CTU, Prague
04.06.2018

Lecture 4: Continuum mechanics III
Variational formulations in statics and dynamics

Anton Tkachuk

Revision: variational calculus and energy methods in statics, Tonti Diagram

Canonical variational principles of elastostatics: Hu-Washizu and Hellinger-Reissner

Principle of virtual work for dynamics

Least (stationary) action and Hamilton's principles

University of Stuttgart
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Department of Civil and
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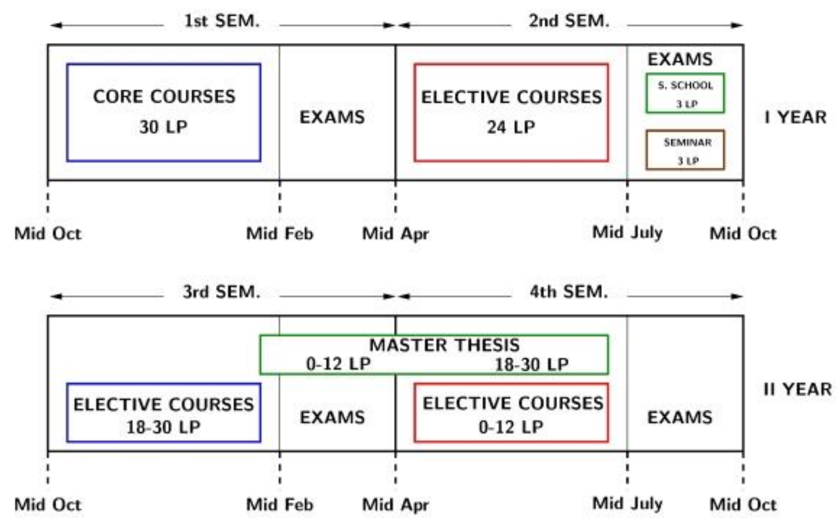
Baustatik und Baudynamik

Module structure

Core Modules: 1st Semester (Winter Term) – 30 Credits

- Continuum Mechanics
- Computational Mechanics of Materials
- **Computational Mechanics of Structures**
- Discretization Methods
- Introduction to Scientific Programming
- Structural Dynamics and Optimization
- Engineering Materials I

>10 lectures on dynamics and locking



Elective Modules: 2nd and 3rd Semester

- **Advanced Computational Mechanics of Structures**
- **Variational methods in Structural Dynamics**
- Foundations of Single and Multiphasic Materials
- Micromechanics of Materials and Homogenization Methods
- Boundary Elements in Statics and Dynamics
- Applied Scientific Programming
- Numerical Modelling of Concrete Structures
- Advanced Materials and Smart Structures
- Optimal Control
- **Implementation and Algorithm for Finite Elements**
- Introduction to the Continuum Mechanics of Multi-Phase Materials
- Smart Systems and Control
- **Computational Methods for Shell Analysis**
- **Computational Contact mechanics**

…

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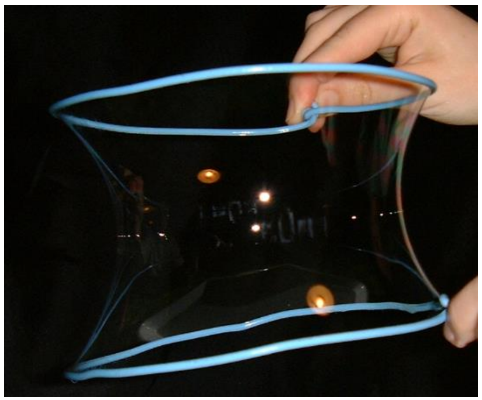
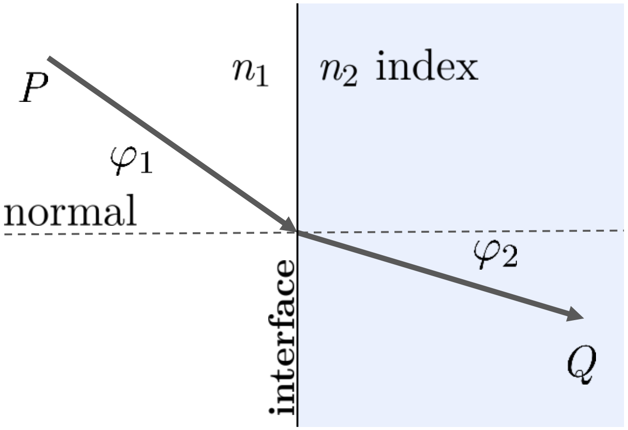
Motivation for single-field variational principles

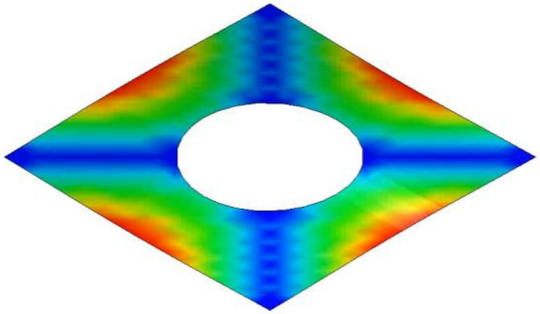
- many physical laws can stated as maximum, minimum or saddle point problem



soap film equilibrates at a minimal surface



Fermat's principle of geometrical optics



extremes of Rayleigh quotient define eigenmode
used in L11, L12 L16

- vehicle for structural optimization and optimal control
- popular starting point for multi-body dynamics — introduced in Lecture 5
- natural ways to introduce constraints — used in L5, L20, L21, L24
- theoretical basis for standard finite elements — treated in Lecture 6

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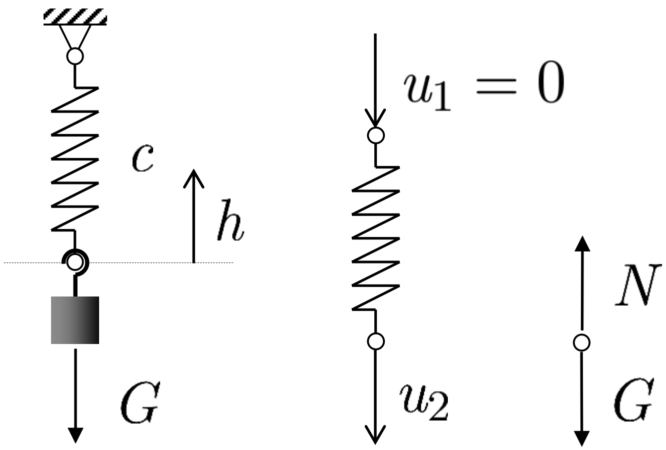
reminder: energy methods

one-dimensional spring, vector mechanics (i)



KINEMATICS $\Delta u = u_2 - u_1 =: u_2$

MATERIAL $N = c\Delta u$

EQUILIBRIUM $N - G = 0$

SOLUTION $cu_2 = G \Rightarrow u_2 = \frac{G}{c}$

alternative form of equilibrium, energy method (ii)

principle of minimum potential energy (PMPE)

KINEMATICS $\Delta u = u_2 - u_1 =: u_2$

MATERIAL $\Pi^{\text{int}} = \frac{1}{2} c\Delta u^2$

EQUILIBRIUM

$$\Pi = \Pi^{\text{int}} + \Pi^{\text{ext}} \rightarrow \min$$

$$\Pi^{\text{ext}} = Gh = -Gu$$

reminder: energy methods

solution via differential calculus (necessary condition for minimum)

$$\frac{\mathrm{d}\Pi}{\mathrm{d}u_2} = 0 \qquad \frac{\mathrm{d}\Pi^{\text{int}}}{\mathrm{d}u_2} = c u_2 \qquad \frac{\mathrm{d}\Pi^{\text{ext}}}{\mathrm{d}u_2} = -G$$

SOLUTION $$c u_2 - G = 0 \Rightarrow u_2 = \frac{G}{c}$$

Example

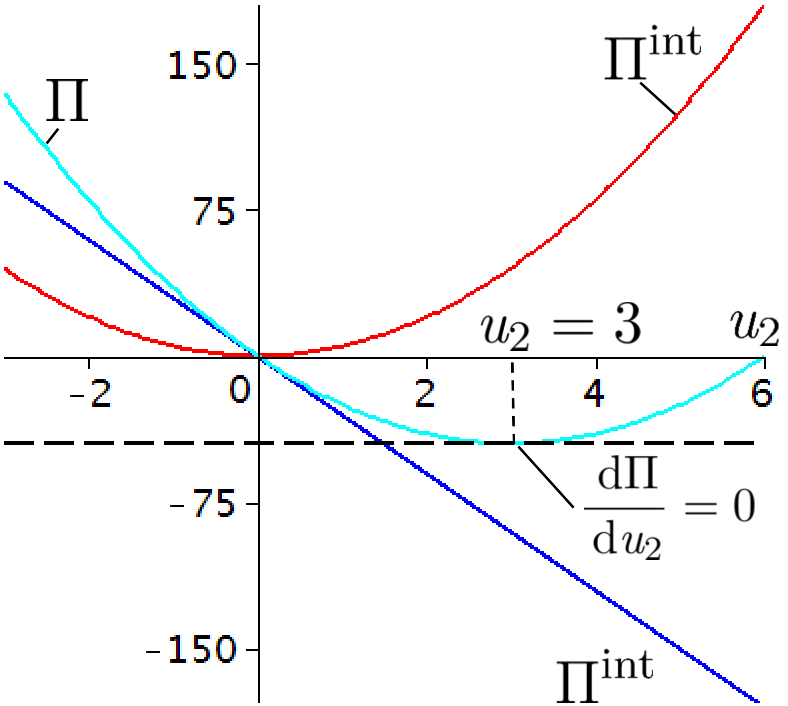
$$c = 10 \qquad G = 30$$

$$\Pi^{\text{int}} = 5u_2^2 \qquad \Pi^{\text{ext}} = -30u_2$$

$$\Pi = 5u_2^2 - 30u_2$$

$$\frac{\mathrm{d}\Pi}{\mathrm{d}u_2} = 10u_2 - 30$$

SOLUTION $$u_2 = 3$$



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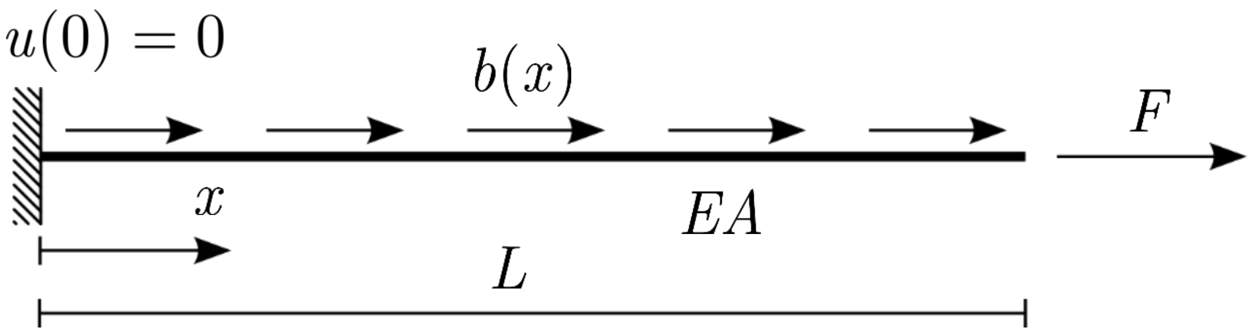
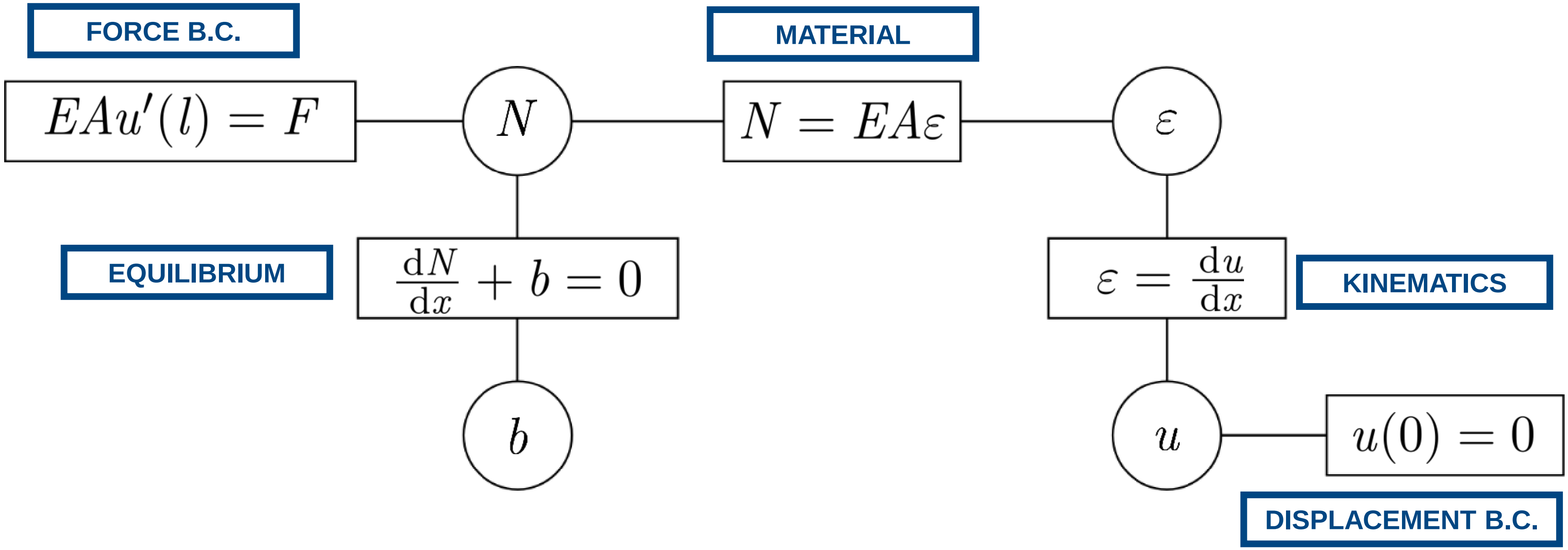
model problem: one-dimensional elasticity



Tonti diagram



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reminder: energy methods in 1D

$$\Pi = \Pi^{\text{int}} + \Pi^{\text{ext}} \rightarrow \min$$

$$\Pi^{\text{int}} = \int_0^L \frac{1}{2} EA \left(u'\right)^2 \, \mathrm{d}x \qquad \Pi^{\text{ext}} = -\int_0^L u b \, \mathrm{d}x - u(l) F$$

necessary conditions for minimum

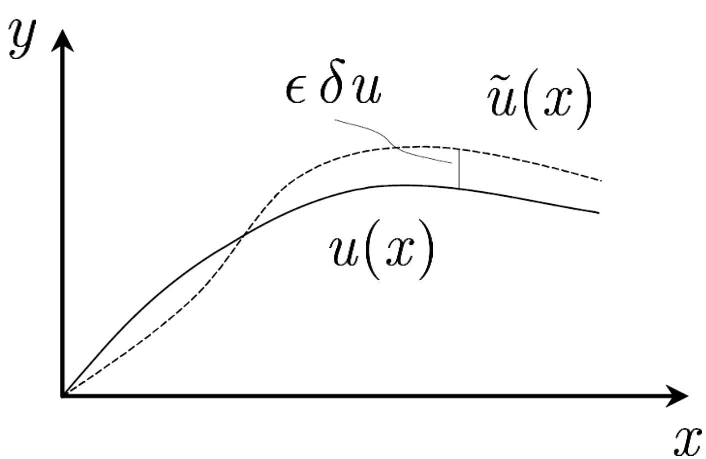
u possible solution

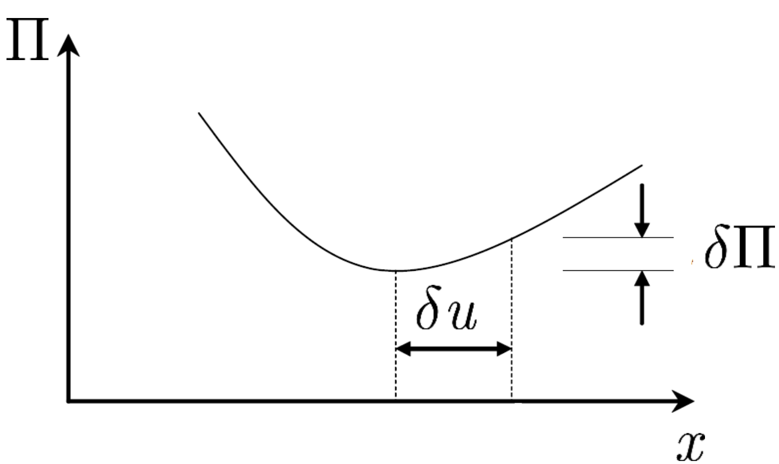
δu variation (perturbation direction)

ϵ scaling of the perturbation

$$\Pi(\tilde{u}) = \Pi(u + \epsilon\,\delta u) \geq \Pi(u)$$

$$\delta\Pi = \lim_{\epsilon \to 0} \frac{\Pi(u + \epsilon\,\delta\, u) - \Pi(u)}{\epsilon} \stackrel{!}{=} 0$$





rules for operating with first variation

$$\delta\left(\Pi_1+\Pi_2\right)=\delta\Pi_1+\delta\Pi_2$$

$$\delta(\alpha\Pi)=\alpha\delta\Pi$$

linearity (α - independent of u)

$$\delta\left(\Pi_1\Pi_2\right)=\Pi_2\delta\Pi_1+\Pi_1\delta\Pi_2$$

product rule

$$\delta\left(F(u)\right)=\frac{\partial F}{\partial u}\delta u$$

chain rule

$$\delta\int_\Omega F(u)\,\mathrm{d}V=\int_\Omega\delta F(u)\,\mathrm{d}V\qquad\delta(u')=(\delta u)'$$

interchangeable with integral and derivative

first variation of PMPE

$$\delta\Pi = \delta\left(\Pi^{\text{int}} + \Pi^{\text{ext}}\right) = \delta \int_0^L \frac{1}{2} EA\left(u'\right)^2 \,\mathrm{d}x - \delta \int_0^L u b \,\mathrm{d}x - \delta(u(l)F)$$

$$= \int_0^L EAu'\delta u' \,\mathrm{d}x - \int_0^L b\delta u \,\mathrm{d}x - F\delta u(l) \stackrel{!}{=} 0$$

…equivalent to principle of virtual work (PVW)

integrating by parts $\int_0^L EAu'\delta u' \,\mathrm{d}x = (EAu'\delta u)|_{x=0}^{x=L} - \int_0^L EAu''\delta u \,\mathrm{d}x$

$$\delta\Pi = \int_0^L \delta u\left(EAu'' + b\right)\,\mathrm{d}x + (EAu' - F)\delta u(l) \stackrel{!}{=} 0$$

and using the main lemma of variational calculus leads to Lagrange equation

EQUILIBRIUM	$EAu'' + b = 0$
FORCE B.C.	$EAu' - F = 0$

covered in detail in Lecture L5

…equivalent to strong form of boundary value problem

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differential calculus vs. variational calculus

	differential calculus	**variational calculus**
problem description via	function	functional
necessary condition for extremum	first derivative = 0	first variation = 0
result	one single value (extremum)	function (extremal function)
type of extremum determined by	second derivative	second variation

Motivation for multi-field variational principles

- theoretical basis for finite element technology, namely for

  … hybrid-mixed finite elements — treated in Lecture 7

  … reciprocal mass matrices — derived in Lecture 8

  … enhanced assumed strain elements — introduced in Lecture 9

  … template finite elements — out of scope of this summer school

  … hierarchic variational formulations

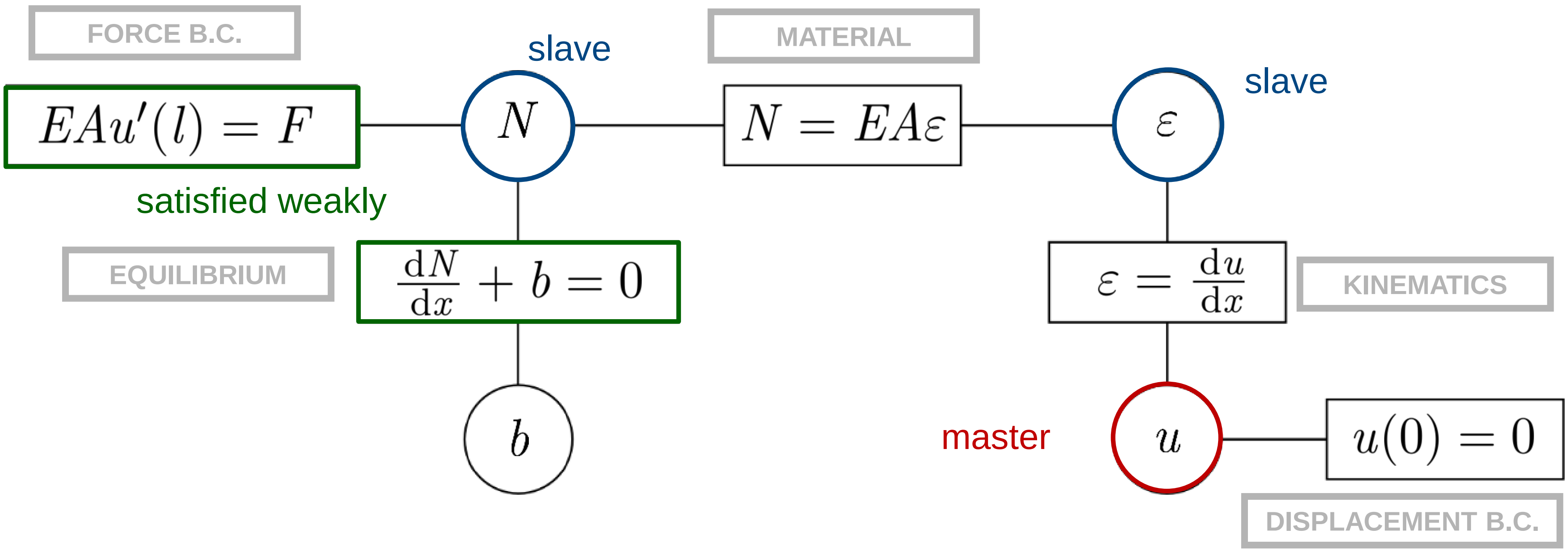
  … some discontinuous Galerkin formulations

- still an area of active research — (objective)
- mathematical beauty — (subjective)

Idea: identify master and slave fields on Tonti diagram for PMPE

$$\Pi[u] = \Pi^{\text{int}} + \Pi^{\text{ext}} \rightarrow \min$$

$$\delta\Pi[u] = \int_0^L \delta u \left(EAu'' + b\right)\, \mathrm{d}x + (EAu' - F)\delta u(l) \stackrel{!}{=} 0$$



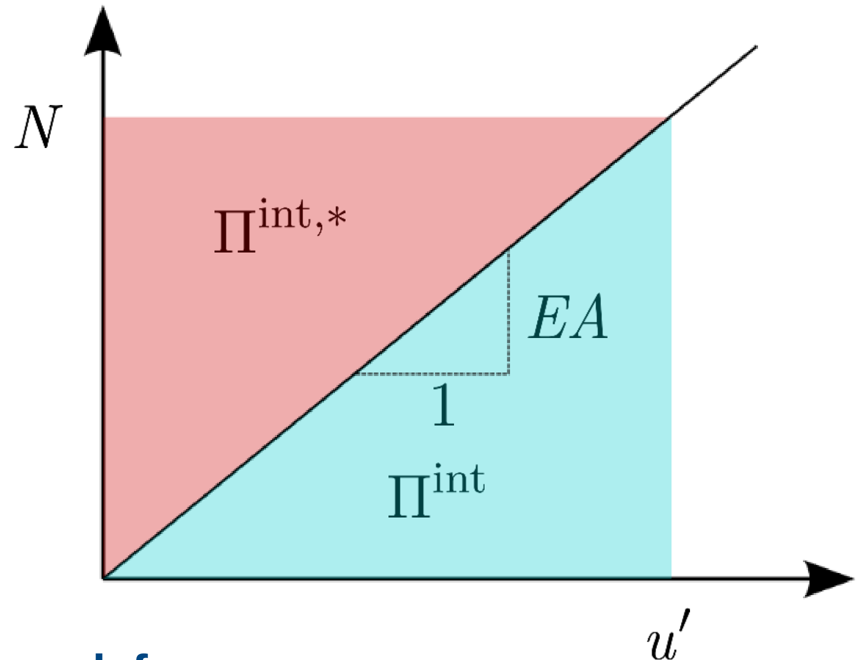
Canonical variational principles of elastostatics

Idea 1: in Hellinger-Reissner principle force is used for the energy computation

$$\Pi[u,N] = \Pi^{\text{int},*} + \Pi^{\text{ext}} \rightarrow \text{stat.}$$

Complementary energy is computed via Legendre transformation

$$\Pi^{\text{int},*} = \int_0^L \left(N u' - \frac{N^2}{2EA} \right) \mathrm{d}x$$



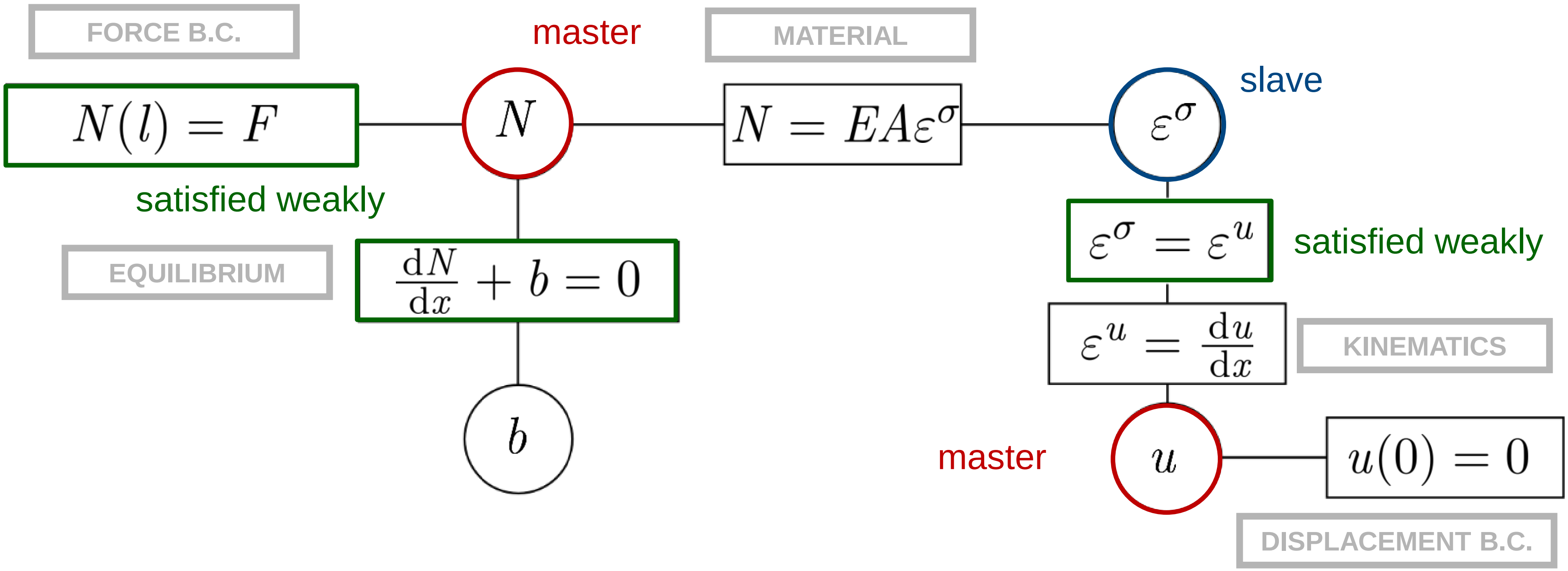
Note: the functional of internal energy is not any more convex, but concave in direction N and convex in direction $u' = N/EA \rightarrow$ saddle point problem

First variation contains several equation in weak form

$$\delta\Pi[u,N] = \int_0^L \left[\delta u \left(N' + b \right) + \delta N \left(\frac{N}{EA} - u' \right) \right] \mathrm{d}x + (N - F)\delta u(l) \stackrel{!}{=} 0$$

Interpretation of Hellinger-Reissner principle with Tonti diagram

$$\delta\Pi[u,N] = \int_0^L \left[\delta u\,(N' + b) + \delta N \left(\frac{N}{EA} - u'\right)\right] \mathrm{d}x + (N - F)\delta u(l) \overset{!}{=} 0$$



used for hybrid-mixed finite elements in Lecture 7

Idea 2: in Hu-Washizu all field equations are satisfied weakly

$$\Pi[u,\varepsilon,N] = \Pi^{\text{int},\circ} + \Pi^{\text{ext}} \rightarrow \text{stat.}$$

Internal energy is computed on strain field

$$\Pi^{\text{int},\circ} = \int_0^L \left(\frac{1}{2} EA\varepsilon^2 + \underbrace{N}_{\text{Lagrange multiplier}} \underbrace{(u' - \varepsilon)}_{\text{weak form of compatibility}} \right) \mathrm{d}x$$

First variation contains all equations in weak form

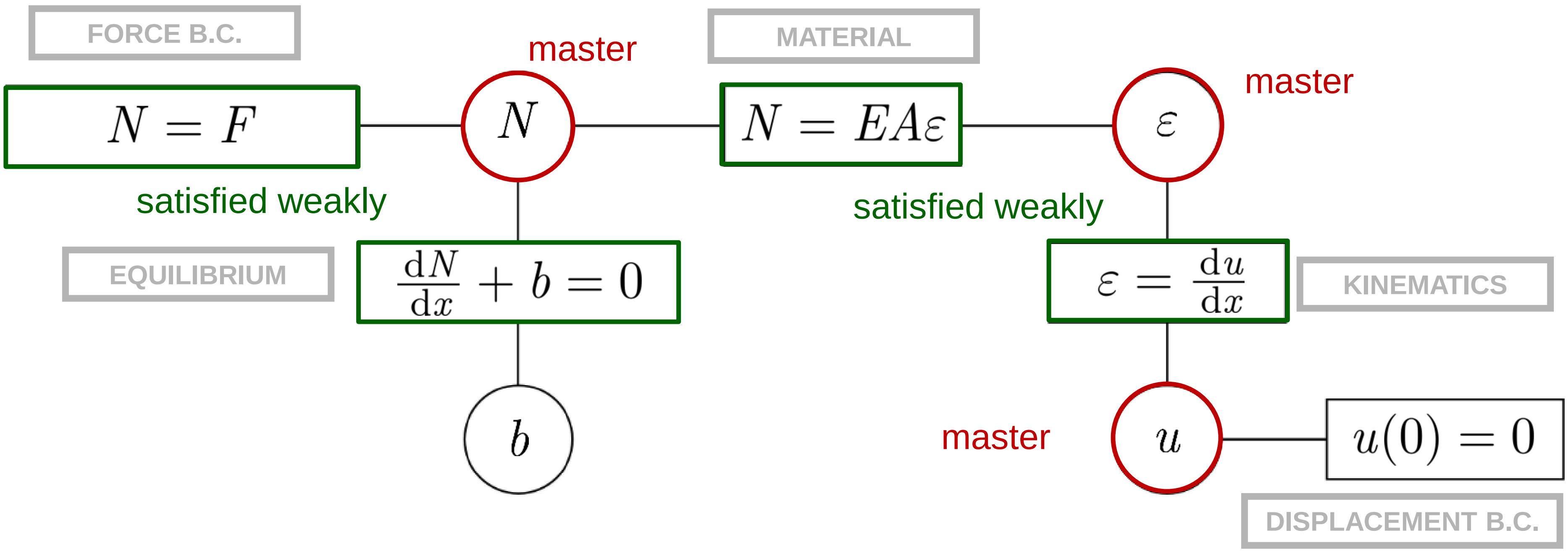
$$\delta\Pi[u,\varepsilon,N] = \int_0^L \left[-\delta u \left(N' + b \right) + \delta N \left(u' - \varepsilon \right) + \delta\varepsilon \left(EA\varepsilon - N \right) \right] \mathrm{d}x$$

$$+ (N - F)\delta u(l) \stackrel{!}{=} 0$$

Interpretation of Hu-Washizu principle with Tonti diagram

$$\delta\Pi[u,\varepsilon,N] = \int_0^L \left[-\delta u\,(N' + b) + \delta N\,(u' - \varepsilon) + \delta\varepsilon\,(EA\varepsilon - N)\right]\,\mathrm{d}x$$

$$+ (N - F)\delta u(l) \overset{!}{=} 0$$



used for enhanced assumed strain finite elements in Lecture 13

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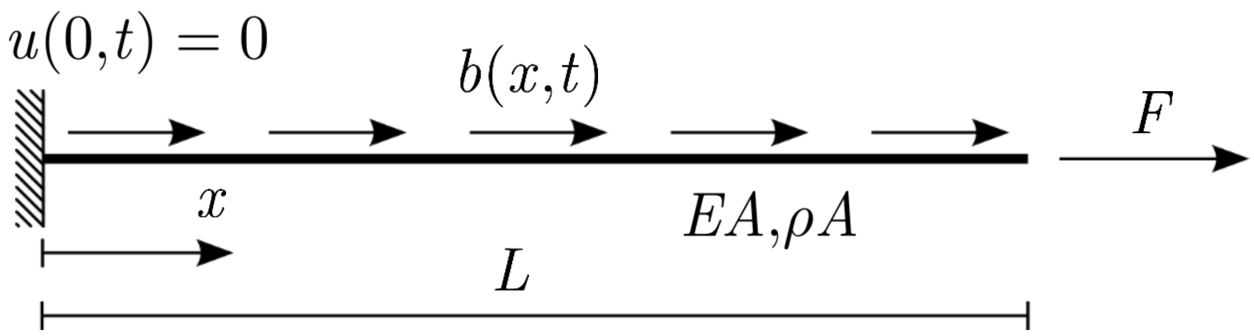
Motivation for variational principles in dynamics

- theoretical basis for

    … consistent mass matrix derivation core of Lecture 8

    … dynamic equation of motion

- one should differentiate between weak forms only in space or in space-time



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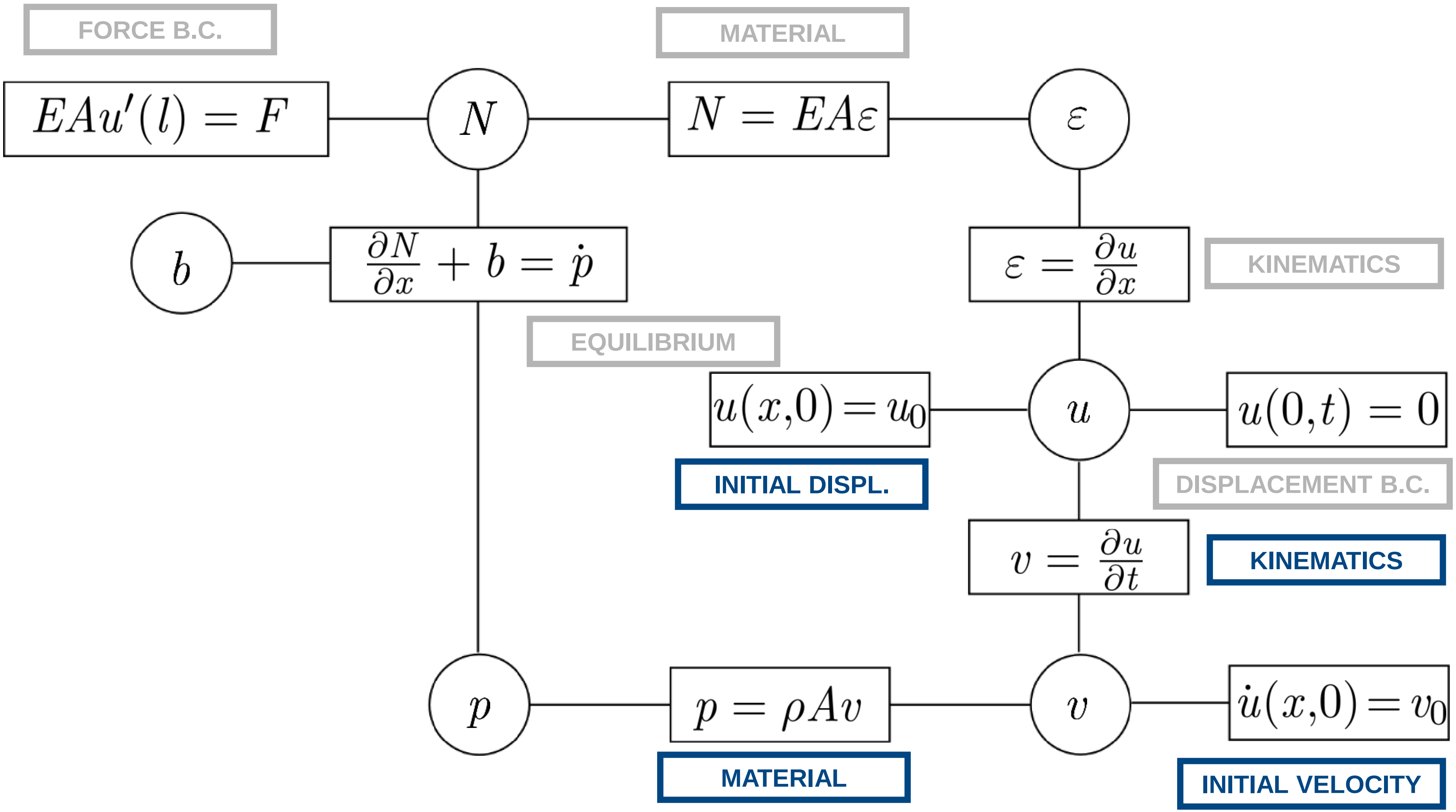
Strong form of dynamic problem via Tonti diagram



Including virtual work of d’Alembert force

$$\delta W = \delta W^{\text{kin}} + \delta W^{\text{int}} + W^{\text{ext}}$$

$$= \int_0^L \underbrace{\rho A \ddot{u}\, \delta u}_{\text{virtual work of d'Alembert force}} \, \mathrm{d}x + \int_0^L EAu' \, \delta u' \, \mathrm{d}x - \int_0^L b\delta u \, \mathrm{d}x - F\delta u(l) \stackrel{!}{=} 0$$

Note: the term of the virual work of d’Alembert force does not possess an potential if only spatial derivative is taken.

Least (stationary) action

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Principle in time w.r.t. to single field u

$$\mathcal{A} = \int_{t_0}^{t_{\text{end}}} (T - \Pi)\,\mathrm{d}t \to \text{stat.} \quad \text{with fixed limit states } u(x,t_0),\, u(x,t_{\text{end}})$$

where kinetic energy is defined with

$$T = \int_0^L \frac{1}{2}\rho A \dot{u}^2\,\mathrm{d}x$$

integrating by parts the first variation of the kinetic energy

$$\delta \int_{t_0}^{t_{\text{end}}} T\,\mathrm{d}t = \delta \int_{t_0}^{t_{\text{end}}} \int_0^L \frac{1}{2}\rho A \dot{u}^2\,\mathrm{d}x\,\mathrm{d}t = \int_{t_0}^{t_{\text{end}}} \int_0^L \rho A \dot{u}\,\delta\dot{u}\,\mathrm{d}x\,\mathrm{d}t =$$

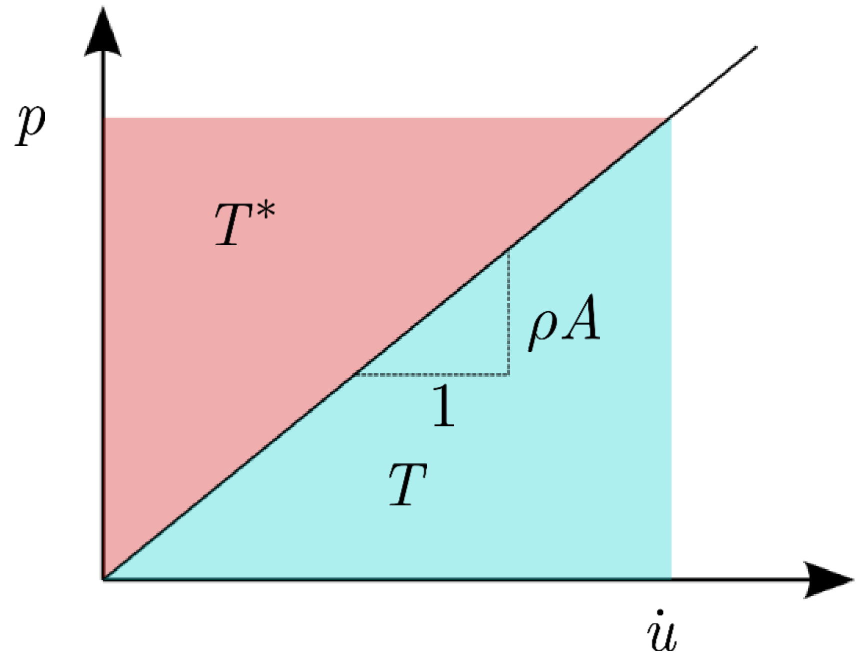
$$= \int_0^L \underbrace{\rho A \dot{u}\,\delta u\,\mathrm{d}x\Big|_{t_0}^{t_{\text{end}}}}_{\text{zero due to fixed limit states}} - \int_{t_0}^{t_{\text{end}}} \int_0^L \underbrace{\rho A \ddot{u}\,\delta u\,\mathrm{d}x}_{\text{virtual work of d'Alembert force}}\,\mathrm{d}t$$

Idea: weakly enforce the compatibility between linear momentum and displacements

Complementary kinetic energy is computed

via Legendre transformation

$$T^* = \int_0^L \left(p\dot{u} - \frac{p^2}{2\rho A} \right) \mathrm{d}x$$



Principle in time w.r.t. to fields $(\dot{u}, p)$

$$\mathcal{H}[u,p] = \int_{t_0}^{t_{\text{end}}} (T^* - \Pi)\, \mathrm{d}t \to \text{stat.}$$

with fixed limit states $u(x,t_0)$, $u(x,t_{\text{end}})$

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Lecture 4: Continuum mechanics III
Variational formulations in statics and dynamics

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Appendix A: Definitions

Principle of minimum potential energy states that among all the admissible displacements which satisfy the prescribed displacement boundary conditions, the actual displacement make the total potential energy stationary.

Virtual work principle states that in the state of equilibrium the virtual work for all the admissible virtual displacements is zero.

Hellinger-Reissner principle states that among the admissible displacements which satisfy the prescribed displacement boundary conditions and the admissible stresses, the actual solution makes the functional given in the principle stationary.

Hu-Washizu principle states that among the admissible displacements which satisfy the prescribed displacement boundary conditions and the admissible stresses and strains, the actual solution makes the functional given in the principle stationary.

Least (stationary) action principle states that among all the admissible displacements which satisfy the prescribed displacement boundary conditions and prescribed displacements on time limits, the actual solution makes the action functional stationary.

Hamilton's principle states that among all the admissible displacements which satisfy the prescribed displacement boundary conditions and prescribed displacements on time limits and admissible linear momentum, the actual solution makes the action functional stationary.

Appendix B: Recommended literature

Washizu, K.. *Variational methods in elasticity and plasticity*. Pergamon press, 3rd edition 1982. (must have mast read, translations are available to Russian, first edition from 1968)

Tabarrok, C., and F. P. Rimrott. *Variational methods and complementary formulations in dynamics*. Vol. 31. Springer Science & Business Media, 2013. (first edition 1994)

Lanczos, Cornelius. *The variational principles of mechanics*. Courier Corporation, 2012. (first edition from 1949)