Lecture 9

Covariant electrodynamics

1. Lorentz Group

Consider Lorentz transformations – pseudo-orthogonal transformations in 4-dimentional vector space (Minkowski space)

Notation:
T = transposition $(x_1, x_2, x_3, x_4)^T = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

4-vectors: $x = (ct, \mathbf{r})^T \equiv (t, \mathbf{x})^T, \quad q = (\omega/c, \mathbf{q})^T$ **(1)**

with contravariant components (upper indices)

$$x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z;$$
$$q^0 = \omega/c,\ q^1 = q_x,\ q^2 = q_y,\ q^3 = q_z \qquad (2)$$

Note: covariant components (lower indices) are defined by

$$x_0 = ct, \quad x_1 = -x, \quad x_2 = -y, \quad x_3 = -z \qquad (3)$$
$$q_0 = \omega/c,\ q_1 = -q_x,\ q_2 = -q_y,\ q_3 = -q_z$$

The transformation between covariant and contravariant components is

$$x_\mu = \sum_{\nu=0}^{3} g_{\mu\nu} x^\nu =: g_{\mu\nu} x^\nu \qquad \mu = 0, 1, 2, 3 \qquad (4)$$

1. Lorentz Group

Pseudometric tensor:

$$(g_{\mu\nu}) = (g^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{5}$$

For any space-time vector we get:

$$\sum_{\mu,\nu=0}^{3} g_{\mu\nu} x^\mu x^\nu = \sum_{\mu=0}^{3} x^\mu x_\mu = \sum_{\mu=0}^{3} x'^\mu x'_\mu . \tag{6}$$

where the vector *x'* is the result of a Lorentz transformation $\Lambda^\mu{}_\nu$:

$$\boxed{x'^\mu = \sum_\nu \Lambda^\mu{}_\nu x^\nu ;} \qquad \mu, \nu = 0, 1, 2, 3 \tag{7}$$

such that

$$\sum_{\mu=0}^{3} x^\mu x_\mu = c^2 t^2 - \mathbf{r}^2 = \text{const.} \tag{8}$$

- **Pseudo-orthogonality relation:**

$$\sum_{\mu,\nu} g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma} \tag{9}$$

which implies that the pseudometric tensor is Lorentz invariant !

1. Lorentz Group

In matrix form:

$$\Lambda^T g \Lambda = g \qquad \text{(10)}$$

where $g^2 = 1_4$ **, and 1_4 is the 4x4 unitary matrix** ➔ $(g\Lambda^T g)\Lambda = 1_4$ **(11)**

The inverse Lorentz transformation reads:

$$\Lambda^{-1} = g\Lambda^T g. \qquad \text{(12)}$$

From (9), (6) we obtain:

$$\sum_\mu x'^\mu x'_\mu = \sum_{\mu\nu} g_{\mu\nu} x'^\mu x'^\nu = \sum_{\mu,\nu,\rho,\sigma} g_{\mu\nu} \Lambda^\mu{}_\rho x^\rho \Lambda^\nu{}_\sigma x^\sigma = \sum_{\rho\sigma} g_{\rho\sigma} x^\rho x^\sigma = \sum_\rho x_\rho x^\rho. \qquad \text{(13)}$$

For the transformation in x^1 direction with velocity $\beta = v/c$
the transformation matrix is given by

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{(14)}$$

with determinant

$$\det \Lambda = \gamma^2 - \gamma^2\beta^2 = \gamma^2(1-\beta^2) = 1 \qquad \text{(15)}$$

$\gamma^2 = 1/(1-\beta^2)$

1. Lorentz Group

Properties of Lorentz transformations:

1) Two Lorentz transformations

$$x'^{\mu} = \sum_{\nu} \Lambda^{\mu}{}_{\nu} x^{\nu}, \quad x''^{\rho} = \sum_{\mu} \Lambda'^{\rho}{}_{\mu} x'^{\mu}, \tag{16}$$

applied successively

$$x''^{\rho} = \sum_{\mu,\nu} \Lambda'^{\rho}{}_{\mu} \Lambda^{\mu}{}_{\nu} x^{\nu} = \sum_{\nu} \Lambda''^{\rho}{}_{\nu} x^{\nu} \tag{17}$$

appear as a single Lorentz transformation. In matrix form:

$$\Lambda''^{T} g \Lambda'' = (\Lambda'\Lambda)^{T} g (\Lambda'\Lambda) = \Lambda^{T} \underbrace{\Lambda'^{T} g \Lambda'}_{g} \Lambda = \Lambda^{T} g \Lambda = g, \tag{18}$$

$$\Lambda'^{T} g \Lambda' = \Lambda^{T} g \Lambda = g \tag{19}$$

2) The neutral element is the 4x4 unitary matrix 1_4 for a Lorentz boost with $v=0$

3) For each Λ exists the inverse transformation:

$$\det\,(\Lambda^{T} g \Lambda) = \det\,\Lambda^{T}\,\det\,g\,\det\,\Lambda = (\det\,\Lambda)^{2}\det\,g = \det\,g = -1, \tag{20}$$

$$(\det\,\Lambda)^{2} = 1 \;\Rightarrow\; \det\,\Lambda = \pm 1 \neq 0. \tag{21}$$

4) The matrix multiplication is associative ➔ the Lorentz transformation is associative also, (i.e. the order in which the operations are performed does not matter as long as the sequence of the operands is not changed.)

2. Lorentz Group: scalars, vectors, tensors

1) Lorentz-Scalars transform as:

$$\Psi \overset{\Lambda}{\rightarrow} \Psi' = \Psi \qquad (22)$$

Lorentz transformation

Ψ: e.g. electric charge

2) Lorentz-Vectors are defined by the transformation properties as:

$$A^\mu \rightarrow A'^\mu = \sum_\nu \Lambda^\mu{}_\nu A^\nu \qquad (23)$$

→ Transformation of covariant components:

$$\begin{aligned} A'_\mu &= \sum_\nu g_{\mu\nu} A'^\nu = \sum_{\nu,\rho} g_{\mu\nu} \Lambda^\nu{}_\rho A^\rho \\ &= \sum_{\nu,\rho,\sigma} g_{\mu\nu} \Lambda^\nu{}_\rho g^{\rho\sigma} A_\sigma = \sum_\sigma (g\Lambda g)_\mu{}^\sigma A_\sigma \\ &= \sum_\sigma (\Lambda^{-1})^\sigma{}_\mu A_\sigma, \end{aligned} \qquad (24)$$

2. Lorentz Group: vectors, tensors

3) Lorentz-Tensor

Transformation of the covariant components of a Lorentz tensor is defined as:

$$T'^{\mu\nu} = \sum_{\rho,\sigma} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} T^{\rho\sigma} \qquad (25)$$

→ Transformation of contravariant-covariant components:

$$T_{\mu}{}^{\nu} = \sum_{\rho} g_{\mu\rho} T^{\rho\nu}, \quad T^{\mu\nu} = \sum_{\rho,\sigma} g^{\mu\rho} g^{\nu\sigma} T_{\rho\sigma}. \qquad (26)$$

$$\Longrightarrow \quad T'_{\mu}{}^{\nu} = \sum_{\rho,\sigma} (\Lambda^{-1})^{\rho}{}_{\mu} \Lambda^{\nu}{}_{\sigma} T_{\rho}{}^{\sigma}. \qquad (27)$$

4) Higher tensor products (Kronecker products)

e.g. tensor of 3rd order: $C_{\mu}{}^{\nu}{}_{\rho} = A_{\mu} T^{\nu}{}_{\rho}$ **(28)**

A_{μ} is a vector-component
$T^{\nu}{}_{\rho}$ is a tensor-component

2. Lorentz Group: vectors, tensors

5) Covariant trace

Consider e.g. the trace of a tensor of 2nd order:

$$\Psi = \sum_\mu T_\mu{}^\mu \tag{29}$$

After Lorentz transformation it has to transform as a scalar:

$$\Psi' = \sum_\mu T'^{\,\mu}_{\mu} = \sum_{\mu,\nu,\rho} (\Lambda^{-1})^\nu{}_\mu \Lambda^\mu{}_\rho T_\nu{}^\rho = \sum_{\nu,\rho} \delta^\nu_\rho T_\nu{}^\rho = \sum_\nu T_\nu{}^\nu = \Psi \tag{30}$$

Cf. Eq.(27)

6) Fundamental Levi-Civita-Pseudo-tensor (or permutation tensor)

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } (\mu,\nu,\rho,\sigma) \text{ is an even permutation of } (0,1,2,3) \\ -1 & \text{if } (\mu,\nu,\rho,\sigma) \text{ is an odd permutation of } (0,1,2,3) \\ 0 & \text{in all other cases} \end{cases} \tag{31}$$

Pseudo-tensor,
since $$\epsilon^{\alpha\beta\gamma\delta} = \sum_{\mu,\nu,\rho,\sigma} g^{\alpha\mu} g^{\beta\nu} g^{\gamma\rho} g^{\delta\sigma} \epsilon_{\mu\nu\rho\sigma} = \det g \, \epsilon_{\alpha\beta\gamma\delta} = -\epsilon_{\alpha\beta\gamma\delta} \tag{32}$$

After Lorentz transformation we get:

$$\epsilon'^{\mu\nu\rho\sigma} = \sum_{\alpha,\beta,\gamma,\delta} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\rho{}_\gamma \Lambda^\sigma{}_\delta \epsilon^{\alpha\beta\gamma\delta} = \det \Lambda \, \epsilon^{\mu\nu\rho\sigma} \tag{33}$$

2. Lorentz Group: vectors, tensors

- **Covariant differentiation:**

$$\partial'_\mu \Psi' =: \frac{\partial \Psi'}{\partial x'^\mu} = \sum_\nu \frac{\partial \Psi}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\mu} = \sum_\nu (\Lambda^{-1})^\nu{}_\mu \frac{\partial \Psi}{\partial x^\nu} \qquad (34)$$

Since Ψ' is a Lorentz scalar and $$x^\mu = \sum_\rho (\Lambda^{-1})^\mu{}_\nu x'^\nu \qquad (35)$$

➢ **the 4-derivative is a covariant 4-vector:** $$\partial_\mu \psi = \frac{\partial \psi}{\partial x^\mu} \qquad (36)$$

Note: **the total derivative is a scalar:** $$\mathrm{d}\psi = \sum_\mu \mathrm{d}x^\mu \frac{\partial \psi}{\partial x^\mu} = \sum_\mu \mathrm{d}x^\mu \partial_\mu \psi \qquad (37)$$

➢ **The 4-divergence of a Lorentz vector is a Lorentz scalar:**

$$\sum_\nu \partial'_\nu A'^\nu = \sum_{\mu,\nu,\sigma} (\Lambda^{-1})^\mu{}_\nu \Lambda^\nu{}_\rho \partial_\mu A^\rho = \sum_{\mu,\rho} \delta^\mu_\rho \partial_\mu A^\rho = \sum_\mu \partial_\mu A^\mu \qquad (38)$$

2. Lorentz Group: vectors, tensors

If one introduces the 4-vector $$A_\mu = \partial_\mu \Psi = \frac{\partial \Psi}{\partial x^\mu}, \tag{39}$$

ít follows from (38): $$\sum_{\mu,\nu} g^{\mu\nu} \partial_\mu A_\nu = \sum_{\mu,\nu} g^{\mu\nu} \partial_\mu \partial_\nu \Psi = \sum_{\mu,\nu} g^{\mu\nu} \partial'_\mu \partial'_\nu \Psi' \tag{40}$$

Thus the D'Alembert-Operator

$$\sum_{\mu,\nu} g^{\mu\nu} \partial_\mu \partial_\nu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2, \quad \nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{41}$$

is also invariant under Lorentz transformations, i.e.

$$\sum_{\nu\sigma} g^{\nu\sigma} \frac{\partial^2}{\partial x^\nu \partial x^\sigma} A_\mu = \sum_{\nu\sigma} g^{\nu\sigma} \partial_\nu \partial_\sigma A_\mu = \sum_\nu \partial_\nu \partial^\nu A_\mu \tag{42}$$

In general: The scalar product of two Lorentz vectors is a Lorentz scalar:

$$\sum_\mu A'^\mu B'_\mu = \sum_\mu \sum_{\sigma,\rho} \Lambda^\mu{}_\rho A^\rho (\Lambda^{-1})^\sigma{}_\mu B_\sigma = \sum_{\rho,\sigma} A^\rho \delta^\sigma_\rho B_\sigma = \sum_\rho A^\rho B_\rho. \tag{43}$$

3. Vector current

The continuity equation:

$$\boxed{\vec{\nabla}\cdot\mathbf{j}+\frac{\partial\rho}{\partial t}=0.} \qquad (44)$$

New notation: $j^0=c\rho;\quad j^1=j_x;\quad j^2=j_y;\quad j^3=j_z$ **(45)**

ρ is the charge density and j_1, j_2, j_3 are the 3-components of the currect
In covariant notation we get:

$$\sum_\mu \partial_\mu j^\mu=\sum_\mu \frac{\partial}{\partial x^\mu}j^\mu=0. \qquad (46)$$

- **In a system with a charge distribution at rest we have:**

$$j'^0=c\rho_0,\quad j'^1=j'^2=j'^3=0. \qquad (47)$$

As a covariant component of a 4-vector, j_μ' must be also invariant under Lorentz transformations with velocity $\beta=v/c$ in x_1-direction:

$$x^0=\gamma(\beta x'^1+x'^0);\quad x^1=\gamma(x'^1+\beta x'^0);\quad x^2=x'^2;\quad x^3=x'^3, \qquad (48)$$

and transformes as

with

$$\boxed{\begin{array}{l} j^0=c\gamma\rho_0;\quad j^1=\gamma\rho_0 v;\quad j_2=0;\quad j_3=0. \\ \rho=\gamma\rho_0. \end{array}} \qquad (49)$$

3. Vector current

The Lorentz transformation of the volume from frame Σ to Σ' reads:

$$dV = \frac{dV_0}{\gamma} \tag{50}$$

One obtains in any frame the charge invariance:

$$Q = \int_V \mathrm{d}V\,\rho = \int \frac{\mathrm{d}V_0}{\gamma}\gamma\rho_0 = \int \mathrm{d}V_0\rho_0 \tag{51}$$

4. The four-potential

- **Consider the vector potential *A* and scalar potential Φ**

The Lorentz gauge reads:

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0 \tag{52}$$

The wave equation:

$$\sum_\mu \partial_\mu \partial^\mu \mathbf{A} = \mu_0 \mathbf{j}; \quad \sum_\mu \partial_\mu \partial^\mu \Phi = -\frac{\rho}{\epsilon_0} \tag{53}$$

ε_0 is the electric constant, μ_0 is the magnetic constant.

One can also write in covariant notation:

$$(A^\mu) = \left(\frac{1}{c}\Phi, \mathbf{A}\right)^T \tag{54}$$

using that $\epsilon_0 \mu_0 = c^{-2}$ **(55)**

Then Eq. (53) reads:

$$\sum_\nu \partial_\nu \partial^\nu A^\mu = \mu_0 j^\mu \tag{56}$$

The Lorentz covariance of Eq. (52) can be also written in the form:

$$\sum_\mu \partial_\mu A^\mu = \sum_\mu \frac{\partial}{\partial x^\mu} A^\mu = 0 \tag{57}$$

4. The four-potential

Consequences:

- **Lorentz invariance of the continuity equation and gauge condition:**

If in the frame Σ

$$\sum_\mu \frac{\partial}{\partial x^\mu} j^\mu = 0; \quad \sum_\mu \frac{\partial}{\partial x^\mu} A^\mu = 0 \tag{57}$$

then in the frame Σ' we also have

$$\sum_\mu \frac{\partial}{\partial x'^\mu} j'^\mu = 0; \quad \sum_\mu \frac{\partial}{\partial x'^\mu} A'^\mu = 0. \tag{58}$$

Since eq. (53) is covariant, the same result will be in Σ and Σ'

$$\sum_\nu \partial'_\nu \partial'^\nu A'_\mu = \mu_0 j'_\mu, \tag{59}$$

thus

$$\sum_{\nu,\rho} \Lambda^\mu{}_\nu \partial_\rho \partial^\rho A^\nu = \sum_\rho \partial_\rho \partial^\rho A'^\mu = \sum_\rho \partial'_\rho \partial'^\rho A'^\mu = \mu_0 \sum_\nu \Lambda^\mu{}_\nu j^\nu = \mu_0 j'^\mu. \tag{60}$$

This is the covariance of classical electrodynamics!

5. Plane-waves

Consider plane-waves in vacuum in the inertial system Σ:

$$A^\mu = A^{(0)\mu} \exp[\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \omega t)] = A^{(0)\mu} \exp\left(-\mathrm{i}\sum_\lambda k_\lambda x^\lambda\right) \tag{61}$$

with $$k^0 = \frac{\omega}{c}; \quad k^1 = k_x; \quad k^2 = k_y; \quad k^3 = k_z. \tag{62}$$

Due to the covariance of the homogenuous wave equation

$$\sum_\nu \partial_\nu \partial^\nu A^\mu = 0 \tag{63}$$

a transformation to another system Σ' gives:

$$A'^\mu(x'^\rho) = \sum_\nu \Lambda^\mu{}_\nu A^\nu(x^\lambda) = \left(\sum_\nu \Lambda^\mu{}_\nu A^{(0)\nu}\right) \exp\left(-\mathrm{i}\sum_\lambda k_\lambda x^\lambda\right) = A'^{(0)\mu} \exp(-\mathrm{i}\sum_\rho k'_\rho x'^\rho) \tag{64}$$

Accordingly, the phase of the plane wave should be also invariant:

$$\sum_\lambda k_\lambda x^\lambda = \sum_\lambda k'_\lambda x'^\lambda \tag{65}$$

as in case of a point-like excitation, where the wave fronts are spherical surfaces moving with the velocity of light c.

5. Plane-waves

Since the phase in Eq. (64) is an invariant scalar product, the k_μ must be the covariant components of a 4-vector, which transform as:

$$k'_x = \gamma\left(k_x - \frac{v}{c^2}\omega\right); \quad k'_y = k_y; \quad k'_z = k_z; \qquad (66)$$

$$\omega' = \gamma(\omega - v k_x).$$

Using the dispersion relation:

$$\frac{\omega}{k} = c = \frac{\omega'}{k'}, \qquad (67)$$

and denoting the angles ϕ and ϕ' with the direction of k and k′ with respect to the direction of v (i.e. the *x*-direction in case of (28)) we obtain:

$$\boxed{\omega' = \gamma\omega(1 - \beta\cos\phi)} \qquad (68)$$

$$\cos\phi' = \frac{k}{k'}\gamma(\cos\phi - \beta) = \frac{\cos\phi - \beta}{1 - \beta\cos\phi}. \qquad (69)$$

5. Plane-wave

Equation (68) describes the Doppler-effect, which apart from a longitudinal effect,

$$\omega' = \omega \frac{1 \mp \beta}{\sqrt{1-\beta^2}} = \omega \left[1 \mp \beta + \mathcal{O}(\beta^2)\right] \tag{70}$$

for $\beta << 1$ and $\phi = 0, \pi$,
also implies a transversal effect

$$\omega' = \gamma\omega = \frac{\omega}{\sqrt{1-\beta^2}} = \omega \left[1 + \frac{\beta^2}{2} + \mathcal{O}(\beta^4)\right] \tag{71}$$

for $\phi = \pm\pi/2$

which is a typical relativistic phenomenon. This effect was shown experimentally in 1938 within the investigation of radiation from moving H-atoms. Well known is the longitudinal effect in the radiation of distant galaxies (red shift) which demonstrates that these objects are moving away from us.

Furthermore, both phenomena describe the relativistic aberration of light, i.e. the apparent change of position of light sources due to the motion of the earth around the sun relative to stars.

6. Transformation of the fields E and B

Once knowing the fields A and Φ, one can compute the fields E and B via

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}; \quad \mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}} \tag{72}$$

Let's rewrite Eq. (72) in covariant form with coordinates x_μ and the components of the 4-potential A_μ . For example we obtain:

$$E_x = -\partial_1(cA^0) - c\partial_0 A^1 = c(\partial_0 A_1 - \partial_0 A_1), \quad B_1 = \partial_2 A^3 - \partial_3 A^2 = -(\partial_2 A_3 - \partial_3 A_2). \tag{73}$$

Eq. (73) suggests to introduce the antisymmetric field-strength tensor of second rank

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} = -F_{\nu\mu} \tag{74}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y & \frac{1}{c}E_z \\ -\frac{1}{c}E_x & 0 & -B_z & B_y \\ -\frac{1}{c}E_y & B_z & 0 & -B_x \\ -\frac{1}{c}E_z & -B_y & B_x & 0 \end{pmatrix} \tag{75}$$

6. Transformation of the fields E and B

The contravariant components are

$$F^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} = \begin{pmatrix} 0 & -\frac{1}{c}E_x & -\frac{1}{c}E_y & -\frac{1}{c}E_z \\ \frac{1}{c}E_x & 0 & -B_z & B_y \\ \frac{1}{c}E_y & B_z & 0 & -B_x \\ \frac{1}{c}E_z & -B_y & B_x & 0 \end{pmatrix} \tag{76}$$

Now we know the transformation properties of the fields E and B since the contravariant components transform as (25)

$$F'^{\mu\nu} = \sum_{\rho,\sigma} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} F^{\mu\sigma} \tag{77}$$

For the special Lorentz boost (14) we obtain:

$$E'_x = E_x; \quad B'_x = B_x; \quad E'_y = \gamma(E_y - vB_z); \quad B'_y = \gamma(B_y + \tfrac{v}{c^2}E_z);$$
$$E'_z = \gamma(E_z + vB_y); \quad B'_z = \gamma(B_z - \tfrac{v}{c^2}E_y). \tag{78}$$

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(14): $\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

transformation in x^1 direction with velocity $\beta = v/c$

6. Transformation of the fields E and B

For a Lorentz-Boost with velocity v in arbitrary direction holds that the parallel components (in direction of v) are conserved:

$$\mathbf{E}'_{\|} = \mathbf{E}_{\|}; \quad \mathbf{B}'_{\|} = \mathbf{B}_{\|}, \tag{79}$$

while the transverse components transform as:

$$\mathbf{E}'_{\perp} = \gamma \left[\mathbf{E}_{\perp} + (\mathbf{v} \times \mathbf{B}) \right]; \quad \mathbf{B}'_{\perp} = \gamma \left[\mathbf{B}_{\perp} - \frac{1}{c^2} (\mathbf{v} \times \mathbf{E}) \right] \tag{80}$$

The inversion

$$\mathbf{E}_{\perp} = \gamma \left[\mathbf{E}'_{\perp} - (\mathbf{v} \times \mathbf{B}') \right]; \quad \mathbf{B}_{\perp} = \gamma \left[\mathbf{B}'_{\perp} + \frac{1}{c^2} (\mathbf{v} \times \mathbf{E}') \right] \tag{81}$$

is obtained – in analogy to the coordinate transformation - by replacing $\mathbf{v} \rightarrow -\mathbf{v}$. Equations (80) and (81) demonstrate the connection between the electromagnetic fields E and B in different frames.

7. Maxwell equations

Now we can rewrite the Maxwell equations for the electromagnetic field in covariant form. We focus on the case of the vacuum and recall the Maxwell equations in conventional notation:

$$\mathrm{rot}\,\mathbf{E} = -\frac{\partial \mathrm{B}}{\partial t}, \quad \mathrm{div}\,\mathrm{B} = 0, \tag{82}$$

$$\mathrm{div}\,\mathbf{E} = \frac{1}{\epsilon_0}\rho, \quad \mathrm{rot}\,\mathrm{B} = \mu_0\left(\mathbf{j} + \epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right). \tag{83}$$

The equations (82) are the homogenuous Maxwell equations. They can be fulfilled by introducing scalar and vector potentials.

The equations (83) describe the ‚creation‘ of the fields from electric charges and currents. As seen before these equations can be written in covariant form of 4-tensor structure. The components of the field strength appear in the field-strength tensor (76), i.e. we have to express the equations in terms of this tensor.

7. Maxwell equations

Indeed, we can construct two independent covariant equations with first order derivatives:

$$\sum_\mu \partial_\mu F^{\mu\nu} \qquad \text{and} \qquad \sum_\mu \partial_\mu (\dagger F)^{\mu\nu}, \tag{84}$$

where $(\dagger F)_\mu$ is the dual tensor to F_μ . The first equation we can evaluate by writing the field-strength tensor in the form (84) via the potentials and take care of the Lorentz gauge :

$$\sum_\mu \partial_\mu F^{\mu\nu} = \sum_\mu \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \sum_\mu \partial_\mu \partial^\mu A^\nu = \quad \mu_0 j^\mu \tag{85}$$

These equations apparently correspond to the inhomogenuous Eqs. (83). When rewriting the dual tensor in terms of the potentials we obtain:

$$(\dagger F)^{\mu\nu} = \frac{1}{2}\sum_{\rho,\sigma} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = \frac{1}{2}\sum_{\rho,\sigma} \epsilon^{\mu\nu\rho\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho) = \sum_{\rho,\sigma} \epsilon^{\mu\nu\rho\sigma} \partial_\rho A_\sigma . \tag{86}$$

By calculating the 4-divergence of (86) we find that due to the antisymmetry of the Levi-Civita tensor $\epsilon^{\mu\nu\rho\sigma}$ and the exchange properties of the derivatives we get:

$$\partial_\mu (\dagger F)^{\mu\nu} = 0 \tag{87}$$

7. Maxwell equations

The advantage of the Maxwell equations (85) and (87) is that they are gauge independent. Indeed, a change in the gauge for the potentials implies

$$\tilde{A}_\mu = A_\mu + \partial_\mu \chi \qquad \textbf{(88)}$$

where $\chi(x)$ is an arbitrary scalar field. In fact, the field-strength tensor does not depend on such gauge transformations as seen from

$$\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu \chi - \partial_\nu \partial_\mu \chi = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}. \qquad \textbf{(89)}$$

This implies that the 4-potentials A_μ und $\tilde{A}_\mu$ describe the same physical fields. The invariance of (85) and (87) with respect to gauge transformations (88) is the starting point of the *standard model of elementary particle physics*.

8. Coulomb field

The field of a charge *q* at rest in the frame Σ′ is:

$$\mathbf{E}'(\mathbf{r}') = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}'}{r'^3}; \qquad \mathbf{B}'(\mathbf{r}') = 0 \qquad (90)$$

In a frame moving with velocity v = (*v*, 0, 0) relative to Σ′ we obtain:

$$\begin{aligned} E_x &= E'_x = \frac{q}{4\pi\epsilon_0}\frac{\gamma(x-vt)}{(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}}, \\ E_y &= \gamma E'_y = \frac{q}{4\pi\epsilon_0}\frac{\gamma y}{(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}}, \\ E_z &= \gamma E'_z = \frac{q}{4\pi\epsilon_0}\frac{\gamma z}{(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}}. \end{aligned} \qquad (91)$$

Here *x′*, *y′*, *z′* has been written explicitly as a function of *x*, *y*, *z* using the Lorentz transformation of the coordinates. The field appears as in Σ′ as a central field but is no longer isotropic. The factor γ^2 in the square root differentiates the *x*-direction relative the *y*- and *z*-direction.

8. Coulomb Field

An observer measures also the magnetic field:

$$\mathrm{B} = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E}), \tag{92}$$

since the charge q is moving and thus generating a current. For illustration we consider the case $\gamma >> 1$:

i) Close to the x-axis ($y, z \approx 0$) we get

$$E_x \approx \frac{1}{\gamma^2}\frac{q}{4\pi\epsilon_0}\frac{1}{(x-vt)^2}, \quad E_y = E_z = 0, \tag{93}$$

which implies a reduction of the field strength by a factor γ^{-2} .

ii) In the plane parallel to the $y - z$-plane through q we get:

$$E_x = 0, \quad E_y = \frac{\gamma y}{(y^2+z^2)^{3/2}}, \quad E_z = \frac{\gamma z}{(y^2+z^2)^{3/2}}, \tag{94}$$

which implies an enhancement of the transverse field strength by a factor of γ .
➔The radial field lines are thus thinner in the direction of motion whereas they are enhanced in transverse direction:

Covariant electrodynamics

Summary:

The basic equations of electrodynamics are covariant with respect to Lorentz transformations and have the same form in all inertial systems thus following the Einstein principle of relativity.