Flexibility Method

In 1864 James Clerk Maxwell published the first consistent treatment of the flexibility method for indeterminate structures. His method was based on considering deflections, but the presentation was rather brief and attraced little attention. Ten years later Otto Mohr independently extended Maxwell's theory to the present day treatment. The flexibility method will sometimes be referred to in the literature as Maxwell-Mohr method.

With the flexibility method equations of compatibility involving displacements at each of the redundant forces in the structure are introduced to provide the additional equations needed for solution. This method is somewhat useful in analyzing beams, framse and trusses that are statically indeterminate to the first or second degree. For structures with a high degree of static indeterminacy such as multi-story buildings and large complex trusses stiffness methods are more appropriate. Nevertheless flexibility methods provide an understanding of the behavior of statically indeterminate structures.

The fundamental concepts that underpin the flexibility method will be illustrated by the study of a two span beam. The procedure is as follows

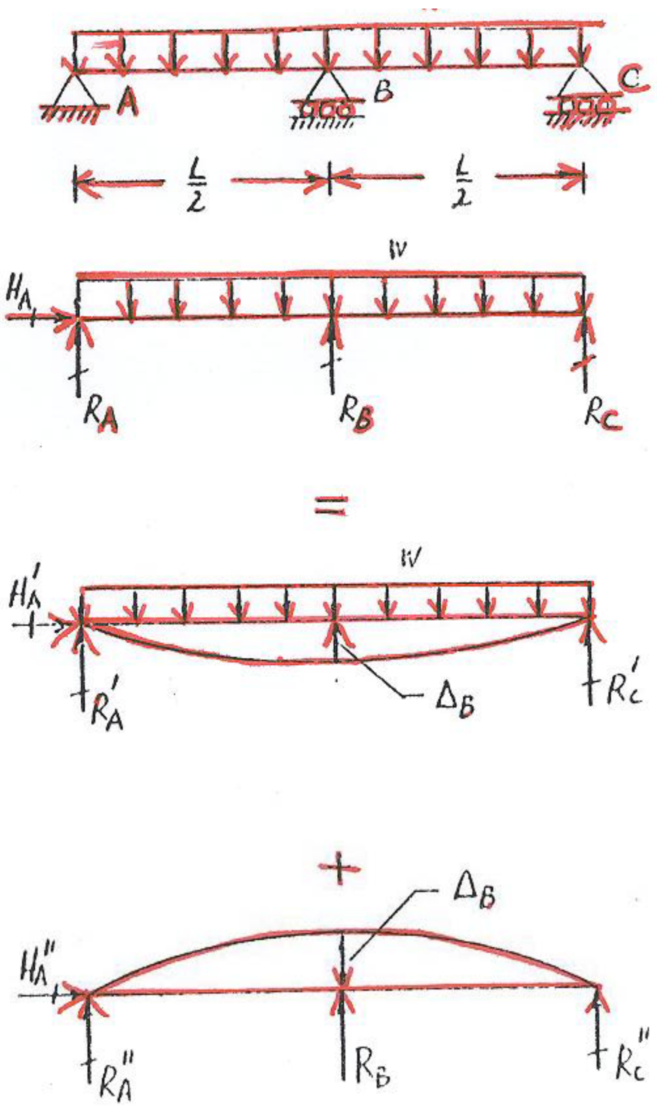
1. Pick a sufficient number of redundants corresponding to the degree of indeterminacy

2. Remove the redundants

3. Determine displacements at the redundants on released structure due to external or imposed actions

4. Determine displacements due to unit loads at the redundants <u>on the released structure</u>

5. Employ equation of compatibility, e.g., if a pin reaction is removed as a redundant the compatibility equation could be the summation of vertical displacements in the released structure must add to zero.



original beam

FBD

released beam

$$\Delta_B = \frac{5wL^4}{384EI}$$

$$\Delta_B = \frac{R_B L^3}{48EI}$$

The beam to the left is statically indeterminate to the first degree. The reaction at the middle support R_B is chosen as the redundant.

The released beam is also shown. Under the external loads the released beam deflects an amount Δ_B.

A second beam is considered where the released redundant is treated as an external load and the corresponding deflection at the redundant is set equal to Δ_B.

$$R_B = \left(\frac{5}{8}\right) w\,L$$

A more general approach consists in finding the displacement at *B* caused by a unit load in the direction of R_B. Then this displacement can be multiplied by R_B to determine the total displacement

Also in a more general approach a consistent sign convention for actions and displacements must be adopted. The displacements in the released structure at *B* are positive when they are in the direction of the action released, i.e., upwards is positive here.

The displacement at *B* caused by the unit action is

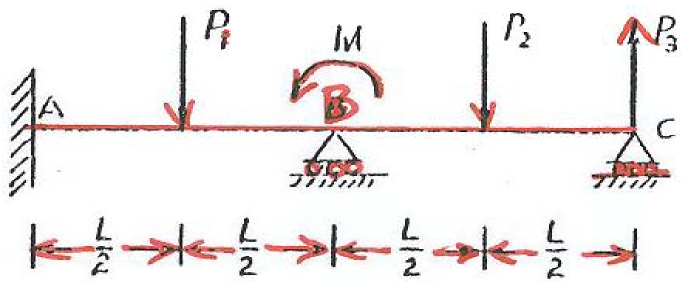
$$\delta_B = \frac{L^3}{48EI}$$

The displacement at *B* caused by R_B is $\delta_B\, R_B$. The displacement caused by the uniform load *w* acting on the released structure is

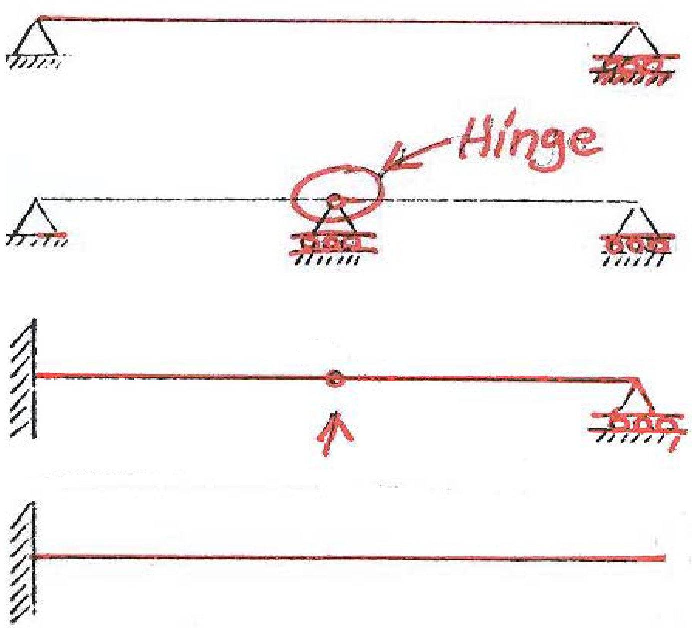
$$\Delta_B = -\frac{5\, w\, L^4}{384\, EI}$$

Thus by the compatibility equation

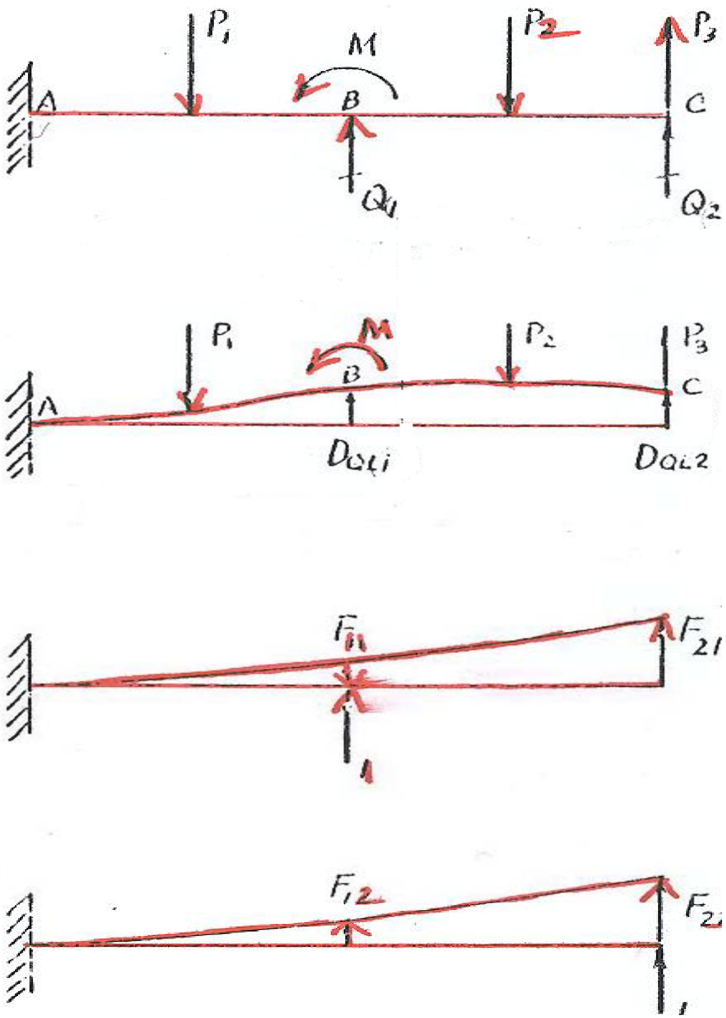
$$\Delta_B + \delta_B\, R_B = 0 \qquad\qquad R_B = -\frac{\Delta_B}{\delta_B} = \left(\frac{5}{8}\right) w\, L$$



If a structure is statically indeterminate to more than one degree, the approach used in the preceeding example must be further organized and more generalized notation is introduced.



Consider the beam to the left. The beam is statically indeterminate to the second degree. A statically determinate structure can be obtained by releasing two redundant reactions. Four possible released structures are shown.



The redundants chosen are at *B* and *C*. The redundant reactions are designated Q_1 and Q_2.

The released structure is shown at the left with all external and internal redundants shown.

D_{QL1} is the displacement corresponding to Q_1 and caused by only external actions on the released structure

D_{QL2} is the displacement corresponding to Q_2 caused by only external actions on the released structure.

Both displacements are shown in their assumed positive direction.

We can now write the compatibility equations for this structure. The displacements corresponding to Q_1 and Q_2 will be zero. These are labeled D_{Q1} and D_{Q2} respectively

$$D_{Q1} = D_{QL1} + F_{11}Q_1 + F_{12}Q_2 = 0$$

$$D_{Q2} = D_{QL2} + F_{21}Q_1 + F_{22}Q_2 = 0$$

In some cases D_{Q1} and D_{Q2} would be nonzero then we would write

$$D_{Q1} = D_{QL1} + F_{11}Q_1 + F_{12}Q_2$$

$$D_{Q2} = D_{QL2} + F_{21}Q_1 + F_{22}Q_2$$

The equations from the previous page can be written in matrix format as

$$\{D_Q\} = \{D_{QL}\} + [F]\{Q\}$$

where:

$\{D_Q\}$ - matrix of actual displacements corresponding to the redundant
$\{D_{QL}\}$ - matrix of displacements in the released structure corresponding to the redundant action [Q] and due to the loads
[F] - flexibility matrix for the released structure corresponding to the redundant actions [Q]
{Q} - matrix of redundant

$$\{D_Q\} = \begin{Bmatrix} D_{Q1} \\ D_{Q2} \end{Bmatrix} \qquad \{D_{QL}\} = \begin{Bmatrix} D_{QL1} \\ D_{QL2} \end{Bmatrix} \qquad \{Q\} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$$

The vector [Q] of redundants can be found by solving for them from the matrix equation on the previous overhead.

$$[F][Q] = [D_Q] - [D_{QL}]$$

$$[Q] = [F]^{-1}\left([D_Q] - [D_{QL}]\right)$$

To see how this works consider the previous beam with a constant flexural rigidity *EI*. If we identify actions on the beam as

$$P_1 = 2P \quad M = PL \quad P_2 = P \quad P_3 = P$$

Since there are no displacements corresponding to Q_1 and Q_2, then

$$D_Q = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The vector $[D_{QL}]$ represents the displacements in the released structure corresponding to the redundant loads. These displacements are

$$D_{QL1} = \frac{13PL^3}{24EI} \qquad D_{QL2} = \frac{97PL^3}{48EI}$$

The positive signs indicate that both displacements are upward. In a matrix format

$$\left[D_{QL}\right] = \frac{PL^3}{48EI}\begin{bmatrix} 26 \\ 97 \end{bmatrix}$$

The flexibility matrix [F] is obtained by subjecting the beam to unit load corresponding to Q_1 and computing the following displacements

$$F_{11} = \frac{L^3}{3EI} \qquad F_{21} = \frac{5L^3}{6EI}$$

Similarly subjecting the beam to unit load corresponding to Q_2 and computing the following displacements

$$F_{12} = \frac{5L^3}{6EI} \qquad F_{22} = \frac{8L^3}{3EI}$$

The flexibility matrix is

$$[F] = \frac{L^3}{6EI}\begin{bmatrix} 2 & 5 \\ 5 & 16 \end{bmatrix}$$

The inverse of the flexibility matrix is

$$[F]^{-1} = \frac{6EI}{7L^3}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}$$

As a final step the redundants [Q] can be found as follows

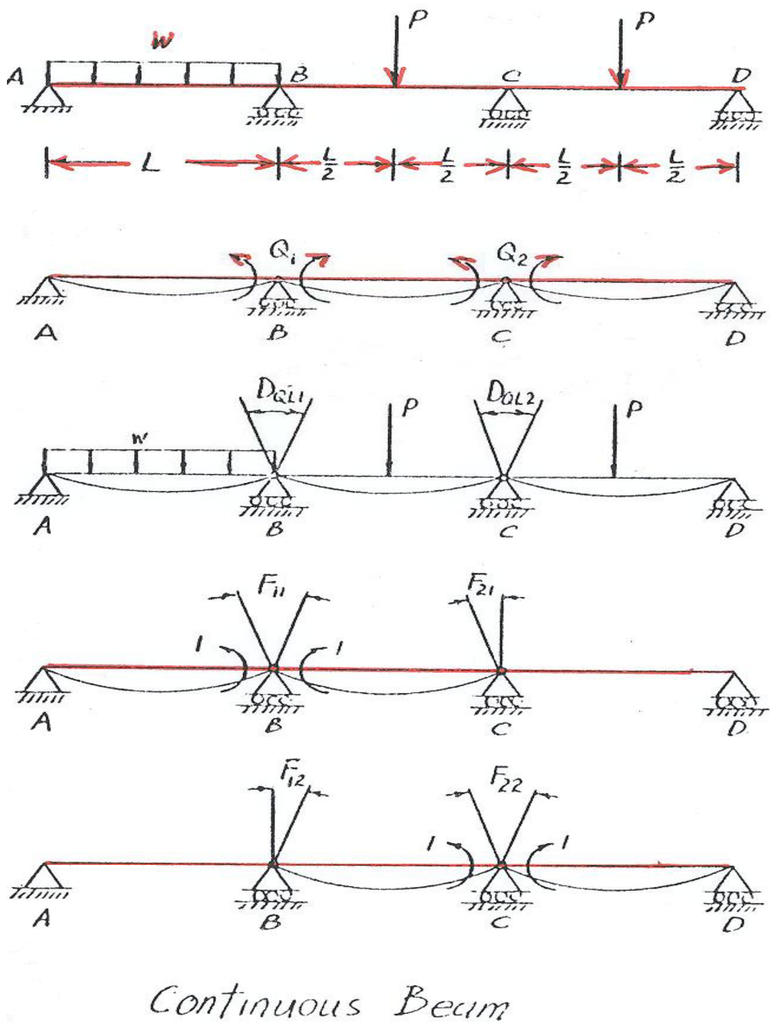
$$[Q] = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = [F]^{-1}\left([D_Q] - [D_{QL}]\right)$$

$$= \left(\frac{6EI}{7L^3}\right)\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \left(\frac{PL^3}{48EI}\right)\begin{bmatrix} 26 \\ 97 \end{bmatrix}\right\}$$

$$= \left(\frac{P}{56}\right)\begin{bmatrix} 69 \\ -64 \end{bmatrix}$$

The redundants have been obtained. The other unknown reactions can be found from the released structure. Displacements can be computed from the known reactions on the released structure and imposing the compatibility equations.

Example



A three span beam shown at the left is acted upon by a uniform load *w* and concentrated loads *P* as shown. The beam has a constant flexural rigidity *EI*.

Treat the supports at *B* and *C* as redundants and compute these redundants.

In this problem the bending moments at *B* and *C* are chosen as redundants to indicate how unit rotations are applied to released structures.

Each redundant consists of two moments, one acting in each adjoining span.

The displacements corresponding to the two redundants consist of two rotations – one for each adjoining span. The displacement D_{QL1} and D_{QL2} corresponding to Q_1 and Q_2. These displacements will be caused by the loads acting on the released structure.

The displacement D_{QL1} is composed of two parts, the rotation of end *B* of member *AB* and the rotation of end *B* of member *BC*

$$D_{QL1} = \frac{wL^3}{24EI} + \frac{PL^2}{16EI}$$

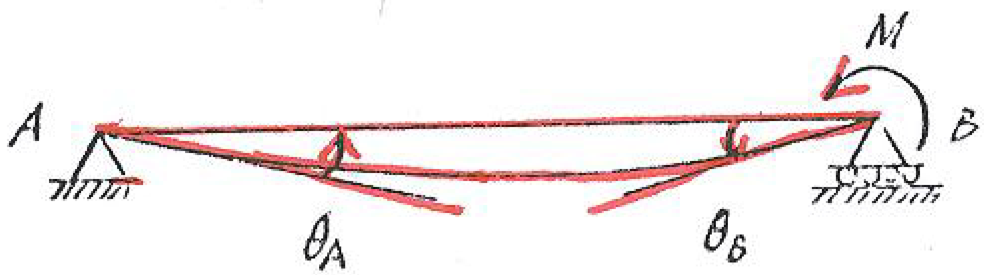
Similarly,

$$D_{QL2} = \frac{PL^2}{16EI} + \frac{PL^2}{16EI} = \frac{PL^2}{8EI}$$

such that

$$D_{QL} = \frac{L^2}{48EI}\begin{bmatrix}(2wL+3P)\\ 6P\end{bmatrix}$$

The flexibility coefficients are determined next. The flexibility coefficient F_{11} is the sum of two rotations at joint B. One in span *AB* and the other in span *BC* (not shown below)



$$F_{11} = \frac{L}{3EI} + \frac{L}{3EI} = \frac{2L}{3EI}$$

Similarly the coefficient F_{21} is equal to the sum of rotations at joint *C*. However, the rotation in span *CD* is zero from a unit rotation at joint *B*. Thus

$$F_{21} = \frac{L}{6EI}$$

Similarly

$$F_{22} = \frac{L}{3EI} + \frac{L}{3EI} = \frac{2L}{3EI}$$

$$F_{12} = \frac{L}{6EI}$$

The flexibility matrix is

$$F = \frac{L}{6EI}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

The inverse of the flexibility matrix is

$$F^{-1} = \frac{2EI}{5L}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$$

As a final step the redundants [Q] can be found as follows

$$[Q] = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = [F]^{-1}\left([D_Q] - [D_{QL}]\right)$$

$$= \frac{2EI}{5L}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \frac{L^2}{48EI}\begin{bmatrix} (2wL + 3P) \\ 6P \end{bmatrix}\right\}$$

$$= -\frac{L}{120}\begin{bmatrix} (8wL + 6P) \\ (-2wL + 21P) \end{bmatrix}$$

and

$$Q_1 = -\frac{wL^2}{15} - \frac{PL}{20} \qquad Q_2 = \frac{wL^2}{60} - \frac{7PL}{40}$$