Spring, 2015

ME 612 – Continuum Mechanics

Lecture 4
Introduction to vectors and tensors

PURDUE ENGINEERING

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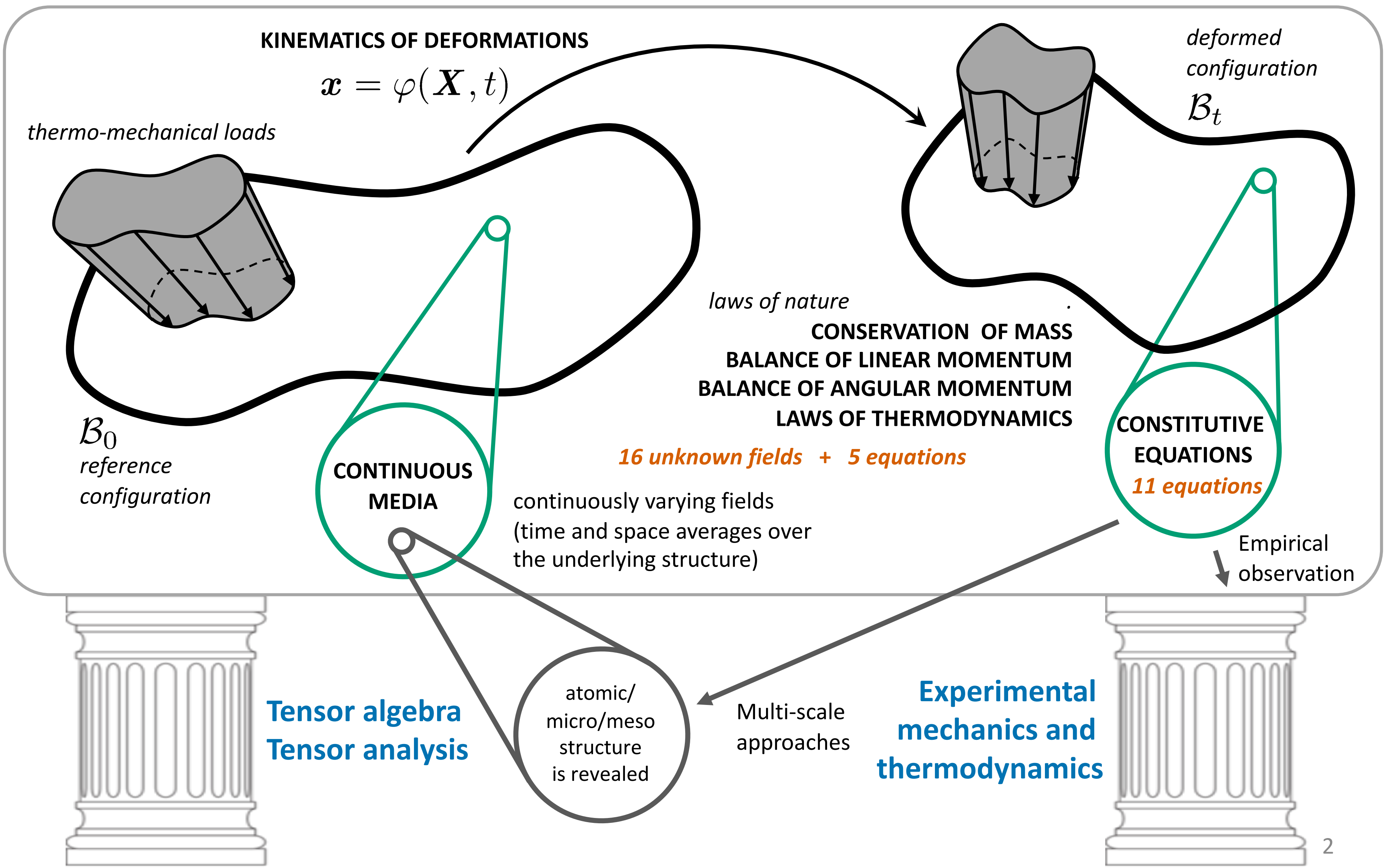
Lecture 4 – Introduction to tensors and vectors



Lecture 4 – Introduction to tensors and vectors

Review (tensor analysis in Cartesian coordinates)

DIY

$$B_{ij} = A_{ji} \Longleftrightarrow \boldsymbol{B} = \boldsymbol{A}^T$$

$$D_{ijk} = A_{ij} v_k \Longleftrightarrow \boldsymbol{D} = \boldsymbol{A} \otimes \boldsymbol{v}$$

$$v_i = A_{ik} u_k \Longleftrightarrow \boldsymbol{v} = \boldsymbol{A}\boldsymbol{u}$$

$$\boldsymbol{A} : \boldsymbol{B} = A_{ij} B_{ij} \qquad \boldsymbol{A} \cdot\cdot \boldsymbol{B} = A_{ij} B_{ji}$$

1-order tensor (vector)

$$v_i \equiv \boldsymbol{v}[\boldsymbol{e}_i] \quad \boldsymbol{v} = v_i \boldsymbol{e}_i$$

2-order tensor

$$A_{ij} \equiv \boldsymbol{T}[\boldsymbol{e}_i, \boldsymbol{e}_j] \qquad \boldsymbol{A} = A_{ij}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)$$

Symmetric, positive-definite 2-order tensor $\lambda_\alpha^{\boldsymbol{S}} \in \mathbb{R}$, $\boldsymbol{\Lambda}_\alpha^{\boldsymbol{S}} \cdot \boldsymbol{\Lambda}_\beta^{\boldsymbol{S}} = \delta_{\alpha\beta}$

$$\boldsymbol{S} = \textstyle\sum_{\alpha=1}^{3} \lambda_\alpha^{\boldsymbol{S}}\, \boldsymbol{\Lambda}_\alpha^{\boldsymbol{S}} \otimes \boldsymbol{\Lambda}_\alpha^{\boldsymbol{S}} = S_{ij}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)$$

$$\sqrt{\boldsymbol{S}} = \textstyle\sum_{\alpha=1}^{3} \sqrt{\lambda_\alpha^{\boldsymbol{S}}}(\boldsymbol{\Lambda}_\alpha^{\boldsymbol{S}} \otimes \boldsymbol{\Lambda}_\alpha^{\boldsymbol{S}})$$

Tensor analysis

Tensor fields

- In continuum mechanics we encounter tensors as spatially and temporally varying fields over a give domain:

$$\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{x}, t) \ , \qquad \boldsymbol{x} = x_i \boldsymbol{e}_i \in \Omega(t)$$

- Partial differentiation of a tensor field (and the comma notation)

$$\frac{\partial s(\boldsymbol{x})}{\partial x_i} \equiv s_{,i}$$

$$\frac{\partial v_i(\boldsymbol{x})}{\partial x_j} \equiv v_{i,j}$$

$$\frac{\partial T_{ij}(\boldsymbol{x})}{\partial x_k} \equiv T_{ij,k}$$

DIY

Comma notation and summation convention

$$(T_{ij}(\boldsymbol{x}) x_j)_{,i} =$$

Tensor analysis (in Cartesian coordinates)

Tensor fields

- Gradient

$$\nabla s = \frac{\partial s(\boldsymbol{x})}{\partial x_i}\boldsymbol{e}_i = s_{,i}\boldsymbol{e}_i = [\nabla s]_i \boldsymbol{e}_i$$

$$\nabla \boldsymbol{B} = \frac{\partial \boldsymbol{B}(\boldsymbol{x})}{\partial x_i} \otimes \boldsymbol{e}_i$$

From tensor of rank m
to tensor of rank m+1

DIY

$$\text{Given } s(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}\cdot\boldsymbol{x} = A_{ij}x_j x_i$$
$$\nabla s = (A_{ij}\delta_{jk}x_i + A_{ij}x_j\delta_{ik})\boldsymbol{e}_k = (A_{ik}x_i + A_{kj}x_j)\boldsymbol{e}_k = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{A}^T\boldsymbol{x}$$

$$\nabla \boldsymbol{T} = \frac{\partial \boldsymbol{T}}{\partial x_k} \otimes \boldsymbol{e}_k = [\nabla \boldsymbol{T}]_{ijk}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k =$$

Tensor analysis (in Cartesian coordinates)

Tensor fields

- Curl

$$\mathrm{curl}\boldsymbol{B} \equiv -\frac{\partial \boldsymbol{B}(\boldsymbol{x})}{\partial x_i} \times \boldsymbol{e}_i$$

DIY

$$\mathrm{curl}\boldsymbol{v} = -\frac{\partial v_i \boldsymbol{e}_i}{\partial x_j} \times \boldsymbol{e}_j = -\frac{\partial v_i}{\partial x_j}(\boldsymbol{e}_i \times \boldsymbol{e}_j) = -\frac{\partial v_i}{\partial x_j}\epsilon_{ijk}\boldsymbol{e}_k = \frac{\partial v_i}{\partial x_j}\epsilon_{kji}\boldsymbol{e}_k =$$

$$\epsilon_{kji} v_{i,j} \boldsymbol{e}_k = [\mathrm{curl}\boldsymbol{v}]_k \boldsymbol{e}_k$$

Tensor analysis (in Cartesian coordinates)

Tensor fields

- Divergence

$$\operatorname{div} \boldsymbol{v} \equiv \frac{\partial \boldsymbol{v}(\boldsymbol{x})}{\partial x_i} \cdot \boldsymbol{e}_i$$

$$\operatorname{div} \boldsymbol{B} \equiv \frac{\partial \boldsymbol{B}(\boldsymbol{x})}{\partial x_i} \boldsymbol{e}_i$$

From tensor of rank m
to tensor of rank m-1

DIY

$$\operatorname{div} \boldsymbol{v} = \frac{\partial (v_i \boldsymbol{e}_i)}{\partial x_j} \cdot \boldsymbol{e}_j = \frac{\partial v_i}{\partial x_j} (\boldsymbol{e}_i \cdot \boldsymbol{e}_j) = \frac{\partial v_i}{\partial x_j} \delta_{ij} = \frac{\partial v_i}{\partial x_i} = v_{i,i}$$

$$\operatorname{div} \boldsymbol{T} = \frac{\partial \boldsymbol{T}}{\partial x_k} \boldsymbol{e}_k = \frac{\partial [T_{ij} (\boldsymbol{e}_i \otimes \boldsymbol{e}_j)]}{\partial x_k} \boldsymbol{e}_k = \frac{\partial T_{ij}}{\partial x_k} (\boldsymbol{e}_i \otimes \boldsymbol{e}_j) \boldsymbol{e}_k = \frac{\partial T_{ij}}{\partial x_k} \boldsymbol{e}_i \delta_{jk} = \frac{\partial T_{ij}}{\partial x_j} \boldsymbol{e}_i$$

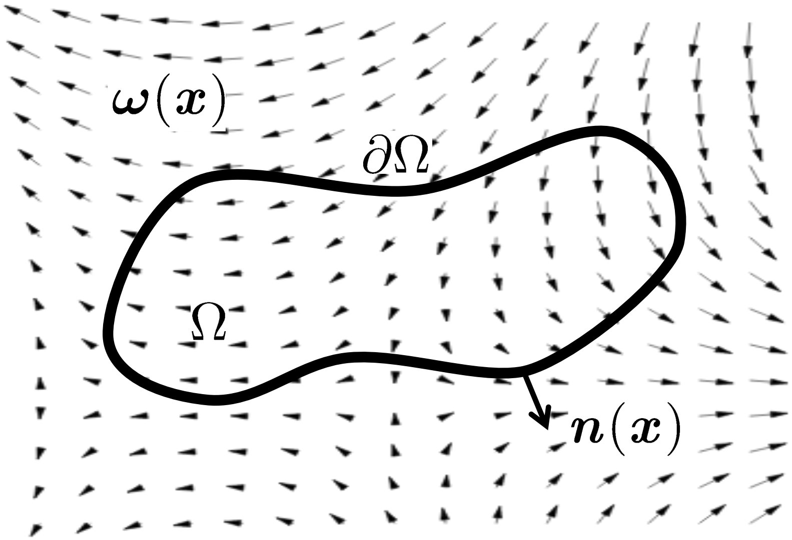
Tensor analysis (in Cartesian coordinates)

Divergence theorem

- Given a vector field $\boldsymbol{\omega}(\boldsymbol{x})$

$$\int_{\partial\Omega} \boldsymbol{\omega} \cdot \boldsymbol{n} \; dA = \int_{\Omega} \mathrm{div}\boldsymbol{\omega} \; dV$$

DIY Cartesian components



- Given a tensor field $\boldsymbol{T}(\boldsymbol{x})$

$$\int_{\partial\Omega} \boldsymbol{T}\boldsymbol{n} \; dA = \int_{\Omega} \mathrm{div}\,\boldsymbol{T} \; dV$$

DIY Cartesian components

Curvilinear coordinate systems

Curvilinear coordinate systems

- Two set of basis vectors at each position in space

  $\{\boldsymbol{g}_i\}$ tangent vectors

  $\{\boldsymbol{g}^i\}$ reciprocal vectors

  defined through $\boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta^i_j$

- Contravariant components

$$\boldsymbol{a} = a^i \boldsymbol{g}_i$$

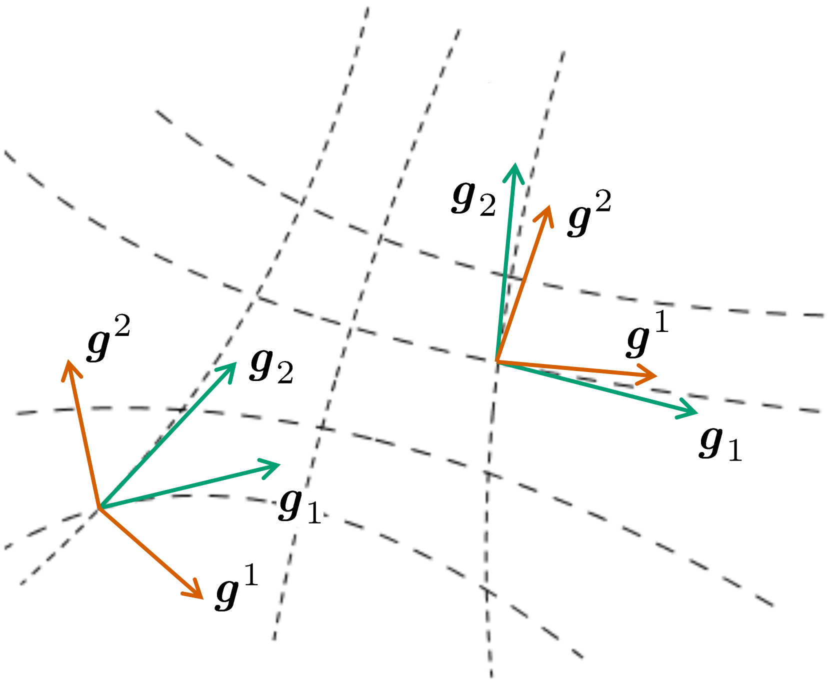
- Covariant components

$$\boldsymbol{a} = a_j \boldsymbol{g}^j$$

- Connection between components

$$a^k = g^{jk} a_j$$
$$a_k = g_{ik} a^i$$

with
$$g^{jk} = \boldsymbol{g}^j \cdot \boldsymbol{g}^k$$
$$g_{ik} = \boldsymbol{g}_i \cdot \boldsymbol{g}_k$$



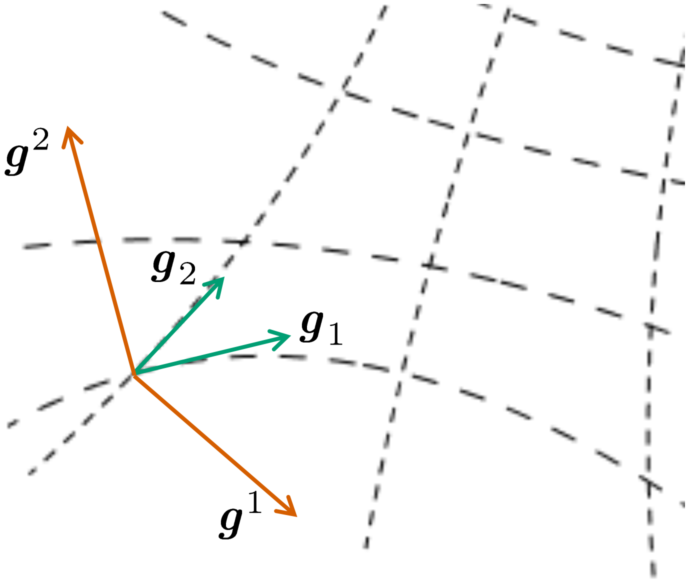
Curvilinear coordinate systems

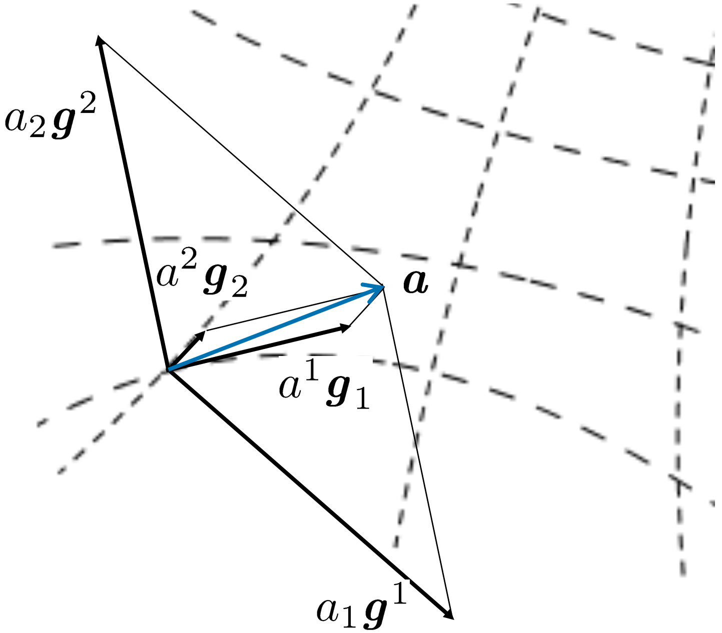
Curvilinear coordinate systems

- Two set of basis vectors at each position in space
  - $\{\boldsymbol{g}_i\}$ tangent vectors
  - $\{\boldsymbol{g}^i\}$ reciprocal vectors

  defined through $\boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta^i_j$

- Covariant and contravariant components: $\boldsymbol{a} = a^i \boldsymbol{g}_i = a_i \boldsymbol{g}^i$





Curvilinear coordinate systems

- Two set of basis vectors
  - $\{\boldsymbol{g}_i\}$ tangent vectors
  - $\{\boldsymbol{g}^i\}$ reciprocal vectors

  defined through
  $$\boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta^i_j$$
  $$\boldsymbol{g}^j \cdot \boldsymbol{g}^k = g^{jk}$$
  $$\boldsymbol{g}_i \cdot \boldsymbol{g}_k = g_{ik}$$

- Covariant, contravariant and mixed components:

$$\boldsymbol{T} = T^{ij}(\boldsymbol{g}_i \otimes \boldsymbol{g}_j) = T_{ij}(\boldsymbol{g}^i \otimes \boldsymbol{g}^j) = T_i^{\cdot j}(\boldsymbol{g}^i \otimes \boldsymbol{g}_j) = T^i_{\cdot j}(\boldsymbol{g}_i \otimes \boldsymbol{g}^j)$$

- Connection between components:

$$T^{mn} = g^{mi}\, T_{ij}\, g^{jn} = g^{mi}\, T_i^{\cdot n} = T^m_{\cdot j} g^{jn}$$

- Metric tensor (it is the identity of a generalized coordinate system):

$\det \boldsymbol{g} = \det[g_{ij}] = g$ (not equal to 1 in general) and symmetric ($g_{ij} = g_{ji}$)

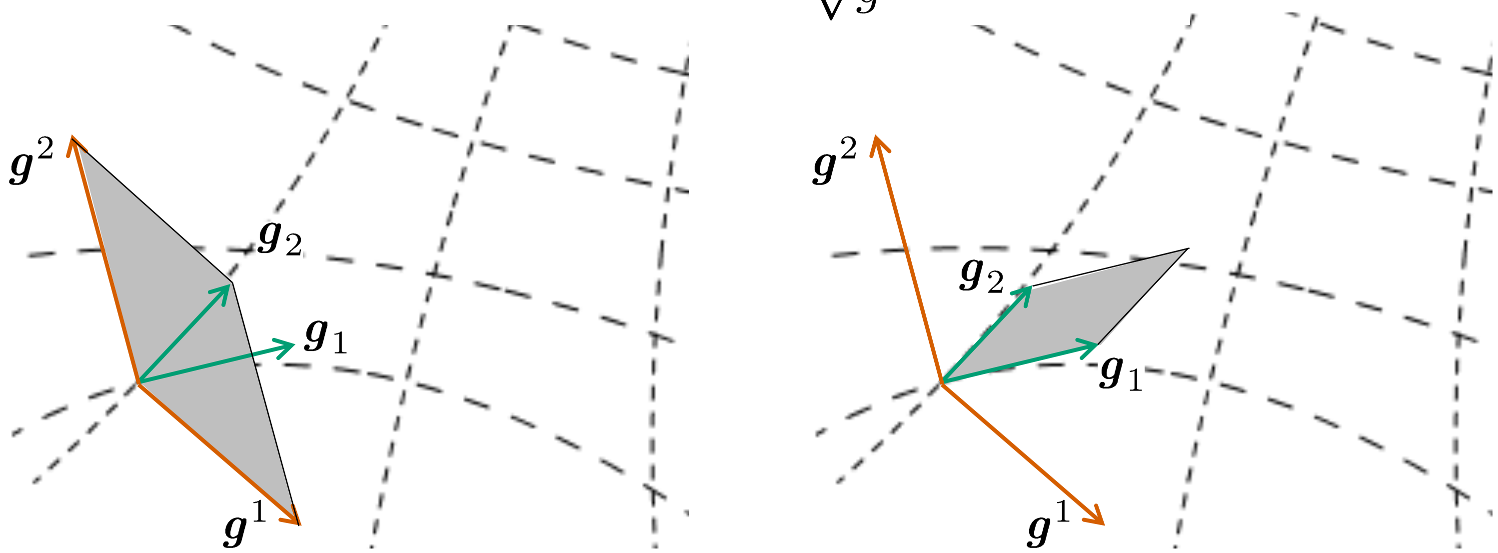
Tensor algebra (in curvilinear coordinates)

Inner product - Dot product

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \boldsymbol{a} \cdot \boldsymbol{b} = (a^i \boldsymbol{g}_i) \cdot (b^j \boldsymbol{g}_j) = a^i b^j (\boldsymbol{g}_i \cdot \boldsymbol{g}_j)$$
$$= a^i b^j g_{ij} = a_i b_j g^{ij} = a_i b^i = a^i b_i$$

Cross product

$$\boldsymbol{a} \times \boldsymbol{b} = \sqrt{g}\ \epsilon_{ijk}\ a^i\ b^j\ \boldsymbol{g}^k = \frac{1}{\sqrt{g}}\ \epsilon^{ijk}\ a_i\ b_j\ \boldsymbol{g}_k$$



Shaded areas are related by the determinant of the metric tensor (and are different in general).

Tensor algebra (in curvilinear coordinates)

Transpose

$$\boldsymbol{B}\boldsymbol{v} = \boldsymbol{v}\boldsymbol{A} \iff \boldsymbol{B} = \boldsymbol{A}^T \iff B_{ij} = A_{ji} \iff B^{ij} = A^{ji}$$
$$\iff B^{i}_{\cdot j} = A_{j}^{\cdot i} \quad (\neq B_{j}^{\cdot i})$$

Contracted multiplication

$$\boldsymbol{C}[\boldsymbol{a}, \boldsymbol{b}] = \mathrm{Cont}_{23}(\boldsymbol{A}[\boldsymbol{a}, \boldsymbol{c}]\boldsymbol{B}[\boldsymbol{d}, \boldsymbol{b}])$$

$$\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B} \iff C_{ij} = A_{ik}B^{k}_{\cdot j} = A_{i}^{\cdot k}B_{kj}$$

Scalar contraction

e.g., $\boldsymbol{A} \cdot \cdot \boldsymbol{B} \equiv \mathrm{Cont}_{12}(\mathrm{Cont}_{23}(\boldsymbol{A}[\boldsymbol{a}, \boldsymbol{c}]\boldsymbol{B}[\boldsymbol{d}, \boldsymbol{b}]))$

$$\boldsymbol{A} : \boldsymbol{B} = A_{ij}B^{ij} = A^{ij}B_{ij} = A^{i}_{\cdot j}B_{i}^{\cdot j} = A_{i}^{\cdot j}B^{i}_{\cdot j}$$

$$\boldsymbol{A} \cdot \cdot \boldsymbol{B} = A_{ij}B^{ji} = A^{ij}B_{ji} = A_{i}^{\cdot j}B_{j}^{\cdot i} = A^{i}_{\cdot j}B^{j}_{\cdot i}$$

Curvilinear coordinate systems

Construction of tangent basis $\{\boldsymbol{g}_i\}$

- Tangent vectors describe how the point in space changes as the coordinates change

$$\boldsymbol{g}_i \equiv \frac{\partial \boldsymbol{x}}{\partial \theta^i}$$ where θ^i are the curvilinear coordinates

$$\boldsymbol{g}_i \equiv \frac{\partial (x^\alpha \boldsymbol{e}_\alpha)}{\partial \theta^i} = \frac{\partial x^\alpha}{\partial \theta^i} \boldsymbol{e}_\alpha$$

where x^α and $\{\boldsymbol{e}^\alpha\}$ are the Cartesian coordinates and basis, resp.
[usually $x^\alpha = x^\alpha(\theta^1, \theta^2, \theta^3)$ are known]

Inversely:

$$\boldsymbol{e}_\alpha \equiv \frac{\partial \boldsymbol{x}}{\partial x^\alpha} = \frac{\partial \boldsymbol{x}}{\partial \theta^j} \frac{\partial \theta^j}{\partial x^\alpha} = \frac{\partial \theta^j}{\partial x^\alpha} \boldsymbol{g}_j$$

with $$\left[\frac{\partial \theta^j}{\partial x^\alpha}\right] = \left[\frac{\partial x^\alpha}{\partial \theta^j}\right]^{-1}$$

- Derivatives of tangent vectors:

$$\frac{\partial \boldsymbol{g}_k}{\partial \theta^i} = \frac{\partial}{\partial \theta^i}\left[\frac{\partial x^\alpha}{\partial \theta^k} \boldsymbol{e}_\alpha\right] = \frac{\partial^2 x^\alpha}{\partial \theta^i \partial \theta^k} \boldsymbol{e}_\alpha = \frac{\partial^2 x^\alpha}{\partial \theta^i \partial \theta^k} \frac{\partial \theta^m}{\partial x^\alpha} \boldsymbol{g}_m = \Gamma^m_{ki} \boldsymbol{g}_m$$

Christoffel symbol of the second kind

Curvilinear coordinate systems

Construction of metric tensor g_{ij}

- By definition

$$\boldsymbol{g}_i \cdot \boldsymbol{g}_j = g_{ij} = \sum_{\alpha=1}^{n_\mathrm{d}} \frac{\partial x^\alpha}{\partial \theta^i} \frac{\partial x^\alpha}{\partial \theta^j}$$

$$\det \boldsymbol{g} = \det[g_{ij}] = g$$

Note: $\sqrt{g} = \det[\frac{\partial x^\alpha}{\partial \theta^i}] = J$

- The metric of the space is the following scalar invariant quadratic form (i.e., the metric is the elementary line element or arc length):

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{r} \\ &= (\mathrm{d}x^\alpha \boldsymbol{e}_\alpha) \cdot (\mathrm{d}x^\beta \boldsymbol{e}_\beta) = \mathrm{d}x^\alpha \ \delta_{\alpha\beta} \ \mathrm{d}x^\beta = \mathrm{d}x^\alpha \mathrm{d}x^\alpha = \\ &= (\mathrm{d}\theta^i \boldsymbol{g}_i) \cdot (\mathrm{d}\theta^j \boldsymbol{g}_j) = \mathrm{d}\theta^i \ g_{ij} \ \mathrm{d}\theta^j \end{aligned}$$

.... elementary volume element: $\mathrm{d}V = \sqrt{g} \ \mathrm{d}\theta^1 \ \mathrm{d}\theta^2 \ \mathrm{d}\theta^3$

Curvilinear coordinate systems

Construction of reciprocal basis $\{\boldsymbol{g}^i\}$

- Reciprocal vectors describe the coordinates change as the point in space changes

$$\boldsymbol{g}^1 = \frac{\boldsymbol{g}_2 \times \boldsymbol{g}_3}{\boldsymbol{g}_1 \cdot (\boldsymbol{g}_2 \times \boldsymbol{g}_3)} \qquad \boldsymbol{g}^2 = \frac{\boldsymbol{g}_3 \times \boldsymbol{g}_1}{\boldsymbol{g}_1 \cdot (\boldsymbol{g}_2 \times \boldsymbol{g}_3)} \qquad \boldsymbol{g}^3 = \frac{\boldsymbol{g}_1 \times \boldsymbol{g}_2}{\boldsymbol{g}_1 \cdot (\boldsymbol{g}_2 \times \boldsymbol{g}_3)}$$

Notice that $\boldsymbol{g}_1 \cdot (\boldsymbol{g}_2 \times \boldsymbol{g}_3) = \sqrt{g}$

- Derivatives of reciprocal vectors:

$$\frac{\partial \boldsymbol{g}^k}{\partial \theta^i} = -\Gamma^k_{im} \boldsymbol{g}^m$$

Christoffel symbol of the second kind

Note 1: $\frac{\partial \boldsymbol{g}^r \cdot \boldsymbol{g}_k}{\partial \theta^i} = \frac{\partial \boldsymbol{g}^r}{\partial \theta^i} \cdot \boldsymbol{g}_k + \boldsymbol{g}^r \cdot \frac{\partial \boldsymbol{g}_k}{\partial \theta^i} = \frac{\partial \delta^r_k}{\partial \theta^i} = 0$

Note 2: Christoffel symbols are zero in Cartesian coordinates

Tensor analysis (in curvilinear coordinates)

Tensor fields

- Gradient

$$
\begin{aligned}
\mathrm{grad}\,\boldsymbol{v} = \nabla \boldsymbol{v} &= \frac{\partial \boldsymbol{v}}{\partial \theta^i} \otimes \boldsymbol{g}^i = \frac{\partial (v_k \boldsymbol{g}^k)}{\partial \theta^i} \otimes \boldsymbol{g}^i \\
&= \frac{\partial v_k}{\partial \theta^i} (\boldsymbol{g}^k \otimes \boldsymbol{g}^i) + v_k (\frac{\partial \boldsymbol{g}^k}{\partial \theta^i} \otimes \boldsymbol{g}^i) \\
&= \frac{\partial v_k}{\partial \theta^i} (\boldsymbol{g}^k \otimes \boldsymbol{g}^i) + v_k (-\Gamma^k_{im} \boldsymbol{g}^m \otimes \boldsymbol{g}^i) \\
&= \left[\frac{\partial v_k}{\partial \theta^i} - \Gamma^j_{ik} v_j \right] (\boldsymbol{g}^k \otimes \boldsymbol{g}^i) \\
&= v_{k,i} (\boldsymbol{g}^k \otimes \boldsymbol{g}^i) = v^k_{,i} (\boldsymbol{g}_k \otimes \boldsymbol{g}^i)
\end{aligned}
$$

Covariant derivative of a covariant component $$v_{k,i} = \frac{\partial v_k}{\partial \theta^i} - \Gamma^j_{ik} v_j$$

Covariant derivate of a contravariant component $$v^k_{,i} = \frac{\partial v^k}{\partial \theta^i} + \Gamma^k_{ji} v^j$$

Tensor analysis (in curvilinear coordinates)

Tensor fields

- Divergence

$$
\begin{aligned}
\operatorname{div}\boldsymbol{T} &= \frac{\partial \boldsymbol{T}}{\partial \theta^k}\boldsymbol{g}^k = \frac{\partial T_{ij}(\boldsymbol{g}^i \otimes \boldsymbol{g}^j)}{\partial \theta^k}\boldsymbol{g}^k \\
&= \frac{\partial T_{ij}}{\partial \theta^k}(\boldsymbol{g}^i \otimes \boldsymbol{g}^j)\boldsymbol{g}^k + T_{ij}(\frac{\partial \boldsymbol{g}^i}{\partial \theta^k} \otimes \boldsymbol{g}^j)\boldsymbol{g}^k + T_{ij}(\boldsymbol{g}^i \otimes \frac{\partial \boldsymbol{g}^j}{\partial \theta^k})\boldsymbol{g}^k \\
&= g^{jk}\left[\frac{\partial T_{ij}}{\partial \theta^k} - \Gamma^m_{ik}T_{mj} - \Gamma^m_{jk}T_{im}\right]\boldsymbol{g}^i = g^{jk}T_{ij,k}\ \boldsymbol{g}^i \\
&= \delta^k_j\left[\frac{\partial T^{ij}}{\partial \theta^k} + \Gamma^i_{mk}T^{mj} + \Gamma^j_{mk}T^{im}\right]\boldsymbol{g}_i = \delta^k_j T^{ij}_{\ ,k}\ \boldsymbol{g}_i = T^{ik}_{\ ,k}\ \boldsymbol{g}_i
\end{aligned}
$$

Orthogonal curvilinear coordinates

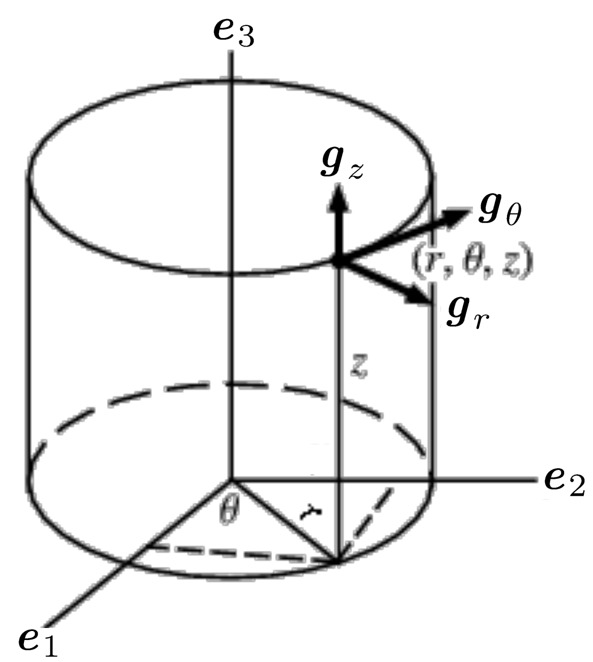
Polar cylindrical coordinates

$$(\theta^1, \theta^2, \theta^3) = (r, \theta, z)$$

$$x^1 = r\cos\theta$$
$$x^2 = r\sin\theta$$
$$x^3 = z$$

$$\boldsymbol{g}_i \equiv \frac{\partial(x^\alpha \boldsymbol{e}_\alpha)}{\partial\theta^i} = \frac{\partial x^\alpha}{\partial\theta^i}\boldsymbol{e}_\alpha$$

Basis vectors are not unit vectors.



$$\Gamma^m_{ki} = \frac{\partial^2 x^\alpha}{\partial\theta^i\partial\theta^k}\frac{\partial\theta^m}{\partial x^\alpha}$$

Most of the Christoffel symbols are zero.

$$\left[\frac{\partial\theta^j}{\partial x^\alpha}\right] = \left[\frac{\partial x^\alpha}{\partial\theta^j}\right]^{-1}$$

$$\boldsymbol{g}_i \cdot \boldsymbol{g}_j = g_{ij} = \sum_{\alpha=1}^{n_\mathrm{d}} \frac{\partial x^\alpha}{\partial\theta^i}\frac{\partial x^\alpha}{\partial\theta^j}$$

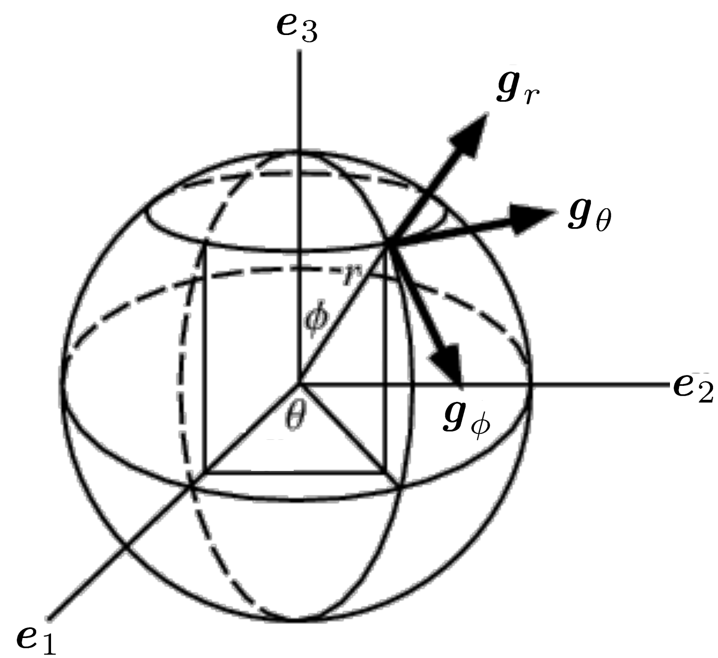
Only diagonal components are non-zero.

Orthogonal curvilinear coordinates

Spherical coordinates

$$(\theta^1, \theta^2, \theta^3) = (r, \phi, \theta)$$

$$x^1 = r \sin\phi \cos\theta$$
$$x^2 = r \sin\phi \sin\theta$$
$$x^3 = r \cos\phi$$



$$\boldsymbol{g}_i \equiv \frac{\partial(x^\alpha \boldsymbol{e}_\alpha)}{\partial \theta^i} = \frac{\partial x^\alpha}{\partial \theta^i} \boldsymbol{e}_\alpha$$

Basis vectors are not unit vectors.

$$\Gamma^m_{ki} = \frac{\partial^2 x^\alpha}{\partial \theta^i \partial \theta^k} \frac{\partial \theta^m}{\partial x^\alpha}$$

Most of the Christoffel symbols are zero.

$$\left[\frac{\partial \theta^j}{\partial x^\alpha}\right] = \left[\frac{\partial x^\alpha}{\partial \theta^j}\right]^{-1} \qquad \boldsymbol{g}_i \cdot \boldsymbol{g}_j = g_{ij} = \sum_{\alpha=1}^{n_\mathrm{d}} \frac{\partial x^\alpha}{\partial \theta^i} \frac{\partial x^\alpha}{\partial \theta^j}$$

Only diagonal components are non-zero.

Lecture 4 – Introduction to tensors and vectors

Any questions?