# Lecture 5: Duality

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# Outline

- Lagrange Dual Function
- Lagrange Dual Problem
- Geometric Interpretation
- Saddle-point Interpretation
- Optimality Conditions
- Perturbation and Sensitivity Analysis
- Examples
- Generalized Inequalities

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# Lagrangian

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, i = 1, 2, \dots, m \\ & h_i(x) = 0, i = 1, 2, \dots, p \end{array}$$

Lagrangian: $L : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}$, with $\mathbf{dom} L = \mathcal{D} \times \mathbf{R}^m \times \mathbf{R}^p$

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

- weighted sum of objective and constraint functions
- $\lambda_i$ is Lagrange multiplier associated with $i$th inequality constraint
- $\nu_i$ is Lagrange multiplier associated with $i$th equality constraint
- $\lambda, \nu$ are called dual variables or Lagrange multiplier vectors

# Lagrange Dual Function

Lagrange dual function: $g : \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}$

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left( f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

- For each $x$ the Lagrangian $L(x, \lambda, \nu)$ is affine in $(\lambda, \nu)$; thus, pointwise infimum over $x \in \mathcal{D}$ yields concave $g(\lambda, \nu)$

- For any $\lambda \geq 0$ and $\nu$, $g(\lambda, \nu) \leq p^*$

- Linear approximation interpretation: Consider the following unconstrained problem with the same optimal point and optimal value as the original one:

$$\text{minimize} \quad f_0(x) + \sum_{i=1}^{m} I_-(f_i(x)) + \sum_{i=1}^{p} I_0(h_i(x))$$

where $I_-$ and $I_0$ are equal to 0 if the argument satisfies the subscript condition and infinity otherwise. For any $\lambda \geq 0$ and $\nu$, $L(x, \lambda, \nu)$ is simply an under-estimator of the above formulation, and therefore minimizing $L(x, \lambda, \nu)$ yields a lower bound $g$ of the original optimal value $p^*$.

# Least-Norm Solution of Linear Equations

$$\begin{array}{ll}\text{minimize} & x^T x \\ \text{subject to} & Ax = b\end{array}$$

- Lagrangian is

$$L(x,\nu) = x^T x + \nu^T (Ax - b)$$

which is a convex quadratic function of $x$. Taking derivative of $L(x,\nu)$ yields

$$\nabla_x L(x,\nu) = 2x + A^T \nu$$

which vanishes when $x = -\frac{1}{2}A^T\nu$.

- Therefore, the Lagrange dual function is

$$g(\nu) = L\left(-\frac{1}{2}A^T\nu, \nu\right) = (-1/4)\nu^T A A^T \nu - \nu^T b$$

which is concave in $\nu$

- Lower bound property:

$$p^* \geq (-1/4)\nu^T A A^T \nu - \nu^T b, \ \forall \nu$$

# Standard Form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

- Lagrangian is

$$L(x, \lambda, \nu) = c^T x - \lambda^T x + \nu^T (Ax - b) = (c^T - \lambda^T + \nu^T A)x - \nu^T b$$

- Lagrange dual function is

$$g(\lambda, \nu) = \begin{cases} -\nu^T b & c^T - \lambda^T + \nu^T A = 0 \\ -\infty & \text{otherwise} \end{cases}$$

which is linear on affine domain and hence concave

- Lower bound property:

$$p^* \geq -\nu^T b, \text{ for any } \nu \text{ such that } c^T + \nu^T A \geq 0$$

# Lagrange Dual and Conjugate Function

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & Ax \preceq b, Cx = d \end{array}$$

Dual function:

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathbf{dom} f_0} \left( f_0(x) + \lambda^T (Ax - b) + \nu^T (Cx - d) \right) \\ &= \inf_{x \in \mathbf{dom} f_0} \left( f_0(x) + (A^T \lambda + C^T \nu)^T x - \lambda^T b - \nu^T d \right) \\ &= -f_0^*(-A^T \lambda - C^T \nu) - \lambda^T b - \nu^T d \end{aligned}$$

- recall definition of conjugate: $f^*(y) = \sup_{x \in \mathbf{dom} f} (y^T x - f(x))$

- simplifies derivation of dual if conjugate of $f_0$ is known

- example: entropy maximization

$$f_0(x) = \sum_{i=1}^n x_i \log x_i, \quad f_0^*(y) = \sum_{i=1}^n e^{y_i - 1}$$

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# Lagrange Dual Problem

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

- called the Lagrange dual problem, associated with the primal problem
- finds best lower bound on $p^*$, obtained from Lagrange dual function
- a convex optimization problem; optimal value denoted by $d^*$
- $\lambda, \nu$ are said to be dual feasible if $\lambda \succeq 0$ and $(\lambda, \nu) \in \mathbf{dom}\ g$
- often simplified by making implicit constraints $(\lambda, \nu) \in \mathbf{dom}\ g$ explicit

example: standard form LP and its dual

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \succeq 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & -b^T \nu \\ \text{subject to} & A^T \nu + c \succeq 0 \end{array}$$

# Weak and Strong Duality

weak duality: $d^* \leq p^*$
- always holds (for convex and nonconvex problems)
- can be used to find nontrivial lower bound for difficult primal problems
- $p^* - d^*$ is referred to as the optimal duality gap, which is always nonnegative

strong duality: $d^* = p^*$
- does not hold in general
- (usually) holds for convex problems
- conditions that guarantee strong duality in convex problems are called constraint qualifications

# Slater's Constraint Qualifications

strong duality holds for a convex problem

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, i = 1, 2, \ldots, m \\ & Ax = b \end{array}$$

if it is strictly feasible, i.e.,

$$\exists x \in \mathbf{int}\mathcal{D}: \ f_i(x) < 0, i = 1, 2, \ldots, m, \ Ax = b$$

- also guarantees that the dual optimum is attained if $d^* = p^* > -\infty$, i.e. there exists a dual feasible $(\lambda^*, \nu^*)$ such that $g(\lambda^*, \nu^*) = d^* = p^*$

- can be sharpened; there also exist many other types of constraint qualifications

- sufficient but not necessary condition; strong duality can hold for convex problems not satisfying Slater's condition, or for nonconvex problems

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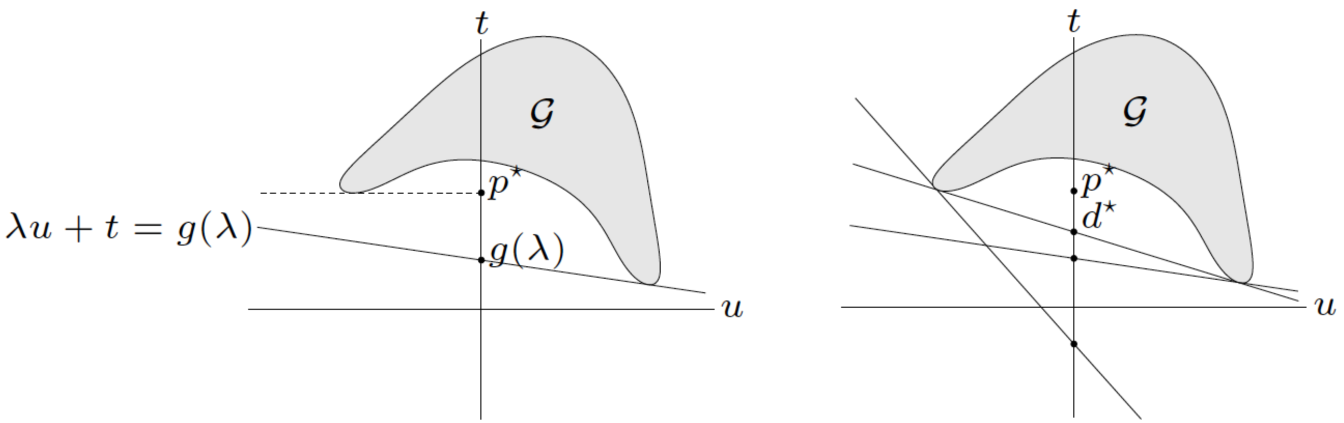
# Geometric Interpretation

for simplicity, consider problem with one constraint $f_1(x) \leq 0$

interpretation of dual function:

$$g(\lambda) = \inf_{(u,t)\in\mathcal{G}} (t + \lambda u), \quad \mathcal{G} = \{(f_1(x), f_0(x)) | x \in \mathcal{D}\}$$



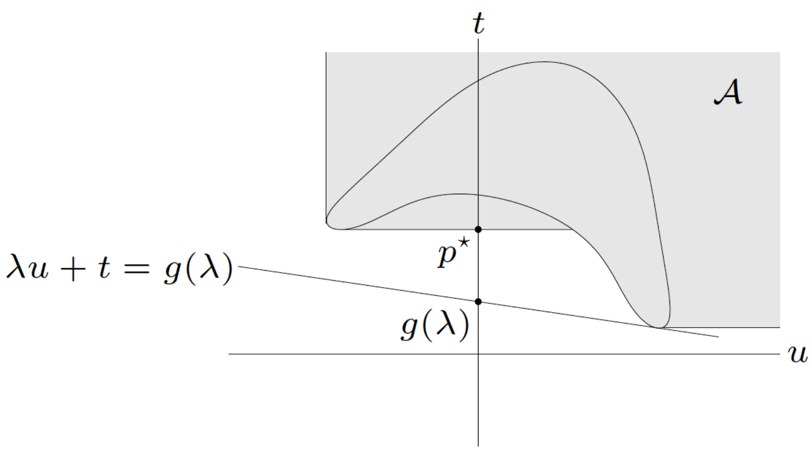

- $\lambda u + t = g(\lambda)$ is (non-vertical) supporting hyperplane to $\mathcal{G}$

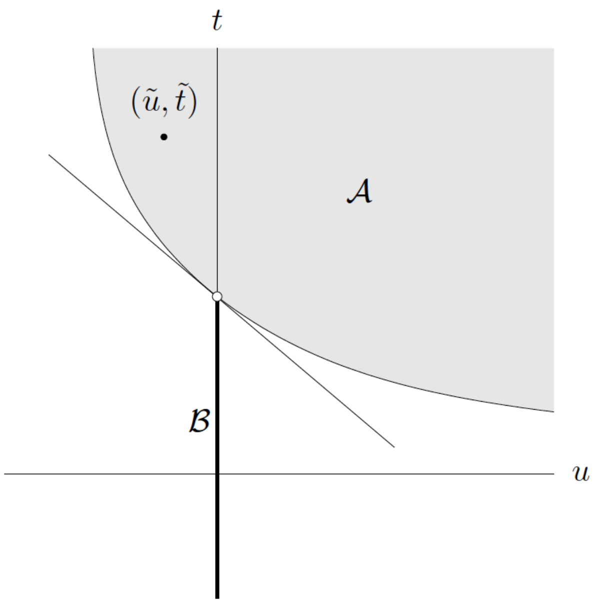
- hyperplane intersects $t$-axis at $t = g(\lambda)$

# Geometric Interpretation

epigraph variation: same interpretation if $\mathcal{G}$ is replaced with

$$\mathcal{A} = \{(u,t) | f_1(x) \le u, f_0(x) \le t \text{ for some } x \in \mathcal{D}\}$$




strong duality

- holds if there is a non-vertical supporting hyperplane to $\mathcal{A}$ at $(0, p^*)$
- for convex problem, $\mathcal{A}$ is convex, hence has supp. hyperplane at $(0, p^*)$
- Slater's condition: if there exist $(\tilde{u}, \tilde{t}) \in \mathcal{A}$ with $\tilde{u} < 0$, then supporting hyperplanes at $(0, p^*)$ must be non-vertical

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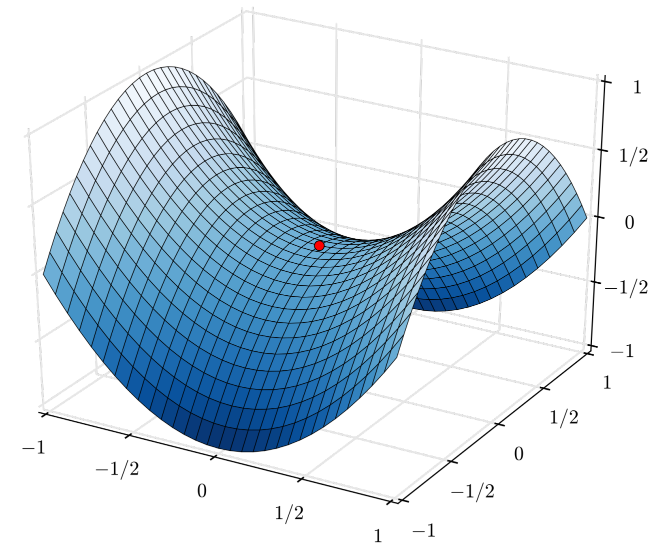
# Max-Min Characterization of Duality

- for simplicity, assume no equality constraints

$$d^* = \sup_{\lambda \succeq 0} \inf_x L(x, \lambda) \text{ and } p^* = \inf_x \sup_{\lambda \succeq 0} L(x, \lambda)$$




- weak duality: $\sup_{\lambda \succeq 0} \inf_x L(x, \lambda) \leq \inf_x \sup_{\lambda \succeq 0} L(x, \lambda)$

- strong duality: $\sup_{\lambda \succeq 0} \inf_x L(x, \lambda) = \inf_x \sup_{\lambda \succeq 0} L(x, \lambda)$

- In general, we have max-min inequality

$$\sup_{z \in Z} \inf_{w \in W} f(w, z) \leq \inf_{w \in W} \sup_{z \in Z} f(w, z)$$

When the equality holds in the above, we say $f$ (and $W, Z$) satisfy the strong max-min property or the saddle-point property. Game interpretation.

- strong max-min property holds only in special cases, e.g., when $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ is the Lagrangian of a problem for which strong duality attains, $W = \mathbf{R}^n$ and $Z = \mathbf{R}^m_+$

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# Complementary Slackness

assume strong duality holds, $x^*$ is primal optimal, $(\lambda^*, \nu^*)$ is dual optimal

$$\begin{aligned} f_0(x^*) = g(\lambda^*, \nu^*) &= \inf_x \left( f_0(x) + \sum_{i=1}^m \lambda_i^* f_i(x) + \sum_{i=1}^p \nu_i^* h_i(x) \right) \\ &\leq f_0(x^*) + \sum_{i=1}^m \lambda_i^* f_i(x^*) + \sum_{i=1}^p \nu_i^* h_i(x^*) \\ &\leq f_0(x^*) \end{aligned}$$

hence the two inequalities hold with equality
- $x^*$ minimizes $L(x, \lambda^*, \nu^*)$
- $\lambda_i^* f_i(x^*) = 0$ for $i \in [1:m]$ (known as complementary slackness)

$$\lambda_i^* > 0 \Rightarrow f_i(x^*) = 0, \quad f_i(x^*) < 0 \Rightarrow \lambda_i^* = 0$$

# Karush-Kuhn-Tucker (KKT) Conditions

KKT conditions (for differentiable $f_i, h_i$):

1. primal constraints: $f_i(x) \leq 0, i \in [1:m]$, $h_i(x) = 0, i \in [1:p]$

2. dual constraints: $\lambda \succeq 0$

3. complementary slackness: $\lambda_i f_i(x) = 0, i \in [1:m]$

4. gradient of Lagrangian w.r.t. $x$ vanishes

$$\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + \sum_{i=1}^{p} \nu_i \nabla h_i(x) = 0$$

If strong duality holds and $x, \lambda, \nu$ are optimal, then they must satisfy the KKT condition

# KKT Conditions For Convex Problems

if $x, \lambda, \nu$ satisfy KKT for a convex problem, then they are optimal:
- from complementary slackness: $f_0(x) = L(x, \lambda, \nu)$
- from 4th condition (and convexity): $g(\lambda, \nu) = L(x, \lambda, \nu)$

hence, $f_0(x) = g(\lambda, \nu)$

if Slater's condition is satisfied:
$x$ is optimal if and only if there exist $\lambda, \nu$ that satisfy KKT conditions
- recall that Slater implies strong duality, and dual optimum is attained
- generalizes optimality condition $\nabla f_0(x) = 0$ for unconstrained problem

# Example: Waterfilling

Consider the problem of allocating a total power of one to a set of $n$ communication channels with noise levels $\alpha_1, \alpha_2, \ldots, \alpha_n$. The goal is to maximize the total communication rate $\sum_{i=1}^n \log(1 + x_i/\alpha_i)$, i.e.,

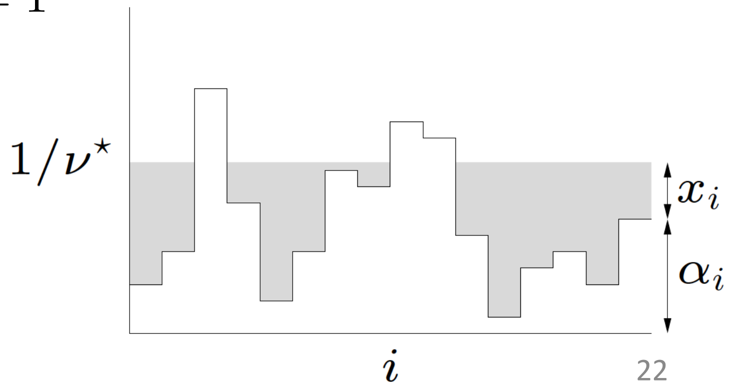
$$\begin{array}{ll} \text{minimize} & -\sum_{i=1}^n \log(\alpha_i + x_i) \\ \text{subject to} & x \succeq 0, \mathbf{1}^T x = 1 \end{array}$$

By KKT conditions, $x$ is optimal iff $x \succeq 0$, $\mathbf{1}^T x = 1$, and $\exists \lambda, \nu$ such that

$$\lambda \succeq 0, \quad \lambda_i x_i = 0, \quad \frac{1}{x_i + \alpha_i} + \lambda_i = \nu$$

- if $\nu < 1/\alpha_i$: $\lambda_i = 0$ and $x_i = 1/\nu - \alpha_i$
- if $\nu \geq 1/\alpha_i$: $\lambda_i = \nu - 1/\alpha_i$ and $x_i = 0$
- determine $\nu$ from $\mathbf{1}^T x = \sum_{i=1}^n \max\{0, 1/\nu - \alpha_i\} = 1$

interpretation
- $n$ patches; level of patch is at height $\alpha_i$
- flood are with unit amount of water
- resulting level is $1/\nu^*$



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# Perturbation and Sensitivity Analysis

unperturbed problem and its dual

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, i = 1, 2, \ldots, m \\ & h_i(x) = 0, i = 1, 2, \ldots, p \end{array}$$

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

perturbed problem and its dual

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq u_i, i = 1, 2, \ldots, m \\ & h_i(x) = v_i, i = 1, 2, \ldots, p \end{array}$$

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) - u^T \lambda - v^T \nu \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

- $x$ is primal variable; $u, v$ are parameters
- $p^*(u, v)$ is optimal value as a function of $u, v$
- what can we say about $p^*(u, v)$ based on the solution of the unperturbed problem and its dual?

# Global Sensitivity Result

assume strong duality holds for unperturbed problem, and $\lambda^*, \nu^*$ are dual optimal for unperturbed problem. Apply weak duality to perturbed problem:

$$\begin{aligned} p^*(u,v) &\geq g(\lambda^*, \nu^*) - u^T\lambda^* - v^T\nu^* \\ &= p^*(0,0) - u^T\lambda^* - v^T\nu^* \end{aligned}$$

- $\lambda_i^*$ large: $p^*$ increases greatly if we tighten constraint $i$ (choose $u_i < 0$)

- $\lambda_i^*$ small: $p^*$ doesn't decrease much if we loosen constraint $i$ (choose $u_i > 0$)

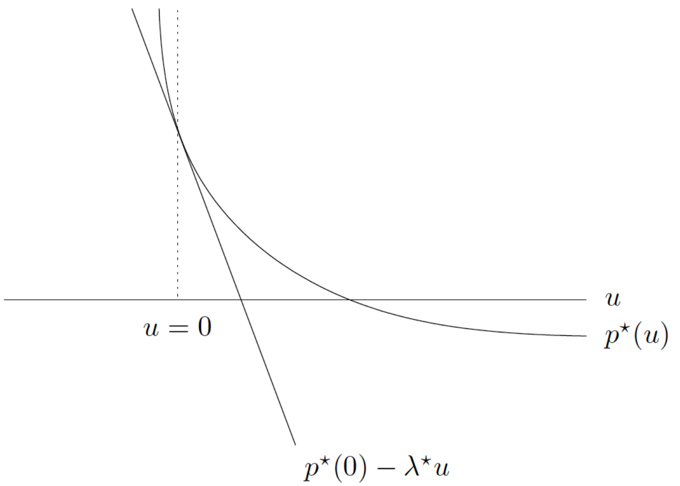
- $\nu_i^*$ large and positive: $p^*$ increases greatly if we take $v_i < 0$
$\nu_i^*$ large and negative: $p^*$ increases greatly if we take $v_i > 0$

- $\nu_i^*$ small and positive: $p^*$ doesn't decrease much if we take $v_i > 0$
$\nu_i^*$ small and negative: $p^*$ doesn't decrease much if we take $v_i < 0$

# Local Sensitivity Result

if (in addition) $p^*(u, v)$ is differentiable at $(0, 0)$, then

$$\lambda_i^* = -\frac{\partial p^*(0,0)}{\partial u_i}, \quad \nu_i^* = -\frac{\partial p^*(0,0)}{\partial v_i}$$




This gives us a quantitative measure of how active a constraint is at the optimum $x^*$:

- $f_i(x^*) < 0$ and $\lambda_i^* = 0$ (complementary slackness): constraint $i$ is inactive and can be tightened or loosened a small amount without affecting the optimal value

- $f_i(x^*) = 0$ and $\lambda_i^*$ is small: constraint $i$ is active, but can be tightened or loosened a small amount without much effect on the optimal value

- $f_i(x^*) = 0$ and $\lambda_i^*$ is large: constraint $i$ is active, and loosening or tightening it a bit will have great effect on the optimal value

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# Duality and Problem Reformulations

equivalent formulations of a problem can lead to very different duals

reformulating the primal problem can be useful when the dual is difficult to derive, or uninteresting

common reformulations
- introduce new variables and equality constraints
- make explicit constraints implicit or vice-versa
- transform objective or constraint functions, e.g., replace $f_0(x)$ by $\phi(f_0(x))$ with $\phi$ convex, increasing

# Introducing New Variables and Equality Constraints

$$\text{minimize } f_0(Ax + b)$$

- dual function is constant: $g = \inf_x L(x) = \inf_x f_0(Ax + b) = p^*$
- we have strong duality, but dual is quite useless

reformulated problem and its dual

$$\begin{array}{ll} \text{minimize} & f_0(y) \\ \text{subject to} & Ax + b - y = 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & b^T\nu - f_0^*(\nu) \\ \text{subject to} & A^T\nu = 0 \end{array}$$

dual function follows from

$$\begin{aligned} g(\nu) &= \inf_{x,y}(f_0(y) - \nu^T y + \nu^T Ax + b^T\nu) \\ &= \begin{cases} -f_0^*(\nu) + b^T\nu & A^T\nu = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

# Implicit Constraints

LP with box constraints: primal and dual problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & -\mathbf{1} \preceq x \preceq \mathbf{1} \end{array} \qquad \begin{array}{ll} \text{maximize} & -b^T \nu - \mathbf{1}^T \lambda_1 - \mathbf{1}^T \lambda_2 \\ \text{subject to} & c + A^T \nu + \lambda_1 - \lambda_2 = 0 \\ & \lambda_1 \succeq 0, \lambda_2 \succeq 0 \end{array}$$

Reformulation with box constraints made implicit

$$\begin{array}{ll} \text{minimize} & f_0(x) = \begin{cases} c^T x & -\mathbf{1} \preceq x \preceq \mathbf{1} \\ \infty & \text{otherwise} \end{cases} \\ \text{subject to} & Ax = b \end{array}$$

dual function $g(\nu) = \inf_{-\mathbf{1} \preceq x \preceq -\mathbf{1}} (c^T x + \nu^T (Ax - b))$

$$= -b^T \nu - \|A^T \nu + c\|_1$$

dual problem: maximize $-b^T \nu - \|A^T \nu + c\|_1$

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# Semidefinite Program

Primal SDP ($F_i, G \in \mathbf{S}^k$)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n - G \preceq 0 \end{array}$$

- Lagrange multiplier is matrix $Z \in \mathbf{S}^k$
- Lagrangian $L(x, Z) = c^T x + \mathbf{tr}(Z(x_1 F_1 + \cdots + x_n F_n - G))$
- dual function

$$g(Z) = \inf_x L(x, Z) = \begin{cases} -\mathbf{tr}(GZ) & \mathbf{tr}(F_i Z) + c_i = 0, i = 1, \ldots, n \\ -\infty & \text{otherwise} \end{cases}$$

dual SDP:

$$\begin{array}{ll} \text{maximize} & -\mathbf{tr}(GZ) \\ \text{subject to} & Z \succeq 0, \mathbf{tr}(F_i Z) + c_i = 0, i = 1, \ldots, n \end{array}$$

$p^* = d^*$ if primal SDP is strictly feasible ($\exists x$ with $x_1 F_1 + \cdots + x_n F_n \prec G$)