{r}}e^{-j\frac{2\pi mx}{\Lambda_x}}\\&=\sum_{m=-\infty}^{\infty}S(m)e^{j\beta_x x}e^{j\beta_y y}e^{j\beta_z z}e^{-j\frac{2\pi mx}{\Lambda_x}}\end{aligned}$$

Here the plane wave term $e^{j\vec{\beta}\cdot\vec{r}}$ is brought inside of the summation.

EMPossible

Slide 20

20

Rearrange the Fourier Series (2 of 2)

β_x can be combined with the last complex exponential.

$$E(x,y)=\sum_{m=-\infty}^{\infty} S(m)e^{j\beta_x x}e^{j\beta_y y}e^{j\beta_z z}e^{-j\frac{2\pi m x}{\Lambda_x}}$$

$$=\sum_{m=-\infty}^{\infty} S(m)e^{j\beta_y y}e^{j\beta_z z}e^{j\left(\beta_x-\frac{2\pi m}{\Lambda_x}\right)x}$$

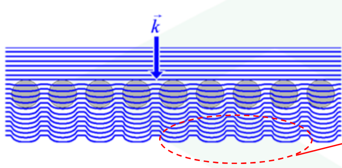
Now let $k_{x,m}=\beta_x-\frac{2\pi m}{\Lambda_x}$ and $k_{y,m}=\beta_y,\ k_{z,m}=\beta_z$

$$E(x,y)=\sum_{m=-\infty}^{\infty} S(m)e^{j\vec{k}(m)\bullet\vec{r}} \qquad \vec{k}(m)=\left(\beta_x-\frac{2\pi m}{\Lambda_x}\right)\hat{a}_x+\beta_y\hat{a}_y+\beta_z\hat{a}_z$$

EMPossible Slide 21

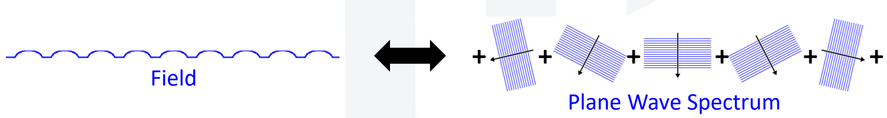
21

The Plane Wave Spectrum



$$E(x,y)=\sum_{m=-\infty}^{\infty} S(m)e^{j\vec{k}(m)\bullet\vec{r}}$$

Terms were rearranged to show that a periodic field can also be thought of as an infinite sum of plane waves at different angles. This is the "plane wave spectrum" of a periodic field.



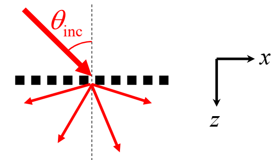
EMPossible Slide 22

22

Longitudinal Wave Vector Components of the Plane Wave Spectrum

The wave incident on a grating can be written as

$$E_{\text{inc}}(x,y) = E_0 e^{j(k_{x,\text{inc}}x + k_{z,\text{inc}}z)} \qquad \begin{aligned} k_{x,\text{inc}} &= k_0 n_{\text{inc}} \sin\theta_{\text{inc}} \\ k_{y,\text{inc}} &= 0 \\ k_{z,\text{inc}} &= k_0 n_{\text{inc}} \cos\theta_{\text{inc}} \end{aligned}$$



Phase matching into the grating leads to

$$k_x(m) = k_{x,\text{inc}} - m\frac{2\pi}{\Lambda_x} \qquad m = \cdots,-2,-1,0,1,2,\cdots$$

Note: k_x is always real.

Each wave must satisfy the dispersion relation.

$$k_x^2(m) + k_z^2(m) = (k_0 n_{\text{grat}})^2$$
$$\downarrow$$
$$k_z(m) = \sqrt{(k_0 n_{\text{grat}})^2 - k_x^2(m)}$$

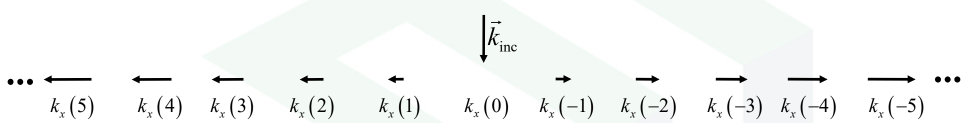
There are two possible solutions here.
1. Purely real k_z
2. Purely imaginary k_z.

EMPossible

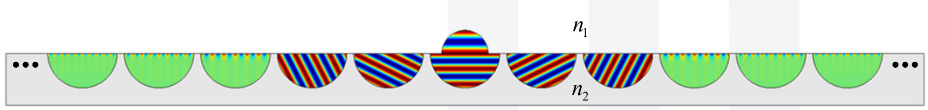
23

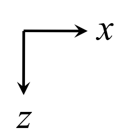
Visualizing Phase Matching into the Grating

The wave vector expansion for the first 11 modes can be visualized as…



Each of these is phase matched into material 2. The longitudinal component of the wave vector is calculated using the dispersion relation in material 2.





k_z is imaginary. The field in material 2 is evanescent.

k_z is real. A wave propagates into material 2.

k_z is imaginary. The field in material 2 is evanescent.

Note: The "evanescent" fields in material 2 are not completely evanescent. They have a purely real k_x so power flows in the transverse direction.

EMPossible

Slide 24

24

Conclusions About the Plane Wave Spectrum

- Fields in periodic media take on the same periodicity as the media they are in.
- Periodic fields can be expanded into a Fourier series.
- Each term of the Fourier series represents a spatial harmonic (plane wave).
- Since there are in infinite number of terms in the Fourier series, there are an infinite number of spatial harmonics.
- Only a few of the spatial harmonics are actually propagating waves. Only these can carry power away from a device. Tunneling is an exception.

EMPossible

Slide 25

25

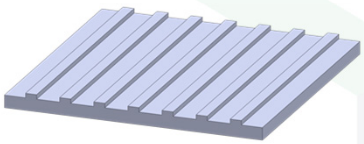
Plane Wave Spectrum from Crossed Gratings

Slide 26

26

Grating Terminology

1D grating
Ruled grating



Requires a 2D simulation

2D grating
Crossed grating

Requires a 3D simulation

EMPossible

Slide 27

27

Diffraction from Crossed Gratings

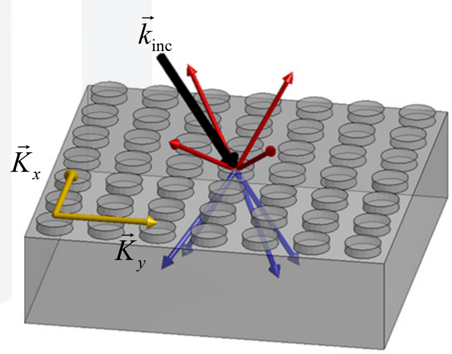
Doubly-periodic gratings, also called crossed gratings, can diffract waves into many directions.

They are described by two grating vectors, $\mathbf{K}_x$ and $\mathbf{K}_y$.

Two boundary conditions are necessary here.

$$k_x(m) = k_{x,\text{inc}} - mK_x \qquad m = \dots,-2,-1,0,1,2,\dots$$
$$k_y(n) = k_{y,\text{inc}} - nK_y \qquad n = \dots,-2,-1,0,1,2,\dots$$

$$\vec{K}_x = \frac{2\pi}{\Lambda_x}\hat{x}$$
$$\vec{K}_y = \frac{2\pi}{\Lambda_y}\hat{y}$$



EMPossible

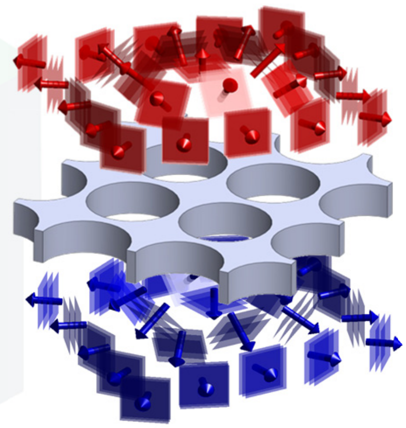
Slide 28

28

Transverse Wave Vector Expansion (1 of 2)

Crossed gratings diffraction in two dimensions, x and y.

To quantify diffraction for crossed gratings, we must calculate an expansion for both k_x and k_y.



$$k_x(m) = k_{x,\text{inc}} - \frac{2\pi m}{\Lambda_x} \qquad m = -\infty,\cdots,-2,-1,0,1,2,\cdots\infty$$

$$k_y(n) = k_{y,\text{inc}} - \frac{2\pi n}{\Lambda_y} \qquad n = -\infty,\cdots,-2,-1,0,1,2,\cdots\infty$$

$$\vec{k}_t(m,n) = k_x(m)\hat{x} + k_y(n)\hat{y}$$

```
% TRANSVERSE WAVE VECTOR EXPANSION
M  = [-floor(Nx/2):floor(Nx/2)]';
N  = [-floor(Ny/2):floor(Ny/2)]';
kx = kxinc - 2*pi*M/Lx;
ky = kyinc - 2*pi*N/Ly;
[ky,kx] = meshgrid(ky,kx);
```

We will use this code for 2D PWEM, 3D RCWA, 3D FDTD, 3D FDFD, 3D MoL, and more.

EMPossible

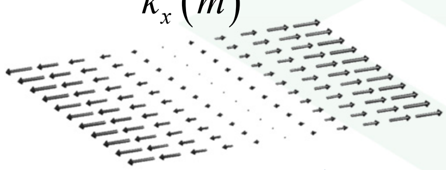
Slide 29

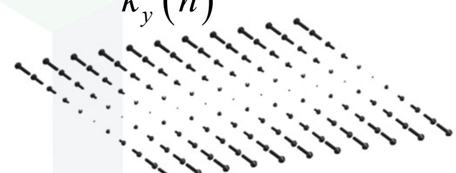
29

Transverse Wave Vector Expansion (2 of 2)

The vector expansions can be visualized this way…

$k_x(m)$ $\qquad$ $k_y(n)$





$$k_t(m,n) = k_x(m)\hat{x} + k_y(n)\hat{y}$$

EMPossible

Slide 30

30

Longitudinal Wave Vector Expansion (1 of 2)

The longitudinal components of the wave vectors are computed as

$$k_{z,\text{ref}}(m,n) = -\sqrt{(k_0 n_{\text{ref}})^2 - k_x^2(m) - k_y^2(m)}$$

$$k_{z,\text{trn}}(m,n) = \sqrt{(k_0 n_{\text{trn}})^2 - k_x^2(m) - k_y^2(m)}$$

```
% Compute Longitudinal Wave Vector Components
k0    = 2*pi/lam;
kzinc = k0*nref;
kzref = -sqrt((k0*nref)^2 - kx.^2 - ky.^2);
kztrn = +sqrt((k0*ntrn)^2 - kx.^2 - ky.^2);
```

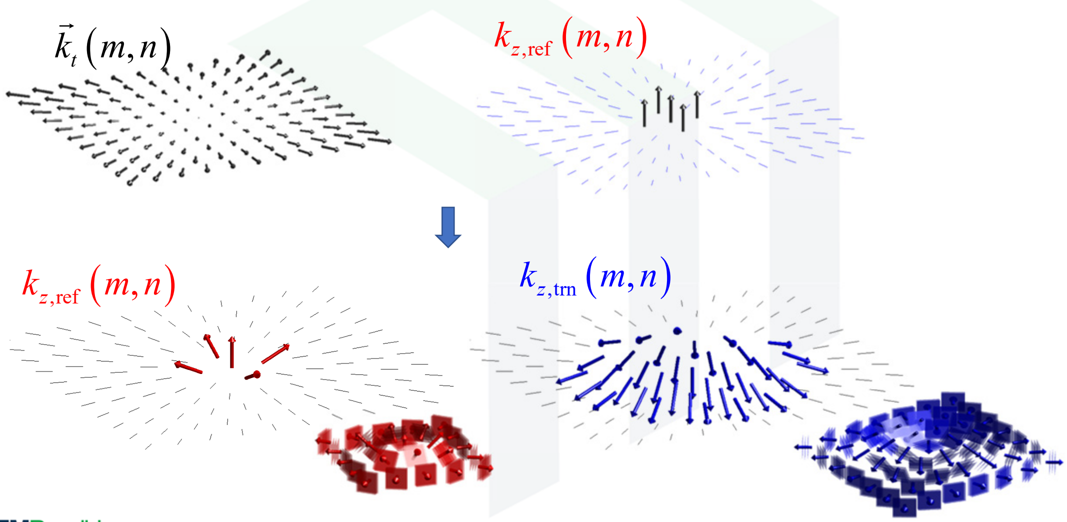
The center few modes will have real k_z's. These correspond to propagating waves. The others will have imaginary k_z's and correspond to evanescent waves that do not transport power.

$\vec{k}_t(m,n)$ $k_{z,\text{ref}}(m,n)$

EMPossible

Longitudinal Wave Vector Expansion (2 of 2)

The overall wave vector expansion can be visualized this way…



EMPossible

Power Flow from Gratings

Slide 33

33

Electromagnetic Power Flow

The instantaneous direction and intensity of power flow at any point is given by the Poynting vector.

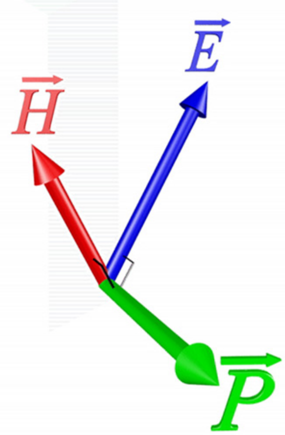
$$\vec{\wp}(\vec{r},t)=\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)$$

The RMS power flow is then

$$\vec{\wp}(\vec{r},\omega)=\frac{1}{2}\text{Re}\left[\vec{E}(\vec{r},\omega)\times\vec{H}^{*}(\vec{r},\omega)\right]$$

This is typically just written in the frequency-domain as

$$\vec{\wp}=\frac{1}{2}\text{Re}\left[\vec{E}\times\vec{H}^{*}\right]$$



EMPossible

Slide 34

34

Wave Expressions

To obtain an expression for the RMS Poynting vector, start by write expression for the electric and magnetic fields.

$$\vec{E}(\vec{r},\omega)=\vec{E}_0 e^{j\vec{k}\bullet\vec{r}} \qquad \vec{H}(\vec{r},\omega)=\frac{\vec{k}\times\vec{E}_0}{\omega\mu}e^{j\vec{k}\bullet\vec{r}}$$

Substituting these into the definition of RMS Poynting vector gives

$$\vec{\wp}=\frac{1}{2}\mathrm{Re}\left[\left(\vec{E}_0 e^{j\vec{k}\bullet\vec{r}}\right)\times\left(\frac{\vec{k}\times\vec{E}_0}{\omega\mu}e^{j\vec{k}\bullet\vec{r}}\right)^*\right]=\frac{1}{2}\mathrm{Re}\left[\left(\vec{E}_0 e^{j\vec{k}\bullet\vec{r}}\right)\times\left(\frac{\vec{k}^*\times\vec{E}_0^*}{\omega\mu^*}e^{-j\vec{k}^*\bullet\vec{r}}\right)\right]$$

$$=\frac{1}{2}\mathrm{Re}\left[\frac{e^{j\vec{k}\bullet\vec{r}}e^{-j\vec{k}^*\bullet\vec{r}}}{\omega\mu^*}\left(\vec{E}_0\times\vec{k}^*\times\vec{E}_0^*\right)\right]=\frac{1}{2}\mathrm{Re}\left[\frac{e^{j\left(\vec{k}-\vec{k}^*\right)\bullet\vec{r}}}{\omega\mu^*}\left|\vec{E}_0\right|^2\vec{k}^*\right]$$

$$=\frac{\left|\vec{E}_0\right|^2}{2}e^{-2\,\mathrm{Im}\left[\vec{k}\right]\bullet\vec{r}}\,\mathrm{Re}\left[\frac{\vec{k}^*}{\omega\mu^*}\right]=\frac{\left|\vec{E}_0\right|^2}{2k_0\eta_0}e^{-2\,\mathrm{Im}\left[\vec{k}\right]\bullet\vec{r}}\,\mathrm{Re}\left[\frac{\vec{k}^*}{\mu_r^*}\right]=\frac{\left|\vec{E}_0\right|^2}{2k_0\eta_0}e^{-2\,\mathrm{Im}\left[\vec{k}\right]\bullet\vec{r}}\,\mathrm{Re}\left[\frac{\vec{k}}{\mu_r}\right]$$

EMPossible

Slide 35

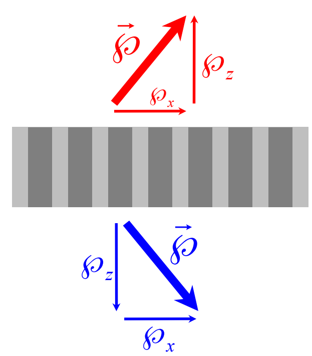
35

Power Flow Away From Grating

To calculate the power flow away from the grating, it is only the z-component of the Poynting vector that is of interest.

$$\vec{\wp}=\frac{\left|\vec{E}_0\right|^2}{2k_0\eta_0}e^{-2\,\mathrm{Im}\left[\vec{k}\right]\bullet\vec{r}}\,\mathrm{Re}\left[\frac{\vec{k}}{\mu_r}\right] \rightarrow \wp_z(z)=\frac{\left|\vec{E}_0\right|^2}{2k_0\eta_0}e^{-2\,\mathrm{Im}\left[k_z\right]z}\,\mathrm{Re}\left[\frac{k_z}{\mu_r}\right]$$

Note: power travelling in the x direction does not transport power into or away from a device that is infinitely periodic along x. This simplifying statement is not true for devices of finite extent.



EMPossible

Slide 36

36

RMS Power of the Diffracted Modes

Recall that the field scattered from a periodic structure can be decomposed into a Fourier series.

$$\vec{E}(x,z)=\sum_{m=-\infty}^{\infty}\vec{S}(m)e^{-jk_x(m)x}e^{-jk_z(m)z}$$

$$k_x(m)=-\frac{2\pi m}{\Lambda_x}$$

$$k_z(m)=\sqrt{(k_0 n)^2-k_x^2(m)}$$

The term $\vec{S}(m)$ is the amplitude and polarization of the m^{th} diffracted mode. Therefore, power flow away from the grating due to the m^{th} diffraction order is

$$\wp_z(z)=\frac{\left|\vec{E}_0\right|^2}{2k_0\eta_0}e^{-2\operatorname{Im}[k_z]z}\operatorname{Re}\left[\frac{k_z}{\mu_r}\right]\rightarrow\wp_z(z,m)=\frac{\left|\vec{S}(m)\right|^2}{2k_0\eta_0}e^{-2\operatorname{Im}[k_z(m)]z}\operatorname{Re}\left[\frac{k_z(m)}{\mu_r}\right]$$

EMPossible

Slide 37

37

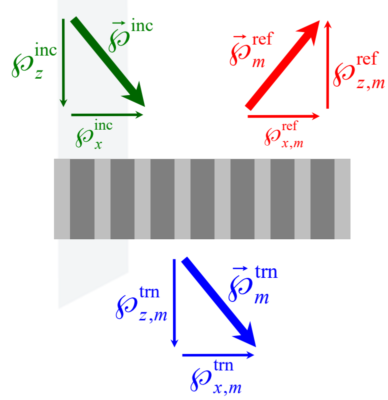
Z Directed Power

From the previous equation, the power flow of the incident, reflected, and transmitted waves into the grating are written as

$$\wp_z^{\text{inc}}(z)=\frac{\left|\vec{S}_{\text{inc}}\right|^2}{2k_0\eta_0}e^{-2\operatorname{Im}\left[k_z^{\text{inc}}\right]z}\operatorname{Re}\left[\frac{k_z^{\text{inc}}}{\mu_{\text{r,inc}}}\right]$$

$$\wp_z^{\text{ref}}(z,m)=\frac{\left|\vec{S}_{\text{ref}}(m)\right|^2}{2k_0\eta_0}e^{2\operatorname{Im}\left[k_z^{\text{ref}}(m)\right]z}\operatorname{Re}\left[\frac{-k_z^{\text{ref}}(m)}{\mu_{\text{r,ref}}}\right]$$

$$\wp_z^{\text{trn}}(z,m)=\frac{\left|\vec{S}_{\text{trn}}(m)\right|^2}{2k_0\eta_0}e^{-2\operatorname{Im}\left[k_z^{\text{trn}}(m)\right]z}\operatorname{Re}\left[\frac{k_z^{\text{trn}}(m)}{\mu_{\text{r,trn}}}\right]$$



EMPossible

Slide 38

38

Definition of Diffraction Efficiency

Diffraction efficiency is defined as the power in a specific diffraction order divided by the applied incident power.

$$\mathrm{DE}(m)=\frac{\wp_z(m)}{\wp_z^{\mathrm{inc}}}$$

Despite the title "efficiency," we don't always want this quantity to be large. We often want to adjust how much power gets diffracted into each mode. So it is neither good nor bad to have high or low diffraction efficiency. It depends on how the grating is being used.

EMPossible

Slide 39

39

Putting it All Together

So far, expressions were derived for the incident power and power in the diffraction orders. Away from the edges of the grating, these are

$$\wp_z^{\mathrm{inc}}(z)=\frac{\left|\vec{S}_{\mathrm{inc}}\right|^2}{2k_0\eta_0}e^{-2\,\mathrm{Im}\left[k_z^{\mathrm{inc}}\right]z}\,\mathrm{Re}\left[\frac{k_z^{\mathrm{inc}}}{\mu_{\mathrm{r,inc}}}\right]$$

$$\wp_z^{\mathrm{ref}}(z,m)=\frac{\left|\vec{S}_{\mathrm{ref}}(m)\right|^2}{2k_0\eta_0}e^{2\,\mathrm{Im}\left[k_z^{\mathrm{ref}}(m)\right]z}\,\mathrm{Re}\left[\frac{-k_z^{\mathrm{ref}}(m)}{\mu_{\mathrm{r,ref}}}\right]$$

$$\wp_z^{\mathrm{trn}}(z,m)=\frac{\left|\vec{S}_{\mathrm{trn}}(m)\right|^2}{2k_0\eta_0}e^{-2\,\mathrm{Im}\left[k_z^{\mathrm{trn}}(m)\right]z}\,\mathrm{Re}\left[\frac{k_z^{\mathrm{trn}}(m)}{\mu_{\mathrm{r,trn}}}\right]$$

The diffraction efficiency of the m^{th} diffraction order was also defined as

$$\mathrm{DE}(m)=\frac{\wp_z(m)}{\wp_z^{\mathrm{inc}}}$$

It is now possible to derive expressions for the diffraction efficiencies of the diffraction orders by combining these expressions.

$$R_{\mathrm{DE}}(m)=\frac{\wp_z^{\mathrm{ref}}(m)}{\wp_z^{\mathrm{inc}}}=\frac{\left|\vec{S}_{\mathrm{ref}}(m)\right|^2}{\left|\vec{S}_{\mathrm{inc}}\right|^2}e^{2\,\mathrm{Im}\left[k_z^{\mathrm{inc}}+k_z^{\mathrm{ref}}(m)\right]z}\,\frac{\mathrm{Re}\left[-k_z^{\mathrm{ref}}(m)/\mu_{\mathrm{r,inc}}\right]}{\mathrm{Re}\left[k_z^{\mathrm{inc}}/\mu_{\mathrm{r,inc}}\right]}$$

$$T_{\mathrm{DE}}(m)=\frac{\wp_z^{\mathrm{trn}}(m)}{\wp_z^{\mathrm{inc}}}=\frac{\left|\vec{S}_{\mathrm{trn}}(m)\right|^2}{\left|\vec{S}_{\mathrm{inc}}\right|^2}e^{-2\,\mathrm{Im}\left[k_z^{\mathrm{trn}}(m)-k_z^{\mathrm{inc}}\right]z}\,\frac{\mathrm{Re}\left[k_z^{\mathrm{trn}}(m)/\mu_{\mathrm{r,trn}}\right]}{\mathrm{Re}\left[k_z^{\mathrm{inc}}/\mu_{\mathrm{r,inc}}\right]}$$

Note that: $\mu_{\mathrm{r,inc}}=\mu_{\mathrm{r,ref}}$

EMPossible

Slide 40

40

Diffraction Efficiency for Magnetic Fields

Diffraction efficiency equations were just derived from electric field quantities.

$\vec{S}(m) \equiv$ Electric field amplitude of the m^{th} diffraction order

$$R_{\text{DE}}(m) = \frac{\wp_z^{\text{ref}}(m)}{\wp_z^{\text{inc}}} = \frac{\left|\vec{S}_{\text{ref}}(m)\right|^2}{\left|\vec{S}_{\text{inc}}\right|^2} e^{2\,\text{Im}\left[k_z^{\text{inc}}+k_z^{\text{ref}}(m)\right]z} \frac{\text{Re}\left[-k_z^{\text{ref}}(m)/\mu_{\text{r,inc}}\right]}{\text{Re}\left[k_z^{\text{inc}}/\mu_{\text{r,inc}}\right]}$$

$$T_{\text{DE}}(m) = \frac{\wp_z^{\text{trn}}(m)}{\wp_z^{\text{inc}}} = \frac{\left|\vec{S}_{\text{trn}}(m)\right|^2}{\left|\vec{S}_{\text{inc}}\right|^2} e^{-2\,\text{Im}\left[k_z^{\text{trn}}(m)-k_z^{\text{inc}}\right]z} \frac{\text{Re}\left[k_z^{\text{trn}}(m)/\mu_{\text{r,trn}}\right]}{\text{Re}\left[k_z^{\text{inc}}/\mu_{\text{r,inc}}\right]}$$

Sometimes Maxwell's equations in terms of the magnetic fields. In this case, the diffraction efficiency equations are

$\vec{U}(m) \equiv$ Magnetic field amplitude of the m^{th} diffraction order

$$R_{\text{DE}}(m) = \frac{\wp_z^{\text{ref}}(m)}{\wp_z^{\text{inc}}} = \frac{\left|\vec{U}_{\text{ref}}(m)\right|^2}{\left|\vec{U}_{\text{inc}}\right|^2} e^{2\,\text{Im}\left[k_z^{\text{inc}}+k_z^{\text{ref}}(m)\right]z} \frac{\text{Re}\left[-k_z^{\text{ref}}(m)/\varepsilon_{\text{r,inc}}\right]}{\text{Re}\left[k_z^{\text{inc}}/\varepsilon_{\text{r,inc}}\right]}$$

$$T_{\text{DE}}(m) = \frac{\wp_z^{\text{trn}}(m)}{\wp_z^{\text{inc}}} = \frac{\left|\vec{U}_{\text{trn}}(m)\right|^2}{\left|\vec{U}_{\text{inc}}\right|^2} e^{-2\,\text{Im}\left[k_z^{\text{trn}}(m)-k_z^{\text{inc}}\right]z} \frac{\text{Re}\left[k_z^{\text{trn}}(m)/\varepsilon_{\text{r,trn}}\right]}{\text{Re}\left[k_z^{\text{inc}}/\varepsilon_{\text{r,inc}}\right]}$$

EMPossible Slide 41

41

Reflectance, Transmittance, and Absorptance

The reflectance R is the sum of the diffraction efficiencies of the reflected modes.

$$R = \sum_{m=-\infty}^{\infty} R_{\text{DE}}(m)$$

The transmittance T is the sum of the diffraction efficiencies of the transmitted modes.

$$T = \sum_{m=-\infty}^{\infty} T_{\text{DE}}(m)$$

The absorptance A is the fraction of power absorbed by the device.

$$A + R + T = 1$$

$A > 0$ materials have loss
$A = 0$ materials have no loss
$A < 0$ materials have gain

EMPossible Slide 42

42

Simplification for TMM

For TMM, the layers are homogeneous so there is no diffraction. Therefore, only the $m = 0$ diffraction order exists.

$\vec{S}(0) \equiv$ Electric field amplitude of the 0^{th} diffraction order

$$R_{\text{DE}}(0) = \frac{\wp_z^{\text{ref}}(0)}{\wp_z^{\text{inc}}} = \frac{\left|\vec{S}_{\text{ref}}(0)\right|^2}{\left|\vec{S}_{\text{inc}}\right|^2} e^{2\,\text{Im}\left[k_z^{\text{inc}} + k_z^{\text{ref}}(0)\right]z} \frac{\text{Re}\left[-\hat{k}_z^{\text{ref}}(0)/\mu_{\text{r,inc}}\right]}{\text{Re}\left[\hat{k}_z^{\text{inc}}/\mu_{\text{r,inc}}\right]}$$

$$T_{\text{DE}}(0) = \frac{\wp_z^{\text{trn}}(0)}{\wp_z^{\text{inc}}} = \frac{\left|\vec{S}_{\text{trn}}(0)\right|^2}{\left|\vec{S}_{\text{inc}}\right|^2} e^{-2\,\text{Im}\left[k_z^{\text{trn}}(0) - k_z^{\text{inc}}\right]z} \frac{\text{Re}\left[k_z^{\text{trn}}(0)/\mu_{\text{r,trn}}\right]}{\text{Re}\left[k_z^{\text{inc}}/\mu_{\text{r,inc}}\right]}$$

Since there is only one diffraction order,

$$\vec{E}_{\text{inc}} = \vec{S}_{\text{inc}}$$
$$\vec{E}_{\text{ref}} = \vec{S}_{\text{ref}}(0)$$
$$\vec{E}_{\text{trn}} = \vec{S}_{\text{trn}}(0)$$
$$R_{\text{DE}} = R_{\text{DE}}(0)$$
$$T_{\text{DE}} = T_{\text{DE}}(0)$$

$\Rightarrow$

$$R = R_{\text{DE}} = \frac{\left|\vec{E}_{\text{ref}}\right|^2}{\left|\vec{E}_{\text{inc}}\right|^2}$$

$$T = T_{\text{DE}} = \frac{\left|\vec{E}_{\text{trn}}\right|^2}{\left|\vec{E}_{\text{inc}}\right|^2} e^{-2\,\text{Im}\left[k_z^{\text{trn}} - k_z^{\text{inc}}\right]z} \frac{\text{Re}\left[k_z^{\text{trn}}/\mu_{\text{r,trn}}\right]}{\text{Re}\left[k_z^{\text{inc}}/\mu_{\text{r,inc}}\right]}$$

$$k_{z,\text{ref}} = -k_{z,\text{inc}}$$

EMPossible

Slide 43

43

Diffraction Efficiency Equations in Lossy LHI Materials Right Against Device (i.e. z = 0)

Equations for Multiple Diffraction Orders

$$R_{\text{DE}}(m) = \frac{\left|\vec{S}_{\text{ref}}(m)\right|^2}{\left|\vec{S}_{\text{inc}}\right|^2} \frac{\text{Re}\left[-k_z^{\text{ref}}(m)/\mu_{\text{r,inc}}\right]}{\text{Re}\left[k_z^{\text{inc}}/\mu_{\text{r,inc}}\right]}$$

$$T_{\text{DE}}(m) = \frac{\left|\vec{S}_{\text{trn}}(m)\right|^2}{\left|\vec{S}_{\text{inc}}\right|^2} \frac{\text{Re}\left[k_z^{\text{trn}}(m)/\mu_{\text{r,trn}}\right]}{\text{Re}\left[k_z^{\text{inc}}/\mu_{\text{r,inc}}\right]}$$

$$R_{\text{DE}}(m) = \frac{\left|\vec{U}_{\text{ref}}(m)\right|^2}{\left|\vec{U}_{\text{inc}}\right|^2} \frac{\text{Re}\left[-k_z^{\text{ref}}(m)/\varepsilon_{\text{r,inc}}\right]}{\text{Re}\left[k_z^{\text{inc}}/\varepsilon_{\text{r,inc}}\right]}$$

$$T_{\text{DE}}(m) = \frac{\left|\vec{U}_{\text{trn}}(m)\right|^2}{\left|\vec{U}_{\text{inc}}\right|^2} \frac{\text{Re}\left[k_z^{\text{trn}}(m)/\varepsilon_{\text{r,trn}}\right]}{\text{Re}\left[k_z^{\text{inc}}/\varepsilon_{\text{r,inc}}\right]}$$

$$R = \sum_m R_{\text{DE}}(m) \qquad T = \sum_m T_{\text{DE}}(m)$$

TMM / Single Diffraction Order

$$R = R_{\text{DE}} = \frac{\left|\vec{E}_{\text{ref}}\right|^2}{\left|\vec{E}_{\text{inc}}\right|^2} \qquad T = T_{\text{DE}} = \frac{\left|\vec{E}_{\text{trn}}\right|^2}{\left|\vec{E}_{\text{inc}}\right|^2} \frac{\text{Re}\left[k_z^{\text{trn}}/\mu_{\text{r,trn}}\right]}{\text{Re}\left[k_z^{\text{inc}}/\mu_{\text{r,inc}}\right]}$$

$$R = R_{\text{DE}} = \frac{\left|\vec{H}_{\text{ref}}\right|^2}{\left|\vec{H}_{\text{inc}}\right|^2} \qquad T = T_{\text{DE}} = \frac{\left|\vec{H}_{\text{trn}}\right|^2}{\left|\vec{H}_{\text{inc}}\right|^2} \frac{\text{Re}\left[k_z^{\text{trn}}/\varepsilon_{\text{r,trn}}\right]}{\text{Re}\left[k_z^{\text{inc}}/\varepsilon_{\text{r,inc}}\right]}$$

EMPossible

Slide 44

44

Diffraction Efficiency Equations in Lossless LHI Materials Right Against Device (i.e. z = 0)

Equations for Multiple Diffraction Orders

$$R_{\text{DE}}(m) = \frac{\left|\vec{S}_{\text{ref}}(m)\right|^2}{\left|\vec{S}_{\text{inc}}\right|^2} \frac{\text{Re}\left[-k_z^{\text{ref}}(m)\right]}{\text{Re}\left[k_z^{\text{inc}}\right]}$$

$$R_{\text{DE}}(m) = \frac{\left|\vec{U}_{\text{ref}}(m)\right|^2}{\left|\vec{U}_{\text{inc}}\right|^2} \frac{\text{Re}\left[-k_z^{\text{ref}}(m)\right]}{\text{Re}\left[k_z^{\text{inc}}\right]}$$

$$T_{\text{DE}}(m) = \frac{\mu_{\text{r,inc}}}{\mu_{\text{r,trn}}} \frac{\left|\vec{S}_{\text{trn}}(m)\right|^2}{\left|\vec{S}_{\text{inc}}\right|^2} \frac{\text{Re}\left[k_z^{\text{trn}}(m)\right]}{\text{Re}\left[k_z^{\text{inc}}\right]}$$

$$T_{\text{DE}}(m) = \frac{\varepsilon_{\text{r,inc}}}{\varepsilon_{\text{r,trn}}} \frac{\left|\vec{U}_{\text{trn}}(m)\right|^2}{\left|\vec{U}_{\text{inc}}\right|^2} \frac{\text{Re}\left[k_z^{\text{trn}}(m)\right]}{\text{Re}\left[k_z^{\text{inc}}\right]}$$

$$R = \sum_m R_{\text{DE}}(m) \qquad T = \sum_m T_{\text{DE}}(m)$$

TMM / Single Diffraction Order

$$R = R_{\text{DE}} = \frac{\left|\vec{E}_{\text{ref}}\right|^2}{\left|\vec{E}_{\text{inc}}\right|^2} \qquad T = T_{\text{DE}} = \frac{\mu_{\text{r,inc}}}{\mu_{\text{r,trn}}} \frac{\left|\vec{E}_{\text{trn}}\right|^2}{\left|\vec{E}_{\text{inc}}\right|^2} \frac{\text{Re}\left[k_z^{\text{trn}}\right]}{\text{Re}\left[k_z^{\text{inc}}\right]}$$

$$R = R_{\text{DE}} = \frac{\left|\vec{H}_{\text{ref}}\right|^2}{\left|\vec{H}_{\text{inc}}\right|^2} \qquad T = T_{\text{DE}} = \frac{\varepsilon_{\text{r,inc}}}{\varepsilon_{\text{r,trn}}} \frac{\left|\vec{H}_{\text{trn}}\right|^2}{\left|\vec{H}_{\text{inc}}\right|^2} \frac{\text{Re}\left[k_z^{\text{trn}}\right]}{\text{Re}\left[k_z^{\text{inc}}\right]}$$

EMPossible