Lecture 5: Choice under uncertainty

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Introduction

- Agents in financial markets operate in a world in which they make choices under risk and uncertainty. Portfolio managers, for example, make investment decisions in which they take risks and expect rewards, based on their own expectations and preferences.
- The theory of how choices under risk and uncertainty are made was introduced by John von Neumann and Oskar Morgenstern in 1944 in their book *Theory of Games and Economic Behavior*. They gave an explicit representation of investor's perferences in terms of an investor's utility function.

Introduction

- If *no uncertainty* is present, the utility function can be interpreted as a mapping between the available alternatives and real numbers indicating the “relative happiness” the investor gains from a particular alternative. If an individual prefers good “A” to good “B”, then the utility of “A” is higher than the utility of “B”. Thus, the utility function characterizes individual’s preferences.
- Von Neumann and Morgenstern showed that if there is *uncertainty*, then it is the expected utility which characterizes the preferences. The expected utility of an uncertain prospect, often called a *lottery*, is defined as the probability weighted average of the utilities of the simple outcomes.

Introduction

- The expected utility theory defines the lotteries by means of the elementary outcomes and their probability distribution.
- In this sense, the lotteries can also be interpreted as random variables which can be discrete, continuous, or mixed, and the preference relation is defined on the probability distributions of the random variables.
- The probability distributions are regarded as *objective*; that is, the theory is consistent with the classical view that, in some sense, the randomness is inherent in Nature and all individuals observe the same probability distribution of a given random variable.

Introduction

- In 1954, a new theory of decision making under uncertainty appeared, developed by Leonard Savage in his book *The Foundations of Statistics*.
- He showed that individual's preferences in the presence of uncertainty can be characterized by an expected utility calculated as a weighted average of the utilities of the simple outcomes and the weights are the subjective probabilities of the outcomes.
- The subjective probabilities and the utility function arise as a pair from the individual's preferences. Thus, it is possible to modify the utility function and to obtain another subjective probability measure so that the resulting expected utility also characterizes the individual's preferences.

$\Rightarrow$ In some aspects, Savage's approach is considered to be more general than the von Neumann-Morgenstern theory.

Introduction

- Another mainstream utility theory describing choices under uncertainty is the state-preference approach of Kenneth Arrow and Gérard Debreu.
- The basic principle is that the choice under uncertainty is reduced to a choice problem without uncertainty by considering state-contingent bundles of commodities. The agent's preferences are defined over bundles in all states-of-the-world and the notion of randomness is almost ignored.
- This construction is quite different from the theories of von Neumann-Morgenstern and Savage because preferences are not defined over lotteries.
- The Arrow-Debreu approach is applied in general equilibrium theories where the payoffs are not measured in monetary amounts but are actual bundles of goods.

Introduction

- In 1992, a new version of the expected utility theory was advanced by Amos Tversky and Daniel Kahneman — the cumulative prospect theory. Instead of utility function, they introduce a value function which measures the payoff relative to a reference point.
- They also introduce a weighting function which changes the cumulative probabilities of the prospect.
- The cumulative prospect theory is a *positive theory*, explaining individual's behavior, in contrast to the expected utility theory which is a *normative theory* prescribing the rational behavior of agents.

Introduction

- It is possible to characterize classes of investors by the shape of their utility function, such as non-satiable investors, risk-averse investors, and so on.
- If all investors of a given class prefer one prospect from another, we say that this prospect *dominates* the other. In this fashion, the first-, second-, and the third-order stochastic dominance relations arise.
- The stochastic dominance rules characterize the efficient set of a given class of investors.

- We start with the well-known St. Petersburg Paradox which is historically the first application of the concept of the expected utility function. As a next step, we describe the essential result of von Neumann-Morgenstern characterization of the preferences of individuals.
- St. Petersburg Paradox is a lottery game presented to Daniel Bernoulli by his cousin Nicolas Bernoulli in 1713. Daniel Bernoulli published the solution in 1734 but another Swiss mathematician, Gabriel Cramer, had already discovered parts of the solution in 1728.

The lottery goes as follows. A fair coin is tossed until a head appears.

- If the head appears on the first toss, the payoff is \$1.
- If it appears on the second toss, then the payoff is \$2. After that, the payoff increases sharply.
- If the head appears on the third toss, the payoff is \$4, on the fourth toss it is \$8, etc.
- Generally, if the head appears on the n-th toss, the payoff is 2^{n-1} dollars.

- At that time, it was commonly accepted that the fair value of a lottery should be computed as the expected value of the payoff.
- Since a fair coin is tossed, the probability of having a head on the n-th toss equals $1/2^n$,

$$P(\text{“First head on trial } n\text{”}) = P(\text{“Tail on trial 1”}) \cdot P(\text{“Tail on trial 2”}) \cdot \ldots \cdot P(\text{“Tail on trial } n\text{-1”}) \cdot P(\text{“Head on trial } n\text{”}) = \frac{1}{2^n}$$

Therefore, the expected payoff is calculated as

$$\begin{aligned}\text{Expected Payoff} &= 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + \ldots + 2^{n-1} \cdot \frac{1}{2^n} + \ldots \\ &= \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} + \ldots \\ &= \infty.\end{aligned}$$

- Because the expected payoff is infinite, people should be willing to participate in the game no matter how large the price of the ticket.
- Nevertheless, in reality very few people would be ready to pay as much as $100 for a ticket.
- In order to explain the paradox, Daniel Bernoulli suggested that instead of the actual payoff, the utility of the payoff should be considered. Thus, the fair value is calculated by

$$\begin{aligned}\text{Fair Value} &= u(1)\cdot\frac{1}{2} + u(2)\cdot\frac{1}{4} + \ldots + u(2^{n-1})\cdot\frac{1}{2^n} + \ldots \\ &= \sum_{k=1}^{\infty}\frac{u(2^{k-1})}{2^n}\end{aligned}$$

where the function $u(x)$ is the utility function. The value is determined by the utility an individual gains.

- Daniel Bernoulli considered utility functions with diminishing marginal utility; that is, the utility gained from one extra dollar diminishes with the sum of money one has.
- In the solution of the paradox, Bernoulli considered logarithmic utility function, $u(x) = \log x$, and showed that the fair value of the lottery equals approximately \$2.
- The solutions of Bernoulli and Cramer are not completely satisfactory because the lottery can be changed in such a way that the fair value becomes infinite even with their choice of utility functions.

The von Neumann-Morgenstern expected utility theory

- The St. Petersburg Paradox shows that the naive approach to calculate the fair value of a lottery can lead to counter-intuitive results.
- A deeper analysis shows that it is the utility gained by an individual which should be considered and not the monetary value of the outcomes.
- The theory of von Neumann-Morgenstern gives a numerical representation of individual's preferences over lotteries.
- The numerical representation is obtained through the expected utility and it turns out that this is the only possible way of obtaining a numerical representation.

Probability	1/2	1/4	1/8	...	$1/2^n$	...
Payoff	1	2	4	...	2^{n-1}	...

Table 1. The lottery in the St. Petersburg Paradox.

- Technically, a lottery is a probability distribution defined on the set of payoffs. In fact, the lottery in the St. Petersburg Paradox is given in Table 15.
- Generally, lotteries can be discrete, continuous and mixed. Table 15 provides an example of a discrete lottery.
- In the continuous case, the lottery is described by the cumulative distribution function (c.d.f.) of the random payoff. Any portfolio of common stocks, for example, can be regarded as a continuous lottery defined by the c.d.f. of the portfolio payoff.
- We use the notation P_X to denote the lottery (or the probability distribution), the payoff of which is the random variable X. The particular values of the random payoff (the outcomes) we denote by lower-case letters, x, and the probability that the payoff is below x is denoted by $P(X \leq x) = F_X(x)$, which is in fact the c.d.f.

Denote by $\mathcal{X}$ the set of all lotteries. Any element of $\mathcal{X}$ is considered a possible choice of an economic agent. If $P_X \in \mathcal{X}$ and $P_Y \in \mathcal{X}$, then there are the following possible cases:

- The economic agent may prefer P_X to P_Y or be indifferent between them, denoted by $P_X \succeq P_Y$.
- The economic agent may prefer P_Y to P_X or be indifferent between them, denoted by $P_Y \succeq P_X$.
- If both relations hold, $P_Y \succeq P_X$ and $P_X \succeq P_Y$, then we say that the economic agent is indifferent between the two choices, $P_X \sim P_Y$.

Sometimes, for notational convenience, we will use $X \succeq Y$ instead of $P_X \succeq P_Y$ without changing the assumption that we are comparing the probability distributions.

The von Neumann-Morgenstern expected utility theory

- A preference relation or a preference order of an economic agent on the set of all lotteries $\mathcal{X}$ is a relation concerning the ordering of the elements of $\mathcal{X}$, which satisfies certain axioms called the axioms of choice[1].
- A numerical representation of a preference order is a real-valued function U defined on the set of lotteries, $U : \mathcal{X} \to \mathbb{R}$, such that $P_X \succeq P_Y$ if and only if $U(P_X) \geq U(P_Y)$,

$$P_X \succeq P_Y \qquad \Longleftrightarrow \qquad U(P_X) \geq U(P_Y).$$

- Thus, the numerical representation characterizes the preference order and allows to compare real numbers.

[1] A detailed description of the axioms of choice is provided in the appendix to this lecture.

The von Neumann-Morgenstern expected utility theory

- The von Neumann-Morgenstern theory states that if the preference order satisfies certain technical continuity conditions, then the numerical representation U has the form

$$U(P_X) = \int_{\mathbb{R}} u(x) dF_X(x) \tag{1}$$

  where $u(x)$ is the utility function of the economic agent defined over the elementary outcomes of the random variable X, the probability distribution function of which is $F_X(x)$.

- Equation (1) is actually the mathematical expectation of the random variable $u(X)$,

$$U(P_X) = Eu(X),$$

  and for this reason the numerical representation of the preference order is, in fact, the expected utility.

The von Neumann-Morgenstern expected utility theory

- In the equivalent numerical representation, it is the utility function $u(x)$ which characterizes U and, therefore, determines the preference order.
- In effect, the utility function can be regarded as the fundamental building block which describes the agent's preferences.

The von Neumann-Morgenstern expected utility theory

- As we explained, lotteries may be discrete, continuous or mixed. If the lottery is discrete, then the the payoff is a discrete random variable and equation (1) becomes

$$U(P_X) = \sum_{j=1}^{n} u(x_j)p_j \qquad (2)$$

  where x_j are the outcomes and p_j is the probability that the j-th outcome occurs, $p_j = P(X = x_j)$.
- The formula for the fair value in the St. Petersburg Paradox given by Daniel Bernoulli has the form of equation (2).
- If the lottery is such that it has only one possible outcome (i.e., the profit is equal to x with certainty), then the expected utility coincides with the utility of the corresponding payoff, $u(x)$.

Types of utility functions

Some properties of the utility function are derived from common arguments valid for investors belonging to a certain category.

- If there are two prospects, one with a certain payoff of \$100 and another, with a certain payoff of \$200, a non-satiable investor would never prefer the first opportunity.
- Therefore, the utility function indicates that $u(200) \geq u(100)$.
- We can generalize that the utility functions of non-satiable investors should be non-decreasing,

Non-decreasing property	$u(x) \leq u(y)$, if $x \leq y$ for any $x, y \in \mathbb{R}$.

- Both outcomes x and y occur with probability one. If the utility function is differentiable, then the non-decreasing property translates as a non-negative first derivative, $u'(x) \geq 0, \; x \in \mathbb{R}$.

Types of utility functions

Other characteristics of investor's preferences can also be described by the shape of the utility function.

- The investor gains a lower utility from a venture with some expected payoff and a prospect with a certain payoff, equal to the expected payoff of the venture; that is, the investor is risk averse.
- Assume that the venture has two possible outcomes — x_1 with probability p and x_2 with probability $1-p$, $p \in [0,1]$.
- Thus, the expected payoff of the venture equals $px_1 + (1-p)x_2$. In terms of the utility function, the risk-aversion property is expressed as

$$u(px_1+(1-p)x_2) \geq pu(x_1)+(1-p)u(x_2), \ \forall x_1, x_2 \text{ and } p \in [0,1] \quad (3)$$

where the left-hand side corresponds to the utility of the certain prospect and the right-hand side is the expected utility of the venture.

Types of utility functions

- By definition, if a utility function satisfies (3), then it is called concave and, therefore, the utility functions of risk-averse investors should be concave,

Concavity	$u(x)$ with support on a set S is said to be a concave function if S is a convex set and if $u(x)$ satisfies (3) for all $x_1, x_2 \in S$ and $p \in [0, 1]$.

- If the utility function is twice differentiable, the concavity property translates as a negative second derivative, $u''(x) \leq 0, \ \forall x \in S$.

- A formal measure of absolute risk aversion is the coefficient of absolute risk aversion defined by

$$r_A(x) = -\frac{u''(x)}{u'(x)}, \tag{4}$$

which indicates that the more curved the utility function is, the higher the risk-aversion level of the investor (the more pronounced the inequality in (3) becomes).

Types of utility functions

Some common examples of utility functions:

1. Linear utility function

$$u(x) = a + bx$$

It always satisfies (3) with equality and represents a risk-neutral investor. If $b > 0$, then it represents a non-satiable investor.

2. Quadratic utility function

$$u(x) = a + bx + cx^2$$

If $c < 0$, then the quadratic utility function is concave and represents a risk-averse investor.

3. Logarithmic utility function

$$u(x) = \log x, \qquad x > 0$$

The logarithmic utility represents a non-satiable, risk averse investor. It exhibits a decreasing absolute risk aversion since $r_A(x) = 1/x$ and the coefficient of absolute risk aversion decreases with x.

Types of utility functions

4. Exponential utility function

$$u(x) = -e^{-ax}, \qquad a > 0$$

The exponential utility represents a non-satiable, risk averse investor. It exhibits a constant absolute risk aversion since $r_A(x) = a$ and the coefficient of absolute risk aversion does not depend on x.

5. Power utility function

$$u(x) = \frac{-x^{-a}}{a}, \qquad x > 0,\ a > 0$$

The power utility represents a non-satiable, risk averse investor. It exhibits a decreasing absolute risk aversion since $r_A(x) = a/x$ and the coefficient of absolute risk aversion decreases with x.

Stochastic dominance

- We noted that different classes of investors can be defined through the general unifying properties of their utility functions.
- Suppose that there are two portfolios X and Y, such that all investors from a given class do not prefer Y to X.
- This means that the probability distributions of the two portfolios differ in a special way that, no matter the particular expression of the utility function, if an investor belongs to the given class, then Y is not preferred by that investor.
- In this case, we say that portfolio X dominates portfolio Y with respect to the class of investors. Such a relation is often called a stochastic dominance relation or a stochastic ordering.

Stochastic dominance

- Let's obtain a criterion characterizing the stochastic dominance, involving only the cumulative distribution functions (c.d.f.s) of X and Y.
- Then, we are able to identify by only looking at distribution functions of X and Y if any of the two portfolios is preferred by an investor from the class.

First-order stochastic dominance

- Suppose that X is an investment opportunity with two possible outcomes — the investor receives \$100 with probability 1/2 and \$200 with probability 1/2.
- Similarly, Y is a venture with two payoffs — \$150 with probability 1/2 and \$200 with probability 1/2.
- A non-satiable investor would never prefer the first opportunity because of the following relationship between the corresponding expected utilities,

$$U(P_X) = u(100)/2 + u(200)/2 \quad \leq \quad u(150)/2 + u(200)/2 = U(P_Y).$$

The inequality arises because $u(100) \leq u(150)$ as a non-satiable investor by definition prefers more to less.

First-order stochastic dominance

- Denote by $\mathcal{U}_1$ the set of all utility functions representing non-satiable investors; that is, the set contains all non-decreasing utility functions.
- We say that the venture X dominates the venture Y in the sense of the first-order stochastic dominance (FSD), $X \succeq_{FSD} Y$, if a non-satiable investor would not prefer Y to X. In terms of the expected utility,

$$X \succeq_{FSD} Y \quad \text{if} \quad Eu(X) \geq Eu(Y), \text{ for any } u \in \mathcal{U}_1.$$

- The condition in terms of the c.d.f.s of X and Y characterizing the FSD order is the following,

$$X \succeq_{FSD} Y \quad \text{if and only if} \quad F_X(x) \leq F_Y(x), \ \forall \ x \in \mathbb{R}. \tag{5}$$

where $F_X(x)$ and $F_Y(x)$ are the c.d.f.s of the two ventures.

First-order stochastic dominance

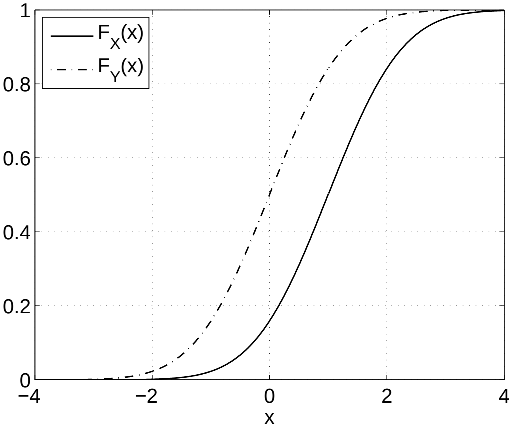

Figure: An illustration of the first-order stochastic dominance condition in terms of the distribution functions, $X \succeq_{FSD} Y$. A non-satiable investor would never invest in Y.

First-order stochastic dominance

- A necessary condition for FSD is that the expected payoff of the preferred venture should exceed the expected payoff of the alternative, $EX \geq EY$ if $X \succeq_{FSD} Y$.
- This is true because the utility function $u(x) = x$ represents a non-satiable investor as it is non-decreasing and, therefore, it belongs to the set $\mathcal{U}_1$.
- Consequently, if X is preferred by all non-satiable investors, then it is preferred by the investor with utility function $u(x) = x$ which means that the expected utility of X is not less than the expected utility of Y, $EX \geq EY$.

First-order stochastic dominance

- In general, the converse statement does not hold.
- If the expected payoff of a portfolio exceeds the expected payoff of another portfolio it does not follow that any non-satiable investor would necessarily choose the portfolio with the larger expected payoff.
- This is because the inequality between the c.d.f.s of the two portfolios given in (5) may not hold.
- In effect, there will be non-satiable investors who would choose the portfolio with the larger expected payoff and other non-satiable investors who would choose the portfolio with the smaller expected payoff.

Second-order stochastic dominance

- For decision making under risk, the concept of first-order stochastic dominance is not very useful because the condition in (5) is rather restrictive.
- If the distribution functions of two portfolios satisfy (5), then a non-satiable investor would never prefer portfolio Y. This conclusion also holds for the sub-category of the non-satiable investors who are also risk-averse.
- Therefore, the condition in (5) is only a *sufficient* condition for this sub-category of investors but is unable to characterize completely their preferences. (See the following example).

Second-order stochastic dominance

- Consider a venture Y with two possible payoffs — \$100 with probability 1/2 and \$200 with probability 1/2, and a prospect X yielding \$180 with probability one.
- A non-satiable, risk-averse investor would never prefer Y to X because the expected utility of Y is not larger than the expected utility of X,

$$Eu(X) = u(180) \geq u(150) \geq u(100)/2 + u(200)/2 = Eu(Y)$$

  where $u(x)$ satisfies property (3) and is assumed to be non-decreasing.
- The distribution functions of X and Y do not satisfy (5). Nevertheless, a non-satiable, risk-averse investor would never prefer Y.

Second-order stochastic dominance

- Denote by $\mathcal{U}_2$ the set of all utility functions which are non-decreasing and concave. Thus, the set $\mathcal{U}_2$ represents the non-satiable, risk-averse investors and is a subset of $\mathcal{U}_1$, $\mathcal{U}_2 \subset \mathcal{U}_1$.
- We say that a venture X dominates venture Y in the sense of second-order stochastic dominance (SSD), $X \succeq_{SSD} Y$, if a non-satiable, risk-averse investor does not prefer Y to X.
- In terms of the expected utility,

$$X \succeq_{SSD} Y \quad \text{if} \quad Eu(X) \geq Eu(Y), \text{ for any } u \in \mathcal{U}_2.$$

- The condition in terms of the c.d.f.s of X and Y characterizing the SSD order is the following,

$$X \succeq_{SSD} Y \quad \Longleftrightarrow \quad \int_{-\infty}^{x} F_X(t)dt \leq \int_{-\infty}^{x} F_Y(t)dt, \ \forall\, x \in \mathbb{R}. \qquad (6)$$

where $F_X(t)$ and $F_Y(t)$ are the c.d.f.s of the two ventures.

Second-order stochastic dominance

- Similarly to FSD, inequality between the expected payoffs is a necessary condition for SSD, $EX \geq EY$ if $X \succeq_{SSD} Y$, because the utility function $u(x) = x$ belongs to the set $\mathcal{U}_2$.
- In contrast to the FSD, the condition in (6) allows the distribution functions to intersect.
- It turns out that if the distribution functions cross only once, then X dominates Y with respect to SSD if $F_X(x)$ is below $F_Y(x)$ to the left of the crossing point. (See the illustration on the next slide).

Second-order stochastic dominance

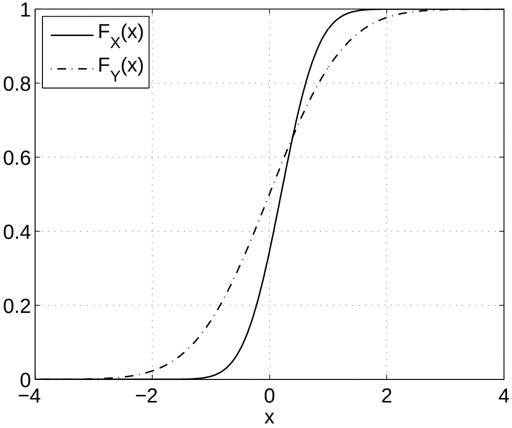

Figure: An illustration of the second-order stochastic dominance condition in terms of the distribution functions, $X \succeq_{SSD} Y$.

Rothschild-Stiglitz stochastic dominance

- Rothschild and Stiglitz (1970) introduce a slightly different order by dropping the requirement that the investors are non-satiable.
- A venture X is said to dominate a venture Y in the sense of Rothschild-Stiglitz stochastic dominance (RSD) ,[2] $X \succeq_{RSD} Y$, if no risk-averse investor prefers Y to X.
- In terms of the expected utility,

$$X \succeq_{RSD} Y \quad \text{if} \quad Eu(X) \geq Eu(Y), \text{ for any concave } u(x).$$

[2] Also called *concave order*.

Rothschild-Stiglitz stochastic dominance

- The class of risk-averse investors is represented by the set of all concave utility functions, which contains the set $\mathcal{U}_2$. Thus, the condition in (6) is only a necessary condition for the RSD but it is not sufficient to characterize the RSD order.
- If the portfolio X dominates the portfolio Y in the sense of the RSD order, then a risk-averter would never prefer Y to X.
- This conclusion holds for the non-satiable risk-averters as well and, therefore, the relation in (6) holds as a consequence,

$$X \succeq_{RSD} Y \quad \Longrightarrow \quad X \succeq_{SSD} Y.$$

Rothschild-Stiglitz stochastic dominance

- The converse relation is not true. If the portfolio Y pays off \$100 with probability 1/2 and \$200 with probability 1/2 then no risk-averse investor would prefer it to a prospect yielding \$150 with probability one,

$$u(150) = u(100/2 + 200/2) \geq u(100)/2 + u(200)/2 = Eu(Y),$$

which is just an application of the assumption of concavity in (3).

- It is not possible to determine whether a risk-averse investor would prefer a prospect yielding \$150 with probability one or the prospect X yielding \$180 with probability one.
- Those who are non-satiable would certainly prefer the larger sum but this is not universally true for all risk-averse investors because we do not assume that $u(x)$ is non-decreasing.

Rothschild-Stiglitz stochastic dominance

- The condition which characterizes the RSD stochastic dominance is the following one,

$$X \succeq_{RSD} Y \quad \Longleftrightarrow \quad \begin{cases} EX = EY, \\ \int_{-\infty}^{x} F_X(t)dt \leq \int_{-\infty}^{x} F_Y(t)dt, \ \forall\, x \in \mathbb{R}. \end{cases} \tag{7}$$

- In fact, this is the condition for the SSD order with the additional assumption that the mean payoffs should coincide.

Third-order stochastic dominance

- We defined the coefficient of absolute risk aversion $r_A(x)$ in equation (4). Generally, its values vary for different payoffs depending on the corresponding derivatives of the utility function. Larger values of $r_A(x)$ correspond to a more pronounced risk-aversion effect.
- A negative second derivative of the utility function for all payoffs means that the investor is risk-averse at any payoff level. The closer $u''(x)$ to zero, the less risk-averse the investor since the coefficient $r_A(x)$ decreases, other things held equal.
- The logarithmic utility function is an example of a utility function exhibiting decreasing absolute risk aversion. The larger the payoff level, the less "curved" the function is, which corresponds to a closer to zero second derivative and a less pronounced risk-aversion property. (See the illustration on the next slide).

Third-order stochastic dominance

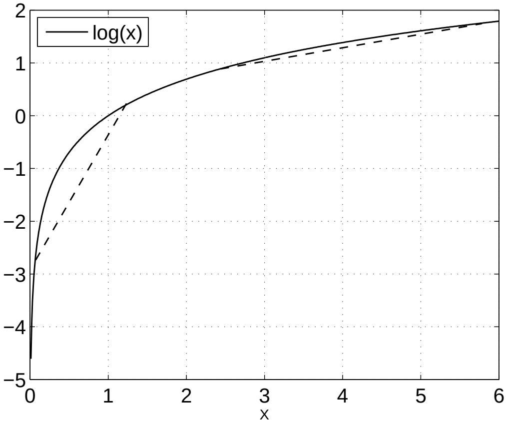

Figure: The graph of the logarithmic utility function, $u(x) = \log x$. For smaller values of x, the graph is more curved while for larger values of x, the graph is closer to a straight line and, thus, to risk neutrality.

Third-order stochastic dominance

- Utility functions exhibiting a decreasing absolute risk aversion are important because the investors they represent favor positive to negative skewness.
- This is a consequence of the decreasing risk aversion — at higher payoff levels such investors are less inclined to avoid risk in comparison to lower payoff levels at which they are much more sensitive to risk taking.
- Technically, a utility function with a decreasing absolute risk aversion has a non-negative third derivative, $u'''(x) \geq 0$, as this means that the second derivative is non-decreasing.

Third-order stochastic dominance

- Denote by $\mathcal{U}_3$ the set of all utility functions which are non-decreasing, concave, and have a non-negative third derivative, $u'''(x) \geq 0$.
- Thus, $\mathcal{U}_3$ represents the class of non-satiable, risk-averse investors who prefer positive to negative skewness.
- A venture X is said to dominate a venture Y in the sense of *third-order stochastic dominance* (TSD), $X \succeq_{TSD} Y$, if an investor with a utility function from the set $\mathcal{U}_3$ does not prefer Y to X.
- terms of the expected utility,

$$X \succeq_{TSD} Y \quad \text{if} \quad Eu(X) \geq Eu(Y), \text{ for any } u \in \mathcal{U}_3.$$

- The set of utility functions $\mathcal{U}_3$ is contained in the set of non-decreasing, concave utilities, $\mathcal{U}_3 \subset \mathcal{U}_2$. Therefore, the condition (6) for SSD is only sufficient in the case of TSD,

$$X \succeq_{SSD} Y \quad \Longrightarrow \quad X \succeq_{TSD} Y.$$

Third-order stochastic dominance

- The condition, characterizing the TSD stochastic dominance, is

$$X \succeq_{TSD} Y \quad \Longleftrightarrow \quad E(X-t)_+^2 \leq E(Y-t)_+^2,\ \forall\, t \in \mathbb{R} \qquad (8)$$

  where the notation $(x-t)_+^2$ means the maximum between $x-t$ and zero raised to the second power, $(x-t)_+^2 = (\max(x-t,0))^2$.
- The quantity $E(X-t)_+^2$ is known as the second lower partial moment of the random variable X. It measures the variability of X below a target payoff level t.
- Suppose that X and Y have equal means and variances. If X has a positive skewness and Y has a negative skewness, then the variability of X below any target payoff level t will be smaller than the variability of Y below the same target payoff level.
- In fact, it is only a matter of algebraic manipulations to show that, indeed, if (6) holds, then (8) holds as well.

Efficient sets and the portfolio choice problem

- Taking advantage of the criteria for stochastic dominance, we can characterize the efficient sets of the corresponding categories of investors.
- The efficient set of a given class of investors is defined as the set of ventures not dominated with respect to the corresponding stochastic dominance relation.
- For example, the efficient set of the non-satiable investors is the set of those ventures which are not dominated with respect to the FSD order.

$\Rightarrow$ By construction, any venture which is not in the efficient set will be necessarily discarded by all investors in the class.

Efficient sets and the portfolio choice problem

The portfolio choice problem of a given investor can be divided into two steps.

1. The first step concerns finding the efficient set of the class of investors which the given investor belongs to. Any portfolio not belonging to the efficient set will not be selected by any of the investors in the class and is, therefore, suboptimal for the investor. The efficient set comprises all portfolios not dominated with respect to the SSD order.
2. The second step involves calculation of the expected utility of the investor for the portfolios in the efficient set. The portfolio which maximizes the investor's expected utility represents the optimal choice of the investor.

Efficient sets and the portfolio choice problem

- The difficulty of adopting this approach in practice is that it is very hard to obtain explicitly the efficient sets.
- That is why the problem of finding the optimal portfolio for the investor is very often replaced by a more simple one, involving only certain characteristics of the portfolios return distributions, such as the expected return and the risk.
- In this situation, it is critical that the more simple problem is consistent with the corresponding stochastic dominance relation in order to guarantee that its solution is among the portfolios in the efficient set.
- Checking the consistency reduces to choosing a risk measure which is compatible with the stochastic dominance relation.

Return versus payoff

- Note that the expected utility theory deals with the portfolio payoff and not the portfolio return.
- Nevertheless, all relations defining the stochastic dominance orders can be adopted if we consider the distribution functions of portfolio returns rather than portfolio profits.
- In the following, we examine the FSD and SSD orders concerning log-return distributions and the connection to the corresponding orders concerning random payoffs.

Return versus payoff

- Suppose that P_t is a random variable describing the price of a common stock at a future time t, $t > 0$ where $t = 0$ is present time. We can assume that the stock does not pay dividends.
- Denote by r_t the log-return for the period $(0, t)$,

$$r_t = \log \frac{P_t}{P_0},$$

  where P_0 is the price of the common stock at present and is a non-random positive quantity.
- The random variable P_t can be regarded as the random payoff of the common stock at time t, while r_t is the corresponding random log-return. Then the random payoff is

$$P_t = P_0 \exp(r_t).$$

- It turns out that, generally, stochastic dominance relations concerning two log-return distributions are not equivalent to the corresponding stochastic dominance relations concerning their payoff distributions.

Return versus payoff

- Consider an investor with utility function $u(x)$ where $x > 0$ stands for payoff. We demonstrate that the utility function of the investor concerning the log-return can be expressed as

$$v(y) = u(P_0 \exp(y)), \ y \in \mathbb{R} \tag{9}$$

  where y stands for the log-return of a common stock and P_0 is the price at present.
- Equation (9) and the inverse,

$$u(x) = v(\log(x/P_0)), \ x > 0 \tag{10}$$

  provide the link between utilities concerning log-returns and payoff.

Return versus payoff

- It turns out that an investor who is non-satiable and risk-averse with respect to payoff distributions may not be risk-averse with respect to log-return distributions.
- The utility function $u(x)$ representing such an investor has the properties
$$u'(x) \geq 0 \quad \text{and} \quad u''(x) \leq 0, \ \forall x > 0,$$
but it does not follow that the function $v(y)$ given by (9) will satisfy them.
- In fact, $v(y)$ also has non-positive first derivative but the sign of the second derivative can be arbitrary.
- Therefore the investor is non-satiable but may not be risk-averse with respect to log-return distributions. (See the figure on the next slide).

Return versus payoff

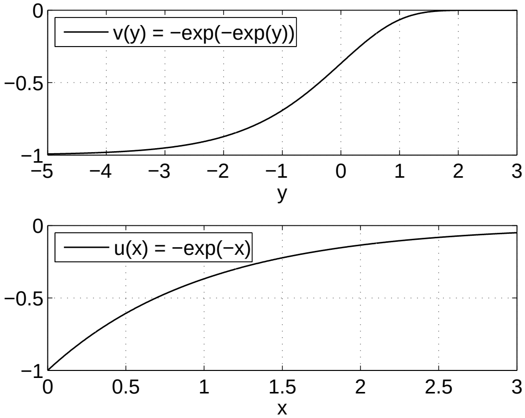

Figure: $u(x)$ represents a non-satiable and risk-averse investor on the space of payoffs and $v(y)$ is the corresponding utility on the space of log-returns. Apparently, $v(y)$ is not concave.

Return versus payoff

- Conversely, an investor who is non-satiable and risk-averse with respect to log-return distributions, is also non-satiable and risk-averse with concerning payoff distributions.
- This is true because if $v(y)$ satisfies the corresponding derivative inequalities, so does $u(x)$ given by (10).
- Consequently, it follows that the investors who are non-satiable and risk-averse on the space of log-return distributions are a sub-class of those who are non-satiable and risk-averse on the space of payoff distributions.

Return versus payoff

- This analysis implies that the FSD order of two common stocks, for example, remains unaffected as to whether we consider their payoff distributions or their log-return distributions,

$$P_t^1 \succeq_{FSD} P_t^2 \qquad \Longleftrightarrow \qquad r_t^1 \succeq_{FSD} r_t^2,$$

  where P_t^1 and P_t^2 are the payoffs of the two common stocks at time $t > t_0$, and r_t^1 and r_t^2 are the corresponding log-returns for the same period.
- However, such an equivalence does not hold for the SSD order. Actually, the SSD order on the space of payoff distributions implies the same order on the space of log-return distributions but not vice versa,

$$P_t^1 \succeq_{SSD} P_t^2 \qquad \Longrightarrow \qquad r_t^1 \succeq_{SSD} r_t^2.$$

Return versus payoff

- Note that these relations are always true if the present values of the two ventures are equal $P_0^1 = P_0^2$.
- Consider, for example, the FSD order of random payoffs. Suppose that P_t^1 dominates P_t^2 with respect to the FSD order, $P_t^1 \succeq_{FSD} P_t^2$. According to the characterization in terms of the c.d.f.s we obtain

$$F_{P_t^1}(x) \leq F_{P_t^2}(x), \quad \forall x \in \mathbb{R}.$$

- Let's represent this inequality in terms of the log-returns r_t^1 and r_t^2:

$$P\left(r_t^1 \leq \log \frac{x}{P_0^1}\right) \leq P\left(r_t^2 \leq \log \frac{x}{P_0^2}\right), \quad \forall x \in \mathbb{R}.$$

- In fact, the above inequality implies that $r_t^1 \succeq_{FSD} r_t^2$ if $P_0^1 = P_0^2$. In case the present values of the ventures differ a lot, it may happen that the c.d.f.s of the log-return distributions do not satisfy the inequality $F_{r_t^1}(y) \leq F_{r_t^2}(y)$ for all $y \in \mathbb{R}$, which means that the FSD order may not hold.

- The conditions for stochastic dominance involving the distribution functions of the ventures X and Y represent a powerful method to determine if an entire class of investors would prefer any of the portfolios.
- For example, in order to verify if any non-satiable, risk-averse investor would not prefer Y to X, we have to verify if condition (6) holds.
- Note that a negative result does not necessarily mean that any such investor would actually prefer Y or be indifferent between X and Y. It may be the case that the inequality between the quantities in (6) is satisfied for some values of the argument, and for others, the converse inequality holds.
- That is, neither $X \succeq_{SSD} Y$ nor $Y \succeq_{SSD} X$ is true. Thus, only a part of the non-satiable, risk-averse investors may prefer X to Y; it now depends on the particular investor we consider.

Probability metrics and stochastic dominance

- Suppose the verification confirms that either X is preferred or the investors are indifferent between X and Y, $X \succeq_{SSD} Y$. This result is only qualitative.
- *If we know that no investors from the class prefer Y to Z, $Z \succeq_{SSD} Y$, then can we determine whether Z is more strongly preferred to Y than X is?*
- The only way to approach these questions is to add a quantitative element through a probability metric since only by means of a probability metric can we calculate distances between random quantities.

Probability metrics and stochastic dominance

- For example, we can choose a probability metric μ and we can calculate the distances $\mu(X, Y)$ and $\mu(Z, Y)$. If $\mu(X, Y) < \mu(Z, Y)$, then the return distribution of X is “closer” to the return distribution of Y than are the return distributions of Z and Y.
- On this ground, we can draw the conclusion that Z is more strongly preferred to Y than X is, on condition that we know in advance the relations $X \succeq_{SSD} Y$ and $Z \succeq_{SSD} Y$.

However, not any probability metric appears suitable for this calculation.

- Suppose that Y and X are normally distributed random variables describing portfolio returns with equal means, $X \in N(a, \sigma_X^2)$ and $Y \in N(a, \sigma_Y^2)$, with $\sigma_X^2 < \sigma_Y^2$. Z is a prospect yielding a dollars with probability one.
- The c.d.f.s $F_X(x)$ and $F_Y(x)$ cross only once at $x = a$ and the $F_X(x)$ is below $F_Y(x)$ to the left of the crossing point because the variance of X is assumed to be smaller than the variance of Y.
- Therefore, according to the condition in (7), no risk-averse investor would prefer Y to X and consequently $X \succeq_{SSD} Y$.
- The prospect Z provides a non-random return equal to the expected returns of X and Y, $EX = EY = a$, and, in effect, any risk-averse investor would rather choose Z from the three alternatives, $Z \succeq_{SSD} X \succeq_{SSD} Y$.

Probability metrics and stochastic dominance

- A probability metric with which we would like to quantify the second-order stochastic dominance relation should be able to indicate that,
    1. $\mu(X, Y) < \mu(Z, Y)$ because Z is more strongly preferred to Y, and
    2. $\mu(Z, X) < \mu(Z, Y)$ because Y is more strongly rejected than X with respect to Z.
- The assumptions in the example give us the information to order completely the three alternatives and that is why we are expecting the two inequalities should hold.

- Let us choose the Kolmogorov metric,

$$\rho(X, Y) = \sup_{x \in \mathbb{R}} |F_X(x) - F_Y(x)|,$$

  for the purpose of calculating the corresponding distances. It computes the largest absolute difference between the two distribution functions.
- Applying it to the distributions in the example, we obtain that $\rho(X,Z) = \rho(Y,Z) = 1/2$ and $\rho(X,Y) < 1/2$.
- As a result, the Kolmogorov metric is capable of showing that Z is more strongly preferred relative to Y but cannot show that Y is more strongly rejected with respect to Z. (See the illustration on the next slide).

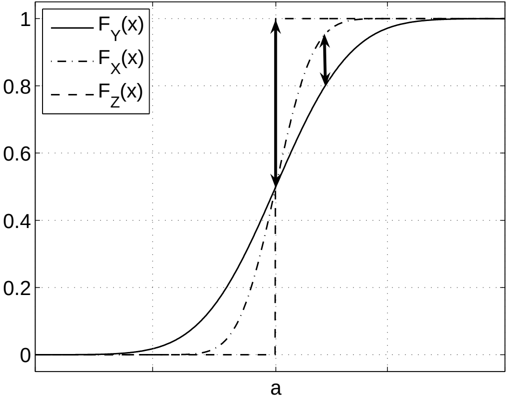

Figure: The distribution functions of two normal distributions with equal means, $EX = EY = a$ and the distribution function of $Z = a$ with probability one. The arrows indicate where the largest absolute difference between the corresponding c.d.f.s is located. The arrow length equals the Kolmogorov distance.

- The example shows that there are probability metrics which are not appropriate to quantify a stochastic dominance order.
- We cannot expect that one probability metric will appear suitable for all stochastic orders, rather, a probability metric may be best suited for a selected stochastic dominance relation.
- Technically, we have to impose another condition in order for the problem of quantification to have a practical meaning.
- The probability metric calculating the distances between the ordered random variables should be bounded. If it explodes, then we cannot draw any conclusions.
- For instance, if $\mu(X, Y) = \infty$ and $\mu(Z, Y) = \infty$, then we cannot compare the investors' preferences.

Probability metrics and stochastic dominance

- Concerning the FSD order, a suitable choice for a probability metric is the Kantorovich metric,

$$\kappa(X, Y) = \int_{-\infty}^{\infty} |F_X(x) - F_Y(x)| dx.$$

  Note that the condition in (5) can be restated as $F_X(x) - F_Y(x) \leq 0,\ \forall x \in \mathbb{R}$.

- Summing up all absolute differences gives an idea how "close" X is to Y which is a natural way of measuring the distance between X and Y with respect to the FSD order.
- The Kantorovich metric is finite as long as the random variables have finite means.

- The RSD order can also be quantified in a similar fashion. Consider the Zolotarev ideal metric,

$$\zeta_2(X, Y) = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{x} F_X(t)dt - \int_{-\infty}^{x} F_Y(t)dt \right| dx.$$

- The structure of this probability metric is directly based on the condition in (7) and it calculates in a natural way the distance between X and Y with respect to the RSD order.
- The requirement that $EX = EY$ in (7) combined with the additional assumption that the second moments of X and Y are finite, $EX^2 < \infty$ and $EY^2 < \infty$, represent the needed sufficient conditions for the boundedness of $\zeta_2(X, Y)$.

- Due to the similarities of the conditions (6) and (8), defining the SSD and the TSD orders, it is reasonable to expect that the Rachev ideal metric is best suited to quantify the SSD and the TSD orders. (See the appendix of this lecture for details).

Svetlozar T. Rachev, Stoyan Stoyanov, and Frank J. Fabozzi
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John Wiley, Finance, 2007.

Chapter 5.