Lecture 8 Notes

Chapter 11. From Randomness to Probability

Learning Outcome

- Recognize probability as long-run relative frequency for repeatable random phenomena.

- Know what is legitimate assignment of probability.

- Understand the Law of Large Numbers

- Perform simple probability calculations using Venn Diagrams, the Complement Rule, and the Addition Rule

Random Phenomena

- A random phenomena is a situation in which we know what outcomes can possibly occur, but we do not know which particular outcome will happen.

- In general, each occasion upon which we observe a random phenomenon is called a trial.

- At each trial, we note results of the random phenomena and call that trail's outcome.

Thus, a random phenomenon consists of trials and each trial has an outcome.

- Outcomes combine to make events. (note: each individual outcome is also an event).

- Collection of all possible outcomes is called a sample space. We denote sample space with the letter S.

Example: Choosing a candidate in an electoral campaign.
Sample space: S = {Yes, No}

Example: Responding to an item on a survey that uses 7-point Likert scale.
Sample Space: S = {1, 2, 3, 4, 5, 6, 7}

Thus, for a random phenomenon, the sample space S is the set of all possible outcomes of each trial.

Event

- An event is an outcome or a set of outcomes of a random phenomenon.
- That is, an event is a subset of the sample space.

Example: Outcomes for (whether or not) choosing a candidate in an electoral campaign

Let Y denote "Yes: for choosing the candidate"

Let N denote "No: for not choosing the candidate"

Suppose we randomly select two eligible voters. Then, the sample space (S) for (whether or not) choosing the candidate in an electoral campaign is S = {YY, YN, NY, NN}.

Let event A denote possible outcomes for when exactly one of the two eligible voters choose the candidate (e.g., select "Yes") in an electoral campaign. Then, A = {YN, NY}.

The Law of Large Numbers

- The Law of Large Numbers (LLN) says that as we repeat a random process over and over, the proportion of times (fraction of times) that an event occurs does settle down to one number. We call this number probability of the event.

This LLN is based on two assumptions:

1. The random phenomenon we are studying must not change (that is, the outcomes must have the same probability for each trial).
2. The events must be independent (that is, the outcome of one trial does not affect the outcomes of the others).

- So, LLN says that as the number of independent trials increases, the long run relative frequency of repeated events get closer and closer to a single value.

- Because this definition is based on repeatedly observing the event's outcome; this definition of probability is often called empirical probability.

- Empirical Probability: For any Event A, P(A) = $\frac{\#\, times\ A\ occurs}{total\ \#\ of\ trails}$ in the long run. P(A) means probability of A.

Independent Trials

When thinking about what happens with combinations of outcomes, things are simplified if the individual trials are independent.

Roughly speaking, this means that the outcome of one trial doesn't influence or change the outcome of another.

- For example, coin flips are independent
- Another example:

  The General Social Survey asked respondents how often they spend their evening with someone in their neighbourhood?

  The participants were asked to choose their response from the following categories:

  Almost Daily, Several Times A Week, Several Times A Month, Once A Month, Several Times A Year, Once A Year, Never

Do you think for a respondent their response (outcome) to this survey item influence or change the outcome of another respondent's in this survey?

Theoretical Probability

Sometimes can argue (in a mathematical model not from observation) what probabilities should be:

For example, roll a die: in theory it is a perfect cube, so each of the 6 faces equally likely to occur:
e.g., P(6)=1/6.

More generally, any time you have equally likely outcomes, prob. of event A is: P(A) = $\frac{Number\ of\ Outocmes\ in\ A}{Number\ of\ Outcomes\ in\ S}$

General Social Survey (GSS) Example

The General Social Survey asked respondents how often they spend their evening with someone in their neighbourhood?

Possible Outcome	Probability (Possible Outcome)
Almost Daily	0.058
Several Times A Week	0.183
Several Times A Month	0.116
Once A Month	0.147
Several Times A Year	0.126
Once A Year	0.094
Never	0.277

- **Suppose A denotes the event:**

An American adult who "almost daily" spends time with someone who lives in his or her neighbourhood.

A: Almost Daily

- **P(A) is the probability for a randomly selected adult who almost daily spends time with someone who lives in his or her neighbourhood.**

- **We have: P(A) = 0.058**

Probability Rules

- **Probability Assignment Rule:** The probability of any event (e.g., A) should be between 0 and 1. That is,

$$0 \leq P(A) \leq 1$$

- **Total Probability Rule:** If S is the sample spaces in a probability model, then P(S) = 1.
  - That means, sum of all probabilities should equal to 1.
- **Complement Rule:** The complement of any event (e.g., A) is the event that does not occur (not A).
  - We denote the complement of event A as A^C
  - In that case, we have: P(A) = 1 - P(A^C)

General Social Survey (GSS) Example

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Almost Daily	0.058
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Several Times A Year	0.126
Once A Year	0.094
Never	0.277

Check that the sum of probabilities equals 1:

0.058 + 0.183 + 0.116 + 0.147 + 0.126 + 0.094 + 0.277 = 1.00
So, P(S) = 1
(actually 1.001 but each probability was subjected to proportion rounding for decimal points).

General Social Survey (GSS) Example

The General Social Survey asked respondents how often they spend their evening with someone in their neighbourhood?

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- **The complement of A is "not A".**

A^C **(Not A): That is, An American adult who does NOT "almost daily" spend time with someone who lives in his or her neighbourhood.**

- **P(Not A) is: Randomly select an American adult. The probability that this person does NOT almost daily spend time with someone who lives in his or her neighbourhood is:**

$$P(A^C) = \text{P(not A)} = 1 - \text{P(A)} = 1 - 0.058 = 0.942$$

The Addition Rule for Disjoint (or Mutually Exclusive) Events

Two events A and B are disjoint if they have no outcomes in common and so can never occur simultaneously.

- If A and B are disjoint, then we have: $P(A \text{ or } B) = P(A) + P(B)$. This is the **addition rule for disjoint events**.

Note: This idea can be extended for more than two events.

General Social Survey (GSS) Example

The General Social Survey asked respondents how often they spend their evening with someone in their neighbourhood?

Possible Outcome	Probability (Possible Outcome)
Almost Daily	0.058
Several Times A Week	0.183
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- **Consider two events such that they do not overlap (disjoint events).**
    - **A: Almost Daily**
    - **SW: Several Times A Week**
    - **A and SW are disjoint events**
- **Probability of "either" A "or" SW,**

  We have:
  P(A or SW) = P(A) + P(SW)
  = 0.058 + 0.183 = 0.241

The General Additive Rule

For any two events A and B: P(A or B) = P(A) + P(B) – P(A and B)

- Add the probability of two events.
- Then, subtract out the probabilities of their intersection.

Example: The General Additive Rule

A retail establishment accepts either the American Express or the VISA credit card. A total of 24% of its customers carry an American Express card, 61% carry a VISA card, and 11% carry both. What percentage of its customers, carry a card that the establishment will accept?

Example: The General Additive Rule

Example - Solution:

A retail establishment accepts either the American Express or the VISA credit card. A total of 24% of its customers carry an American Express card, 61% carry a VISA card, and 11% carry both. What percentage of its customers, carry a card that the establishment will accept?

Let A be the event for carrying American Express; $P(A) = 0.24$
Let V be the event for carrying Visa; $P(V) = 0.61$
Let A and V be the event for carrying both card; $P(A \text{ and } V) = 0.11$
Let A or V be the event for carrying either cards; $P(A \text{ or } V) = ?$

$$P(A \text{ or } V) = P(A) + P(V) - P(A \text{ and } V) = 0.24 + 0.61 - 0.11 = 0.74$$

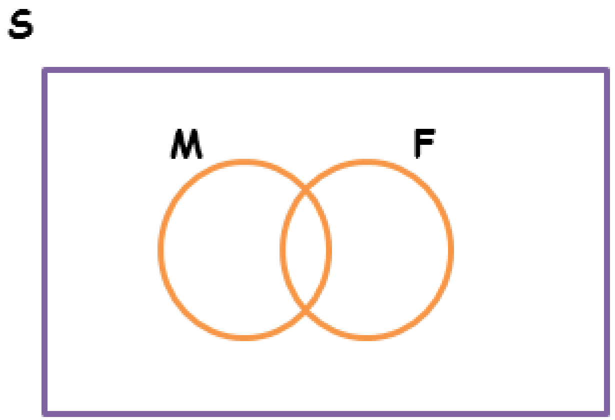
Example (Using Venn Diagram): The General Additive Rule

Among 33 students in a class 17 of them earned A's on the midterm exam, 14 earned A's on the final exam, and 11 did not earn A's on either examination. What is the probability that a randomly selected student from this class earned A's on both exams?

Example (Using Venn Diagram): The General Additive Rule

Example - Solution:

Among 33 students in a class 17 of them earned A's on the midterm exam, 14 earned A's on the final exam, and 11 did not earn A's on either examination. What is the probability that a randomly selected student from this class earned A's on both exams?



We will illustrate solving this problem with the use of a Venn Diagram.

Let M denote earning A on Midterm exam; P(M) = $\frac{17}{33}$ = 0.515

Let F denote earning A on final exam; P(F) = $\frac{14}{33}$ = 0.424

P(neither earning A on M or F) = $\frac{11}{33}$ = 0.333

That means (by using the complement of event "neither earning A on M or F"),

P(either earning A on M or F) = 1 - P(neither earning A on M or F) = 1- $\frac{11}{33}$ = 1 - 0.333 = 0.667 (or $\frac{22}{33}$)

And, we know that P(either earning A on M or F) = P(M) + P(F) - P(earning A on M and F)

Therefore, we can solve for P(earning A on M and F):

$\frac{22}{33} = \frac{17}{33} + \frac{14}{33}$ - P(earning A on M and F) $\Rightarrow$ P(earning A on M and F) = $\frac{31}{33} - \frac{22}{33} = \frac{9}{33}$ = 0.2727

Chapter 12. Probability Rules

Learning Outcomes

- Define and calculate conditional probability
- Understand and use independence
- Apply the General Multiplicative Rule
- Picture Probability with tables, and Venn Diagrams

Probability on Conditions

Consider the following examples:

- Is the probability of perceived mental health the same for both males and for females?
- Is the probability of perceived mental health the same for all ages?
- Is the reported job satisfaction the same for all jobs?

The above are examples of conditional probabilities.

Note: The concept of independence was introduced in lecture 1 when we looked at contingency tables and asked whether the distribution of one variables was the same for each category of another variable. In this lecture we will see how useful an assumption of independence can be in computing probabilities.

Example: Spending Evening with Neighbor and Sex of the Respondents

Frequency Distribution				
Cells contain: -N of cases		SEX		
		1 MALE	2 FEMALE	***ROW TOTAL***
SOCOMMUN	1: ALMOST DAILY	864	1,050	*1,914*
	2: SEV TIMES A WEEK	2,700	3,336	*6,036*
	3: SEV TIMES A MNTH	1,700	2,110	*3,810*
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	5: SEV TIMES A YEAR	1,982	2,166	*4,148*
	6: ONCE A YEAR	1,482	1,602	*3,084*
	7: NEVER	3,506	5,600	*9,106*
	COL TOTAL	*14,390*	*18,543*	*32,933*

Suppose M and A are two possible outcomes.

Let M: Male
Let N: Almost Daily

- The probability that a randomly selected American adult is a male (a marginal probability):

  P(M) =

- The probability that a randomly selected American adult almost daily spends time with someone who lives in their neighbourhood is (a marginal probability):

  P(A) =

- The probability that a randomly selected American adult is a male who almost daily spends time with someone who lives in his neighbourhood is (a joint probability):

  P(M and A) =

Example: Spending Evening with Neighbor and Sex of the Respondents

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Suppose M and A are two possible outcomes.

Let M: Male
Let A: Almost Daily

- The probability that a randomly selected American adult is a male (marginal probability):

$$P(M) = \frac{14390}{32933} \cong 0.44$$

- The probability that a randomly selected American adult almost daily spends time with someone who lives in their neighbourhood is (marginal probability):

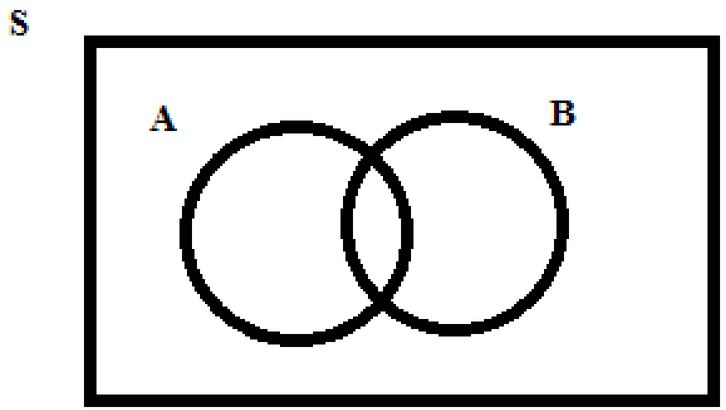
$$P(A) = \frac{1914}{32933} \cong 0.06$$

- The probability that a randomly selected American adult is a male who almost daily spends time with someone who lives in his neighbourhood is (joint probability):

$$P(M \text{ and } A) = \frac{864}{32933} \cong 0.03$$

Conditional Probability

- Consider two events A, and B.
- When $P(A) > 0$, the conditional probability of B given A is: $P(B|A) = \frac{P(B \text{ and } A)}{P(A)}$
- To find probability of the event B given A, we need to restrict our attention to outcomes in A.
- We then find in what fraction of those outcomes B has also occurred.



Conditional Probabilities: Spending Evening with Neighbor and Sex of the Respondents

What if we are given the information that the randomly selected person is a male, would that change the probability that the person spend almost daily with someone in their neighbourhood?

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Suppose M and A are two possible outcomes.

Let M: Male Let A: Almost Daily

- The probability that a randomly selected American adult almost daily spends time with someone who lives in their neighbourhood is:

$$P(A) = \frac{1914}{32933} \cong 0.06$$

- The conditional probability that a randomly selected American adult spends almost daily with someone in their neighborhood given that the person is a male ?:

Example: Spending Evening with Neighbor and Sex of the Respondents

What if we are given the information that the randomly selected person is a male, would that change the probability that the person spend almost daily with someone in their neighbourhood?

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Suppose M and A are two possible outcomes, then

Let M: Male Let A: Almost Daily

- The probability that a randomly selected American adult almost daily spends time with someone who lives in their neighbourhood is (marginal probability):

$$P(A) = \frac{1914}{32933} \cong 0.06$$

- The conditional probability that a randomly selected American adult spends almost daily with someone in their neighborhood given that the person is a male:

$$P(A \text{ given } M) = \frac{864}{14390} \cong 0.06$$

- P(A) = P(A given M) = 0.06 . The unconditional probability of randomly selecting an American adult who spends almost daily with someone in their neighbourhood is unchanged once we condition on the event being male. Therefore, the two events spending evening almost daily with someone in neighbourhood and being a male are independent.

Example: Epidural and Nursing At Six Months

There is some concern that if a woman has an epidural to reduce pain during child birth, the drug can get into the baby's bloodstream, making the baby sleepier and less willing to breastfeed. In 2006, the International Breastfeeding Journal published results of a study conducted at Sydney University. Researchers followed up on 1178 births, noting whether the mother had an epidural and whether the baby was still nursing after six months. The results are summarized in the following contingency table.

Epidural	Breastfeeding at 6 Months	
	Yes	No
Yes	206	190
No	498	284

- [P(B) = 0.52] ≠ [P(B given E) = 0.60]. Therefore, the two events breastfeeding at six months and having epidural are depended (associated).

- **Note:** statistical dependence only means association as we discussed in lecture 4, there are many possible explanations for associations other than cause-and-effect.

Suppose E and B are two possible outcomes, then

Let E: Epidural (Yes)
Let B: Breastfeeding at six months (Yes)

- Randomly select a mother who had epidural. What is the probability that this mom is breastfeeding at six months:

$$P(B \text{ given } E) = \frac{206}{(206+190)=396} \cong 0.52$$

- Randomly select a mother. What is the marginal probability that this mom is breastfeeding at six months:

$$P(B) = \frac{(206+498)=704}{1178} \cong 0.60$$

Example: Epidural and Nursing At Six Months

Conditional Percentages are also displayed in each cell.

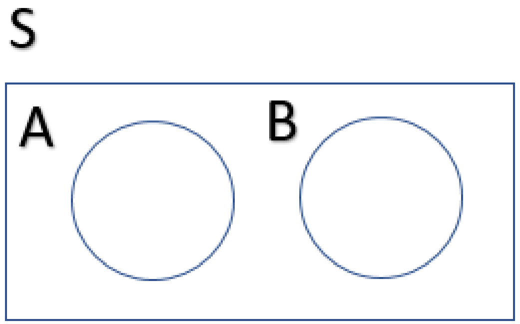
Epidural	Breastfeeding at 6 Months	
	Yes	No
Yes	206 (52%)	190 (48%)
No	498 (64%)	284 (36%)

Conclusion:
Mothers who did not have epidural are more likely to breastfeed at 6 months than those who had epidural.

Disjoint (Mutually Exclusive) Events

- If two events A, B are disjoint, they can't both happen.

Suppose A happens, then P(B|A) must be 0, whatever P(B) is.



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Example:

The conditional probability that a randomly selected American adult spends almost daily with someone in their neighborhood given that this person spends several times a week with someone in their neighbourhood is:

P(Almost Daily given Several Times a Week) = 0

The General Multiplication Rule

- The probability that two events, A and B, both occur is the probability that the event A occurs multiplied by the probability that event B also occurs – that is, by the probability that event B occurs given A occurs.

- Rearrange the equation for conditional probability:

We can express $P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$ as $P(A \text{ and } B) = P(B) \times P(A|B)$

We can express $P(B|A) = \frac{P(B \text{ and } A)}{P(A)}$ as $P(A \text{ and } B) = P(A) \times P(B|A)$

- Note that P(B and A) is the same as P(A and B).

Example: Spending Evening with Neighbor and Sex of the Respondents

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Suppose M and A are two possible outcomes.

Let M: Male
Let N: Almost Daily

- The probability that a randomly selected American adult is a male who almost daily spends time with someone who lives in his neighbourhood is (a joint probability):

P(M and A) = P(M) x P(A|M)

P(A given M) = $\frac{864}{14390} \cong 0.06$

P(M) = $\frac{14390}{32933} \cong 0.44$

P(M and A) = 0.06 x 0.44 ≅ 0.03

Check: P(M and A) = $\frac{864}{32933} \cong 0.03$

Another Conditional Example Using Multiplicative Rule

Consider the contingency table below for applicant's admission outcome (accepted or rejected) to law school for males and females.

Gender	Admission		Total
	Accepted	Rejected	
Male	10	90	100
Female	100	200	300
Total	110	290	400

Randomly select two male applicants to law school. What is probability that they are both rejected?

Another Conditional Example Using Multiplicative Rule

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Randomly select two male applicants to law school. What is probability that they are both rejected?

Let R1 be the event "1st one rejected"
Let R2 be the event "2nd one rejected"

$P(R1 \text{ and } R2) = P(R1) \times P(R2|R1) = 90/100 \times 89/99 \cong 0.81$

Independent Events

- Statistical Independence is a critically important statistical concept often used as an assumption in probability calculations and statistical analysis.

- Two events are independent, if the occurrence of one does not influence or change the occurrence of the other event.

- Events A and B are independent if (and only if) the probability of B is the same when we are given that A has occurred.

That is, two events A and B with $P(A) > 0$, $P(B) > 0$ are **independent** **<u>if and only if</u>** $P(B|A)=P(B)$

That means, if P(B given A) = P(B), the events A and B are independent.

- **<u>If A and B are independent</u>**, then P(A and B) = P(A) x P(B)

Example of Independent Events

The gene for albinism in humans is recessive. That is, carriers of this gene have probability 1/2 of passing it to a child, and the child is albino only if both parents pass the albinism gene. Parents pass their genes independently of each other. If both parents carry the albinism gene, what is the probability that their first child is albino?

Example of Independent Events

The gene for albinism in humans is recessive. That is, carriers of this gene have probability 1/2 of passing it to a child, and the child is albino only if both parents pass the albinism gene. Parents pass their genes independently of each other. If both parents carry the albinism gene, what is the probability that their first child is albino?

$0.5 \times 0.5 = 0.25$

Example of Independent Events

Refer to the example in the previous slide.

If they have two children (who inherit independently of each other), what is the probability that

a) both are albino?

b) neither is albino?

c) exactly one of the two children is albino?

Example of Independent Events

If they have two children (who inherit independently of each other), what is the probability that

a) both are albino? 0.25 x 0.25 = 0.0625

b) neither is albino? (1-0.25) x (1-0.25) = 0.75 x 0.75 = 0.5625

c) exactly one of the two children is albino?

Let S denote the Sample Space for all possible outcomes for whether the two children are Albino or not
S = {(Albino, Albino), (Albino, Not Albino), (Not Albino, Albino), (Not Albino, Not Albino)}
So, the event “exactly one of the two children is albino” = {(Albino, Not Albino), (Not Albino, Albino)}
Which has probability: (0.25 x 0.75) + (0.75 x 0.25) = 0.375

Example of Independent Events

If they have three children (who inherit independently of each other), what is the probability that *at least one* of them is albino?

Example of Independent Events

If they have three children (who inherit independently of each other), what is the probability that *at least one* of them is albino?

_____ _____ _____

At least one means: 1, 2, or 3

In three children, we can have: 0, 1, 2, 3 albino.

Use the idea of the complement of the probability of the event that all three children are *not* albino:
Recall: Probability of a child is albino if both parents carry the albinism gene is 0.25 (see slide #18).
Therefore, the probability of a child is *not* albino if both parents carry the albinism gene is 1- 0.25 = 0.75
So, P(at least one of the three children is albino) = 1 – P(none of the three children is albino)
= 1 – (0.75 x 0.75 x 0.75) = 1 – 0.4219 = 0.5781