Lecture Based
Modules for Bridge Course in Mathematics

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PREFACE

Globalization of the world economy and higher education are driving profound changes in engineering education system. Worldwide adaptation of Outcome Based Education framework and enhanced focus on higher order learning and professional skills necessitates paradigm shift in traditional practices of curriculum design, education delivery and assessment. AICTE has also taken various quality initiatives for strengthening the technical education system in India. These initiatives are essential for promoting quality education in our institutions in the country so that our students passing out from these institutions may match the pace with global standards.

A quality initiative by AICTE is 'Revision of Curriculum'. Recently, AICTE has released an outcome based Model Curriculum for various Undergraduate degree courses in Engineering & Technology which are available on AICTE website. A three-week mandatory induction program is developed as a part of the model curriculum for the first year UG Engineering students which helps students joining the first year of the college from diverse backgrounds to get adjusted in the new environment of the institution.

Education is primarily conceived by students as one simple remembering facts by rote. However, Science education also requires clear understanding of science concepts and a proper logical thinking or a constructive thinking by students. We all know that the students seeking admission in an undergraduate degree engineering program have passed their 10+2 in science but it was felt that a student joining an engineering program after 10+2 require reinforcement of fundamental science concepts i.e. basic science courses in Physics, Chemistry and Mathematics. To support the students, gain better understanding, AICTE decided to initiate the task of development of bridge courses in Physics, Chemistry and Mathematics and it was entrusted to IIT-BHU. These bridge courses aim to accelerate the students' knowledge in these subjects acquired at 10+2 level; and also bridge the gap between the school science syllabus and the level needed to understand their applications to engineering concepts. Therefore, it was decided that after completion of the 3-week mandatory induction programme introduced for the first year UG engineering students, bridge course in basic Physics, Chemistry and Mathematics may be taken up by universities/institutions for the students for the remaining part of the semester. The concerned University/institution has a flexibility to adopt these modules on bridge courses by adjusting teaching hours accordingly.

The lecture based modules in Physics, Chemistry and Mathematics have been developed by a team of respective Course Coordinators from Indian Institute of Technology, Banaras Hindu University. AICTE approved institutions may utilize these modules *'Lecture Based Modules for Bridge Courses - Physics, Chemistry and Mathematics'* for teaching students to help bridge the gap of their studies of 10+2 and UG level.

(Prof. Anil D. Sahasrabudhe)
Chairman, AICTE

ACKNOWLEDGEMENT

Curriculum plays a crucial role in enabling quality learning for our young learners in our society i.e. students. An effective curriculum not only enables a student's learning process & knowledge acquired but also supports students to overcome their inhibitions and aids in their holistic development. AICTE in 2018 released a Model Curriculum for various Undergraduate degree courses in Engineering & Technology. This curriculum is equipped with making students industry ready, allow internships for hands on experience, learn about Constitution of India, Environment science etc. Induction program has been included as a mandatory program for the first year engineering students to get acquainted and get accustomed to this new environment in the college. a curriculum needs to be consistent and sustainable and it has been noticed that students joining an engineering program required to strengthen their concepts in science subjects i..e Physics, Chemistry and Mathematics building a better foundation during the first semester itself. AICTE therefore decided to develop lecture based bridge courses in basic science subjects i..e Physics, Chemistry and Mathematics for students,. The lecture based modules in Physics, Chemistry and Mathematics have been developed by IIT-BHU. This task has been accomplished by a team of respective Course Coordinators under Prof Indrajit Sinha, Department of Chemistry, IIT BHU as Overall Coordinator.

AICTE places on record its acknowledgement and appreciation to Dr. Indrajit Sinha, Department of Chemistry, IIT-BHU as overall coordinator; and respective course coordinators and their team of faculty members at IIT-BHU for developing these lecture based modules for bridge courses:

The faculty team from IIT-BHU:

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Institutions may adjust teaching hours to utilize these modules *'Lecture based Modules for Bridge Courses - Physics, Chemistry and Mathematics'* to bridge the gap of 10+2 and UG level.

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Adviser-I(P&AP), AICTE

Mathematics Modules

(For AICTE Approved Colleges)

Prepared by

Department of Mathematical Sciences

Indian Institute of Technology

(Banaras Hindu University)

Varanasi - 221005

Contents

Preface

The genesis of this module lies in the Induction Program first conceived and started by IIT(BHU) on 2016 on mass scale for about 1000 students. The fact is that the students are overburdened and stressed out due to a hectic high school life. To refresh their creative mind, they were exposed to month long diverse credit courses like Physical Education, Human Values and Creative Practices, as well as several non-credit informal activities. In a welcome step the AICTE has proposed to extend this program to the Engineering Colleges affiliated to them.

Infact, purpose of this module is to bridge the gap between what the students need to know before they can start taking the advanced courses in the college level and what they are actually aware of from the intermediate level. Consequently, after the completion of the 3-weeks induction program, it is proposed that (besides other subjects) bridge courses in basic Physics, Chemistry and Mathematics should be taught to these students for the rest of the semester. The bridge courses will cover typical weaknesses of students in science at the 10+2 level.

The modules in Mathematics are prepared keeping in mind that an hour of discussion will bring all the students in the same stage such that they can cope up with the courses in their college level, that requires the concepts of different topics in Mathematics. The modules are made as interactive sessions between the students and the instructors. Furthermore, we have discussed those topics which harder to understand. At the end of the discussion teacher may also take a small test to understand how much the students followed the class.

In brief the contents of the modules are presented as follows. In Mudule-1, basic concepts of sets, relations and function are discussed. Module -2 describes the definition of limit and discuss some of its properties. After that we introduce the notion of continuity of a function and the concept of the derivative of a function, and their properties.

Module-3, presents the idea of the basics of matrices, types of matrices, operations on matrices, determinants and cofactors, computing inverse of a square matrix, rank and elementary operations with brief discussion on system of linear equations.

Module-4 introduces the idea of the complex numbers and its basic properties. Further, the definition of the complex sets, neighbourhood of a complex number, domain, complex functions, limit of a complex functions and continuity of complex functions are presented in detail with several examples.

Module-5 is devoted to the differential equations and includes the topics as the formation of the differential equations, some special forms of the differential equations and then existence and uniqueness of the first order differential equations. Module-6 focuses on the double and triple integral and describes the method to solve such problems. It includes the other topics as polar equations of conics, directional derivatives, gradients, divergence and curl. Module-7, 8 and 9 presents the basic idea of the trigonometry, probability and statistics respectively.

We are very much grateful to all the faculty members (Prof. L. P. Singh, Prof. Rekha Srivastava, Prof. T. Som, Prof. S. K. Pandey, Prof. S. K. Upadhyay, Prof. S. Das, Prof. S. Mukhopadhyay, Prof. S. Ram, Prof. K. N. Rai, Dr. A. J. Gupta, Dr. Rajeev, Dr. R. K. Pandey, Dr. V. K. Singh, Dr. Sunil Kumar, Dr. Lavanya Shivkumar, Dr. A. Benerjee, Dr. D. Ghosh,

Dr. V. S. Pandey, in the Department of Mathematical Sciences who devoted their valuable time to prepare these modules.

This is to mention that that modules are prepared for the students with an objective to create interest among them in the subject. The references used in preparing these modules are cited at end of each module.

Department of Mathematical Sciences,

IIT(BHU) Varanasi

Module-1

Pretest on Sets, Relations and Functions

Sets

1) Specify the following sets in roster forms
   (a) The set of prime numbers less than less than 20.
   (b) The set of consonants in the word “VARANASI”.
2) Classify the following sets as finite, infinite, empty or singleton
   (a) Even number which is prime.
   (b) Set of teachers in your school.
   (c) Cows which have five legs.
   (d) The collection of integers.
3) Let A, B and C be sets. Show that
   i) $A \cup (B - A) = A \cup B$
   ii) $A \cap B \subseteq A$
4) Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 4, 5, 6\}$ and $C = \{4, 5, 6, 7, 8, 9, 10\}$. Find
   i) $A \cup B \cup C$
   ii) $A \cap B \cap C$

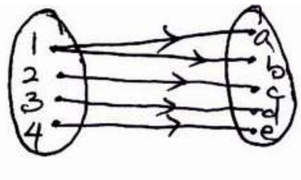
Relations

5) Let R be a relation defined on $A \times A$, where $A = \{1, 2, 3, 4\}$ such that $R = \{(a, b) : a\ divides\ b,\ a, b \in A\}$. Write R.
6) If $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$, which of the following are relations from A to B and why?
   i) $R_1 = \{ (1, 4), (1, 6), (2, 6) \}$
   ii) $R_2 = \{ (2, 4), (2, 5), (3, 5), (3, 6), (3, 4) \}$
   iii) $R_3 = \{ (4, 1), (1, 5), (2, 5) \}$
7) In a set $X = (a_1, a_2, a_3, a_4\}$ of four men, a_1 is younger to other three, a_2 is younger to a_3 and a_4 only, a_3 is younger to a_4 only. Is the relation “is younger to” (i) Reflexive, (ii) Symmetric and (iii) Transitive.
8) Find the domain and range of the relations defined on a set A of real numbers :
   i) For any two elements a and b of A, $a\ R\ b$ iff $2a + 3b = 6$.
   ii) For $a, b \in A, a\ R\ b$ if and only if $a^2 + b^2 = 25$.

Functions

9) Which of these are functions.

(i)



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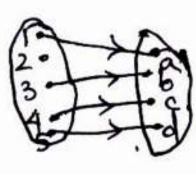

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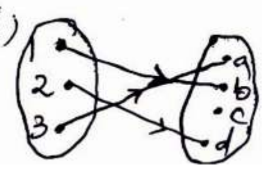


10) Find the domain of the following function defined on set of real numbers: $f(x) = \frac{x}{x-1}$

11) Find the range of $f : \mathbb{R} \to \mathbb{R}$ where $f(x)$ is given by (i) $\frac{x-1}{x+1}$, (ii) $3 - \frac{4}{(x-2)^2+2}$.

12) Is $f(x) = 4x^3 - 7$, a bijective function?

13) If $f(x) = x^2 + 1$, $g(x) = x^3$, then find fog and gof.

Module-1

Set Theory, Relations and Functions

Lectures required -02

In this module, we will discuss about the basic concepts of sets, relations and function. The model is divided into three sections with three subsection each given below:

1. Set Theory
   1.1 Definition and Representation
   1.2 Types of Sets
   1.3 Operation on Sets
2. Relations
   2.1 Definition
   2.2 Types of Relations
   2.3 Partial order and Equivalence Relations
3. Functions
   3.1 Definition and classification
   3.2 Types of functions
   3.3 Composition and Inverse of functions

1. Set Theory

A set is most basic term in mathematics. Hardly any discussion can proceed without sets (class of collections). A set cannot be defined. Set is only primitive idea by which we can say that such element belongs to the set or not.

A set is a collection of well defined distinct objects.

By "well-defined", we mean "being unambiguous", that is, when the idea assigns a unique interpretation that given object in the world at large (abstract or concrete) is either an element of set or it is not.

Notation: $x \in S$, is an element of S.

Example: 1. The set of days in a week.

2. The set of integers i.e. $\mathbb{Z} = \{\dots\dots -3, -2, -1, 0, 1, 2, 3, \dots\dots\}$

The following do not describe a well-defined collection and so are not sets.

Examples: 1. All good books.

2. The fruit which taste good to all.

1.1 **Representations of sets:** Sets can be represented by many ways. The most used are

(i) Roster form
(ii) Set-Builder form

(i) In the *Roster form*, the elements are enumerated as a list and are enclosed between bracket "{ }"

For example, $A = \{1, 2, 3, 4, 5\}$.

$\mathbb{N}$ = Set of natural numbers = {1, 2, 3, ... }

$\mathbb{R}$ = Set of real numbers

$\mathbb{Z}$ = Set of integers = $\{0, \pm 1, \pm 2, \ . \ . \ . \ . \}$

(ii) In *Set-Builder form,* a set is represented by describing its element and terms of one or several characteristics properties that helps us to decide whether given element belong to set or not.
Example:
a) $A = \{ x : x \in \mathcal{R} \; and \; x^2 - 1 = 0\}$.
b) $B = \{ x : x \; is \; an \; even \; integer\}$.

1.2 Types of Sets

There are four types of sets based on the number of elements contain, namely, empty set, singleton set, finite set, infinite sets.

(i) ***Empty Set:*** A set which does not contain any element is called as an *empty set or the null set.* Empty set is denoted by $\emptyset$.
Example:
a) A = Set of vowels in the word "RHYTHM"
b) B = {Number of occurrences of the letter 'U' in the word " ENCYCLOPEDIA"}
c) $C = \{x: x \; is \; an \; even \; prime \; number \; greater \; than \; 2\}$.

(ii) ***Singleton Set:*** A set which contains exactly one element is called a singleton set. Example: $A = \{2\}$

(iii) ***Finite set:*** A set A set which is empty or consists of a definite number of elements is called *finite.*
Example:
a) $A = \{x: x \in \mathbb{R} \; and \; (x-1)(x+4) = 0 \}$
b) $B = \{1, 2, 3, 4, 5\}$

(iv) ***Infinite set:*** A set which is not finite is called *infinite set.*
Example:
a) $A = \{x: x \; is \; a \; prime \; number\}$
b) $B = \{Set \; of \; all \; odd \; integers\}$

Further, given two sets we could compare the two sets and classify accordingly as equal sets (contain same elements), equivalent sets (contain equal number of elements), proper subset, improper subsets, superset.

(v) ***Equal Sets:*** Two sets A and B are said to be *equal* if they have exactly the same elements and we write $A = B$. Otherwise, the sets are said to be *unequal* and we write $A \neq B$.

(vi) ***Subsets***: A set A is said to be a subset of a set B if every element of A is also an element of B. In other words, $A \subseteq B$ if whenever $\in A$, then $a \in B$. If $A \subseteq B$ and $A \neq B$, then A is called a *proper subset of* B, denoted by $A \subset B$ and B is called the *superset* of A.
Example:
a) $A = \{1,2,3\} \; and \; B = \{1,2,3,4,5\}$. Here A is a proper subset of B
b) $A = \{1,3\} \; and \; B = \{4,5,9\}$. Here $A \not\subset B$, since $1 \in A$ and $1 \notin B$.

Apart from these, we have two more sets called the Power Set and Universal Set.

(vii) **Power Set:** A Power set of A is the set of all subsets of A and is denoted by $\mathcal{P}$(A).

Example: If $A = \{1, 2, 3\}$, then $\mathcal{P}(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} = A\}$

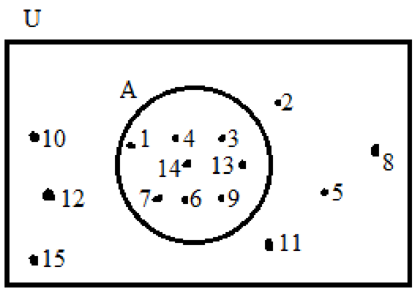
- If A has n-elements, then $\mathcal{P}$ (A) contains 2^n elements.

(viii) **Universal set :** A set that contains all possible sets in a given context is called universal set.

Example: $U = \{1, 2, 3, 4, 5, 6, 7\}$ when $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 5, 6, 7\}$ are subsets of its universal set.

Venn diagrams: Sets and their relationships can also be represented by using diagrams, called the *Venn diagrams.* Venn diagrams are graphic representation of sets as enclosed areas in a plane region. This representation is named after the English Logician John Venn. In Venn diagrams, the elements of the sets are written in their respective circles, while these sets are encompassed in a rectangle by their Universal sets.

Example: A set A = {1,3,4,6,7,9,13,14} from its universal set U = { 1, 2, … 15} can be represented by



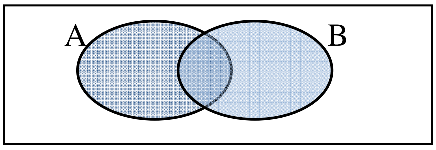
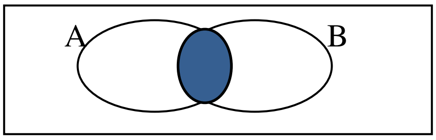
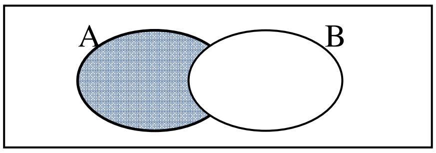
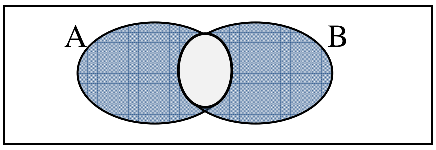
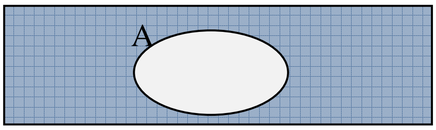
Note: A is contained in the universal set U.

3. Basic Operation on sets:

Given two sets A and B, there are various operations that can be performed namely, then union or join, the intersection or meet, difference, symmetric difference and complement of a set.

Here, except "complement", all the other operations are binary operation, as we need two sets to perform the operation. While the complement is an unary operation with reference to the universal set. In addition, complements can also be viewed as binary operation of difference ($\bar{A} = U - A$) between universal set and the set A.

Historically, we could represent operation using Venn diagram as shown below :

Operations	Venn Diagram Representation
The union of A and B: **$A \cup B=\{x: \ x \in A \ \text{or} \ x \in B\}$**	
The intersection of A and B: **$A \cap B=\{x: \ x \in A \ \text{and} \ x \in B\}$**	
The difference: **$A-B=A \backslash B=\{x: \ x \in A \ \text{and} \ x \notin B\}$**	
The symmetric difference: **$A \Delta B= (A-B) \cap (B-A)$**	
The complement of A: **$A'=\{x \in U; \ x \notin A\}$**	

Partition of a set :

A partition of a set A is a collection of non-overlapping non-empty subsets of A whose union is A.

(i.e.) If $A_1, A_2, \ldots, A_n \subseteq A$ then the collection $\delta = \{ A_1, \ \ldots A_n\}$ is called a partition, if

(i) $A_i \neq \phi, i = 1, 2, \ldots, n$

(ii) $A_i \cap A_j = \phi, \ i \neq j, \ i, j = 1, 2, \ldots n,$ that is A_i*'s* are pairwise disjoint.

(iii) $A_1 \cup A_2 \ldots \cup A_n = A \ (i.e.) \ \bigcup_{i=1}^{n} A_i = A.$

For Example: $A=\{1, 2, 3, 4, 5, 6, 7 \}$, $\delta = \{ \{1, 4, 5\}, \{2, 7\}, \{3, 6, 8\}\}$ is a partition of A.

Principle of inclusion-exclusion:

Let A and B be any two finite sets over a Universal set U, then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, where $n(A)$ represents number of elements in the set A.

As to get number of elements of $A \cup B$, we include number of elements of A and B, and exclude $(A \cap B)$.

2. Relations

Let us consider two sets A and B: $A = \{a_1, a_2, a_3\}$ and $B= \{b_1, b_2, b_3\}$.

If one can indicate a relationship or association between (two or more objects) A and B, then we can say there is a relation between A and B. Relations are given by subsets of the cartesian product of sets $A \times B$. That is, let

$A \times B = \{(a_1, b_1), (a_2, b_2), (a_1, b_3), (a_2, b_1), (a_2, b_2), (a_2, b_3), (a_3, b_1), (a_3, b_2), (a_3, b_3) \}$

Let $R = \{(a_1, b_1), (a_2, b_3), (a_3, b_1)\}$ represents a relation between A and B.

Definition 1: Given two non-empty sets A and B, the **relation** from the set A to set B is defined as the subset of $A \times B$.

If an ordered pair $(a, b) \in R$ then we say $(a \text{ related to } b)$. If $|A| = m$ and $|B| = n$, then $|A \times B| = mn$. Then the number of relations on A and B are 2^{mn}.

Domain and range of relations

Let R be a relation from a set A to set B. Domain is the set consisting of all the first elements of the ordered pairs belonging to R and the range of the relation is the set of all second element of the ordered pair of R.

Therefore, Domain $(R) = \{ a \in A : (a, b) \in R\}$ and Range $(R) = \{ b \in B : (a, b) \in R\}$

Example: $A = \{1, 2, 3\}$, $B = \{a, b\}$ and $R = \{(1, a), (2, b), (3, a), (3, b)\}$.

Domain for $R = \{1, 2, 3\}$

Range of $B = \{a, b\}$

2.2 Types of relations :

i) **Universal relation:** If $R = A \times B$, then R is called as universal relation.

ii) Null/Void/Empty Relation: If $R = \phi$, then R is called empty.

iii) Inverse relation: If R is a relation from A to B, then R^{-1}, inverse of R, is from B to S.

$R^{-1} = \{(b,a) : (a,b) \in R\}$.

Example: If $R = \{(1, a), (1, c), (2, b), (2, a)\} \text{ then } R^{-1} = \{(a, 1), (c, 1), (b, 2), (a, 2)\}$

iv) Reflexive: If a relation R is such that aRa, $\forall a \in A$, then R is said to reflexive relation.

v) Irreflexive: If a relation R is such that $a\not{R}a$ for every $a \in R$, that is, for every element a in A, a is not related to itself.

vi) Non reflexive: If R is such that $\text{for some } a \in A, aRa \text{ and for some } a \in A, a\not{R}a$ (that is, for some element $a \in A$, a is related to itself, while there are some $a \in A$ not related to itself.

(vii) Symmetric relation: A relation R defined on set A is said to be symmetric $\text{if } aRb \Rightarrow bRa, \text{where } a, b \in A$.

Example: For A = {1,2,3}, R = { (1,2), (2,1), (2,2), (1,3), (3,1)}

R is symmetric, since $(1,2) \in R \Rightarrow (2,1) \in R$

$$(2,2) \in R \Rightarrow (2,2) \in R$$

$$(1,3) \in R \Rightarrow (3,1) \in R$$

(viii) Asymmetric Relation: $If\ (a, b) \in R \Rightarrow (b, a) \notin R, for\ a \neq b$

(ix) Antisymmetric Relation: A relation R is said to be antisymmetric, $if\ (a, b) \in R, aRb \text{ and } bRa \Leftrightarrow a = b$.

In other words, if $a \neq b$ then either $a\not{R}b$ or $b\not{R}a$ or both.

(x) Transitive Relation: A relation R on a set A is transitive if for $a, b, c \in A, aRb \text{ and } bRc \text{ then } aRc.$

Example: $R = \{(1,2), (2,3), (1,3)\}$. *Here,* $1R2$ and $2R3 \Rightarrow 1R3$.

2.3 Partial Order and Equivalence Relations

A relation R is said to be *partially ordered* if R is reflexive, anti-symmetric and transitive.

Example: $R = \{ (a,b) : a \text{ divides } b,\ a,b \notin \mathbb{N}\}$.

(i) R is reflexive: Since a divides a $\forall a \in \mathbb{N}$.

(ii) R is antisymmetric:

$aRb \quad \Rightarrow a \text{ divides } b$

$\Rightarrow b=ak,\ k\in \mathbb{N}$

$bRa \quad \Rightarrow b \text{ divides } a$

$\Rightarrow a=bk,$

$\Rightarrow b=ba_1k$

$\Rightarrow k_1k_1 =1.$

$\Rightarrow k =1$ and $k_1 =1.$

$\Rightarrow a = b.$

(iii) R is transitive :

Since $aRb \Rightarrow a \text{ divides } b \Rightarrow b = ak$

$bRc \Rightarrow b \text{ divides } c \Rightarrow c = bk_1$

$\Rightarrow c = akk_1 = ak_2$

$\Rightarrow a \text{ divides } c$

$\Rightarrow aRc.$

$\therefore$ R is a partial order.

Equivalence relation:

A relation R defined on A is called an *equivalence relation*, if R is reflexive, symmetric and transitive.

Given a set A and an equivalence relation R, an equivalence class is subset of X of the form

$[a] = \{x \in A : xRa\}, where\ a \in A$

$[a]$ - contains those elements which are equivalent to a.

- Set of all equivalence classes in A is called a *Quotient set* of A by R (A/R).
- Set of all equivalence classes forms a partition of A.

Example: $A = \{1,2,3,4\}. R = \{(1,1,), (2,2), (3,3), (4,4), (1,3), (2,4), (3,1), (4,2)\}$

R is reflexive, symmetric and transitive. Hence, R is an equivalence relation.

$$[1] = \{1,3\} = [3]$$
$$[2] = \{2,4\} = [4]$$

∴ [1] and [2] form the equivalence classes. Note that {[1], [2]} = {{1,3}, {2,4}} forms a partition of A.

3. Functions

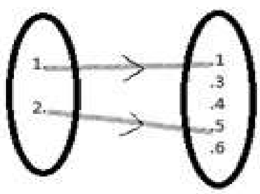
Functions provide us a convenient way to handle a relationship between a variable that depends on the value of another variable. Every function is a relation. However, every relation does not become a function.

Definition : Let A and B be any two non-empty sets, then the rules or correspondence between the elements of A and B is called a function from A to B if to each element of A, there corresponds exactly one element of B. (i.e.) A function $f : A \to B$ is a rule such that every element of A has a unique image in B.

A is called the domain and B is called the co-domain.

Set of image of A (which is a subset of B) is called the range of *f*.

Example: $A = \{1,2\}$, $B = \{1,3,4,5,6\}$ $f : A \to B$ be defined as below:

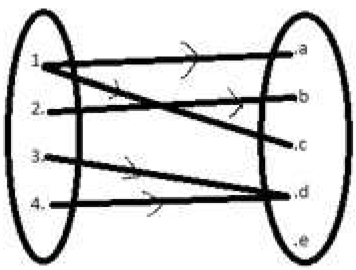


Note *f* is a function.

Domain $(f) = \{1,2\}$

Range $(f) = \{1,5\}$

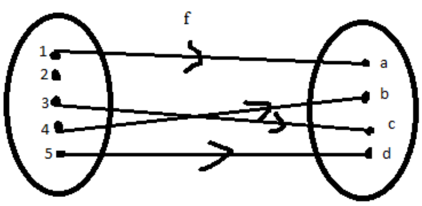
Example 2: $A = \{1,2,3,4\}, B = \{a,b,c,d,e\}$



Here *f* is not a function from A to B, since $f(1) = a$ and $f(1) = c$.

That is, 1 has no unique image.

Example. 3 : $A = \{1,2, 3,4,5\}$, $B = \{a, b, c, d\}$



Here, *f* is not a function from A to B, since the element $2 \in A$ does not have an image in B.

Note : (i) There may be elements in B not related to elements in A but every element of A must have unique image in B.

(iii) If $|A| = m$ and $|B| = n$, then the number of functions from A to B are n^m.

Classification of functions

Functions are broadly classified into algebraic and transcendental functions.

Algebraic function represents polynomial functions and rational functions, while transcendental functions are trigonometric functions, logarithmic and exponential functions.

3.2 Types of Functions

i) One to one function [Injective or Into]:

A function $f : A \to B$ is said to be one to one iff distinct elements of A have distinct images in B (i.e.) $x_1, x_2 \in A, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

ii) Many to one functions :

A function from A to B is said to be many to one iff two or more elements of A have same images in B.

Example: $f(x) = x^2, \quad x \in \mathbb{R}$

Suppose $f(x_1) = f(x_2)$

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = \pm x_2$

$\Rightarrow$ Every element of B has two pre-image in A.

$f(1) = 1^2 = f(-1)$

$\therefore f(x)$ is many to one function.

e.g. : Let $f(x) = x^2, x \in \mathbb{R}^+$ [set of all positive reals].

Here, f is one-to-one

Suppose $f(x_1) = f(x_2) \Rightarrow x_1 = \pm x_2$.

However x_2 or- x_2 does not belong to the domain of positive real numbers.

(iii) Onto functions

A mapping $f: A \to B$ is said to be onto if every element $y \in B$ has some preimage x in A, (i.e.) $\forall y \in B \quad x \in A$ such that $f(x) = y$.

Example: $f(x) = 2x-1, x \in \mathbb{R}$

Let $y = f(x)$

$\Rightarrow 2x-1 = y$

$\Rightarrow x = \frac{y+1}{2} \in \mathbb{R}$

(i.e.) $x \in R$

$f(x) = f\left(\frac{y+1}{2}\right) - 1 = y$

(iv) Bijective Function

A function *f : A* to B is bijective, if *f* is one-to-one and onto.

$f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 4x^3 - 7$.

One-One : $f(x_1) = f(x_2) \Rightarrow 4x_1^3 - y = 4x_2^3 - y$

$\Rightarrow x_1^3 - x_2^3 = 0$

$\Rightarrow (x_1 - x_2)\left[\left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3x_2^2}{4}\right] = 0$

$\Rightarrow x_1 = x_1\ [x_1^2 + x_1x_2 + x_2^2 > 0$

Onto: $y = f(x) = 4x^3 - 7 \Rightarrow x = \left(\frac{y+7}{4}\right)^{1/3}$, since y is real, for every $y \in \mathbb{R}$, there exists an $x \in \mathbb{R}$ such that $f(x)=y$.

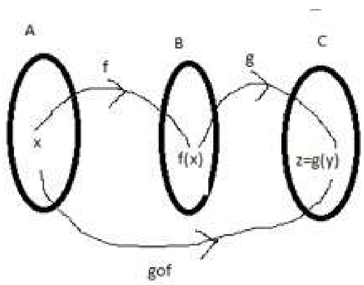
$\therefore f$ is both one-one and onto and hence bijective.

3.3 Composition and Inverse of functions:

Composition of functions:

Let $f: A \rightarrow B$ and $g: B \rightarrow C$ The composite of f and g denoted by $gof: A \rightarrow C$ is defined by $(gof)(x) = g(f(x))$.

Pictorially,



$$z = g(y) = g\big(f(x)\big) = (gof)(x)$$

Similarly, for $f: A \rightarrow C$, $g: B \rightarrow A$

$$fog: B \rightarrow C$$

$$fog(x) = f(g(x))$$

Eg. $A = \{1,2,3\}$, $B = \{a, b\}$, $C = \{x, y\}$

$f: A \rightarrow B$ is $f(1) = a, f(2) = a, f(3) = b$

$g: B \rightarrow C$ is $g(a) = x, g(b) = y$.

$gof: A \to C$ is defined as

$gof(1) = g(f(1)) = g(a) = x$

$gof(2) = g(f(2)) = g(a) = x$

$gof(3) = g(f(3)) = g(b) = y.$

Inverse of a function:

Let $f: A \to B$ and $g: B \to A$ is called the inverse of f if $gof = I_A$ and $fog = I_B$ where, $I_A: A \to A$ defined by $I_A(a) = a$ is called the identity function on A.

That is, $g(f(x)) = x, \forall\, x \in A$

and $f(g(y)) = y, \forall\, y \in B.$

The inverse of f is also denoted by f^{-1}.

Necessary and sufficient condition for inverse of f to exists is that $f: A \to B$ is bijective, that is, f is one-one and onto.

Example: $f(x) = x^4$ and $g(x) = x^{1/4}, x \in \mathbb{R}$

are inverses of each other.

$$(fog)(x) = f(g(x)) = f\left(x^{\frac{1}{4}}\right) = (x^{1/4})^4 = x$$

$$(gof)(x) = g(f(x)) = g(x^4) = (x^4)^{1/4} = x$$

Therefore, $gof = I_{\mathbb{R}} = fog$

i.e, f and g are inverses of each other.

References:

1. P.B. Bhattacharya, S,.K.Jain, S.R. Nagpaul, First Course in Linear Algebra, Wiley, 1983.
2. G. Hadley, Linear Algebra, Narosa Publishing, 1992.
3. J.P. Singh, Discrete Mathematics for Under graduates, Ane Books, 2014.

Assignment on Sets, Relations and Functions

Sets

1) Let A, B and C be sets. Shows the
   i) $(A - B) - C \subseteq A - C$
   ii) $(A - C) \cap (C - A) \neq \phi$
2) Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 4, 5, 6\}$ and $C = \{4, 5, 6, 7, 8, 9, 10\}$. Find
   i) $(A \cup B) \cap C$
   ii) $(A \cap B) \cup C$
3) Show that $\bar{\bar{A}} = A$.
4) Let $A = \{a, b, c\}$, $B = \{x, y\}$ and $C = \{0, 1\}$. Find
   i) $A \times B \times C$
   ii) $B \times B \times B$
   iii) $C \times B \times A$
5) Which of these collections of subsets of partitions of $\{1, 2, 3, 4, 5, 6\}$.
   i) $\{\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}\}$
   ii) $\{\{1\}, \{2, 3, 6\}, \{4\}, \{5\}\}$
   iii) $\{\{1, 4, 5\}, \{2, 6\}\}$
6) Give a formula for the number of elements in union of four sets.
7) How many positive integers not exceeding 1000 are divisible by 7 or 11?

Relations

8) Following relations are defined on $A \times A$, where $A = \{1, 2, 3\}$. Which of the following relations are (i) Reflexive, (ii) Symmetric and (iii) Transitive.
   i) $R_1 = \{(1, 1), (2, 3), (3, 3)\}$
   ii) $R_2 = \{(1, 2), (2, 3), (2, 1)\}$
   iii) $R_3 = \{(1, 2), (2, 3), (1, 3)\}$
   iv) $R_4 = \{\emptyset\}$
9) Give an example of a relation, which is
   i) Reflexive but not symmetric
   ii) Symmetric but not transitive
   iii) Transitive but not reflexive.
10) Let R be a relation on the set of natural numbers such that $R = \{(a, b) : a \; divides \; b, a, b \in N\}$. Show that R is a partial order relation.
11) A relation R is defined on the set $N \times N$, where N is the set of natural numbers by setting $(a, b) R \, (c, d) \Leftrightarrow (a + d) = (b + c),\ a, b, c, d \in N\}$. Show that this relation is an equivalence relation.
12) If R is a relation on the set of an integer $\mathbb{Z}$ defined by $R = \{(a, b) : a - b \; is \; even \; for \; a, b \in \mathbb{Z}\}$. Describe the equivalence classes of $\mathbb{Z}$.

13) Find the domain and range of the relations defined on a set A of real numbers :

iii) For any two elements a and b of A, $a\ R\ b$ iff $2a + 3b = 6$.

iv) For $a, b \in A, a\ R\ b$ if and only if $a^2 + b^2 = 25$.

Functions

14) Which of these are functions.

15) Find the domain of the following function defined on set of real numbers:

i) $f(x) = \frac{1}{x-|x|}$

ii) $f(x) = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$

16) If $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $f(x) = 2x^2 + 3$. Show that $f(x)$ is not one to one function.

17) If $f(x) = x^2 + 1$, $g(x) = x^3$, then find fog and gof.

18) If $f(x) = \log \frac{1-x}{1+x}$, then prove that $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$.

19) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \frac{x}{\sqrt{1+x^2}}$, then show that

(i) $(fofof)(x) = \frac{x}{\sqrt{1+3x^2}}$ and

(ii) $(fofof)(x) \neq [f(x)]^3$

20) If $f(x) = \begin{cases} x^2 - 4x + 3, & x < 2 \quad and \\ x - 4, & x \geq 2 \end{cases}$

$g(x) = \begin{cases} x - 3, & x < 3 \\ x^2 + 2x + 2, & x \geq 3 \end{cases}$.

Determine $f+g$, f/g.

21) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be two functions such that $f(x) = 2x-3$, $g(x) = x^3+5$. Then find $(fog)^{-1}(x)$.

22) Prove that $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = e^x$ is one-one but not onto.

Module-2

Pretest on Differential and Integral Calculus

Q. 1. If f has a derivative at x=c, show that f is continuous at x=c.

Q. 2. The function,

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at …

Q. 3. The function

$$f(x) = \begin{cases} x\sin(\frac{1}{x}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is differentiable at x=0 or not? Give reason for your answer.

Q. 4. Evaluate

$$\lim_{x\to 0}\left(\frac{xe^x - \log(1+x)}{x^2}\right)$$

Q. 5. If f(x) and g(x) are continuous for $0 \le x \le 1$, could f(x) /g(x) possibly be discontinuous at a point of [0, 1]? Give reason for your answer.

Q. 6. Give an example of function f and g, both continuous at x=0, for which the composite $f \circ g$ is discontinuous at x=0.

Q. 7. Suppose that h is integrable and $\int_{-1}^{1} h(r)dr = 0$ and $\int_{-1}^{3} h(r)dr = 6$. Find $\int_{3}^{1} h(r)dr = ?$

Q. 8. For what values of c the following function

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & \text{if } x < 2\,; \\ (c^2-c)x - 8\,, & \text{if } x \ge 2 \end{cases}$$

is continuous everywhere?

Q. 9. The expression $\frac{1}{50}\left(\sqrt{\frac{1}{50}} + \sqrt{\frac{2}{50}} + \cdots + \sqrt{\frac{50}{50}}\right)$ is a Riemann sum approximation for ……

(a) $\int_0^1 \sqrt{\frac{x}{50}}dx$ (b) $\int_0^1 \sqrt{x}dx$ (c) $\frac{1}{50}\int_0^1 \sqrt{\frac{x}{50}}dx$ (d) $\frac{1}{50}\int_0^1 \sqrt{x}dx$ (e) $\frac{1}{50}\int_0^{50} \sqrt{x}dx$

Module-2

Differential and Integral Calculus

Lectures required-02

A fundamental concept in single variable calculus is the concept of the limit of a function. In this module, we first introduce the definition of limit and discuss some of its properties. After that we introduce the notion of continuity of a function and the concept of the derivative of a function, and their properties. Also we discuss some of the results related to continuity and differentiability. We begin this module with some preliminary concepts.

Intervals: A subset 'A' of $\mathbb{R}$ is called an interval if 'A' contains every element lies between any two members of 'A'.

i.e. whenever $a \leq c \leq b$, where $a, b \epsilon A$

$\Rightarrow c \epsilon A$

Open Interval: $(a, b) = \{x \epsilon \ \mathbb{R}/a < x < b\}$

Neighbourhood of a point:

A set $N \subseteq \mathbb{R}$ is called the neighbourhood of a point $a \epsilon \ \mathbb{R}$, if there exists an open interval I containing a and contained in N, i.e. $a \epsilon I \subseteq N$.

Function:

Let A and B be two non-empty sets. A Function f from A to B is a rule of correspondence that assigns to each element x in A, a unique y in B.

A is said to be the domain of f and B, the co-domain of f.

Examples:

(1) The set $\mathbb{R}$ of real numbers is the neighbourhood of each of its points.
$\therefore \forall \ x \epsilon \mathbb{R}$, $\exists$ an open interval$(x - \epsilon , \ x + \epsilon)$, where s. t. $x \epsilon (x - \epsilon , \ x + \epsilon) \subseteq \mathbb{R}$.

(2) $\mathbb{N}, \mathbb{Z}, Q, Q^c$ are not the nbd of any of its points (since these sets do not contain any open interval)

(3) $S = \left\{\frac{1}{n} | n \ \epsilon \mathbb{N} \right\}$ is not nbd of any real no.

Limit point of a Set: Let $S \subseteq \mathbb{R}$ and $\alpha \ \epsilon \ \mathbb{R}$ then α is called a limit point of S if or any $\delta > 0$

$$(\alpha-\delta,\alpha+\delta)\cap S-\{\alpha\}\neq\emptyset$$

i.e. every nbd of α Contains at least one element of S other than α.

Note:

(1) A limit point of a set may or may not be a member of the set.
(2) $\alpha \in \mathbb{R}$ is limit point of $S \subseteq \mathbb{R}$ if every nbd of α contains infinite elements of S.

Example: The set $S=\mathbb{N}$(Natural Number) has no limit point.

$\because$ for any $\alpha \in \mathbb{R}$, any $\delta > 0$

$(\alpha-\delta,\alpha+\delta)\cap S$ if finite.

$\Longrightarrow$ α is not a limit point $\mathbb{N}$

Here α is arbitrary $\Rightarrow$ $\mathbb{N}$ is no limit pt.

Example: The set $\left\{\frac{1}{n}/n \in\mathbb{N}\right\}$ has only one limit point, zero, which is not a member of the set.

Limits

Limit of a function: Let $f(x)$ be defined on an open interval about x_0 except possibly at x_0 itself. We say that limit of $f(x)$ as x approaches x_0 is the number L if for every number $\epsilon > 0$, there exist a corresponding number $\delta > 0$, s. t. for all x,

$$0<|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon$$

OR:

Definition-2: Let $A \subseteq \mathbb{R}$, and let C be a limit point of A for a function $f: A \to \mathbb{R}$, a real no. L is said to be a limit of f at C if, given any $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in A$ and

$$0<|x-c|<\delta,\ \text{then } |f(x)-L|<\varepsilon.$$

Remarks:

(a): The inequality $0<|x-c|$ is equivalent to saying $x \neq c$.

(b): Since the value of δ usually depends on ϵ, we will sometimes write $\delta(\epsilon)$ instead of δ.

Question: Show that a function cannot have two different limits at the same point. That is, if $\lim\limits_{x\to x_0} f(x)=L_1$ and $\lim\limits_{x\to x_0} f(x)=L_2$ then $L_1=L_2$.

Solution: Let, if possible, $f(x)$ tend to limits L_1 and L_2 here for any > 0 , it is possible to choose a $\delta > 0$ such that

$|f(x) - L_1| < \frac{\varepsilon}{2}$ when $0 < |x - x_0| < \delta$

$|f(x) - L_2| < \frac{\varepsilon}{2}$ when $0 < |x - x_0| < \delta$

Now, $|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2|$

$\leq |L_1 - f(x)| + |f(x) - L_2|$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ when $0 < |x - x_0| < \delta$

i.e. $|L_1 - L_2|$ is less than any positive number ϵ (however small) and so must be equal to zero.

Thus $L_1 = L_2$.

<u>Question:</u> Show that: $\lim_{x \to c} f(x) = c^2$ if $f(x) = \begin{cases} x^2 & , x \neq c \\ 1 & , x = c \end{cases}$

Solution: We want to make the difference $|x^2 - c|$ less than a reassigned $\epsilon > 0$ by taking x sufficiently close to c. To do so, we note that

$x^2 - c^2 = (x + c)(x - c)$. Moreover, if $|x - c| < 1$ then

$|x| < |c| + 1$, so that

$|x + c| \leq |x| < |c| < 2|c| + 1.$

Therefore, if $|x - c| < 1$, we have

$$|x^2 - c| = |x + c| \cdot |x - c| < (2|c| + 1) \cdot |x - c| \tag{1}$$

Moreover this last term will be less than ϵ provide we take $|x - c| < \epsilon/(2|c| + 1)$

Consequently, if we choose

$\delta(\epsilon): inf\left\{1, \frac{\epsilon}{2|c|+1}\right\}$

Then if $0 < |x - c| < \delta(\epsilon)$, it will follow first that $|x - c| < 1$ so that (1) is valid, and therefore since $|x - c| < \epsilon/(2|c| + 1)$ that $|x^2 - c^2| < (2|c| + 1) \cdot |x - c| < \epsilon$.

Since we have a way of choosing $\delta(\epsilon)$ for an arbitrary choice of $\epsilon > 0$, we infer that

$$= \lim_{x \to c} f(x) = c^2$$

<u>Exercise:</u> Prove the limit statement $\lim_{x \to -2} f(x) = 4$ if $f(x) = \begin{cases} x^2 & , x \neq -2 \\ 1 & , x = -2 \end{cases}$

Question: Let $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

Use definition of limit to prove that $\lim_{x\to 0} f(x)$ does not exist.

Solution: Let $L \in \mathbb{R}$

<u>Case-I:</u> $L = 0$

Let $\epsilon = \frac{1}{2}$

$\forall\, \delta > 0,\ \exists\, x \epsilon Q'$ Such that $|x - 0| < \delta$ $\quad$ ($\because$ Q' is dense in $\mathbb{R}$)

$\therefore f(x) = 1$

$|f(x) - L| = |1 - 0| = 1 \geq \frac{1}{2}$

$\therefore \exists \quad \epsilon > 0$, namely $\frac{1}{2}$ s.t. $\forall\ \delta > 0, \exists\, x$ s.t. $|x - 0| < \delta$ and $|f(x) - 0| \geq \epsilon$

$\therefore \lim_{x\to 0} f(x) \neq 0$

<u>Case-II:</u> $L \neq 0$

Let $\epsilon = |L|/2 > 0$

$\forall\, \delta > 0, \quad \exists \quad x \epsilon Q$ such that $|x - 0| < \delta$

$\therefore f(x) = 0$

$|f(x) - L| = |0 - L| = |L| > \frac{|L|}{2}$

$\therefore \exists \quad \epsilon > 0$, s.t $\forall\, \delta > 0, \exists\, x$ s.t. $|x - 0| < \delta$ and $|f(x) - L| \geq \epsilon$

$\therefore \lim_{x\to 0} f(x) \neq L$

$\nexists\, L \epsilon \mathbb{R}$ s.t. $\lim_{x\to 0} f(x) = L$

$\therefore \lim_{x\to 0} f(x)$ does not exist.

Continuity

Definition: Let $A \subseteq \mathbb{R}$, Let $f: A \to \mathbb{R}$, and let a ϵA. We say that f is continuous at a if

$$\lim_{x \to a} f(x) = f(a)$$

In other words, the function is continuous at 'a', if for each $\epsilon > 0, \exists\ \delta > 0$ s.t. $|f(x) - f(a)| < \epsilon$, when $|x - a| < \delta$.

Discontinuous functions: A function is said to be discontinuous at a point c of its domain if it is not continuous there at. The point c is then called a point of discontinuity of the function.

Types of discontinuities:

(i) A function is said to have a removable discontinuity at $x = c$ if $\lim_{x \to c} f(x)$ exists but is not equal to the value $f(c)$.

(ii) f is said to have a discontinuity of the first kind at $x = c$ if $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ both exist but are not equal.

(iii) f is said to have a discontinuity of the second kind at $x = c$ if neither $\lim_{x \to c^-} f(x)$ nor $\lim_{x \to c^+} f(x)$ exists.

Question: Suppose that f is continous at x_0 and $f(x_0)$. Prove that there exists an open interval containing x_0 on which $f(x) > 0$

Solution: Since f is continuous at x_0,

$$\lim_{x \to x_0} f(x) = f(x_0)$$

$\therefore$ for $\epsilon = f(x_0) > 0, \exists\ \delta > 0$ s. t.

$\therefore\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < f(x_0)$

$= f(x_0) - f(x_0) < f(x) < f(x_0) + f(x_0)$

$= f(x) > 0$

$\therefore$ for $x \epsilon (x_0 - \delta, x_0 + \delta), f(x) > 0$.

Question: Show that if $f(x)$ is continuous at $x = c$, then so is $|f(x)|$. Is the converse true?

Solution: Let $f(x)$ be continuous at $x = c$

i.e. $\lim_{x \to c} f(x) = f(c)$

i.e. $\forall \epsilon > 0, \exists \delta > 0$ s.t.

$|x - x_0| < \delta \Rightarrow |f(x) - f(c)| < \epsilon \qquad \forall x$

$\Rightarrow \big||f(x) - f(c)|\big| < |f(x) - f(c)| < \epsilon$

(by triangular inequality)

i.e. $\forall \epsilon > 0, \exists \delta > 0$ s.t.

$|x - x_0| < \delta \Rightarrow \big||f(x) - f(c)|\big| < \epsilon \qquad \forall x$

i.e. $\lim_{x \to c} |f(x)| = |f(c)|$

$\therefore |f(x)|$ is continuous at $x = c$.

The Converse is not true.

i.e. $|f(x)|$ is continuous at $x = c \not\Rightarrow f(x)$ is continuous at $x = c$

e.g. Consider, $f(x) = \begin{cases} 1, & x \leq 2 \\ -1, & x > 2 \end{cases}$

$\therefore |f(x)| = 1$

$|f(x)|$ is a constant function $\Rightarrow$Continuous of $x = 2$

But $\lim_{x \to 2} f(x) \neq f(2) = 1$

$\therefore$ for $\epsilon = 1 > 0$

$\forall \delta > 0, \exists\ x > 2 \text{ s.t. } |x - x_0| < \delta$

$\because |f(x) - 1| = |-1 - 1| = 2 > \epsilon$

$\therefore \exists \epsilon > 0$ s.t $\forall \delta > 0 \ \exists \text{ s.t.} |x - x_0| < \delta$ and $|f(x) - 1| \geq \epsilon$

$\therefore \lim_{x \to 2} f(x) \neq 1$

So, $f(x)$ is not continuous at $x = 2$

<u>Question:</u> Let $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$

a) Show that f is continuous at $x = 0$.
b) Show that f is not continuous at every non-zero value of x.

Solution:

a) $f(0) = 0$

To show: $\lim\limits_{x \to 0} f(x) = 0$

Let $\epsilon > 0$ be given.

$$|f(x) - 0| < \epsilon \Rightarrow \begin{cases} |x - 0| < \epsilon, & \text{if } x \text{ is rational} \\ 0 < \epsilon, & \text{if } x \text{ is irrational} \end{cases}$$

Choosing $\delta = \epsilon$,

$|x - 0| < \delta, \quad |x - 0| < \epsilon,$

$|x - 0| < \delta \ |x - 0| < \epsilon$

$$|x - 0| < \delta \Rightarrow \begin{cases} |x - 0| < \epsilon, & \text{if } x \text{ is rational} \\ 0 < \epsilon, & \text{if } x \text{ is irrational} \end{cases}$$

$\Rightarrow |f(x) - 0| < \epsilon$

$\therefore \lim\limits_{x \to 0} f(x) = 0$

$\therefore f$ is continuous at $x = 0$.

Let $x_0 \neq 0$

To show: f is not continuous at $x = x_0$.

Let if possible, $\lim\limits_{x \to x_0} f(x) = f(x_0)$

Case-1: $x_0 \epsilon Q$:

$\because x_0 \epsilon Q \Rightarrow f(x_0) = x_0 \neq 0$

for $\epsilon \frac{f(x_0)}{2}$

$\exists \ \delta$ s. t. $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$ (exists because of denseness of Q in $\mathbb{R}$)

For $x \epsilon Q'$ s.t. $|x - x_0| < \delta$

$|f(x) - f(x_0)| = |0 - f(x_0)| = |f(x_0)| < \frac{|(x_0)|}{2}$

which is a contradiction

$\therefore \lim\limits_{x \to x_0} f(x) \neq f(x_0)$

<u>Case-2:</u> $x_0 \epsilon Q'$:

$\therefore f(x) = 0$

for $\epsilon = \frac{|(x_0)|}{2} > 0$

$\exists \ \delta_1$ s. t. $|x - x_0| < \delta_1 \Rightarrow |f(x) - 0| < \epsilon$

Let δ =min $\{\delta_1, \ \epsilon\}$

$\therefore |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$

for $x \, \epsilon \, Q$ s.t. $|x - x_0| < \delta$

$|x - 0| < \frac{|x_0|}{2} \Rightarrow \ |x| < \frac{|x_0|}{2}$ (i)

Further, $\big||x| - |x_0|\big| < |x - x_0| < \delta < \epsilon$

$\therefore \ \big||x| - |x_0|\big| < \frac{|x_0|}{2}$

$\therefore \ |x| - \frac{|x_0|}{2} < |x| < |x_0| + \frac{|x_0|}{2}$

$\therefore \ |x| > \frac{|x_0|}{2}$ (ii)

by (i) & (ii), $|x| < \frac{|x_0|}{2}$ and $|x| > \frac{|x_0|}{2}$

which is a contradiction.

$\therefore \lim_{x \to x_0} f(x) \neq f(x_0)$

$\therefore f$ is not continuous at $x \neq 0$

<u>Intermediate Value Theorem:</u> If a function f is continuous on $[a, b]$ and $f(a) \neq f(b)$, then it assumes every value between $f(a)$ and $f(b)$.

<u>Proof:</u> Let A be any number between $f(a)$ & $f(b)$. We shall show that $\exists$ a number $c \ \in [a, b]$ s.t. $f(c) = A$. Consider a function $\emptyset$ defined on $[a, b]$ s.t.

$\emptyset(x) = f(x) - A$

clearly $\emptyset$ is continuous on $[a, b]$.

also, $\emptyset(a) = f(a) - A \qquad and \qquad \emptyset(b) = f(b) - A$

so that $\emptyset(a)$ and $\emptyset(b)$ are of opposite signs.($\because$ A lies between $f(a)$ & $f(b)$)

Thus the function $\emptyset$ is continuous on $[a,b]$ and $\emptyset(a)$ & $\emptyset(b)$ are of opposite signs therefore, $\exists$ $c \in (a,b)$ s. t. $\emptyset(c) = 0 \Rightarrow f(c) - A = 0 \Rightarrow f(c) = A$.

Differentiability

Definition: Let $I \subseteq \mathbb{R}$ be an interval, let $f: I \to \mathbb{R}$, and let $c \in I$. We say that a real number L is the derivative of f at c if given any $\epsilon > 0,\ \exists\ \delta(\epsilon) > 0$

s.t. if $x \epsilon\ I$ satisfies $0 < |x - c| < \delta(\epsilon)$, then

$$\left|\frac{f(x) - f(c)}{x - c} - L\right| < \epsilon$$

In this case we say that f is differentiable at c, and we write $f'(c)$ for L.

In other words, the derivative of f at c is given by the limit

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided this limits exists.

Example: Show that the function $f(x) = x^2$ is derivable on $[0,1]$.

Let x_0 be any point of $(0,1)$ then

$$f'(x_0) = \lim_{x \to x_0} \frac{x^2 - {x_0}^2}{x - x_0} = \lim_{x \to x_0} (x + x_0) = 2x_0 (\text{exist finitely})$$

at the end points, we have

$$f'(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{x^2}{x} = \lim_{x \to 0+} x = 0\ (\text{exist finitely})$$

$$f'(1) = \lim_{x \to 1-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1-} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1-} (x + 1) = 2\ (\text{exist finitely})$$

Thus the function is differentiable in $[0,1]$.

Example: A function f is defined as:

$$f(x) = \begin{cases} x^2\ Sin\ \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is derivable at $x = 0$ but $\lim_{x\to 0} f'(x) \neq f'(0)$

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{x}$$

$$= \lim_{x\to 0} \frac{x \sin\frac{1}{x}}{1} = 0 \Rightarrow f \text{ is differentiable at } x = 0.$$

From elementary calculus, we know that for $x \neq 0$

$$f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$$

Clearly, $\lim_{x\to 0} f'(x)$ does not exist and therefore, there is no possibility of $\lim_{x\to 0} f'(x)$ being equal to $f'(0)$.

Thus $f'(x)$ is not continuous at $x = 0$ but $f'(0)$ exists.

Theorem: If $f: I \to \mathbb{R}$ has a derivative at $c \in I$, then f is continuous at c.

Proof: for all $x \in I$, $x \neq c$, we have

$$f(x) - f(c) = \left(\frac{f(x) - f(c)}{x - c}\right)(x - c)$$

Since $f'(c)$ exists, so

$$\lim_{x\to c} \big(f(x) - f(c)\big) = \lim_{x\to c} \left(\frac{f(x) - f(c)}{x - c}\right) . \left(\lim_{x\to c}(x - c)\right)$$

$$= f'(c) \times 0$$

Therefore, $\lim_{x\to c} f(x) = f(c)$ so that f is continuous at c.

<u>Rolle's Theorem:</u> Suppose that f is continuous on a closed interval $I = [a, b]$, that the derivative f' exists at every point of the open interval (a, b) and that $f(a) = f(b)$. Then $\exists$ at least one point c in (a, b) s. t. $f'(c) = 0$.

<u>OR</u>

If a function f defined on $[a, b]$ is

(i) Continuous on $[a, b]$,

(ii) derivable on (a, b), and
(iii) $f(a) = f(b)$

then $\exists$ at least one real no. c between a & b $a < c < b$ s. t. $f'(c) = 0$

Proof: Do your-self.

Lagrange's Mean Value Theorem:

If a function f defined on $[a, b]$ is

(i) Continuous on $[a, b]$ and
(ii) derivable on (a, b),

then $\exists$ at least one real no. c between a and b $\big(C \in (a,\ b)\big)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Cauchy's Mean Value Theorem:

If two functions $f,\ g$ defined on $[a, b]$ are

(i) Continuous on $[a, b]$
(ii) derivable on (a,b), and
(iii) $g'(x) \neq 0$, $\forall\, x \in (a, b)$

then $\exists$ at least one real no. c between a & b s.t.

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Theorem: Let $f: I \to \mathbb{R}$ be differentiable on the interval I. Then:

(a) f is increasing on I iff $f'(x) \geq 0$ $\quad \forall\, x \in I$
(b) f is decreasing on I iff $f'(x) \leq 0$ $\quad \forall\, x \in I$

Theorem: Let I be on open interval and let $f: I \to \mathbb{R}$ have a second derivative on I. Then f is a convex function on I iff $f''(x) \geq 0\ \forall \quad x \in I$

Assignment

Q.1. Show that

a) Interval (Open/closed) is nbd of all of its members except the end points.
b) A non-empty finite set is not a nbd of any point.
c) Superset of a nbd of a point x is also a nbd of x.
d) If M and N are nbds of a point x, then that $M \cap N$ is also a nbd of x.

Q.2. If a function f is continuous on a closed interval $[a, b]$ and $f(a)$ & $f(b)$ are of opposite signs $(f(a).f(b) < 0)$, then there exists at least are point $\alpha \in (a,\ b)$ s. t. $f(\alpha) = 0$.

Q.3. Show that the function defined by

$$f(x) = \begin{cases} x\, Sin\, \dfrac{1}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

is continuous at $x = 0$

Q.4. A function f is defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} -x^2 & if\ \ x \leq 0 \\ 5x - 4 & if\ 0 < x \leq 1 \\ 4x^2 - 3x & if\ 1 < x < 2 \\ 3x + 4 & if\ \ x \geq 2 \end{cases}$$

Examine f for continuity at $x = 0, 1, 2$. Also discuss the kind of discontinuity, if any.

Q.5. Is the function, where $f(x) = \frac{x - |x|}{x}$ continuous?

Q.6. Show that $f(x) = |x| + |x - 1|, \qquad \forall\, x \in \mathbb{R}$

is continuous but not derivable at $x = 0$ and $x = 1$.

Q.7. Show that

$$f(x) = \begin{cases} x\, sin\, \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is continuous but not derivable at the origin.

Q.8. Show that

$$f(x) = \begin{cases} 0 & , \quad \text{if } x \leq 0 \\ x & , \quad \text{if } x > 0 \end{cases}$$

is continuous but not derivable at $x = 0$.

Q.9. Show that for any real no. k, the polynomial $f(x) = x^3 + x + k$ has exactly one real root.

Q.10. Verify Rolle's theorem for the function $f(x) = x^3 - 9x$.

Q.11. Use Intermediate value theorem to show that there is a root of $sinx = x^2 - x$ in the interval (1, 2).

References:

- G.B. Thomas, M.D. Weir, J.R. Hass, Thomas' Calculus, Pearson Publication.
- R.G. Bartle, D.R. Sherbert, Introduction to Real Analysis, Wiley Publication.

Riemann Integrals

The Riemann integral, as it is called today, is the one fundamental topic usually discussed in introductory calculus. In this module, we introduce the concept of Riemann integrals and discuss some of its properties. Throughout this module, it is assumed that we are working with a bounded function f on a closed interval $[a,\ b]$.

Partition of $[a,\ b]$: Let $[a,\ b]$ be a closed interval and let $x_0\ x_1, \dots \dots \dots\ x_n$ are the points of $[a,\ b]$ s.t.

$a = x_0 < x_1 < \dots\dots\dots\dots x_{n-1} < x_n = b$

The define a set

$$P = \{a = x_0,\ x_1\ x_2, \dots \dots \dots \dots \dots \dots x_{r-1},\ x_r\ \dots \dots \dots \dots \dots \dots ., x_n = b\}$$

is called a partition of $[a,\ b]$ with $(n+1)$ points; and $I_r = [x_{r-1}\ ,\ x_r\ ]$ be the r^{th} sub-interval of $[a,\ b]$ obtained by the points x_{r-1} & x_r of P. i. e.

$I_1 = [x_0,\ x_1],\ \ I_2 = [x_1,\ x_2],\ \dots\dots\dots\dots, I_{r=} [x_{r-1}\ ,\ x_r\ ], \dots \dots \dots \dots, I_n = [x_{n-1}\ ,\ x_n\ ]$

<u>**Length of r^{th} interval**</u>: = $l\ (I_r) = |I_r| = |x_r - x_{r-1}|$

<u>**Norm of Partition:**</u> Let P be a partition of $[a, b]$ and $I_r\ = [x_{r-1}\ ,\ x_r\ ],$ be the r^{th} sub-interval and $I(I_r) = \Delta x_r = |\ x_r\ \ x_r - 1,|$

The norm of P is denoted by $\mu(P)$ or$\|P\|$ and defined as:

$\|P\| = Max\ \{\Delta x_i\ \ |\ \ \ i = 1\ \ to\ \ \ n\}$

$e.g.\ \ \ I = [0,1]$

Let $P = \{0 = x_0,\ x_1,\ x_2,\ x_3 = 1\}$

$= \left\{0,\ \frac{1}{2}\ ,\ \frac{2}{3}\ ,\ 1\right\}$

$I_1 = \left[0,\ \frac{1}{2}\right], \qquad I_2 = \left[\frac{1}{2},\ \frac{2}{3}\right], \qquad I_3 = \left[\frac{2}{3}\ ,\ 1\right]$

$l\ (I_1) = \frac{1}{2} \qquad , \qquad l\ (I_2) = \frac{1}{6}, \qquad l\ (I_3) = \left|1 - \frac{2}{3}\right| = \frac{1}{3}$

So $\|P\| = \max\left\{\ \frac{1}{2}\ ,\ \frac{1}{6}\ ,\ \frac{1}{3}\right\} = \frac{1}{2}$

<u>**Refinement of Partition**</u>**:** Let P and P^* are two partitions of $[a, b]$ s. t. $P \subseteq P^*$ then P^* is called the refinement or finer than P.

$P = \left\{0,\ \frac{1}{2}\ ,\ \frac{2}{3}\ ,\ 1\right\}$

$P^* = \left\{0, \frac{1}{4}\ ,\ \frac{1}{2}\ , \frac{2}{3}\ , 1\right\}$

$P^{**} = \left\{0, \frac{1}{4}\ , \frac{1}{2}\ , \frac{2}{3}\ , \frac{3}{4}, 1\right\}$

$\|P\| = \frac{1}{2}$, $\quad \|P^*\| = \frac{1}{2}$, $\quad \|P^{**}\| = \frac{1}{4}$

<u>Ex</u>: $\mathrm{P} = \left\{0,\ \frac{1}{2}\ ,\ \frac{2}{3}\ ,\ 1\right\}$

$\mathrm{P}^* = \left\{0, \frac{1}{2}\ ,\ \frac{2}{3}\ , \frac{3}{4}\ , 1\right\}$

$\|P\| = \frac{1}{2}$, $\quad \|P^*\| = \frac{1}{2}$

Note:

(1) $\|P\| \geq \|P^*\|$

(2) Let $f(x)$ be a bounded function defined on $[a,\ b]$.

$\exists\ m, M \ \in \mathbb{R}$ s. t.

$m \leq f(x) \leq M$

Now, $I_r = [x_{r-1}\ ,\ x_r\ ]$ be the r^{th} sub-interval.

$f(x)$ is also bounded on I_r.

let $M_r = \sup\limits_{x \in [x_{r-1}\ ,\ x_r\ ]} \ f(x)$

$M_r = \inf\limits_{x \in [x_{r-1}\ ,\ x_r\ ]} \ f(x)$

Then

$$m \ \leq mr \leq Mr \ \leq M$$

<u>Darboux Upper Sum and Lower Sum:</u>

Let $f: [a,\ b] \rightarrow \mathbb{R}$ be a bounded function on $[a,\ b]$. Let us take a partition P of $[a,\ b]$ defined by P= $\{x_0,\ x_1,\ \dots \dots \dots \dots \dots x_n\}$ where $a = x_0 < x_1 \ldots\ldots\ldots\ldots < x_n = b$.

Since f is bounded on $[a,\ b]$, f is bounded on $[x_{r-1}\ ,\ x_r\ ]$, for $r = 1, 2, \dots \dots \dots, n$.

Let $M = \sup\limits_{x \in [a,\ b]} f(x)$ $\quad , \quad$ $m = \inf\limits_{x \in [a,\ b]} f(x)$

$$M_r = \sup_{x \in [x_{r-1},\ x_r]} f(x) \quad , \qquad m_r = \inf_{x \in [x_{r-1},\ x_r]} f(x)$$

$$r = 1, 2, \dots \dots \dots n$$

Then

$M_1 \Delta x_1 + M_2 \Delta x_2 + \ \dots\dots\dots\dots M_n \Delta\, x_n = \sum_{r=1}^{n} M_r\ \Delta x_r$

called Darboux upper sum over the partition P and is denoted by $\mathrm{U}(\mathrm{P}.f)$.

$\sum_{r=1}^{n} m_r\, \Delta x_r = m_1 \Delta x_1 + \ \dots\dots\dots\dots m_n \Delta\, x_n$

is called the lower Darboux sum corresponding partition P and is denoted by L. (P. f)

Note: The Family of all partitions of $[a,\ b]$ is denoted by P $[a,\ b]$ and the partition

P $= \{x_0,\ x_1,\ x_2,\ x_3 \dots \dots \dots x_n\}$ is amember of P $[a,\ b]$.

<u>Upper and Lower Riemann Integral:</u>

Upper integral of f on $[a,\ b]$ is denoted by $\int_a^{\bar{b}} f(x)$ and defined by

$\int_a^{\bar{b}} f(x)\ dx = inf\{\ U(P,f): P \in P[a,\ b]\ \}$

Similarly, Lower integral of f on $[a,\ b]$ is denoted by

$\int_{\bar{a}}^{b} f(x)\ dx = Sup\ \{\ L\ (P,f): P \in P[a,\ b]\ \}$

f is said to be Riemann integrable on $[a,\ b]$ if

$$\int_{\bar{a}}^{b} f(x)\ dx = \int_a^{\bar{b}} f(x)\ dx$$

The common value of $\int_{\bar{a}}^{b} f$ or $\int_a^{\bar{b}} f$ is called the Riemann integral of f on $[a,\ b]$ and is denoted by $\int_a^b f(x)\ dx$

<u>Example:</u> A function f defined on $[0,\ 1]$ by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

Show that f is not integrable on$[a,\ b]$

Solution: $\because$ Range set of $f(x) = \{0\ ,\ 1\} \implies f$ is bounded on [0, 1].

Let us take a partition P of $[0,\ 1]$ defined by P= $\{x_0,\ x_1,\ \dots\dots\dots\dots x_n\}$, where $0 = x_0 < x_1 <$…………< $x_n = 1$.

Let Let $M = \sup_{x\,\epsilon\,[0,\ 1]} f(x)$, $m = \inf_{x\,\epsilon\,[0,\ 1]} f(x)$

$M_r = \sup_{x\,\epsilon\,[x_{r-1},\ x_r]} f(x)$, $m_r = \inf_{x\,\epsilon\,[x_{r-1},\ x_r]} f(x)$

$r = 1, 2, \dots\dots\dots n$

Then $\mathrm{M} = 1, m = 0, M_r = 1, m_r = 0$

for $r = 1, 2, \dots\dots\dots.., n$

$$U(P, f) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1})$$

$$= 1\ (x_1 - x_0) + 1\ (x_2 - x_1) + \cdots + 1\ (x_n - x_{n-1})$$

$$= x_n - x_0 = 1 - 0 = 1$$

$$L\ (P, f) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1})$$

$$= 0$$

Let us consider the set $P\ [0,\ 1]$ of all partitions of $[0,\ 1]$

Let the set $\{L\ (P, f) : P\ \epsilon\ P[0,\ 1]\} = \{0\}$

$\sup\{L\ (P, f) : P\ \epsilon\ P[0,\ 1]\} = \{0\}$ i. e. $\int_{\mathbf{0}}^{1} f(x)dx$

and $\{U\ (P, f) : P\ \epsilon\ P[0,\ 1]\} = \{1\}$

$= \inf\{U\ (P, f) : P\ \epsilon\ P[0,\ 1]\} = \{1\}$ i. e. $\int_{0}^{\bar{\mathbf{1}}} f(x)dx = 1$

Since $\int_{\mathbf{0}}^{1} f(x)\,dx \neq \int_{0}^{\bar{\mathbf{1}}} f(x)\,dx$

f is not integrable on $[a,\ b]$

<u>Theorem:</u>(Condition for integrability)

Let a function $f: [a,\ b] \to \mathbb{R}$ be bounded on $[a,\ b]$. Then f is integrable on $[a,\ b]$ iff for each $\epsilon > 0$, $\exists$ a partition P of $[a,\ b]$ s. t.

$$U(P.f) - L(P.f) < \epsilon$$

Proof: Do yourself.

Result:

(1) Let a function $f: [a,\ b] \to \mathbb{R}$ be monotone on $[a,\ b]$. Then f is integrable on $[a,\ b]$.
(2) Let a function $f: [a,\ b] \to \mathbb{R}$ be continuous on $[a,\ b]$. Then f is integrable on $[a,\ b]$.
(3) Let a function $f: [a,\ b] \to \mathbb{R}$ be bounded on $[a,\ b]$ and let f be continuous on $[a, b]$ except for a finite no. finite no. of points in $[a,\ b]$. Then f is $R -$ integrable on $[a, b]$

Example: $f(x) = sgn\ x, \quad x \epsilon\ [-2,\ 2]$

$$f(x) = \begin{cases} -1 & , \quad if\ -2 \le x < 0 \\ 0 & , \quad if\ x = 0 \\ 1 & , \quad if\ 0 < x \le 2 \end{cases}$$

f is bounded on $[-2,\ 2]$, since $|f(x)| \le 1$, for all $x \epsilon\ [-2,\ 2]$. f is continuous on $[-2,\ 2]$ except at only one point 0. Therefore f is R-integrable on $[-2,\ 2]$(by previous result – III).

Example: $f: [0,\ 1] \to [0,\ 1]$ s.t.

$$f(x) = \frac{2^k-1}{2^k} \text{ if } x \epsilon \left[\frac{2^{k-1}-1}{2^{k-1}}, \frac{2^k-1}{2^k}\right] \text{ then find } \int_0^1 f(x)dx = ?$$

Solution: $\Delta x_k = x_k - x_{k-1} = \frac{2^k-1}{2^k} - \frac{2^{k-1}-1}{2^{k-1}} = \frac{1}{2^k}$

$$= \overline{\int_0^1} f(x)\, dx = \lim_{k\to\infty} \sum_k M_k \Delta x_k = \lim_{k\to\infty} \sum_k \frac{2^k-1}{2^k} \cdot \frac{1}{2^k}$$

$$= \lim_{k\to\infty} \sum_k \frac{2^k-1}{2^{2k}}$$

$$= \lim_{k\to\infty} \sum \left(\frac{1}{2^k} - \frac{1}{2^{2k}}\right)$$

$$= \ s_1 = \frac{1}{2} - \frac{1}{4}$$

$$= \ s_2 = \frac{1}{4} - \frac{1}{16}$$

$$= \ s_3 = \frac{1}{8} - \frac{1}{64} \ldots\ldots.$$

So $\lim_{k\to\infty} \sum M_k \Delta x_k = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots\ldots\ldots\ldots$

$$= \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \ldots\ldots\ldots\ldots..$$

$$= \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{1}{2} \times \frac{4}{3} = \frac{2}{3}$$

$$\overline{\int_0^1} f(x)\,dx = \frac{2}{3}$$

Similarly, $\underline{\int_0^1} f(x)\,dx = \frac{2}{3}$

So, $\int_0^1 f(x)\,dx = \frac{2}{3}$.

<u>Assignment</u>

Q.1. Show that

- **a)** $U\,(P, f) \geq L\,(P,\ f)$
- **b)** $U\,(P, f) \geq U\,(P^*,\ f)$
- **c)** $L\,(P, f) \leq L\,(P^*,\ f)$
- **d)** $U(P, f) \geq\ L\,(P^*,\ f)$
- **e)** $U\,(P^*,\ f) \geq L\,(P^*,\ f) \geq L\,(P,\ f)$

Q.2. Discuss Riemann integrability of function

$$f(x) = [x]\ , \qquad\qquad x \in [0,\ 2].$$

Q.3. Let f: [0, 1] → R is the function $f(x) = x^2$. For any ε >0, choose a partition

$$P = \{0 = x_0,\ x_1\ x_2, \dots\dots\dots\dots\dots\dots x_{r-1},\ x_r\ \dots\dots\dots\dots\dots\dots., x_n = 1\}$$

such that $x_i - x_{i-1} < \varepsilon/2$ for all $1 \leq i \leq n$.

Show that

$$U(P.f) - L(P.f) < \epsilon$$

Hence, f is Reimann integrable.

<u>References:</u>

- G.B. Thomas, M.D. Weir, J.R. Hass, Thomas' Calculus, Pearson Publication.
- R.G. Bartle, D.R. Sherbert, Introduction to Real Analysis, Wiley Publication.

Module-3

Pretest on Matrices and Determinant

1) Find $3A - 2B$, if $A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 7 & 1 \\ 5 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -6 & 0 & 2 \\ 1 & 5 & 3 \\ 8 & 2 & 1 \end{bmatrix}$

2) Find $A(B + C)$, if $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -5 & 7 \\ 0 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \\ 6 & 2 \\ 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & -8 \\ 3 & 1 \\ 10 & 4 \end{bmatrix}$

3) Classify the type of matrix :

i) $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 4 & -7 \\ 5 & -7 & 6 \end{bmatrix}$, ii) $\begin{bmatrix} 0 & -4 & -5 \\ 4 & 0 & -7 \\ 5 & 7 & 0 \end{bmatrix}$, iii) $\begin{bmatrix} 4 & 2 & 6 \\ 0 & 5 & -3 \\ 0 & 0 & 1 \end{bmatrix}$,

iv) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, v) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4) Determine the matrices X & Y from the equation:

X + Y = $\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, X - Y = $\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

5) If $A = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 3 & 4 \\ -1 & 5 \end{pmatrix}$ does AB exists?

6) If A and B are two square matrices of same order. Is $(A + B)^2 = A^2 + 2AB + B^2$

7) Apply the properties of determinants and calculate :

i) A= $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$, ii) B= $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$, and iii) C = $\begin{vmatrix} 2 & 3 & 4 \\ 2 & a+3 & b+4 \\ 2 & c+3 & d+4 \end{vmatrix}$.

Module-3
Matrices and Determinants

Lectures required -02

In this module, we discuss about the basics of matrices, types of matrices, operations on matrices, determinants and cofactors, computing inverse of a square matrix, rank and elementary operations with brief discussion on system of linear equations.

1. Matrices and Determinants

1. Matrices and Determinants:

A matrix is defined to be a rectangular array of a number assigned into rows and columns. A set of "mn" elements arranged in rectangular formation containing m-rows and n-columns is called m×n matrix

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$, where a_{ij} are elements of the matrix A.

1.1 Type of Matrices :

There are around 12 types of matrix.

i) **Row matrix** : When m=1, the matrix with one row is called row matrix or vector.

ii) **Column matrix** : When there is only one column i.e. n=1, the matrix is called a column matrix.

iii) **Square matrix :** When m=n, that is number of rows is equal to number of columns.

iv) **Triangular Matrix :** In a square matrix, when the elements above the principal diagonal or below the principal diagonal are all zero, the matrix is called triangular matrix.

v) **Diagonal Matrix :** In a square matrix, when the elements above and below the principal diagonal is zero i.e. matrix is filled with zero elements except on the main diagonal e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

vi) **Scalar matrix :** Scalar matrix is a diagonal matrix in which all the elements along the main diagonal are equal.

e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

vii) **Unit matrix** (Identity Matrix) : A scalar matrix with all the elements on the diagonal are equal to 1.

e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

viii) **Null or Zero Matrix** : If all the elements in matrix are zero, then it is called zero matrix

e.g. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

ix) **Symmetric matrix** : A square matrix A = (a_{ij}) is called symmetric matrix if $a_{ij}= a_{ji}$ for all i and j.

e.g. $\begin{bmatrix} 2 & 3 & 5 \\ 3 & 6 & -7 \\ 5 & -7 & 4 \end{bmatrix}$

x) **Skew symmetric matrix** : A square matrix A = (a_{ij}) is called a skew symmetric if $a_{ij}= -a_{ji}$ for all i and j, i≠j and $a_{ii}= 0$ for all i.

e.g. $\begin{bmatrix} 0 & 4 & 7 \\ -4 & 0 & -5 \\ -7 & 5 & 0 \end{bmatrix}$

xi) **Singular matrix** : If $|A|$=0 (det(A)), then the matrix A is called singular.

xii) **Non-singular matrix** : If $|A| \neq 0$, then the matrix A is called non-singular.

1.2 Operations on Matrices

(i) Addition of two matrices:

Addition of two matricesA=$[a_{ij}]_{m\times n}$and B=$[b_{ij}]_{m\times n}$ can be defined only when both A and B have same more same order.

$$C = [c_{ij}]_{m\times n} (= A + B), where\ c_{ij} = a_{ij} + b_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$$

is the sum of A and B.

Example: A = $\begin{bmatrix} 2 & 1 & 5 \\ -1 & 6 & 2 \end{bmatrix}$, B = $\begin{bmatrix} 0 & 5 & -2 \\ 3 & 4 & 1 \end{bmatrix}$

C = A + B = $\begin{bmatrix} 2+0 & 1+5 & 5-2 \\ -1+3 & 6+4 & 2+1 \end{bmatrix}$ = $\begin{bmatrix} 2 & 6 & 3 \\ 2 & 10 & 3 \end{bmatrix}$

(ii) Multiplication of matrices by scalar :

If k is a real or complex number, and $A = [a_{ij}]$ is a m×n matrix, then the matrix $B = [b_{ij}]$ where $b_{ij} = ka_{ij}$ for $1 \leq i \leq m, 1 \leq j \leq n$ is called a scalar multiplication of A by k and written as $B = kA$.

e.g. : If A $=\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ then 2A $=\begin{bmatrix} 4 & 6 \\ 8 & 10 \end{bmatrix}$

(iii) Multiplication of two matrices :

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{r\times s}$ be two matrices. The product AB is defined only when n=r, (i.e.) no. of columns of A = no. of rows of B.

$C = AB = [c_{ij}]_{m\times n}$ where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}.b_{kj}$, for $1 \leq i \leq m, 1 \leq j \leq s$.

A $=\begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$, B $=\begin{bmatrix} 3 & 1 \\ 2 & 7 \end{bmatrix}$

AB $= \begin{bmatrix} 1\times 3 + 2\times 2 & 1\times 1 + 2\times 7 \\ 0\times 3 + 5\times 2 & 0\times 1 + 5\times 7 \end{bmatrix}$

$=\begin{bmatrix} 7 & 15 \\ 10 & 35 \end{bmatrix}$

Note AB ≠ BA

BA $= \begin{bmatrix} 3\times 1 + 1\times 0 & 3\times 2 + 1\times 5 \\ 2\times 1 + 7\times 0 & 2\times 2 + 7\times 5 \end{bmatrix}$

$=\begin{bmatrix} 3 & 11 \\ 2 & 39 \end{bmatrix}$

(iv) Transpose of matrix :

If $A = [a_{ij}]_{m\times n}$, then the n×m-matrix $B = [b_{ij}]_{n\times m}$ is defined as $b_{ij} = a_{ji}$, $1\leq i \leq m, 1 \leq j \leq n$, is obtained by interchanging the rows and columns is called the transpose of A, denoted by A^T.

e.g. : If A$=\begin{bmatrix} 2 & 3 & 0 \\ 4 & 7 & 5 \end{bmatrix}_{2\times 3}$, $A^T=\begin{bmatrix} 2 & 4 \\ 3 & 7 \\ 0 & 5 \end{bmatrix}_{3\times 2}$

1.3 Determinants and Cofactors

Let $A = [a_{ij}]_{m\times n}$ be a square matrix. If we delete the row and column containing the element a_{ij}, we obtain a square matrix of order n-1.The determinant of this square matrix of order n-1 is called the <u>Minor</u> of the element a_{ij} and is denoted by M_{ij}.

Cofactor $(a_{ij}) = (-1)^{i+j}M_{ij}$ and is denoted by A_{ij}. If A=$[a_{ij}]$ is a square matrix of order n, the matrix $\begin{bmatrix} A_{11}A_{21} & \cdots & A_{n1} \\ A_{12} & . \quad \cdots & . \\ . & . \quad \cdots & . \\ . & . \quad \cdots & . \\ . & . \quad \cdots & . \\ A_{1n}A_{2n} & \cdots & A_{nn} \end{bmatrix}$ is called the adjoint of A, denoted by adj(A).

1.4 Inverse of a square matrix :

A square matrix A of order n is said to be invertible, if there exists a square matrix B of order n such that $AB = BA = I_n$ and B is called the inverse of A and is denoted by A^{-1}.

If A is non-singular square matrix, then A^{-1} exists and $A^{-1} = \frac{1}{|A|}.adj\ (A)$.

1.5 Rank of a Matrix :

Rank of a matrix A is said to be r, if A satisfies the following conditions:

i) There exists an $r \times r$ submatrix whose determinant is non-zero.

ii) The determinant of every (r+1)×(r+1)submatrix is zero.

In other words, order of the determinant of the largest submatrix of A which does not vanish is called the rank of the matrix and is denoted by $r(A)$.

Note that $r(A) \leq min\ (m, n)$ where A is of order m×n.

e.g. Find r(A) using determinants of minors.

$$A=\begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$$

Since A is 3 × 3, $r(A) \leq 3$

$$|A|=\begin{vmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{vmatrix}$$

= 2(2-4)-3(6+2)+4(6+1)

=2(-2)-3(8)+4(7)

= -4 – 24 + 28 = 0

∵ $|A|$ =0, r(A)≤2.

Consider the submatrix order 2×2 :

$$\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} = 4-12 \neq 0.$$

Since determinant of 2×2 order matrix is not equal to zero, $r(A) = 2$.

1.6 Elementary Row(Column) Operations:

Let A be an m×n order matrix. An elementary row (column) operation on A is one of the following three types:

i) Interchange of any two rows(columns) denoted by $R_i \leftrightarrow R_i$ ($C_i \leftrightarrow C_j$).

ii) Multiplication of row(column) by a non-zero element c, denoted by $R_i \rightarrow cR_i$ ($C_i \rightarrow cC_i$).

iii) Addition of any multiple of one row(column) with other row $R_i \rightarrow R_i + kR_j$ ($C_i \rightarrow C_i + kC_j$).

By applying any of these elementary operations, the rank of matrix is not affected. Hence, "By successive application of elementary row and column operations, any non-zero m×n matrix A can be reduced to a diagonal matrix D in which the diagonal entries are either 0 or 1 and all the 1's precede all the zeros on the diagonal.

In other words, the non-zero m×n matrix is equivalent to a matrix of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where I_r is the $r \times r$ - identity matrix and 0 is the zero matrix. This is called the canonical form of the matrix.

Two matrices A and B of the same order are said to be equivalent if B can be obtained from A by a finite number of elementary transformation.

Definition (Rank of a matrix) :

If A is a m×n matrix, then the unique non-negative integer r such that A ~ $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is said to be the rank of A. The matrix is called as the canonical form of A.

Example: Find the rank of A = $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 4 & 2 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 4 & 2 \end{bmatrix}$$

A is 3×4 matrix.∴ Clearly $r(A) \leq 3$.

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -4 & -2 \\ 0 & 3 & 4 & 2 \end{bmatrix} \qquad R_2 \to R_2 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -4 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4/3 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_2 \to R_2/(-3)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4/3 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1,\ C_4 \to C_4 - C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_3 \to C_3 - 4/3\, C_2,\ C_4 \to C_4 - 2/3\, C_2$$

$$A \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

∴ r(A) = 2.

1.7 Solving systems of equations

Given a system of equations of the form $AX = B$, where A is an m×n coefficient matrix, X -unknown vector of (n×1) order and B-a vector of (m×1) order.

$$A = \begin{bmatrix} a_{11} a_{12} & \cdots & a_{1n} \\ a_{21} a_{22} & \cdots & a_{2n} \\ . \quad . & \cdots & . \\ . \quad . & \cdots & . \\ . \quad . & \cdots & . \\ a_{m1} a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{bmatrix}$$

The m×(n+1) matrix, denoted [A|B] is called the augmented matrix of the system

$$[A|B]=\begin{bmatrix} a_{11}a_{12} & \cdots & a_{1n}b_1 \\ a_{21}a_{22} & \cdots & a_{2n}b_2 \\ . \quad . & \cdots & . \quad . \\ . \quad . & \cdots & . \quad . \\ . \quad . & \cdots & . \quad . \\ a_{m1}a_{m2} & \cdots & a_{mn}b_m \end{bmatrix}$$

If the system has atleast one solution, then the system is called consistent, otherwise the system is said to be inconsistent.

A system AX=B

(i) is consistent iff $r(A) = rank\ ([A|B])$.

(ii) has a unique solution iff $r(A) = r([A|B]) = n$, the number of unknowns (In this case m ≥ n).

(iii) has infinitely many solutions if and only if $r(A) = r([A|B]) < min\{m, n\}$.

Remark1: If $m = n$ and the $r(A) = n$, then the $r(A) = r([A|B]) = n$ and hence the system $AX = B$ has a unique solution and the solution is given by $X = A^{-1}B$.

Remark 2: To test whether the system $AX = B$, when $m = n$, is consistent or not, and if it is consistent, then to find the solutions of the system, we can use elementary row operation to the augmented matrix $[A|B]$ and reduce A in $[A|B]$ to a triangular matrix.

Note:

i) If $|A| \neq 0$, then the system has a unique solution, ∴ It is consistent.

ii) If $|A| = 0$ and if $r(A) = r([A|B])$, then the system has infinitely many solutions and hence consistent.

iii) If $|A| = 0$ and if $r(A) \neq r([A|B])$ then the system has no solution and hence the system is inconsistent

References:

1. J.P. Singh, Discrete Mathematics for Under graduates, Ane Books, 2014.
2. P.B. Bhattacharya, S,.K.Jain, S.R. Nagpaul, First Course in Linear Algebra, Wiley, 1983.
3. G. Hadley, Linear Algebra, Narosa Publishing, 1992.

Assignment on Matrices and Determinants

1) If A = $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that A satisfies $A^2-5A+7I = 0$. Hence determine A^3 and A^{-1}.
2) Find the inverse of $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix}$, if it exists, using adjoint.
3) Find the rank of matrix using elementary row/column operation: $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$
4) Find rank of the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 4 & 2 \end{bmatrix}$ by examining the determinant of minors.
5) For what value of λ and μ, the system of equations:

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

(i) Consistent

(ii) Consistent with unique solution

(iii) Inconsistent

6) Apply the properties of determinants and calculate :

i) A= $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$, ii) B= $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$, and iii) C = $\begin{vmatrix} 2 & 3 & 4 \\ 2 & a+3 & b+4 \\ 2 & c+3 & d+4 \end{vmatrix}$.

Module-4

Pretest on Complex Numbers

1. What are the roots of Find the value of $x^2 + 1 = 0$?
2. Find real numbers x and y such that $3x + 2iy - ix + 5y = 7 + 5i$.
3. Perform each of the indicate operations

   (a) $(3 + 2i) + (-7 - i)$ (b) $(-7 - i) - (3 + 2i)$
4. Find the value of (i) $(1 + i)^3$ (ii) $(-2 + 3i)(4 - i)$
5. What is the cube roots of unity ?
6. Find $Re(z)$ and $Im(z)$ of following complex number z

   (a) $z = \left(\frac{i}{3-i}\right)\left(\frac{1}{2+3i}\right)$ (b) $z = \frac{1}{(1+i)(1-2i)(1+3i)}$
7. Find all solutions of the equation $w^2 = (-1 + i)^5$.
8. Find the real and imaginary parts u and v of given complex functions of x and y

   (a) $f(z) = 6z - 5 + 9i$ (c) $f(z) = -3z - 2\bar{z} - i$

   (b) $f(z) = z^3 - 2z + 6$ (d) $f(z) = z^2 - \bar{z}^2$

Module-4
Complex Numbers

Lectures required -03

Note: We use := to abbreviate "defined by" or "written as".

Definition of complex numbers:

Consider ordered pairs of real numbers (x, y). The word '*ordered*' means that $(x, y), (y, x)$ are distinct unless $x = y$. We denote the set of all ordered pairs of real numbers by$\mathbb{C}$. We shall call $\mathbb{C}$ as the set of all complex numbers. In $\mathbb{C}$, we define addition $(+)$ and multiplication (× or juxtaposition) between two such ordered pairs $(x_1, y_1), (x_2, y_2)$ by

$$(x_1\, y_1) + (x_2\, y_2) = (x_1 + x_2,\ y_1 + y_2) \tag{1}$$

and

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 - \ y_1 y_2,\ x_1\, y_2 +\ y_1 x_2). \tag{2}$$

If $z_1 \ = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then we say that

$$z_1 \ = z_2 \ \Leftrightarrow \quad x_1 = x_2 \ \text{ and } y_1 = y_2.$$

In particular, $z = (x\, y) = (0, 0) \ \Leftrightarrow \quad x = 0$ and $y = 0$.

We can easily check the following simple properties for equality of ordered pairs making it an equivalence relation: For $z_1,\ z_2\ and\ z_3$ in $\mathbb{C}$,

I. $z_1 \ = z_2$
II. $z_1 \ = z_2 \Rightarrow z_2 = \ z_1$
III. $z_1 \ = z_2$ and $z_2 \ = z_3 \ \Rightarrow z_1 \ = z_3$.

The associative and commutative laws for addition and the multiplication and distributive laws etc., follow easily from the properties of field of real numbers $\mathbb{R}$. Further, it is clear from equations (1) and (2) that (0,0) is the additive identity, (1,0) is the multiplicative identity, $(-x, -y)$ is the additive inverse of $z = (x, y)$ and

$$\frac{1}{z} := \left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2}\right)$$

is the multiplicative inverse of $z \ = (x, y) \neq (0,0)$.

- There is a unique complex Number, say z_3, such that $z_1 \ + z_3 \ = \ z_2$. If $z_j = (x_j, y_j)$ $(j = 1, 2, 3)$, then $z_3 \ = (x_2 - x_1,\ y_2 - y_1)$ and is denoted by $z_2 \ - z_1$. $[Subtraction]$

- for $z_2 \neq (0,0)$, there is a unique z_3 such that $z_1 = z_2\, z_3$. In fact , z_3=$z_1 . (1/z_2)$ Since $z_2\, z_3 = z_2 . z_1 . (1/z_2) = (z_2 . (1/z_2)). z_1 = 1. z_1 = z_1$. The complex number z_3 is otherwise written as $z_3 = z_1 / z_2$.

The symbol commonly used for a complex number is not (x, y) but $x + iy$, x, y real. Following Euler, we define $i := (0,1)$ in the complex number system $\mathbb{C}$ of ordered pairs. We write the real number x as $(x, 0)$. Then accoridng to (2)

i^2= (0, 1) (0 1)= $(-1,0)$, $\quad i^3 = i^2 . i = (-1,0)(0,1) = (0,-1)$,

and $i^4 = i^2 . i^2 = (1,0)$. Also, $x + iy = (x,0) + (0,1)(y,0) = (x,y)$.The above discussion shows that $\mathbb{C}$ is also a field.Further,writing a real number x as $(x,0)$ and noting that

$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$ and $(x_1, 0)(x_2, 0) = (x_1 x_2, 0)$,

$\mathbb{R}$ Turns out to be a subfield of $\mathbb{C}$. The association $x \mapsto (x,0)$ shows that we can always treat $\mathbb{R}$ as a subset of $\mathbb{C}$. Complex numbers of the form $(x,0)$ are said to be *purely real or just real*. Those of the form $(0,y)$ are said to be *purely imaginary* whenever $y \neq 0$. In particular, we have with the above identification of $\mathbb{R}$, $i^2 = -1$. Every (complex number) $z = (x,y) \in \mathbb{C}$, denoted now by $x + iy$, admits a unique representation.

$$(x,y) = (x,0) + (0,1)(y,0) = x + iy, \text{ with } x, y \in \mathbb{R}.$$

'Zero' viz. $(0,0) = 0 + i0$ is the only complex number both real and purely imaginary. The conjugate of a complex number $z = x + iy$ is the complex number $\bar{z} := x - i\,y$. Note that $z = \bar{z}$ if and only if $x + iy = x - iy$, i.e $y = 0$ i. e.z is purely real. The inverse or reciprocal z^{-1} of a complex number $z = x + iy \neq 0$ is

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\left(\frac{y}{x^2 + y^2}\right),$$

which was defined earlier as the multiplicative inverse of z. we call x and y the *real part* and *imaginary part* of $z = x + iy$, respectively. We write

Re $z := x$ and Im $z := y$; Re $z = \frac{z+\bar{z}}{2}$ and Im z $= \frac{z-\bar{z}}{2i}$.

We know that ordered pairs of real numbers represent points in the geometric place referred to a pair of rectangular axes. We then call all the collection of ordered pairs as $\mathbb{R}^2$ and the two axes as the x-*axis* and $y - axis$. Because $(x,0) \in \mathbb{R}^2$ corresponds to real numbers, the x- *axis* called the *real axis* and since $iy = (0,y) \in \mathbb{R}^2$ is purely imaginary, the *y-axis* is called the *imaginary axis*.

Now, we can visualize $\mathbb{C}$ as a plane with $x + iy$ as points in $\mathbb{R}^2$ and we simply refer to it as the finite complex plane or simply complex plane. Depending on the problems on hand, we use $x + iy$ or (x,y), to represent a complex number.

Theorem 1: The field $\mathbb{C}$ cannot be totally ordered in consistence with the usual order on $\mathbb{R}$. (Total ordering means that if $a \neq b$ then either $a < b$ or $a > b$.

Proof: Suppose that such a total ordering exists on $\mathbb{C}$. Then for $i \in \mathbb{C}$ we would have either $i > 0 \;\; or\; i < 0$ since $i \neq 0$.This means that in either case

$$-1 = i \cdot i = (-i)(-i) > 0$$

which is not true in $\mathbb{R}$. This observation shows that such an ordering in impossible in $\mathbb{C}$.

Theorem 1 means that the expressions $z_1 > z_2$ or $z_1 < z_2$ have to meaning unless $z_1$and z_2 are real.

Concepts of modulus / absolute value:

The modulus or absolute value of $x \in \mathbb{R}$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

As it stands, there is no natural generalized of $|\cdot|$ to $\mathbb{C}$, because, as we have seen in Theorem 1, there is no total ordering on $\mathbb{C}$. However we interpret $|x|$ geometrically as the distance from x to the origin (zero) of the real line. It is this fact which leads us to define the modulus of a complex number $z = x + iy \in \mathbb{C}$ by $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$.

Circles Suppose $z_0 = x_0 + iy_0$, Since

$|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ is the distance between the points $z = x + iy$ and

$z_0 = x_0 + iy_0$, the points $z = x + iy$ that satisfy the equation

$$|z - z_0| = \rho, \qquad \rho > 0. \tag{3}$$

Lie on a circle of a radian ρ centered at the point z_0 see Figure (1)

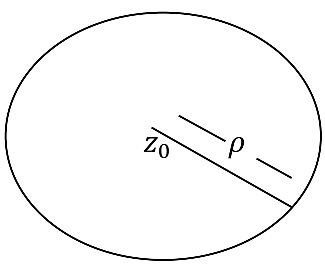


Fig 1

$|z - z_0| = \rho$

Circle of raids ρ

Example 1: Two circles

(a) $|z| = 1$ is an equation of a unit circle centered at the origin.

(b) by rewriting $|z - 1 + 3i| = 5$ as $|z - (1 - 3i)| = 5,$ we see from (3)

That the equation describes a circle of radius 5 centered at the point $z_0 = 1 - 3i.$

Disks and Neighborhoods:

The points z that satisfy the in equality $|z - z_0| \leq \rho$ can be either on the circle $|z - z_0| = \rho$ or within the circle. We say that the set of points defined by $|z - z_0| \leq \rho$ is a **disk** of radius ρ centered at z_0. But the points z that satisfy the strict inequality $|z - z_0| < \rho$ lie within, and not on a circle of radius ρ centered at the point z_0. This set is called a **neighborhood** of z_0. Occasionally, we will need to use a neighborhood of z_0 that also excludes z_0 such a neighborhood is defined by the simultaneous inequality $0 < |z - z_0| < \rho$ and is called a **deleted neighborhood** of z_0. For example, $|z| < 1$ defines a neighborhood of the origin, where as $0 < |z| < 1$ defines a deleted neighborhood of the origin; $|z - 3 + 4i| < 0.01$ defines a neighborhood of $3 - 4i,$ whereas the inequality $0 < |z - 3 + 4i| < 0.01$ define a deleted neighborhood of $3 - 4i.$

Open Sets: A point z_0 is said to be a interior point of a set S of the complex plane if there exists some neighborhood of z_0 that lies entirely within S. if every point of z of a set S in an **interior point**, then S is said to be an open set see Figure 2 For example Re $(z) > 1$ defines a *right half plane*, which is an **open set**. If we choose, for example $z_0 = 1.1 + 2i$, then a neighborhood of z_0 lying entirely in the set is defined by $|z - (1.1 + 2i)| < 0.05$ See Fig. 3.On the other hand, the set S of points in complex plan defined by Re $(z) \geq 1$ *not* an open set because every neighborhood of a point lying on the line $x = 1$ must contain a points in S and points not in S. See Fig 1.18

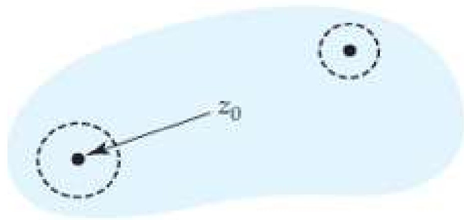


Figure 2 Open set

$$|z-(1.1+2i)|<0.05$$

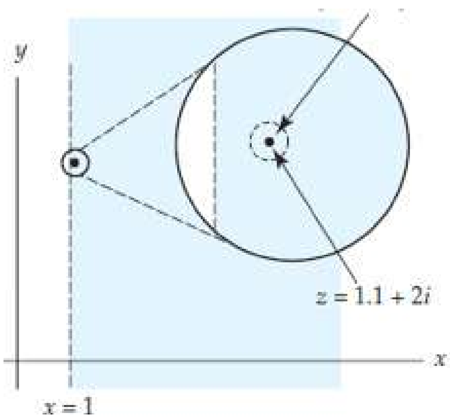


Figure 3 Open set with magnified view of point near $x = 1$

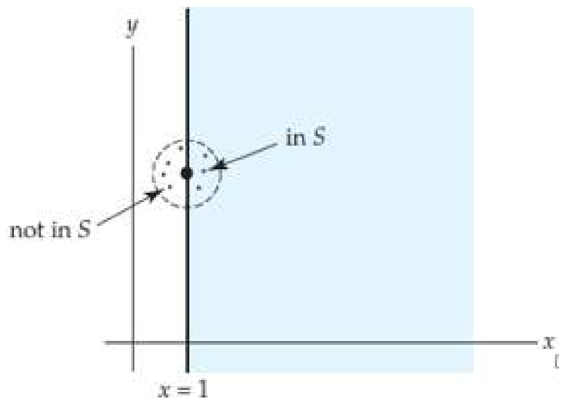


Figure 4 Set S not Open

If every neighborhood of a point z_0 of a set *S* contains at least one point of *S*, and at least one point not in *S* then, z_0 is said to be **boundary point** of *S.* A point z that is neither an interior point nor a boundary point of a set *S* is said to be an **exterior point** of S; in other words, z_0 is an exterior point of a set *S* if there exists some neighborhood of that Contains no points of *S*. Figure 5 shows a typical set *S* with Interior, boundary, and exterior.

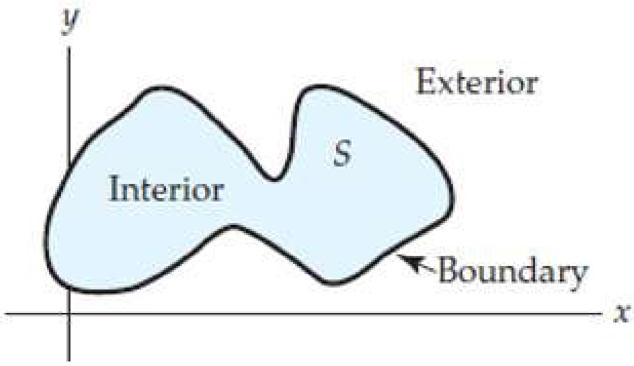
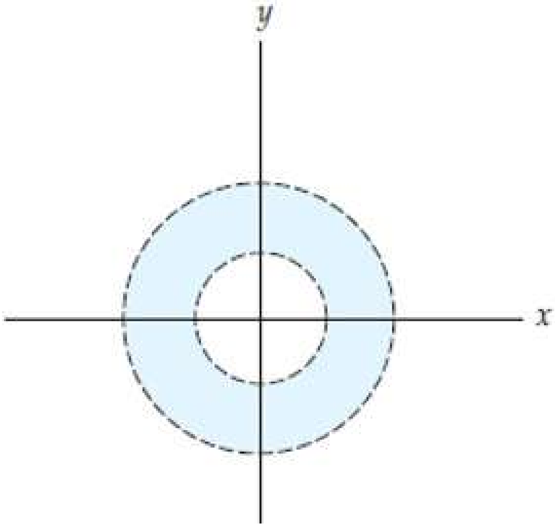


Figure 5 Interior, boundary, and exterior of set *S*

Annulas: The set S_1 of points satisfying the inequality $\rho_1 < |z - z_0|$ lie exterior to the circle of radius ρ_1 centered at z_0, whereas the set S_2 of points satisfying $|z - z_0| < \rho_2$ lie interior to the circle of radius ρ_2 centered at z_0. Thus, if $0 < \rho_1 < \rho_2$, the list of points satisfying the simultaneous inequality

$$\rho_1 < |z - z_0| < \rho_2 \tag{4}$$

is the intersection of the sets $S_1$and S_2. This intersection is an open circular ring centered at z_0. Figure 6 illustrates such a ring centered at the origin. The set defined by (4) is called an open **circular annulus**. By allowing $\rho_1 = 0$, we obtain a deleted neighborhood of z_0.



$1 < |z| < 2$; Interior of circular ring

Figure 6

Domain: If any pair of points z_1 and z_2 in a set S can be connected by a polygonal line that consists of a finite number of line segments joined end to end that lies entirely in the set, then the S is said to be **connected**. See Figure 7. An open connected set is called a **domain**. Figure 3 is connected and so are domain. The set of numbers z satisfying Re $(z) \neq 4$ is an open set but is not connected since it is not possible to join points on either side of the vertical line $x = 4$ by a polygonal line without leaving the set (bear in mind that the points on the line $x = 4$ are not in the set). A neighborhood of a point z_0 is a connected set.

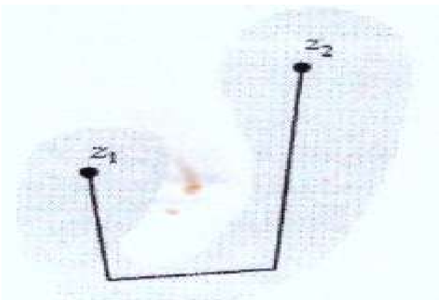


Figure 7 Connected set.

Regions:

A region is set of points in the complex plane with all, some, or one of its boundary points. Since an open set does not contain any boundary points, it is automatically a region. A region that contains all its boundary points is said to be **closed**. The disk defined by $|z - z_0| \leq \rho$ is an example of a closed region and is referred to as a **closed disk**. A neighborhood of a point z_0 defined by $|z - z_0| < \rho$ is an open set or an open region and is said to be an **open disk**. If the center z_0 is deleted from either a closed disk or an open disk, the regions defined by $0 < |z - z_0| \leq \rho$ or $|z - z_0| < \rho$ are called **punctured disks**. A punctured open disk is the same as a deleted neighborhood of z_0. a region can be neither open nor closed; the annular region defined by the inequality $1 \leq |z - 5| < 3$ contains only some of its boundary points (the points lying on the circle$|z - 5| = 1$), and so it is neither open nor closed. In (4) we defined a circular annular region; in a more general interpretation, an annulus or annular region may have the appearance shown in figure 8.

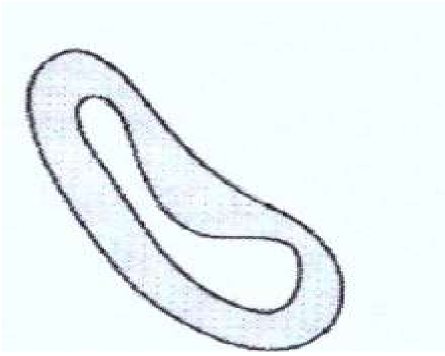


Figure 8 Annular region

<u>Bounded Sets:</u> Finally, we say that a set s in the complex plane is bounded if there exists a real number $R > 0$ such that $|z| < R$ every z in S. That is, s is bounded if it can be completely enclosed within some neighborhood of the origin. In Figure 9, the set S shown in color is bounded because it is contained entirely within the dashed circular neighborhood of the origin. A set is unbounded if it is not bounded. For example, the set in Figure 6 is bounded, whereas the sets in Figures 10, 11, and 12 are unbounded.

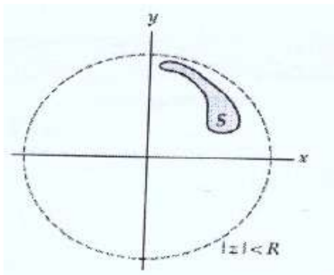


Figure 9 The set S is bounded Since some neighborhood of the origin encloses S entirely.

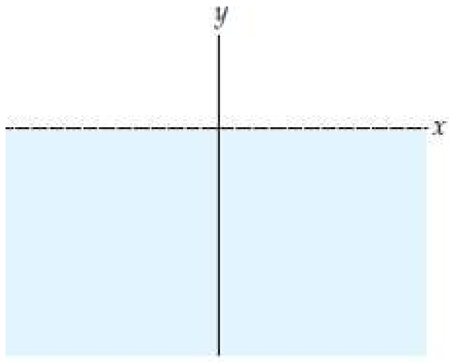


Figure 10 $\text{Im}(z) < 0$; lower half plan

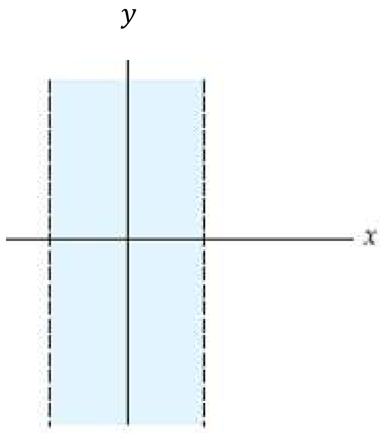


Figure 11. $-1 < Re(z) < 1$; infinite vertical strip

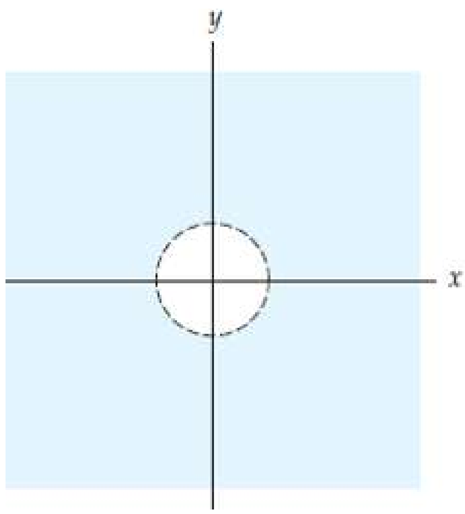


Figure 12. $|z| > 1$; exterior of unit circle

Complex Function: One of the most important concepts in matchematics is that of a function. You may recall from previous courses that a function is a certain kind of corresspondence between two sets; more specifically:

A function f from a set A to a set B is a rule of correspondence that assign to each element in a one and only one element in *B*.

We often think of a function as a rule or a machine that accepts inputs from the set *A* and returns outputs in the set *B*. In elementary calculus we studied functions whoe inputs and outputs were real numbers. such functions are called **real-valued functions** of a real variable. In this section we begin our study of functions whose inputs and outputs are complex numbers. Naturally, we call these functions **complex functions of a complex variable**, or **complex functions** for short. As we will see, many interesting and useful complex functions are simply generalizations of well-known functions from calculus.

Function: Suppose that f is a function from the set *A* to the set *B* . If f assigns to the element a in *A* the element b in *B*, then we say that $b = f(a)$. The set *A*-the set of inputs is called the domain of f and the set of images in *B* the set outputs is called the range of f. We denote the domain and range of a function f by Dom (f) and Range (f), respectively. As an example, consider the "squaring" function $f(x) = x^2$ defined for the real variable *x*. Since any real number can be squared, the domain of f is the set **R** of all real numbers. That is , Dom (*f*) = *A* = **R.** The range of *f* consists of all real numbers x^2 where x is a real number. Of course, $x^2 \geq 0$ for all real x, and it is easy to see from the graph of f that the range of f that the range of f is the set of all nonnegative real numbers. Thus, Range (f) is the interval $(0, \infty)$. The range of f need not be the same as the set B. For instance, because the interval $(0, \infty)$ is a subset of both **R** and the set **C** of all complex numbers, *f* can be viewed as a function from ***A***=***R*** *to* ***B***=***R*** or f can be viewed as a function from **A**=**R** to **B**=**C.** In both cases, the range of f is contained in but not equal to the set *B*.

As the following definition indicates, a complex function is a function whose inputs and outputs are complex numbers.

Definiton 1 Complex function: A complex function is a function f whose domain and range are subsets of the set C of complex numbers.

A complex function is also called a complex-valued function of a complex variable. for the most part we will use the usual symbols $f, g,$ and h to denote complex functions. In addition, inputs to a complex function *f* will typically be denoted by the variable z and outputs by the variable $w = f(z)$. When referring to a complex function we will use three notations interchangedably, for example, $f(z) = z - i, w = z - i,$ or, simply, the function $z - i$. Throughout this text the notation $y = f(x)$ will be reserved to represent a real-valued function of a real variable x.

Example1. Complex function: (a) The expression $z^2 - (2 + i)z$ can be evaluated at any complex number z and always yields a single complex number, and so $f(z) = z^2 - (2 + i)z$ define a complex function. Values of *f* are found by using the arithmetic operations for complex numbers . For instance, at the points $z = i$ and $z = 1 + i$ we have:

$f(i) = (i)^2 - (2 + i)(i) = -1 - 2i + 1 = -2i$

and $f(1 + i) = (1 + i)^2 - (2 + i)(1 + i) = 2i - 1 - 3i = -1 - i$

(b) The expression $g(z) = z + 2\,Re(z)$ defines a complex function .Some values of g are:

$g(i) = i + 2\,Re\,(i) = i + 2(0) = I$

and $g(2 - 3i) = 2 - 3i + 2\,Re\,(2 - 3i) = 2 - 3i + 2(2) = 6 - 3i.$

Real and Imaginary Parts of a Complex Function: It is often helpful to express the inputs and the outputs a complex function in terms of their real and imaginary parts. If $w = f(z)$ is a complex function, then the image of a complex number $z = x + iy$ under f is a complex number $w = u + iv$. By simplifying the expression $f(x + iy)$, we can write the real variables u and v in terms of the real variables x and y. For example, by replacing the symobl z with $x + iy$, we can express any complex function $w = z^2$, we obtain:

$$w = u + iv = (x + iy)^2 = x^2 - y^2 + 2xyi. \tag{5}$$

From (5) the real variables u and v are given by $u = x^2 - y^2$ and $v = 2xy$ respectively. This example illustrates that, if $w = u + iv = (x + iy)$ is a complex function, then both u and v are real functions of the two real variables x and y. That is, by setting $z = x + iy$, we can express any complex function $w = f(z)$ in terms of two real funtions as:

$$f(z) = u(x, y) + iv(x, y). \tag{6}$$

The functions $u(x,\ y)$ and $v(x,\ y)$ in (6) are called the real and imaginary parts of *f* , respectively.

Example 2. Real and imaginary parts of a function:

Find the real and imaginary parts of the functions: (a) $f(z) = z^2 - (2 + i)z$ and

(b) $g(z) = z + 2\,Re\,(z)$.

Solution. In each case, we replace the symbol z by $x + iy$, then simplify.

(a) $f(z) = (x + iy)^2 - (2 + i)(x + iy) = x^2 - 2x + y - y^2 + (2xy - x - 2y)i.$
So ,$u(x, y) = x^2 - 2x + y - y^2$ and $v(x,\ y) = 2xy - x - 2y.$
(b) Since $g(z) = x + iy + 2\,Re\,(x + iy) = 3x + iy,$ we have $u\,(x, y) = 3x$ and $v\,(x, y) = y.$

Every complex function is completely determined by the real functions $u\,(x, y)$ and $v(x,\ y)$ in (6). Thus, a complex function $w = f(z)$ can be defined by arbitarily specifying two real functions $u(x,\ y)$ and $v(x,\ y)$, even though $w = u + iv$ we not be obtainable through familiar operations performed soley on the symbol z. for example, if we take, say, $u(x,\ y) = xy^2$ and $v(x,\ y) = x^2 - 4y^3$, then $f(z) = xy^2 + i(x^2 - 4y^3)$ define a complex function. in order to find the value of f at the point $z = 3 + 2i$, we substitute $x = 3$ and $y = 2$ into the expression for f to obtain $f(3 + 2i) = 3 \cdot 2^2 + i(3^2 - 4 \cdot 2^3) = 12 - 23i$.

Limit of a Real Function $f(x)$: The limit of f as x tends x_0 exists and is equal to L if for every $\varepsilon > 0$ there exists $a\ \delta > 0$ such that $|f(x) - L| < \varepsilon$ (7)

Whenever $0<|x - x_0| < \delta$.

The geometirc interpretation of (7) is shown in figure 13. In this figure we see that the graph of the function $y = f(x)$ over the interval $(x_0 - \delta,\ x_0 + \delta)$, excluding the point $x = x_0$, shown in color on the $x -$ axis is mapped onto the set shown in black in the interval $(L - \varepsilon, L + \varepsilon,)$ on the $y -$axis. for the limit to exist, the relationship exhibited in figure 13 must exist for any choice of $\varepsilon > 0$. We also see in Figure 13 that if a smaller ε is chosen, then a smaller δ may be needed.

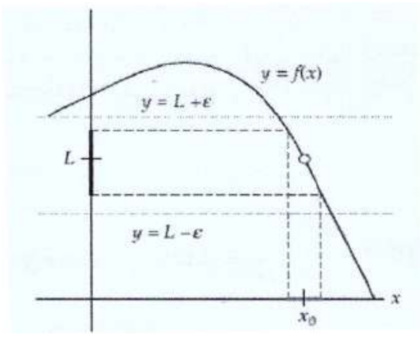
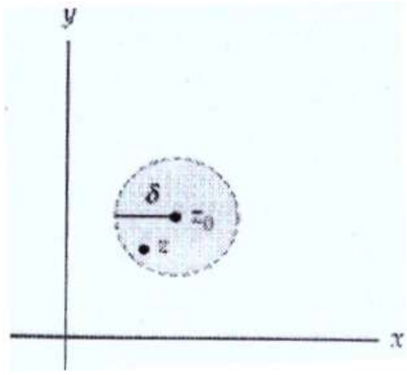


Figure 13.

Complex Limits: A complex limit is, in eassence, the same as a real limit exept that it is based on a notion of "close" in the complex plane. Because the distance in the complex plane between two points z_1 and z_2 is given by the modulus of the difference of z_1 and z_2, the precise definiton of a complex limit will involve $|z_2 - z_1|$. For example, the phrase "$f(z)$ can be made arbitarily close to the complex number L," can be stated precisely as: for every $\varepsilon > 0$, z can be chosen so that $|f(z) - L| < \varepsilon$. Since the modulus of a complex number is a real number, both ε and δ still represent small positive real numbers in the following definiton of a complex limit. The complex analogue of (7) is:

Definition 2 Limit of a complex funtion: Suppose that a compex funtion $f$is defined in a deleted neighborhood of z_0 and suppose that L is a complex number. The limit of f as z tends to z_0 existsand is equal to L, written as $\lim_{z \to z_0} f(z) = L$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.



(a) Figure 14. Deleted δ −neighborhood of z_0

Complex and real limits have many common properties, but there is at least one very imprtant differnce. For real function $\lim_{x \to x_0} f(x) = L$ if and only if $\lim_{x \to {x_0}^+} f(x) = L$ and $\lim_{x \to {x_0}^-} f(x) = L$. That is, there are two directions from which x can approach x_0 on the real line , from the right (denoted by $x \to {x_0}^+$) or from the left (denoted by $x \to {x_0}^-$) The real limit exists ifr and only if these two one –sided limits have the same value. For example, consider the real function defined by:

$$f(x) = \begin{cases} x^2 , & x < 0 \\ x - 1, & x \geq 0 \end{cases}$$

The limit of f as x approaches to 0 does not exist since $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} = x^2 = 0$,
but $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} f(x - 1) = -1$. See Figure 15.

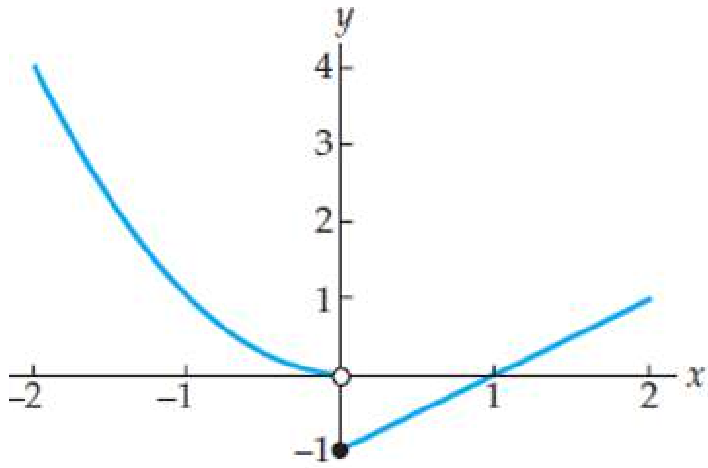


Figure 15. The limit of f does not exist as x approaches 0.

For limits for complex functions, z is allowed to approach z_0 from any direction in the complex plane, that is, along any curve or path through z_0. See Figure 16.In orer that $\lim_{z \to z_0} f(z)$ exists and equals L along every possible curve through z_0. Put in a negative way:

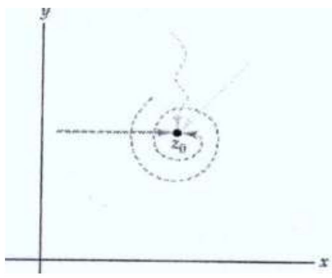


Figure 16. Different ways to approach z_0 in a limit.

<u>Example 3.</u> A Limit That does Not Exist

Show that $\lim_{z \to 0} \frac{z}{\bar{z}}$ does not exist.

<u>Solution:</u> We show that this limit does not exist by finding two different ways of letting z approach 0 that yield different values of $\lim_{z \to 0} \frac{z}{\bar{z}}$ .First, we let z approach 0 along the real axis. That is we consider complex numbers of the form $z = x + 0i$ where the real number x is approaching 0. for these points we have:

$$\lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{x \to 0} \frac{x+0i}{x-0i} = \lim_{x \to 0} 1 = 1. \tag{8}$$

On the other hand, if we let $z = 0 + iy$ where the real number y is approaching 0. For this approach we have:

$$\lim_{z\to 0}\frac{z}{\bar{z}} = \lim_{x\to 0}\frac{0+iy}{0-iy} = \lim_{y\to 0}(-1) = -1. \qquad (9)$$

Since the values in (8) and (9) are not the same, we conclude that $\lim_{z\to 0}\frac{z}{\bar{z}}$ does not exist.

<u>Example 4 An Epsion-Delta Prof of a Limit:</u>

Prove that $\lim_{z\to 1+i}(2+i)z = 1+3i.$

Solution According to definition 2.; $\lim_{z\to 1+i}(2+i)z = 1+3i,$ if, for every $\varepsilon > 0,$ there is a $\delta > 0$ such that $|(2+i)z-(1+3i)| < \varepsilon$ when ever $0 < |z-(1+i)| < \delta.$ proving that the limit exists requires that we find an appropriate value of δ for a given value of ε . In other words, for a given value of ε we must find a positive number δ with the property that if $0 < |z-(1+i)| < \delta,$ then $|(2+i)z-(1+3i)| < \varepsilon.$ One way of finding δ is to "work backwards." The idea is to start with the inequality:

$$|(2+i)z-(1+3i)| < \varepsilon \qquad (10)$$

and then use properties of complex numbers and the modulus to manipulate this inequality until it involves the expression $|z-(1+i)|$. Thus, a natural first step is to factor $(2+i)$ out of the left-hand side of (10):

$$|(2+i)|.\,|z-\frac{(1+3i)}{(2+i)}| < \varepsilon. \qquad (11)$$

Because $|(2+i)| = \sqrt{5}$ and $\frac{1+3i}{2+i} = 1+i,$ (11)is equivalent to:

$$\sqrt{5}.\,|z-(1+i)| < \varepsilon \text{ or } |z-(1+i)| < \frac{\varepsilon}{\sqrt{5}}. \qquad (12)$$

Thus, (12) indicates that we should take $\delta = \frac{\varepsilon}{\sqrt{5}}.$ Keep in mind that the choice of δ is not unique. Our choice of $\delta = \frac{\varepsilon}{\sqrt{5}}$ is a result of the particular algebraic manipulations that we employed to obtain (12). Having found δ we now present the formal proff that $\lim_{z\to 1+i}(2+i)z = 1+3i.$ that does not indicate how the choice of δ was made:

Given $\varepsilon > 0,$ let $\delta = \frac{\varepsilon}{\sqrt{5}}$. if $0 < |z-(1+i)| < \delta,$ then we have $|z-(1+i)| < \frac{\varepsilon}{\sqrt{5}}.$ Multiplying both sides of the last inequality by $|1+i| = \sqrt{5}$ we obtain:

$$|(2+i)|.\,|z-(1+i)| < \sqrt{5}.\frac{\varepsilon}{\sqrt{5}}. \text{ or } |(2+i)z-(1+3i)| < \varepsilon.$$

Therefore, $|(2+i)z-(1+3i)|<\varepsilon$ whenever $0<|z-(1+i)|<\delta$, So, according to Definition 2., we have proven that $\lim_{z\to 1+i}(2+i)z=1+3i$.

Real Multivariable Limits: The epsilon-delta proof from Example 4 illustrates the important fact that alghouth the theory of complex limits is based on Definition 2, this definition does not provide a convenient method for computing limits. We now present a practical method for computing complex limits in Theorem 2. In addition to being a useful computational tool, this theorem also establishes an important connection between the complex limit of $f(z)=u(x,y)+ivf(x,y)$ and the real limits of the real- valued function of two real variables $u\,(x,y)$ and $v(x,\ y)$. Since every complex function is completely determined by the real functions u and v.

Before stating Theorem 2. we recall some of the important concepts regarding limits of real-valued functions of two real variables $F(x,y)$. The following definition of $\lim_{(x,y)\to(x_0,y_0)} F(x,y)=L$ is analosous to both equation (7) and Definiton 2.

Limit of the real function $F(x,y)$:

The limit of F as (x,y) tends to (x_0,y_0) exists and is equal to the real number L if for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$|F(x-y)-L|<\varepsilon \text{ whenever } 0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\ \delta. \tag{13}$$

The expression $\sqrt{(x-x_0)^2+(y-y_0)^2}$ in (13) represents the distance between the points (x,y) and (x_0,y_0) in the Cartesian plane. Using (13), it is relatively easy to prove that:

$$\lim_{(x,y)\to(x_0,y_0)} 1=1, \quad \lim_{(x,y)\to(x_0,y_0)} x=x_0, \quad and \lim_{(x,y)\to(x_0,y_0)} y=y_0 \tag{14}$$

If $\lim_{(x,y)\to(x_0,y_0)} F(x,y)=L$ and $\lim_{(x,y)\to(x_0,y_0)} G(x,y)=M$, then (13) can also be used to show:

$$\lim_{(x,y)\to(x_0,y_0)} cF\,(x,y)=cL,\ \ c \text{ a real constant}, \tag{15}$$

$$\lim_{(x,y)\to(x_0,y_0)} (F(x,y)\pm G(x,y)=L\pm M, \tag{16}$$

$$\lim_{(x,y)\to(x_0,y_0)} F(x,y)\cdot G(x,y)=L\cdot M, \tag{17}$$

and $$\lim_{(x,y)\to(x_0,y_0)} \frac{F(x,y)}{G(x,y)}=\frac{L}{M}\ , M\neq 0. \tag{18}$$

Limits involving polynomial expressions in x and y can be easily computed using the limits in (14) combined with properties (15)-(18). For example,

$$\lim_{(x,y)\to(1,2)}(3xy^2 - y) = 3\left(\lim_{(x,y)\to(1,2)} x\right)\left(\lim_{(x,y)\to(1,2)} y\right)\left(\lim_{(x,y)\to(1,2)} y\right) - \lim_{(x,y)\to(1,2)} y$$

$$= 3 \cdot 1 \cdot 2 \cdot 2 - 2 = 10.$$

In general, if $p(x, y)$ is a two-variable polynomial function, then (14)-(18) can be used to show that

$$\lim_{(x,y)\to(x_0, y_0)} p(x, y) = p(x_0, y_0) \tag{19}$$

If $p(x, y)$and $q(x, y)$ are two-variable polynomial functions and $q\ (x_0, y_0) \neq 0$, then equations (19) and (18) give:

$$\lim_{(x,y)\to(x_0, y_0)} \frac{p(x,y)}{q(x,y)} = \frac{p(x_0,y_0)}{q(x_0,y_0)} \tag{20}$$

We now present Theorem 2., which relates real limits of $u(x, y)$and $v(x, y)$ with the complex limit of $f(z) = u(x, y) + iv(x, y)$.

<u>Theorem 2. Real and imaginary parts of a Limit:</u>

Suppose that $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$, $and\ L = u_0 + iv_0$. Then $\lim_{z\to z_0} f(z) = L$ if and only if

$$\lim_{(x,y)\to(x_0, y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y)\to(x_0, y_0)} v(x, y) = v_0\,.$$

Theorem 2. has many uses. First and foremost , it allows us to compute many complex limits by simply computing a pair of real limits.

<u>Example 5: Using Theorem 2. to compute a Limit:</u>

Use Theorem 2. to compute $\lim_{z\to(1+i)} (z^2 + i)$.

<u>Solution:</u> Since $f(z) = z^2 + i = x^2 - y^2 + (2xy + 1)i$, we can apply Theorem 2. with

$u(x,\ y) = x^2 - y^2$, $v(x, y) = 2xy + 1$, and $z_0 = 1 + i$. Identifying $x_0 = 1$ and $y_0 = 1$,we find u_0 and v_0 by computing the two real limits:

$u_0 = \lim_{(x,y)\to(1,1)} (x^2 - y^2)$ and $v_0 = \lim_{(x,y)\to(1,1)} (2xy + 1)$.

Since both of these limits involve only multivariable polynomial functions, we can use (19) to obtain:

$$u_0 = \lim_{(x,y)\to(1,1)} (x^2 - y^2) = 1^2 + 1^2 = 0$$

$$v_0 = \lim_{(x,y)\to(1,1)} (2xy + 1) = 2 \cdot 1 \cdot 1 + 1 = 3$$

and so $L = u_0 + iv_0 = 0 + i(3) = 3i.$ Therefore, $\lim_{z\to(1+i)} (z^2 + i) = 3i.$

Continuity of Complex Functions:

The definition of continuity for a complex function is, in essence, the same as that for a real function. That is a complex function f is continuous at a point z_0 if the limit of f as z approaches z_0 exists and is the same as the value of f at z_0. This gives the following definition for complex functions.

Definition 3. Continuity of a Complex Function:

A complex function f is continuous at a point z_0 if $\lim_{z\to(z_0)} f(z) = f(z_0)$.

Analogous to real function, if a complex f is continuous at a point , then the following three conditions must be met.

Criteria for Continuity at a Point:

A complex function f is continuous at a point z_0 if each of the following three conditions hold:

(i) $\lim_{z\to z_0} f(z)$ exists,
(ii) f is defined at z_0, and
(iii) $\lim_{z\to z_0} f(z) = f(z_0)$.

If a complex function f is not continious at a point z_0 then we say that f is discontinuous at z_0. for example, the function $f(z) = \frac{1}{1+z^2}$ is discountinuous at $z = i$ and $z = -i$.

Example 6 Checking Continuity at a Point:

Consider the function $f(z) = z^2 - iz + 2$. In order to determine if f is continuous at, say, the point $z_0 = 1 - i,$ we must find $\lim_{z\to z_0} f(z)$ and $f(z_0)$, then chek to see whether these two complex values are equal. From Theorem 2.2 and the limits in (15) and (16) we obtain:

$$\lim_{z\to z_0} f(z) = \lim_{z\to 1-i} (z^2 - iz + 2) = (1 - i)^2 - i(1 - i) + 2 = 1 - 3i.$$

Furtheromre, for $z_0 = 1 - i$ we have:

$f(z) = f(z_0)$, we conclude that $f(z) = z^2 - iz + 2$ is a continuous at the point $z_0 = 1 - i$.

Continuity of a Real Function $F(x\,,y)$

A function *F* is continuous at a point $(x_0\,, y_0)$ if

$$\lim_{(x,y)\to(x_0\,,y_0)} F(x,y) = F(x_0\,, y_0). \tag{21}$$

Theorem 3. Real and Imaginary Parts of a Continuous Function:

Suppose that $f(z) = u(x, y + iv(x,y)$ and $z_0 = x_0 + iy_0$. Then the complex function f is continuous at the point z_0 if and only if both real functions u and v are contiunous at the point (x_0, y_0).

Proof: Assume that the complex function f is continuous at z_0. Then form Definition 3 we have:

$$\lim_{z\to z_0} f(z) = f(z_0) = u(x_0, y_0) + iv(x_0, y_0). \tag{22}$$

By Theorem 2., this implies that:

$$\lim_{(x,y)\to(x_0\,,y_0)} u(x,y) = u(x_0, y_0) \text{ and } \lim_{(x,y)\to(x_0\,,y_0)} v(x,y) = v(x_0, y_0) \tag{23}$$

Therefore, from (21), both u and v are continuous (x_0, y_0). Conversely, if u and v are continuous at (x_0, y_0), then

$$\lim_{(x,y)\to(x_0\,,y_0)} u(x,y) = u(x_0, y_0) \text{ and } \lim_{(x,y)\to(x_0\,,y_0)} v(x,y) = v(x_0, y_0).$$

It then follows form Theorm 2. that $\lim_{z\to z_0} f(z) = u(x_0, y_0) + iv\,(x_0, y_0) = f(z_0)$. Therefore, f is continuous by Definition 3.

Example 7.Checking Continuity Using Theorem 3:

Show that the function $f(z) = \bar{z}$ is continuous on C.

Solution: According to Theorem 3, $f(z) = \bar{z} = \overline{x + iy} = x - iy$ is continuous at $z_0 = x_0 + iy_0$ if both $u(x,\ y) = x$ and $v(x,\ y) = -y$ are continuous at (x_0, y_0). Because u and v are two-variable polynomial functions, it follows from (19) that:

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = x_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x,y) = -y_0.$$

This implies that u and v are continuous at $(x_0\ , y_0)$, and, therefore, that f is continuous at $z_0 = x_0 + iy_0$ by the Theorem 3. Since $z_0 = x_0 + iy_0$ was an arbitary point, we conclude that the function $f(z) = \bar{z}$ is continuous on C.

Theorem 4 Properties of Complex Limits

Suppose that f and g are complex functions. If $\lim_{z\to z_0} f(z) = L$ and $\lim_{z\to z_0} g(z) = M$, then

(i) $\lim_{z\to z_0} cf(z) = cL$, c a complex constant,

(ii) $\lim_{z\to z_0} \big(f(z) \pm g(z)\big) = L \pm M$,

(iii) $\lim_{z\to z_0} f(z).\,g(z) = L.M$, and

(iv) $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{L}{M}$, provided $M \neq 0$.

Proof of (i) Each part of Theorem 4 follows from Theorem 2 and the analogous property (15)-(18). We will prove part (i) and leave the remaining parts as exercises.

Let $f(z) = u(x,y) + iv(x,y), z_0 = x_0 + iy_0,\ L = u_0 + iv_0$ and $c = a + ib$.

Since $\lim_{z\to z_0} f(z) = L$, it follows from Theorem 2 that $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$. By (15) and (16) ,we have

$$\lim_{(x,y)\to(x_0,y_0)} au(x,y) - bv(x,y) = au_0 - bv_0$$

and

$$\lim_{(x,y)\to(x_0,y_0)} au(x,y) + bv(x,y) = au_0 + bv_0.$$

However , $\text{Re}\big(cf(z)\big) = au(x,y) - bv(x,y)$ and $\text{Im}\big(cf(z)\big) = bu(x,y) + av(x,y)$.Therefore, By Theorem 2,

$$\lim_{z\to z_0} cf(z) = (au_0 - bv_0) + i(au_0 + bv_0) = cL.$$

The algebraic properties of complex limits from Theorem 4 can also be restated in terms of continuity of complex functions.

<u>Theorem 5:</u> <u>Properties of Continuous Function:</u>

If f and g are continuous at the point z_0, then the following functions are continuous at the point z_0:

(i) cf, c a complex constant,
(ii) $f + g$,
(iii) $f \cdot g$, and
(iv) $\frac{f}{g}$ provided $g(z_0) \neq 0$.

<u>Proof of (ii)</u> We prove only (ii); proofs of the remaining parts are similar. Since f and g are continuous at z_0 we have that $\lim_{z \to z_0} f(z) = f(z_0)$ and $\lim_{z \to z_0} g(z) = g(z_0)$. From Theorem 4 (ii), it follows that $\lim_{z \to z_0} \left(f(z) + g(z_0)\right) = f(z_0) + g(z_0)$. Therefore, $f + g$ is continuous at z_0 by Definition 3.

Assignment

1. Draw the pair of points $z = a + ib$ and $\bar{z} = a - ib$ in the complex plane if

$a > 0,\ b > 0;\ \ a > 0,\ b < 0;\ a < 0,\ b > 0;$ and $a < 0,\ b < 0.$

2. Consider the complex number $z_1 = 4 + i,\ z_2 = -2 + i,\ z_3 = -2 - 2i,\ z_4 = 3 - 5i.$

(a) Use four different sketches to plot the four pairs of points $z_1,\ iz_1;\ z_2,\ iz_2$; $z_3,\ iz_3$; and z_4, iz_4.

(b) In general, how would you describe geometrically the effect of multiplying a complex number $z = x + iy$ by i ? By $-i$?

3. Under what circumstances does $|z_1 + z_2| = |z_1| + |z_2|$?

4. Without doing any calculations, explain why the inequalities $|\mathrm{Re}\, z| \leq |z|$ and $|\mathrm{Im}\, z| \leq |z|$ hold for all complex numbers z.

5. Find the modulus of the given complex number.

(a). $(1 - i)^2$ (b). $\frac{2i}{3-4i}$

(c). $i(2-i)-4\left(1+\frac{1}{4}i\right)$ (d). $\frac{1-2i}{1+i}+\frac{2-i}{1-i}$.

6. Sketch the graph of the given equation in the complex plane.

(a) $|z-4+3i|=5$ (b) $|z+2+2i|=2$

(c) $|z+3i|=2$ (d) $|2z-1|=4$

(e) Re $(z)=5$ (f) $\text{Im}(z)=-2$

In problems 7 -18 sketch the set *S* of points in the complex plane satisfying the given inequality. Determine whether the set is (a) open (b) closed (c) a domain, (d) bounded, or (e) connected.

7. $\text{Re}(z)<-1$ **8.** $|\text{Re}(z)|>2$

9. $|\text{Im}(z)|>3$ **10.** $\text{Re}\left((2+i)z+1\right)>0$

11. $2<\text{Re}(z-1)<4$ **12.** $-1\le \text{Im}(z)<4$

13. $Re\ (z^2)>0$ **14.** $\text{Im}(z)<\text{Re}(z)$

15. $2<|z-i|<3$ **16.** $1\le |z-1-i|<2$

17. $2\le |z-3+4i|\le 5$ **18.** $|z-i|>1$

In Problems 19-25, show that the function f is continuous at the given point.

19. $f(z)=z^2-iz+3-2i\ ;\ z_0=2-i$

20. $f(z)=z^3-\frac{1}{z}\ ;\ z_0=3i$

21. $f(z)=\frac{z^3}{z^3+3z^2+z};\ z_0=i$

22. $f(z)=\frac{z-3i}{z^2+2z-1};\ z_0=1+i$

23. $f(z)=\begin{cases}\frac{z^3-1}{z-1} & |z|\neq 1\\ 3 & |z|=1\end{cases}\ ;\ z_0=1$

24. $f(z)=\begin{cases}\frac{z^3-1}{z^2+z+1} & |z|\neq 1\\ \frac{-1+i\sqrt{3}}{2} & |z|=1\end{cases}\ ;\ z_0=\frac{1+i\sqrt{3}}{2}$

25. $f(z) = \bar{z} - 3Re(z) + i\ ;\ z_0 = 3 - 2i$

Refrences:

1. Dennis G. Zill and Patrick D. Shanahan. A First Course in Complex Analysis with Applications, Jones and Bartlett publishers Sudbury, Massachusetts
2. S. Ponnusamy. Foundations of Complex Analysis, Norasa publishers.

Module-5

Pretest on Differential Equations

Note: Find the correct answer in the following questions given below.

1. The differential equation

$\frac{d^2y}{dx^2} + (\frac{dy}{dx})^3 + e^{y'} = 0$ is of

(i) Order 1 and degree 3
(ii) Order 3 and degree 1
(iii) Order 2 and degree is not defined
(iv) Order 2 and degree 3

2. The differential equation of the family of circles touching the x axis at origin is

(i) $x\,\frac{dy}{dx} = \frac{2y}{x^2-y^2}$
(ii) $\frac{dy}{dx} = \frac{2xy}{x^2-y^2}$
(iii) $y\,\frac{dy}{dx} = \frac{2x}{x^2-y^2}$
(iv) None of these

3. Find the family of curves for which the slope of the tangent at any point (x, y) on it is

(i) $x^2 - y^2 = cx$
(ii) $x - y = \frac{c}{x}$
(iii) $x^2 + y^2 = cx$
(iv) $x + y = \frac{c}{x}$

4. The integrating factor of the differential equation $(1-y^2)\frac{dx}{dy} + yx = ay,\quad (-1 < y < 1)$,

(i) $\frac{1}{\sqrt{y^2-1}}$
(ii) $\frac{1}{y^2-1}$
(iii) $\frac{1}{1-y^2}$
(iv) $\frac{1}{\sqrt{1-y^2}}$

5. General solution of the differential equation

$\frac{dy}{dx} = \frac{1+y^2}{\tan^{-1}y - x}$ is

(i) $x = (\tan^{-1}y - 1) + c\,e^{-\tan^{-1}y}$
(ii) $y = (\tan^{-1}x - 1) + c\,\tan^{-1}x$
(iii) $y = (\tan^{-1}x - 1) + c\,e^{\tan^{-1}x}$
(iv) None of these.

Module-5

Differential Equations

Lectures required-03

An equation involving the dependent variable, the independent variable/variables and derivatives of dependent variable *w.r.t.* independent variable/variables is called a differential equation.

Ex. (a) $\frac{dy}{dx} = 3y + x^3$

(b) $x\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = u^2$

(c) $\frac{d^2y}{dx^2} + 3y^2 = sin\, x$

The equations (a) and (c) are ordinary differential equations because these equations consist of one independent variable *x*. Eq. (b) is a partial differential equation because it contains one dependent variable u and two independent variables *x* and *y*.

In this section, we shall discuss about ordinary differential equations only. Before going further, let us discuss the following definitions:

Order: The order of a differential equation is the order of the highest derivative occurring in the equation.

Ex. (i) The order of $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^3 + y = 0$ is two.

(ii) The order of $\frac{d^3y}{dx^3} + y^2 + e^{y'} = 0$ is three.

Degree: The degree of a differential equation is the highest power of the highest order derivative occurring in the given differential equation if the differential equation is a polynomial equation in derivatives.

Ex. (i) The degree of $\left(\frac{dy}{dx}\right)^3 + \frac{dy}{dx} + sin\, y = 0$ is three.

(ii) The degree of $y''' + y^2 + e^{y'} = 0$ is not defined because the given equation is not a polynomial equation in its derivatives.

(iii) The degree of $\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} + sin^2\, y = 0$ is two.

1. Formation of differential equations of first order

We first study how to form a differential equation if the complete primitive (or general solution) is known.

Let the primitive of the differential equation be

$$f(x, y, a) = 0 \tag{1.1}$$

Where a is an arbitrary constant.

Differentiating Eq. (1.1), we obtain a relation between x, y, a and $\frac{dy}{dx}$ of the form

$$\phi\left(x, y, a, \frac{dy}{dx}\right) = 0 \tag{1.2}$$

Eliminating a between the Eqs. (1.1) and (1.2), we obtain the following equation:

$$\psi\left(x, y, \frac{dy}{dx}\right) = 0 \tag{1.3}$$

Eq. (1.3) is a differential equation of the first order.

Example: Find the differential equation of all circles which pass through the origin and whose centres are on the *x* axis.

We know that the equation of any circle passing through the origin and whose centre is on the *x* axis is given by

$$x^2 + y^2 + 2gx = 0, \tag{1.4}$$

Where g is an arbitrary constant.

Differentiating Eq. (1.4) *w.r.t. x*, we get

$$2x + 2y\frac{dy}{dx} + 2g = 0. \tag{1.5}$$

From Eq. (1.4), we have

$$2g = -(x^2 + y^2)/x \tag{1.6}$$

Substituting the expression of $2g$ from Eq. (1.6) in Eq. (1.5), we get

$$2xy\frac{dy}{dx} + x^2 - y^2 = 0,$$

Which is a differential equation whose primitive is given in Eq.(1.4).

Exercise:

(1) Find the differential equation corresponding to the family of curves $y = c(x - c)$, where c is an arbitrary constant.

(2) Find the differential equation of all circles which pass through the origin and whose centres are on the *y* axis.

(3) Find the differential equation corresponding to the family of curves $y^2 = 4a(x + a)$, where a is an arbitrary constant.

2. Differential Equations reducible to linear form

The students have already studied about the differential equation of first order and first degree and their solutions by the method of separation of variables, method of solving homogeneous and linear equations. Therefore, let us discuss first the differential equations reducible to the linear form:

The equation of the form

$$\frac{dy}{dx} + Py = Qy^n \tag{2.1}$$

is known as Bernoulli's equation, where P and Q are functions of x alone.

Equation (2.1) can be reduced to the linear form by dividing the equation by y^n and putting y^{-n+1} equal to v. That is if are divide (2.1) by y^n, we get

$$y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q \tag{2.2}$$

If we take $y^{-n+1} = v$, (2.2) becomes

$$\frac{1}{(-n+1)}\frac{dv}{dx} + Pv = Q, \qquad n \neq 1,$$

Which is a linear differential equation in v and x.

Example: Solve $x\frac{dy}{dx} + y = xy^3$.

Dividing by y^3, we have

$$xy^{-3}\frac{dy}{dx} + y^{-2} = x$$

Now, taking $y^{-2}=v$, we have

$$-2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$$

This gives

$$-\frac{1}{2}x\frac{dv}{dx} + v = x \qquad \Rightarrow \frac{dv}{dx} - \frac{2}{x}v = -2$$

Which is a linear differential equation.

The solution of this linear equation is

$$v.IF = \int -2.IF\,dx + C,$$

Where IF (Integrating factor) = $e^{\int \frac{-2}{x} dx} = \frac{1}{x^2}$ and C is an arbitrary constant.

Hence, the solution of linear differential equation is

$$\frac{v}{x^2} = \frac{2}{x} + C$$

And the final solution of given differential equation becomes

$$(2 + cx)xy^2 = 1.$$

An equation of the form $f'(y)\frac{dy}{dx} + Pf(y) = Q$, where P and Q are function of x only can also be reduced to linear form.

If we take $f(y) = v$, so $f'(y)\frac{dy}{dx} = \frac{dv}{dx}$ and the differential equation becomes:

$$\frac{dv}{dx} + Pv = Q$$

Which is linear in v and x. This equation can be easily solved.

Ex. Reduce $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$ in linear form.

Dividing above differential equation by $\cos^2 y$, we get

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$$

If $\tan y = v$, then $\sec^2 y \frac{dy}{dx} = \frac{dv}{dx}$. Hence, the above equation becomes

$$\frac{dv}{dx} + 2xv = x^3$$

Which is linear in v and x.

Exercise

(i) $\frac{dy}{dx} + \frac{1}{x} \sin 2y = x^2 \cos^2 y$

(ii) $\frac{dy}{dx} + y \sin x = y^3 \cos 2x$

(iii) $\frac{dy}{dx} = e^{x-y} (e^x - e^y)$

(iv) $x\frac{dx}{dy} + 3y = x^3 y^2$

3. Exact Differential Equation

A differential equation is said to be exact if it can be derived from its general solution directly by differentiating without any subsequent multiplication, elimination, etc.

Thus, the differential equation

$$Mdx + Ndy = 0 \tag{3.1}$$

is exact if there exists a function $f(x,y)$ such that

$$d\,[f(x,y)] = Mdx + Ndy,$$

where M & N are functions of x and y.

Theorem: A necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{3.2}$$

If the equation (1) is exact, then its general solution is

$$\int Mdx + \int (term\ is\ N\ not\ containing\ x)dy = c,$$

[Treating y as constant]

where c is an arbitrary constant.

Example: $(x^2 - 4\,xy - 2y^2)\,dx + (y^2 - 4\,xy - 2x^2)$

$$\frac{\partial M}{\partial y} = -4x - 4y \qquad \& \qquad \frac{\partial N}{\partial x} = -4y - 4x$$

Hence, the given equation is exact and its solution is

$$\int (x^2 - 4xy - 2y^2)dx + \int y^2 dy \ = c \tag{3.3}$$

In first integral of (3), we take y as a constant and after integration, we have

$$\frac{x^3}{3} - 4y\frac{x^2}{2} - 2y^2x + \frac{y^3}{3} = C$$

$\Rightarrow x^3 + y^3 - 6xy(x + y) = C'$, $\quad C'$ is arbitrary constant.

4. Integrating factor

Sometimes the equation Mdx + $Ndy = 0$ is not exact. But, it can be made exact by multiplying it with a function of x and y. Such a function is known as integrating factor (IF). In general, the differential $Mdx + Ndy = 0$ has an infinite number of integrating factor. But, there is not a unique method to find them. Therefore, we discuss some rules to find integrating factor for Eq. (3.1).

(i) By inspection: Sometimes an integrating factor can be found by rearranging the terms of the given equation and/or by dividing by a suitable function of x & y.

Ex: Solve $y\,(2xy + e^x)dx = e^x dy$

$$\Rightarrow (2xy^2 + ye^x)dx - e^x dy = 0$$

$$\Rightarrow 2xdx + \frac{ye^x dx - e^x dy}{y^2} = 0$$

$$= d\left(x^2 + \frac{e^x}{y}\right) = 0 \qquad \text{(Exact)}$$

$$= x^2 + e^x/y = c.$$

(ii) If the equation $Mdx + Ndy = 0$ can be written in the form

$$f_1(xy)ydx + f_2(xy)xdy = 0,$$

and $Mx - Ny \neq 0$, then $\frac{1}{Mx-Ny}$ is an integrating factor.

Ex.: Find I.F. of $y(1+xy)dx + x(1-xy)dy = 0.$

Clearly $M = (1+xy)y$ & $N = (1-xy)y.$

In this case $Mx - Ny = 2x^2y^2 \neq 0$

Hence I.F. $= \frac{1}{Mx-Ny} = \frac{1}{2x^2y^2}$.

(iii) If M and N are homogeneous in the equation $Mdx + Ndy = 0$ and $Mx + Ny \neq 0$, then $\frac{1}{Mx+Ny}$ is an integrating factor of the equation.

Ex.: Find integrating factor of

$$(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$$

Clearly $M = x^2y - 2xy^2$ & $N = -(x^3 - 3x^2y)$ and M & N are homogeneous functions.

Now, $Mx + Ny = x^2y^2 \neq 0.$

Hence, I.F. $= \frac{1}{x^2y^2}$.

(iv) If $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of x alone, say $f(x)$, then $e^{\int f(x)dx}$ is an integrating factor of the equation $Mdx + Ndy = 0$.

Ex. : Find integrating factor of $(x^2+y^2 + x)dx + xydy = 0$.

Clearly, $M = x^2+y^2 + x$ & $N = xy$

Here, $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{xy}(2y-y)=\frac{1}{x}=f(x)$.

Hence, I.F. = $e^{\int \frac{1}{x}dx}=e^{\log x}=x$.

(v) If $\frac{1}{N}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$ is a function of y alone, say $f(y)$, then $e^{\int f(y)dy}$ is an integrating factor of the equation $Mdx + Ndy = 0$.

(vi) Let $x^h y^k$ be the I.F. of the equation

$$x^a y^b(mydx + nxdy) + x^r y^s(pydx + qxdy) = 0,$$

where a, b, m, n, r, s, p, q are constants and h, k are unknowns.

Multiplying the diff. equation by the I.F., we get

$$(mx^{a+h}y^{b+k+1} + px^{r+h}y^{s+k+1})dx + (nx^{a++1h}y^{b+k} + qx^{r+h+1}y^{s+k})dy = 0$$

Since this equation must be an exact. Therefore, the condition

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x},$$

provides

$$m(b+k+1)x^{a+h}y^{b+k} + p(s+k+1)x^{r+h}y^{s+k} = n(a+h+1)x^{a+h}y^{b+k} + (r+h+1)x^{r+h}y^{s+k}.$$

This will be true only when

$$\left.\begin{aligned} m(b+k+1)&=n\,(a+h+1)\\ p(s+k+1)&=q\,(r+h+1)\end{aligned}\right\} \tag{4.1}$$

Eqs. in (4.1) determine the values of h and k.

Ex. : Find the integrating factor of

$$(2ydx + 3ydy) + 2xy\,(3ydx + 4xdy) = 0.$$

Let $x^h y^k$ be I.F. of the equation. Therefore, the equation

$$(2x^h y^{k+1} + 6x^{h+1}y^{k+2})dx + (3x^{h+1}y^k + 8x^{h+2}y^{k+2})\,dy = 0$$

must be exact.

Clearly, $M = 2x^h y^{k+1} + 6x^{h+1} y^{k+2}$ & $N = 3x^{h+1} y^k + 8x^{h+2} y^{k+2}$.

By the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, we get

$$2(k+1)\, x^h y^k + 6(k+2)\, x^{h+1} y^{k+1} = 3(h+1)\, x^h y^k + 8(h+2)\, x^{h+1} y^{k+1},$$

which gives $3h-2k= -1$ and $4h-3k= -2$.

After solving these equations, we get

$h=1$ & $k=2$.

Hence, I.F. $= x^h y^k = xy^2$

5. Differential equations of the first order but not of the first degree.

The general first order differential equation of the degree > 1 is

$$a_0\left(\frac{dy}{dx}\right)^n + a_1\left(\frac{dy}{dx}\right)^{n-1} + a_2\left(\frac{dy}{dx}\right)^{n-2} + \cdots + a_{n-1}\left(\frac{dy}{dx}\right) + a_n = 0 \qquad (5.1)$$

where $a_0, a_1, a_2, \ldots a_n$ are functions of x and y.

In this section, we shall discuss the following solutions of Eq. (5.1).

(i) Equation solvable for $\frac{dy}{dx}$

Let $\frac{dy}{dx} = p$

then Eq. (5.1) becomes

$$a_0 p^n + a_1 p^{n-1} + \ldots\ldots a_{n-1} p + a_n = 0. \qquad (5.2)$$

Suppose (5.2) is solvable for p then it can be written as :

$$(p-f_1(x,y))\,(p-f_2(x,y))\ldots\ldots(p-f_n(x,y))=0 \qquad (5.3)$$

Equating each factor of Eq. (5.3) to zero, we get n equations of the first order and first degree. Suppose the solutions of resulting n equations are respectively,

$$F_1(x,y,c_1)=0,\ F_2(x,y,c_2)=0 \ldots\ldots F_n(x,y,c_n)=0. \qquad (5.4)$$

In Eq. (5.4), $c_1, c_2, \ldots\ldots, c_n$ are arbitrary constants of integration.

Without loss of generality, we can replace the arbitrary constants, $c_1, c_2, c_3, \ldots. c_n$ by single arbitrary constant C. Therefore, the n solutions given in Eq. (5.4) can be put in the form

$$F_1(x,y,c)=0,\ F_2(x,y,c)=0 \ldots\ldots F_n(x,y,c)=0 \qquad (5.5)$$

The solution of Eq. (5.2) can evidently be put in the form

$$F_1(x,y,c)\ F_2(x,y,c)\ F_3(x,y,c)\ldots\ldots F_n(x,y,c)=0 \qquad (5.6)$$

Ex. 1 $4x\left(\frac{dy}{dx}\right)^2 = (3x-a)^2$

$\Rightarrow 4x\,p^2 = (3x-a)^2$

$\Rightarrow\ p = \pm\frac{3x-a}{2\sqrt{x}}$

$\Rightarrow\ dy = \pm\left\{\frac{3}{2}\sqrt{x} - \frac{ax^{-1/2}}{2}\right\}dx$

$\Rightarrow\ y+c = \pm\left(x^{3/2} - ax^{1/2}\right)$

$\Rightarrow\ (y+c) = \pm\sqrt{x}(x-a)$

$\Rightarrow\ (y+c)^2 = x\,(x-a)^2$ (general solution)

(ii) **Equation solvable for *y***

Suppose the given differential equation $f(x,y,p)=0$ is solvable for y. Thus it can be put in the form

$$y = F(x,p), \qquad (5.7)$$

where $p = \frac{dy}{dx}$.

Differentiating Eq. (5.7) w.r.t. x, we get an equation in the form:

$$p = \emptyset\left(x, p, \frac{dp}{dx}\right). \qquad (5.8)$$

In Eq. (5.8) only two variables p and x are present . It may be possible to find the solution of Eqn. (5.8) in the form:

$$\psi(x, p, c) = 0, \qquad (5.9)$$

where c is arbitrary constant.

The elimination of p between Eqs. (5.7) and (5.9) gives us the required solution.

If elimination is not possible, the Eqs. (5.7) and (5.9) may be taken as solution which give x and y in terms of parameter p.

Ex. 1. Solve $y+px = x^4p^2$.

$$\Rightarrow \quad y = x^4p^2-px \tag{5.10}$$

Diff. w.r.t. x and taking p for dy/dx, we get

$$P = 4x^3p^2+2x^4p\frac{dp}{dx} - p\text{-}x\frac{dp}{dx}$$

$$2P(1\text{-}2x^3p)+x\left(\frac{dp}{dx}\right)(1\text{-}2x^3p) = 0$$

$$\Rightarrow \quad (1\text{-}2x^3p)\left[2p + x\frac{dp}{dx}\right] = 0 \tag{5.11}$$

The second factor of Eq. (5.11) contains $\frac{dp}{dx}$, so from this equation, we have

$$2p + x\frac{dp}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{p}dp + 2\frac{dx}{x} = 0 \qquad \Rightarrow \qquad p = c/x^2 \tag{5.12}$$

Eq. (5.10) and Eq. (5.12) gives

$$xy \pm c = c^2x$$

which is the solution of the problem.

(iii) Equations solvable for *x*.

If the given equation is solvable for y, then we can put that equation in the form

$$x = f(y,p) \tag{5.13}$$

Differentiating (1) w.r.t. y and writing $1/p$ for $\frac{dx}{dy}$, we get

$$\frac{1}{p} = \emptyset\left(y, p, \frac{dp}{dy}\right) \tag{5.14}$$

Suppose that the solution of Eq. (5.14) is possible and let the solution be

$$\psi(y, p, c) = 0 \tag{5.15}$$

where c is arbitrary constant.

The elimination of p between Eqs (5.13) and (5.14) gives us the required solution. If the elimination of p is not possible, then we solve (5.13) and (5.15) to express x and y in terms of parameter p.

Ex. Solve $y = 2px + p^2 y$

$$\Rightarrow \quad 2x = -py + y/p \tag{5.16}$$

Diff. w.r.t. y, we get

$$\frac{2}{p} = -p - y\frac{dP}{dy} + \frac{1}{p} - \frac{y}{p^2}\frac{dP}{dy},$$

$$\Rightarrow \quad (1 + 1/p^2)\left[p + y\left(\frac{dP}{dy}\right)\right] = 0. \tag{5.17}$$

Neglecting first factor of Eq. (5.17) which does not involve dp/dy, we have

$$p + y\left(\frac{dP}{dy}\right) = 0$$

$$\Rightarrow \; py = c. \tag{`5.18}$$

Eqs. (5.16) and (5.18) give

$$2xc - y^2 + c^2 = 0,$$

Which is the required solution.

(iv) Clairaut's equation

A differential equation of the form

$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$ is known as Clairaut's equation.

If $\frac{dy}{dx} = p$, then we can write Clairaut's equation as:

$$y = xp + f(p) \tag{5.19}$$

To solve Eq (5.19), we differente (5.19) w.r.t. x and writing p for $\frac{dy}{dx}$, we have

$$p = p + x\frac{dp}{dx} + f'\left(\frac{dp}{dx}\right)$$

$$\Rightarrow \; [x + f'(p)]\left(\frac{dp}{dx}\right) = 0 \tag{5.20}$$

Neglecting first factor of Eq (5.20) which does not involve $\frac{dp}{dx}$, we get

$$\frac{dp}{dx} = 0$$

$$\Rightarrow \; p = c \qquad (5.21)$$

From Eqs. (5.19) and (5.21), the required solution of Clairaut's equation is

$$y = xc + f(c)$$

Ex.: Solve $y = 4xp + 16y^3p^2$

Multiplying the above equation by y^3 , this equation becomes

$$y^4 = 4xy^3p + 16y^6p^2. \qquad (5.22)$$

Taking $y^4=v,$ the Eq (5.22) becomes

$$v = x\frac{dv}{dx} - \left(\frac{dv}{dx}\right)^2. \qquad (5.23)$$

which is Clairaut's equation.

Solution of Eq. (5.23) is $v = xc - c^2$.

Finally, $y^4 = xc - c$ is the solution of given differential equation.

6. Initial value problem

The differential equation

$$y' = f(x,y) \qquad (6.1)$$

with $y(x_0) = y_0$ is an initial value problem (IVP).

There are three possibilities of existence of solution of an initial value problem given in Eq (1). These possibilities are illustrated by the following example:

$\Rightarrow$ The initial value problem

$$|y'| + |y| = 0, \qquad y(0)=1 \qquad (6.2)$$

has no solution. Clearly, $y=0$ is the only solution of differential equation and this solution does not satisfy the initial condition $y(0)=1$.

$\Rightarrow$ The initial value problem

$$y' = x, \qquad y(0)=1 \qquad (6.3)$$

has only one solution, i.e. $y = x^2/2+1$.

⇒ Now, consider the following initial value problem

$$y' = \frac{y-1}{x}, \qquad y(0)=1 \tag{6.4}$$

It can be written as :

$$\frac{dy}{y-1} = \frac{dx}{x} \Rightarrow \log(y-1) = \log x + \log c$$

$\Rightarrow\; y-1 = xc,$

where c is arbitrary constant.

Here, we cannot determine c by using initial condition $y(0)=1$. Therefore, the initial value problem (6.4) has infinite solution.

This discussion leads the following two fundamental questions:

(i) Under what condition an initial value problem of the form (6.1) has at least one solution?

(ii) Under what condition an initial value problem of the form (6.1) has unique solution?

The answers of these questions are discussed in the following theorems:

Theorem 1 (Existence theorem)

If $f(x,y)$ is continuous at all points (x,y) in a rectangle

$$R:\; |x-x_0| < a\,, \qquad |y-y_0| < b$$

and bounded in R, say

$|f(x,y)| \le k$ for all (x,y) in R

then the initial value problem (1) has at least one solution $y(x)$ which is defined for all x in the interval

$|x-x_0| < \gamma$, where $\gamma = \min(a, \frac{b}{k})$

Theorem 2 (Uniqueness theorem)

Suppose $f(x,y)$ is continuous at all points (x,y) in a rectangle

$$R:\; |x-x_0| < a\,; \qquad |y-y_0| < b$$

and bounded in *R,* i.e.

$$|f(x,y)| \leq k \quad \text{for all } (x,y) \text{ in } R.$$

where k is a constant.

Let $f(x,y)$ satisfy the Lipschitz condition in *R, i.e.*

$$|f(x,y_2) - f(x,y_1)| \leq M|y_2 - y_1|;$$

where M is constant.

Then the initial value problem (1) has unique solution *y(x).* This solution is defined for all *x* in the interval $|x - x_0| < \gamma$, where $\gamma = \min(a, \frac{b}{k})$.

Remark : If *f(x,y)* satisfies the condition

$$\left|\frac{\partial f}{\partial y}\right| \leq M \qquad (*)$$

for all values of *f(x,y)* in *R* then for the same constant M the Lipchitz condition is also satisfied.

Proof : By Mean value theorem

$$\frac{f(x,y_2)-f(x,y_1)}{y_1-y_2} = \left.\frac{\partial f}{\partial y}\right|_{y=\bar{y}}; \qquad y_1 < \bar{y} < y_2$$

where (x,y_2) and (x,y_1) are in *R*.

This gives

$$|f(x,y_2) - f(x,y_1)| \leq M|y_1 - y_2|$$

which is Lipschitz condition. This shows that Lipschitz condition can be replace by (*) in the existence & uniqueness theorem. The condition (*) is stronger condition.

Ex. 1 Consider the ODE (IVP)

$$y' = 1+y^2,\ y(0) = 0 \qquad (6.5)$$

Consider the rectangle R=$\{(x,y) |\ |x| \leq 1,\ |y| \leq 1\}$

Clearly, $f(x,y)=1+y^2$ and $\left|\frac{\partial f}{\partial y}\right| = 2y$ are continuous in *R*. Hence *f(x,y)* satisfies a Lipschitz condition in *R*. Hence IVP (6.5) has a unique solution in a nhd of 0.

Ex. 2 Consider the function $f(x,y)=x^2|y|$ in the rectangle R=$\{(x,y) |\ |x| \leq 1, |y| \leq 1\}$

Since $|f(x,y_1)-f(x,y_2)| \leq \left|x^2|y_1|-x^2|y_2|\right|$

$= \left|x^2|y_1|-x^2|y_2|\right|$

$\leq x^2|y_1-y_2|$

$\leq |y_1-y_2|$

$\forall (x,y_1)$ and (x,y_2) in R. Hence, *f(x,y)* satisfies a Lipschitz condition with Lipschitz constant 1.

Now, we observe that $\frac{\partial f}{\partial y}$ does not exist at any point in *R* for *a≠0.*

[Note : $f(x,y)=x^2|y|$ is continuous in *R*].

Ex.3 Discuss the existence and uniqueness of solution of the IVP

$$y'=y^{1/3}, \qquad y(0)=0 \qquad (6.6)$$

First of all, we show that $f(x,y)=y^{1/3}$ does not satisfy a Lipschitz condition in any rectangle centred at (0,0). To show this, let M>0 be given, if we choose y_1, y_2 such that

$$0 < y_1, y_2 < \left(\frac{1}{3M}\right)^{3/2}$$

then $y_1^{2/3}+y_2^{2/3}+y_1^{1/3}y_2^{1/3} < \frac{1}{M}$ and

$$\left|y_1^{1/3}-y_2^{1/3}\right| \left|y_1^{2/3}+y_2^{2/3}+y_1^{1/3}y_2^{1/3} < \frac{\left|y_1^{1/3}-y_2^{1/3}\right|}{M}\right|$$

Hence, $|f(x,y_2)-f(x,y_1)| = \left|y_1^{1/3}-y_2^{1/3}\right| > M|y_1-y_2|$.

Thus, *f(x,y)* does not satisfy a Lipschitz condition. Hence, the above problem may have unique solution or many solutions. Possible solutions of above problem are:

$y(x) = 0,$

$y(x) = \left(\frac{2x}{3}\right)^{3/2},$

$y(x) = -\left(\frac{2x}{3}\right)^{3/2}.$

Note: The existence and uniqueness theorems do not tell us anything about the uniqueness of the solution if Lipschitz condition is not satisfied.

Ex. 4 Discuss the existence and uniqueness of the solution of IVP

$y' = xy - \sin y, \qquad y(0) = 2$

Solution: Here $f(x,y) = xy - \sin y$

and $\frac{\partial f}{\partial y} = x - \cos y$

Hence, $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous in R^2. Therefore, both are continuous in any rectangle centred at (0,2) i.e. $|x| \leq a$ and $|y-2| \leq b$. So the above problem has a unique solution in the nhd of 0.

Assignment

1. Solve

(i) $(x^2y^2 + xy + 1)ydx - (x^2y^2 - xy + 1)xdy = 0$

(ii) $(x^2y + 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

(iii) $y(1+xy)dx + x(1-xy)dy = 0$

(iv) $(xy^2 - x^2)dx + (3x^2y^2 + x^2y - 2x^2 + y^2)dy = 0$

(v) $(2x^2y^2 + y)dx - (x^3y - 3x)dy = 0$

(vi) $(x^2y^2 + xy + 1)ydx - (x^2y^2 - xy + 1)xdy = 0$

2. Discuss the existence and uniqueness solution for IVP

$y' = y^{1/3} + x, \qquad y(1) = 0.$

3. Discuss the existence and uniqueness solution for the IVP

$y' = \frac{2y}{x}, \qquad y(x_0) = y_0.$ Also find the solution.

Reference Books:

1. Erwin Kreyszig, Advanced engineering mathematics, Wiley India (P) Ltd. John Wiley & sons.
2. Earl A. Coddington, An Introduction to Ordinary Differential Equations, prentice-Hall.
3. S.G. Deo, V. Lakshmikantham, V. Raghavendra; Textbook of Ordinary Differential Equations, Second Edition, Tata McGraw-Hill Education Pvt. Ltd.

Module-6

Pretest on Analytical geometry and Vector Algebra

Polar Coordinates

1. Find a polar coordinates of the point (-1, -1)
2. Identify the polar equation of the circle $(x-1)^2 + y^2 = 1$.
3. Identify the polar equation of the x-axis, y-axis and the straight line $y = x$.
4. Identify a rough sketch of the curves $r = \cos\theta$ and $r = 1 + \cos\theta$. Algebraically identify the common points of these two curves.

Directional Derivative

1. Find both the partial derivatives of the function $f(x, y) = \sqrt{x^2 + y^2}$ at $(0, 0)$. Is the function continuous at (0, 0)? Justify your answer mathematically.

Surface integral

1. Evaluate $\iint_R x^2 y \ dA$ where r is the region bounded by the lines $y = x,\ y = 3x,\ \&\ x + y = 2$.

2. Change the order of integration

$$\int_0^2 \int_{\frac{x^2}{4}}^{3-x} f(x, y) dy dx$$

3. With the help of polar double integration, evaluate the area inside the curve $r = 1 - \cos\theta$

Curl and Gradient

1. Consider the function

$$f(x, y, z) = x^2 y + y^2 z + z^2 x$$

Execute the expression of
$Curl\ (Grad\ f(x, y, z))$

2. Find the unit normal vector on the cone $z^2 = 4(x^2 + y^2)$ at the point $(1, 0, 2)$.

<u>Module-6</u>

<u>Analytical geometry and Vector Algebra</u>

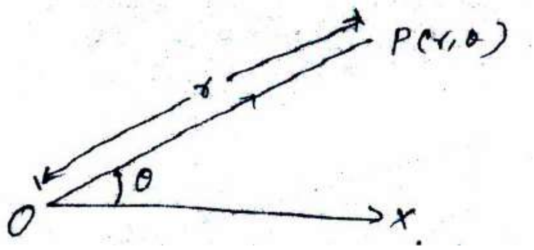
Lectures required -03

Polar Equation of Conics

In discussing polar coordinate system we always refer to the Cartesian coordinate system such that we can tally with our existing knowledge on curves

Definition of Polar coordinates

$$\begin{cases} Fix\ an\ origin\ O \\ Fix\ a\ ray\ from\ O \end{cases}$$



Origin 0 Initial ray : OX

also called

Pole

$$\begin{cases} \theta \geq 0 \rightarrow anticlockwise \\ \quad < 0 \rightarrow clockwise \end{cases}$$

$$\begin{cases} r \geq 0 \rightarrow along\ \overrightarrow{OP} \\ < 0 \rightarrow reverse\ direction\ of\ \overrightarrow{OP} \end{cases}$$

$P(\ r,\ \theta\ )$

Directed distance from O to P

Directed angle from initial ray to OP.

The value of

r can be +ve, can be –ve and any real number

θ can be +ve, can be –ve and any real number

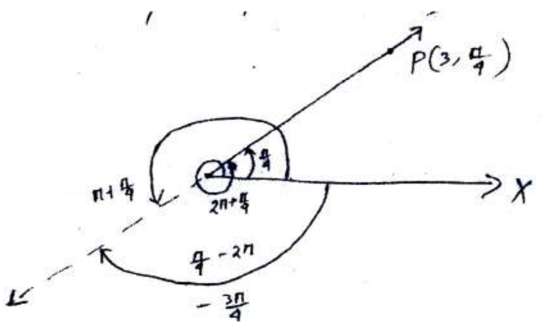
Note : In the cartesian coordinate system a point has ONE and ONLY ONE pair of cartesian co-ordinates. However in the polar coordinate system of a point has INFINITELY MANY pairs of polar coordinates.

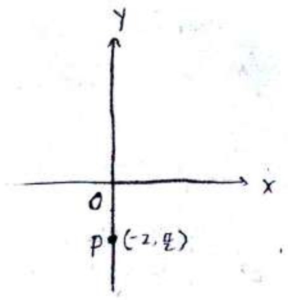
Example 1 Find all the polar coordinates of the point $P\left(3, \frac{\pi}{4}\right)$.

Coordinates of P are

$\left(3, 2n\pi + \frac{\pi}{4}\right)$, n = 0, ±1, ±2,….

$\left(-3, 2n\pi - \frac{3\pi}{4}\right)$, n = 0, ±1, ±2,….





Example 2 Plot the point $\left(-2, \frac{\pi}{2}\right)$. Then find all of its polar coordinates.

The point $\left(-2, \frac{\pi}{2}\right)$ is located at P.

All the polar coordinates of P are

$$\left(-2, \frac{3\pi}{2} + 2n\pi\right) \text{ and } \left(-2, \frac{3\pi}{2} + (2n+1)\pi\right),\ n = 0, \pm1, \pm2, \ldots.$$

Example 3 Find the polar coordinates of $\left(\sqrt{3}, -1\right)$ in $0 \le \theta \le 2\pi$ and $r \ge 0$.

$$\left(\sqrt{3}, -1\right) \Rightarrow r = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2,\ \sin\theta = -\frac{1}{2} \text{ and } \cos\theta = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta = \frac{11\pi}{6}$$

$\therefore$ Polar coordinates of $\left(\sqrt{3}, -1\right)$ in $0 \le \theta \le 2\pi$ and $r \ge 0$ are $\left(2, \frac{11\pi}{6}\right)$.

Example 4 Find the polar coordinates of (5, -12) in $\pi \le \theta < 2\pi$ and $r \ge 0$

$$(5, -12) \Rightarrow r = \sqrt{5^2 + (-12)^2} = 13,\ \sin\theta = -\frac{12}{13} \text{ and } \cos\theta = \frac{5}{13}$$

$$\Rightarrow \theta = -\tan^{-1}\left(\frac{12}{5}\right)$$

$\therefore$ Polar coordinates of (5, -12) in $\pi \le \theta \le 2\pi$ and $r \ge 0$ are $\left(13, -\tan^{-1}\left(\frac{12}{5}\right)\right)$

Example 5 Find the polar coordinates of $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ in $-\pi \le \theta < 2\pi$ and $r \ge 0$.

$$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \Rightarrow r = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = -1,\ \sin\theta = -\frac{1}{2} \text{ and } \cos\theta = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta = \frac{7\pi}{6} \text{ or } \theta = -\frac{5\pi}{6}$$

Polar coordinates of $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ in $-\pi \le \theta < 2\pi$ and $r \le 0$ are $\left(-1, \frac{7\pi}{6}\right)$ or $\theta = \left(-1, -\frac{5\pi}{6}\right)$.

Example 6 (Identifying the graph)

Graph the sets of points whose polar coordinates satisfy the following conditions

(a) $r = 1$

(b) $1 \le r \le 2$

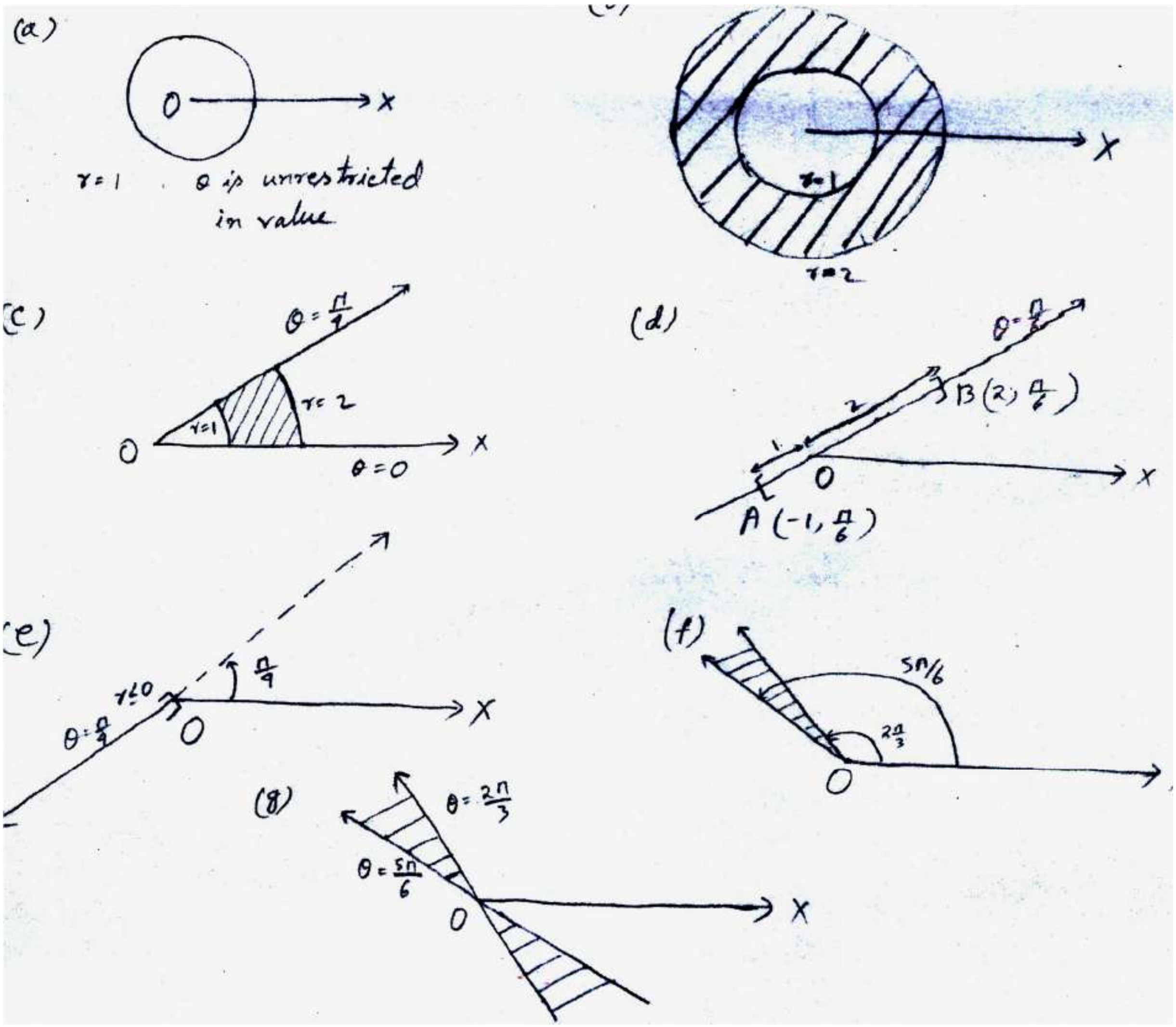
(c) $1 \le r \le 2, \quad 0 \le \theta \le \frac{\pi}{4}$

(d) $-1 \le r \le 2, \quad \theta = \frac{\pi}{6}$

(e) $r \leq 0$, $\theta = \frac{\pi}{4}$

(f) $\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$, $r \geq 0$

(g) $\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$



Note $r = a$: Circle of radius $|a|$ centred at O

$\left.\begin{matrix} r = -2 \\ r = 2 \end{matrix}\right\}$ Circle of radius 2 centred at O

$\theta = \theta_0$: Line through O making an angle θ_0 with the initial ray

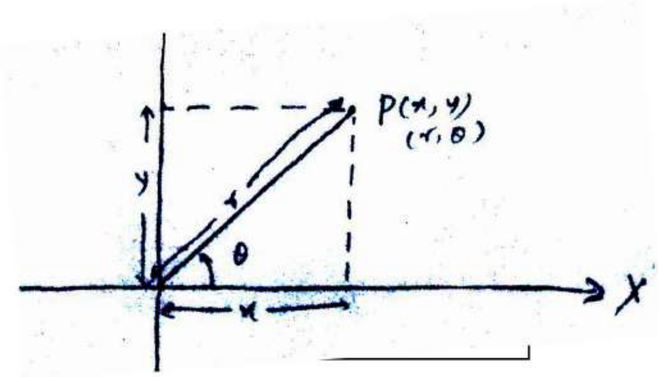
Question A polar

Representation of x-axis : $\theta = 0$ +ve x-axis : $\theta = 0$, $r > 0$.

Representation of y-axis : $\theta = \frac{\pi}{2}$ +ve y-axis : $\theta = \frac{\pi}{2}$, $r > 0$.

In order to represent the points on the plane, now we have two coordinate systems. Let's now see their interrelation

Relating Polar and Cartesian coordinates :



If (r, θ) is given we can obtain the corresponding cartesian coordinate by

$$\boxed{\begin{array}{l} x = r\cos\theta \\ y = r\cos\theta \end{array}}$$

Question If (x,y) is given, how to obtain the corresponding polar coordinates?

$r^2 = x^2 + y^2$

↓

Two values of r : $+\sqrt{x^2+y^2}$, $-\sqrt{x^2+y^2}$

↓

$\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$, letting $r \neq 0$ [If r = 0 then θ can be any value]

↓

Obtain unique $\theta \in (0, 2\pi)$ corresponding to a particular r.

or $\theta \in (-\pi, \pi)$

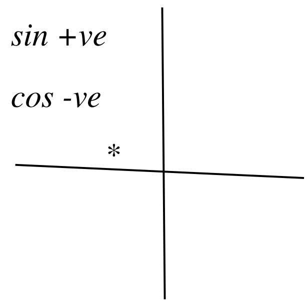
Example 7 Find the polar coordinates of the point whose cartesian coordinates (-1, 1).

Solution $(x,y) = (-1, 1) \Rightarrow x = -1, y = 1$

$r^2 = x^2 + y^2 \Rightarrow r^2 = 2 \Rightarrow r = \sqrt{2}, -\sqrt{2}$

$r = \sqrt{2}$: $\cos\theta = \frac{x}{r} = -\frac{1}{\sqrt{2}}$, $\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}}$

$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$



$\therefore$ A polar coordinates of the cartesian coordinates $(-1, 1)$ is $\left(\sqrt{2}, \frac{3\pi}{4}\right)$

Example 8 Converting cartesian to polar equation

xy – equation $\longrightarrow$ rθ equation

Given $(x-2)^2 + y^2 = 4$

Use the relation

$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\}$

Solution $(r\cos\theta-2)^2 + y^2\sin^2\theta = 4$

$\Rightarrow r^2 - 4r\cos\theta + 4 = 4$

$\Rightarrow r(r - 4\cos\theta) = 0$

$\Rightarrow r = 0 \quad or \quad r = 4\cos\theta$

$\Rightarrow r = 4\cos\theta.$

Example 9 Converting polar to cartesian coordinate

rθ equation $\longrightarrow$ xy – equation Type equation here.

(a) $r = 1 + 2r\cos\theta$

(b) $r = 1 - \cos\theta$

Don't use

$r = \pm\sqrt{x^2 + y^2}$

Use the relation

$r^2 = x^2 + y^2$

$\cos\theta = = \frac{x}{r}$

$\sin\theta = \frac{y}{r}$

Solution: **(a)** $r = 1+2x \Rightarrow r^2 = (1+2x)^2 \Rightarrow x^2+ y^2 = 1+ 4x + 4x^2 \Rightarrow y^2-3x^2- 4x -1 = 0$

(b) $r = 1- \cos\theta \Rightarrow r^2 = r - \cos\theta \Rightarrow x^2+ y^2 = r - x$

$\Rightarrow (x^2+ y^2 + x)^2 = r^2$

$\Rightarrow x^4+ y^4 + x^2+2x^2y^2+2y^2x + 2x^3 = x^2 + y^2$

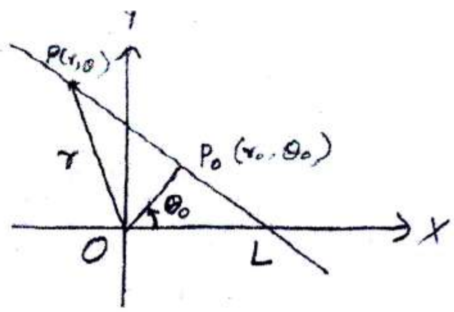
$\Rightarrow x^4+ y^4 + 2x^2y^2+2xy^2 + 2x^3 - y^2 = 0$

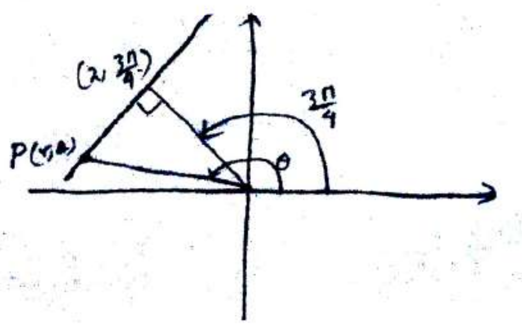
Conic Section in Polar Coordinates

Lines Equation of the line L :

$$\boxed{r\cos(\theta-\theta_0) = r_0}$$

where $r_0 \geq 0$, $P_0(r_0, \theta_0)$ is the foot of perpendicular from the origin to the line L.





Example 10 Find the equation of the line in which the point $\left(2,\frac{3\pi}{4}\right)$ is the foot of perpendicular from the origin

$$r\cos\left(\theta-\frac{3\pi}{4}\right)=2$$

Cartesian $r\cos\theta\left(-\frac{1}{\sqrt{2}}\right)+r\sin\theta\left(-\frac{1}{\sqrt{2}}\right)=2$

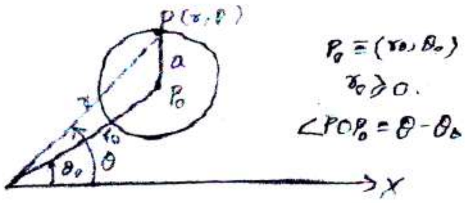
$$\Rightarrow y-x=2\sqrt{2}$$

Note $\quad Ax+By+C=0, \qquad (A,B)\neq(0,0)$

$$R\,(A\cos\theta+B\sin\theta)+C=0$$

$$\Rightarrow r\cos(\theta-\theta_0)=\underbrace{r_0}_{-C/\sqrt{A^2+B^2}}$$

Circle Equation of a circle with centre $P_0(r_0,\theta_0)$ and radius a



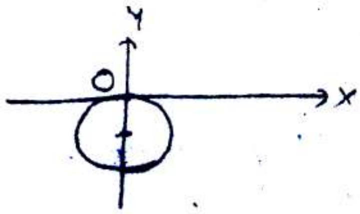
$$2r\,r_0\cos(\theta-\theta_0)=r^2+r_0^2-a^2$$

$$\Rightarrow \boxed{a^2=r^2+r_0^2-2r\,r_0\cos(\theta-\theta_0)}$$

Example 11 Find the polar equation of a circle with centre $\left(+1,-\frac{\pi}{2}\right)$ and radius 1.

Center : $\left(1,-\frac{\pi}{2}\right)$

Radius : 1

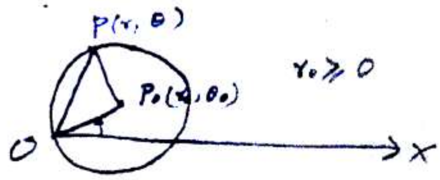


$$1=r^2+1-2r\cos\left(\theta+\frac{\pi}{2}\right)$$

$$\Rightarrow r=-2\sin\theta$$

Particular cases

Circles passing through origin



$$a^2=r^2+a^2-2ra\cos(\theta-\theta_0)$$

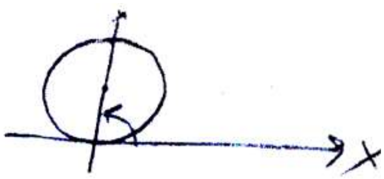
$$\Rightarrow r=2a\cos(\theta-\theta_0)$$

$$a^2=r^2+a^2-2ra\cos(\theta-\theta_0) \qquad x^2+y^2+2y=0$$

$$\Rightarrow r=2a\cos(\theta-\theta_0) \qquad \Rightarrow x^2+(y+1)^2=1$$

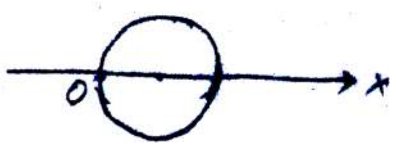
Subcases (i) Centre lies on the positive y-axis

$$\theta_0=\frac{\pi}{2}$$



$r = 2a\ cos(\theta - \frac{\pi}{2}) = 2a\ sin\ \theta$

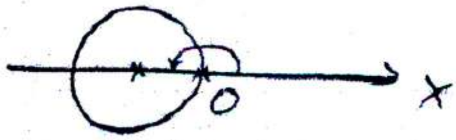
(ii) Centre lies on the positive x-axis



$\theta_0 = 0$

$r = 2a\ cos(\theta - 0) = 2a\ cos\ \theta$

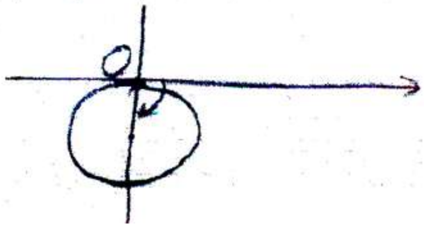
(iii) Centre lies on the negative x-axis



$\theta_0 = \pi$

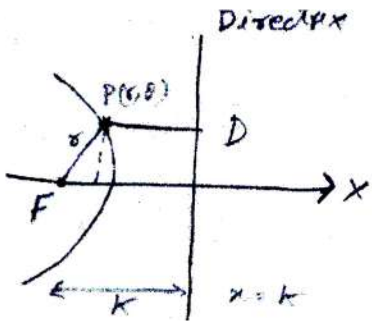
$r = 2a\ cos(\theta - \pi) = -2a\ cos\ \theta$

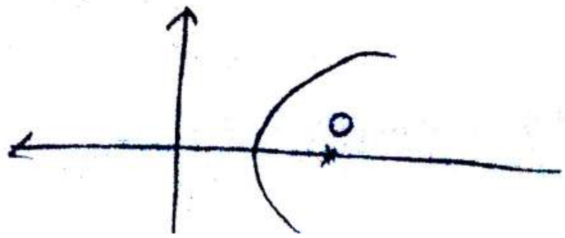
(iv) Centre lies on the negative y- axis



$\theta_0 = \frac{\pi}{2}$

$r = 2a\ cos(\theta + \frac{\pi}{2}) = -\ 2a\ sin\ \theta$





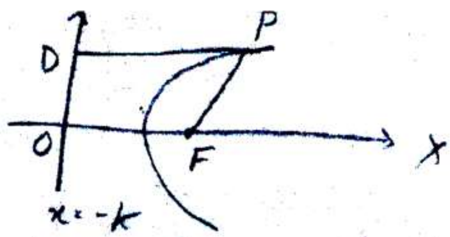
Conic Sections

$\frac{FP}{PD} = e \Rightarrow \frac{r}{k - r cos\theta} = e \Rightarrow r = e\ (\ k - r\ cos\theta)$

$\Rightarrow r = \frac{ke}{1 + e cos\theta}$, where $x = k > 0$ is the vertical directrics

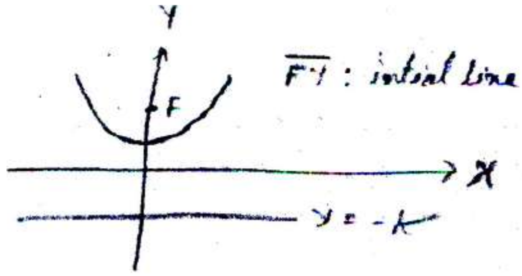
If the directrix is the line $x = -\ k,\ k > 0,$ i.e. directrics lies on the left of the origin

$r\ = \frac{ke}{1 - e cos\theta}$



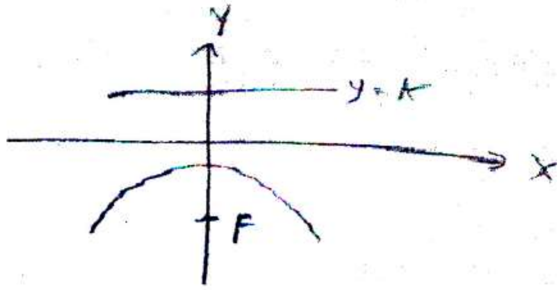
If $y = -k,\ k > 0,$ is the directrics

$$r = \frac{ke}{1-e\sin\theta}$$



If $y = -k,\ k > 0,$ is the directrics

$$r = \frac{ke}{1+e\sin\theta}$$



Examples 12 (Polar equation of a hyperbola)

Find the polar equation of the hyperbola with

eccentricity e = $\frac{3}{2}$ and directrix x=2

$K= > 0,$ e = $\frac{3}{2}$

$$r = \frac{ke}{1+e\cos\theta} = \frac{3}{1+\frac{3}{2}\cos\theta}$$

$$= \frac{6}{2+3\cos\theta}$$

Example 13 (Finding a directrix)

Find the directrix of a parabola $r = \frac{25}{10+10\cos\theta}$

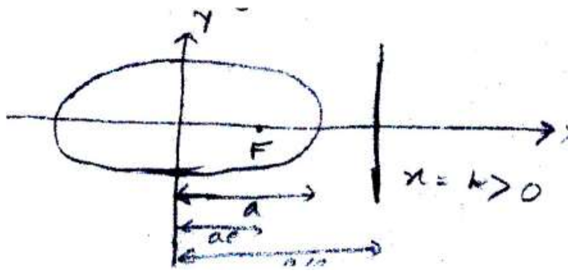
$$r = \frac{25/10}{1+\cos\theta} : x = 5/2$$

*Find the ellipse with eccentricity e and semi-major axis a

$$k = \frac{a}{e} - ae$$

$$\Rightarrow\ ke = a(1-e^2)$$

equation $r = \frac{a(1-e^2)}{1+e\cos\theta}$



a/e

Particular Case

e = 0 ⇒ r = a : circle

Directional Derivative

Gradient

The gradient vector of a function *f(x,y)* at a point $p_0(x_0,y_0)$ is the vector

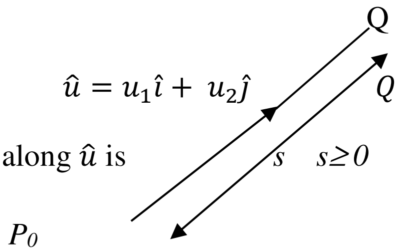
$$\vec{\nabla} f = f_x|_{(x_0,y_0)}\hat{\imath} + f_y|_{(x_0,y_0)}\hat{\jmath}$$

Notation $\vec{\nabla} f$, *grad(f)*

This is the direction in which functional rate of changes is maximum but the question is how to measure the rate of changes of a function along any direction. We have already learnt that functional rate of change along the x-axis & y-axis and are given by first order partial derivatives. This rate of change of a function along a particular direction is known as directional derivative.

Directional derivative $\hat{u} = u_1\hat{\imath} + u_2\hat{\jmath}$ $Q = P_0 + s_1\hat{u}$

Rate of change of *f(x,y) at* $P_0(x_0,y_0)$ and along $\hat{u}$ is



$$\underset{Q\to P}{\text{Lt}} \frac{f(Q)-f(P)}{|\vec{PQ}|}$$

$$= \underset{s\to 0}{\text{Lt}} \frac{f(x_0+su_1, y_0+su_2)-f(x_0,y_0)}{s}$$, this limit (if exists) is called directional derivative of *f* at p_0 along $\hat{u}$

Notation $\left(\frac{dt}{ds}\right)_{\hat{u}.P_0}$ or $(D_{\hat{u}}f)_{P_0}$

Attention! The given direction in the def. is a unit vector.

Example 1 Let $f(x,y) = \begin{cases} \frac{x^2y}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Find the directional derivative of *f(x,y)* at *(0,0)* in the direction of the vector $\hat{u} = \left(\frac{1}{\sqrt{2}}\right)\hat{\imath} + \left(\frac{1}{\sqrt{2}}\right)\hat{\jmath}$

Solution

$$\underset{s\to 0}{\text{Lt}} \frac{f\left(0+\frac{s}{\sqrt{2}}, 0+\frac{s}{\sqrt{2}}\right)-f(0,0)}{s}$$

$$\underset{s\to 0}{\text{Lt}} \frac{\frac{s/2\sqrt{2}}{\sqrt{s^2}}-0}{s}$$

$$\underset{s\to 0}{\text{Lt}} \frac{s^2}{2\sqrt{2|s|}} = 0 \qquad \text{along} \quad \hat{\imath} + \hat{\jmath} \text{ is } \frac{1}{\sqrt{2}}$$

$\therefore$ Required directional derivative is 0.

Let's relook the definition of **directional derivative for a differentiable function**

$$(D_{\hat{u}})_{P_0} = \underset{s\to 0}{\text{Lt}} \frac{f(x_0+su_1, y_0+su_2)f(x_0, y_0)}{s},$$

$$= \underset{s\to 0}{\text{Lt}} \frac{f_x(x_0,y_0)su_1 + f_y(x_0, y_0)su_2 + \epsilon_1 su_1 + \epsilon_2 su_2}{s} \quad \begin{bmatrix} where\ \ \epsilon_1 \to 0 \\ \epsilon_2 \to 0 \\ as\ \ s \to 0 \end{bmatrix}$$

$$= f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

$$= \left(f_x(x_0, y_0)\hat{\imath} + f_y(x_0, y_0)\hat{\jmath}\right).(x_1\hat{\imath}, x_2\hat{\jmath})$$

$$= (grad\ f)_{P_0}.\hat{u}$$

Example 2 Find the directional derivative of the function $f(x,y) = 2xy - 3y^2$ at $P_0(5,5)$ in the direction $\vec{A} = 4\hat{\imath} + 3\hat{\jmath}$

Solution $f_x(x,y) = 2y$, $f_y(x,y) = 2x - 6y$: both the functions are continuous everywhere

$\therefore$ the function is differentiable everywhere

$$\Rightarrow \left(D_{\vec{A}}\right)_{P_0} = \left(D_f\right)_{P_0}.\frac{\vec{A}}{|\vec{A}|}$$

$$= \left(f_x(5,5)\hat{\imath} + f_y(5,5)\hat{\jmath}\right).\left(\frac{4}{5}\hat{\imath} + \frac{3}{5}\hat{\jmath}\right) = -4$$

Attention! If the direction is given by $\vec{a}$ of any length ($\neq 0$) then

$$(D_{\vec{a}}f)_{\rho_0} = \left(\frac{dt}{ds}\right)_{\rho_0} \left(\vec{\nabla}f\right)_{\rho_0} \frac{\vec{a}}{|\vec{a}|} \quad (f \text{ is differentiable function})$$

Properties of Directional Derivative

$$D_{\hat{u}}f = \vec{\nabla}f.\hat{u} = \left|\vec{\nabla}f\right|.|\hat{u}|Cos\ \theta, \qquad \theta = \langle\vec{\nabla}f.\hat{u}\rangle$$

$$= \left|\vec{\nabla}f\right|Cos\ \theta$$

1. When $cos\ \theta = 1$ or $\theta = 0$, i.e. $\hat{u} = \frac{\vec{\nabla}f}{|\vec{\nabla}f|}$ then $D_{\hat{u}}f$ has maximum value. Therefore the function increases most rapidly along the gradient direction.

The directional derivative along $\vec{\nabla}f$ is $\left|\vec{\nabla}f\right|$.

2. When $\cos\theta = -1$ or $\theta=\pi$, i.e. $\hat{u} = -\vec{\nabla}f$. Then $D_{\hat{u}}f$ has minimum value.

$\therefore$ The function decreases most rapidly along $-\vec{\nabla}f$

The directional derivative along $\overline{-\vec{\nabla}f}$ is $-\left|\vec{\nabla}f\right|$

3. When $\cos\theta = 0$ or $\theta=\pi/2,\ 3\pi/2$, i.e. $\hat{u} \perp \vec{\nabla}f$ then $D_{\hat{u}}f = 0$. Therefore, any direction orthogonal to $\vec{\nabla}f$ is a direction of zero change in *f*.

Example 3 Find the direction in which $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$

a) Increases most rapidly at the point (1, 1).
b) Decreases most rapidly at (1, 1).
c) What are the directions of zero change in *f* at (1, 1)?

Solution a) The function increases most rapidly in the direction of $\vec{\nabla}f$ at (1, 1).

$$(\vec{\nabla}f)\,(1,1) = (x\hat{\imath} + y\hat{\jmath})_{(1,1)} = \hat{\imath} + \hat{\jmath}$$

$\therefore$ Direction is $\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}$

b) The function decreases most rapidly in direction of $-\vec{\nabla}f$

$$\left(-\vec{\nabla}f\right)_{(1,1)} = -\hat{\imath} - \hat{\jmath}$$

$\therefore$ Required Direction is $\frac{-\hat{\imath}-\hat{\jmath}}{|-\hat{\imath}-\hat{\jmath}|} = -\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath}$

c) The directions of zero change at (1, 1) are the directions orthogonal to $\vec{\nabla}f$.

$\therefore$ Required Direction is $\hat{n} = -\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}$

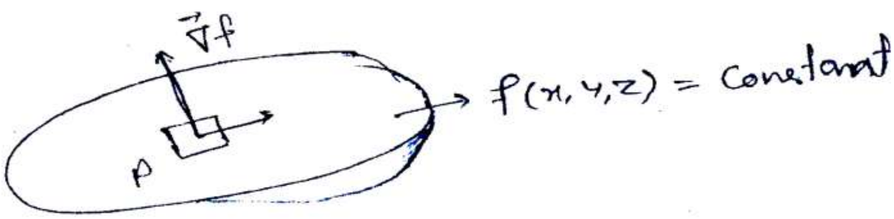
$$\& -\hat{n} = \frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath}$$

Gradient and directional derivative for function of three variables.

Gradient - $\quad \vec{\nabla}f = f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}$

Directional derivative - $\quad D_{\hat{u}}f = \vec{\nabla}f.\hat{u} = f_x u_1 + f_y u_2 + f_z u_3$

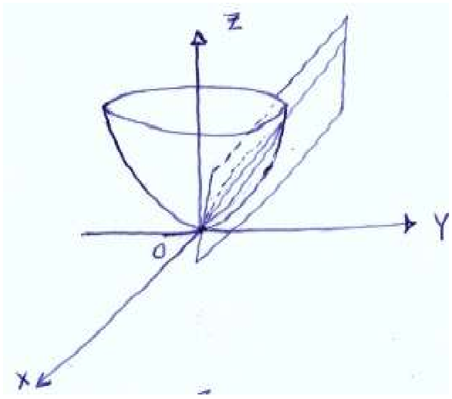
Observation -



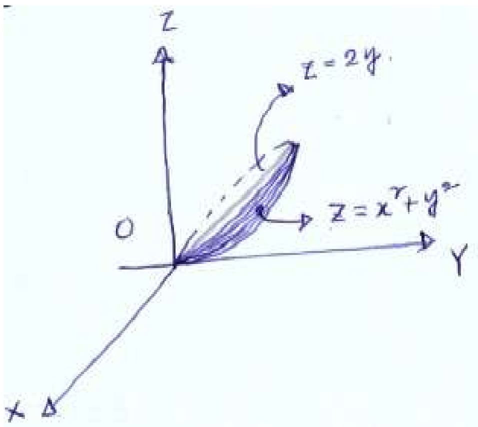
Change of order of integration in triple integral.

Example. Let D be the region bounded by the parabolid $z = x^2 + y^2$ and the plane $z = 2y$. write the triple integral to evaluate the vol. of D in the order of $dz\ dx\ dy$, $dy\ dx\ dz$, $dx\ dzdy$.

Step 1.



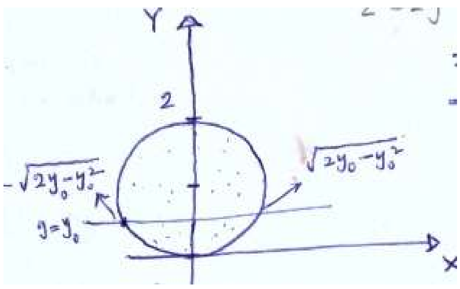
Step 2.



Step 3. $\boldsymbol{dz\,\overline{dxdy}}$

Eliminate z from $z = 2y$ & $z = x^2 + y^2$

$$x^2 + y^2 - 2y = 0$$
$$x^2 + (y-1)^2 = 1.$$



$$\mathrm{I} = \int_{y=0}^{2} \int_{x=-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int_{z=x^2+y^2}^{2y} dz\ dx\ dy$$

$\boldsymbol{dy\ \overline{dxdz}}$

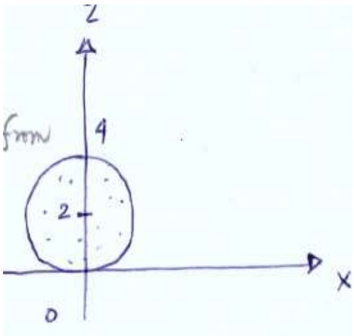
Eliminate y from $z = 2y$ & $z = x^2 + y^2$

$$z = x^2 + \frac{z^2}{4}$$

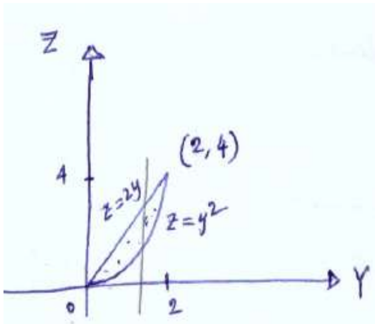
$$z^2 - 4z + x^2 = 0$$

$$(z-2)^2 + x^2 = 2^2$$

$$I = \int_{z=0}^{4} \int_{x=-\sqrt{4z-z^2}}^{\sqrt{4z-z^2}} \int_{y=z/2}^{\sqrt{z-x^2}} dy \quad dx\, dz$$



$\boldsymbol{dx\ \overline{dzdy}}$



$x = 0$, gives

$z = 0^2 + y^2$

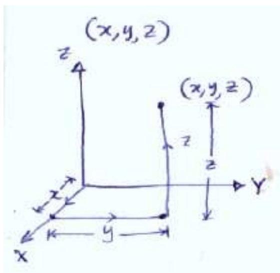
$I = \int_{y=0}^{2} \int_{z=y^2}^{2y} \int_{x=-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx \quad dz\, dy$

Here elimination will not work since the projection consists of projections of two surfaces $z = 2y$ & $z = x^2 + y^2$ where as in the previous two we projected the curve on $z = 2\,y$ itself.

<u>Triple integral in cylindrical & spherical coordinate system:</u> First we shall introduce the coordinate systems then we will go to find the integral w.r.t those coordinate system.

Let's suppose you have a point in the 3D space with Cartesian coordinates (x, y, z). We can also locate this point in two different mamaers.

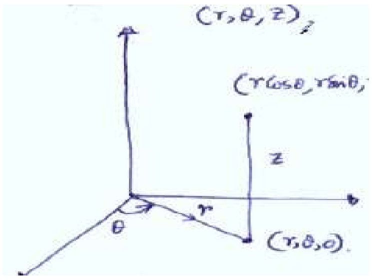
<u>Cartesian</u>



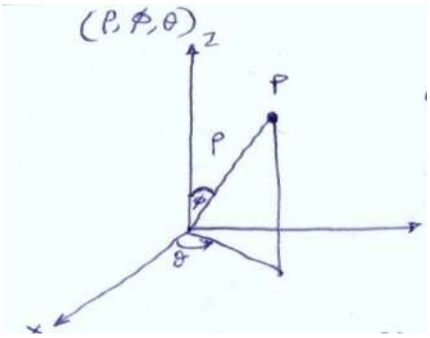
Cylindrical

Cylindrical=Polar (r, θ) in $x, y -$ plane

$+z$ Cartesian, $\theta \epsilon [-\pi,\ \pi]$ or $[0,\ 2\pi]$



Spherical



$$P = |\overrightarrow{0P}| \geq 0$$

$$\phi = \angle(\overrightarrow{0P}, +\text{ve z axis})\ \epsilon[0,\ \pi]$$

$$\theta = \text{cylindrical},\ \theta \epsilon [-\pi,\ \pi]\ \text{ Or } [0,\ 2\pi]$$

$x = \rho \sin\phi \cos\theta,\quad y = \rho \sin\phi \sin\theta,\quad z = \rho\cos\phi$

$\rho = \sqrt{x^2 + y^2 + z^2}$, unlike r, ρ is always ≥ 0.

Volume/ Triple integrals w.r.t three coordinate system:

$\iiint_D f(x, y, z)dv$ => To evaluate over D$\subset 3d -$ space, where D is a bounded region.

As we have defined triple integral for Cartesian coordinates, we have to discretize the region D through some elementary regions whose boundaries are given by constant values of the variables. Well firs then we try to find the surfaces those are given by constant values of the variables.

Cartesian

$x = a\ \rightarrow$ plane

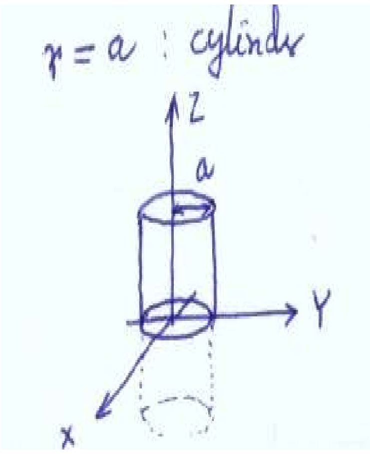
$y = b \rightarrow$ plane

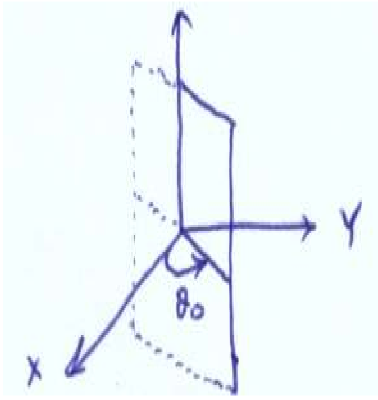
z = c→ plane

$dv = dx\ dy\ dz$

Cylindrical

$\theta = \alpha \rightarrow$ plane



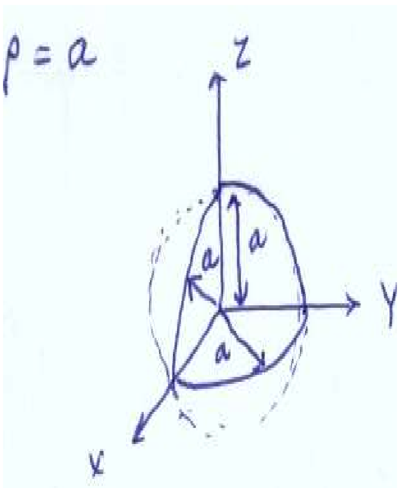


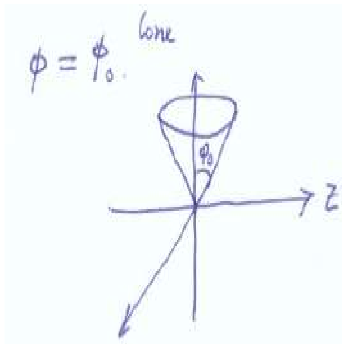
$z = c \rightarrow$ plane

$dv = r\, dr\, d\theta\, dz$

Spherical

$\phi = \phi_{0.}$ Cone



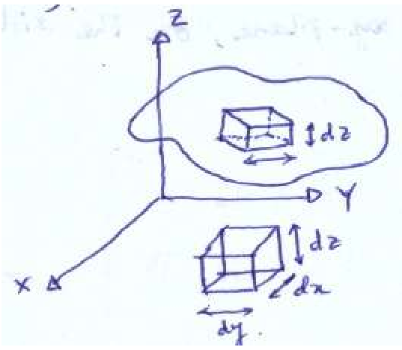


θ = Constant : plane

$dv = \rho^2 \sin\phi \;\, d\rho\, d\phi\, d\theta$

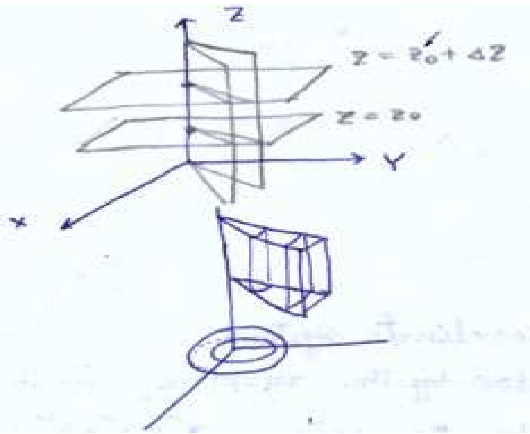
Volume elements (dv):

Cartesian



$dv = dx\, dy\, dz$

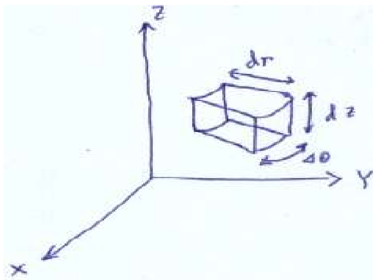
Cylindrical



$z \epsilon\, [z_0\,, \quad z_0\, + \Delta z]$

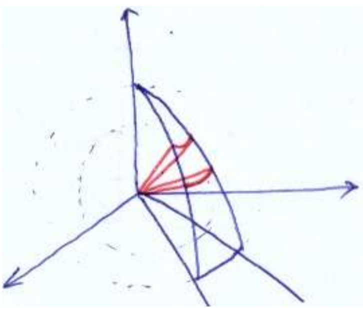
$r \epsilon\, [r_0\,,\ r_0\, + \Delta r]$

$\theta\ \epsilon\ \ [\theta_0\,,\ \theta_0\, + \Delta\theta]$



$dv = r\, dr\, d\theta\, dz$

Spherical

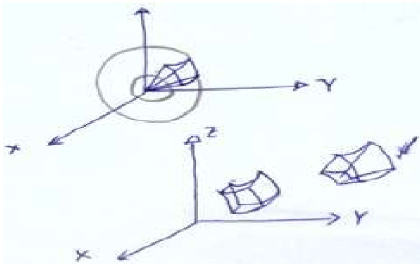


Steps, $\phi\, \epsilon[\phi_0\,,\, \phi_0\, + \Delta\phi]$

$1^{st}\ \theta\ \epsilon\ \ [\theta_0\,,\, \theta_0\, + \Delta\theta]$

$2^{nd}\ \rho = \rho_0$

$3^{rd}\ \ \phi\, \epsilon[\phi_0\,,\, \phi_0\, + \Delta\phi]$

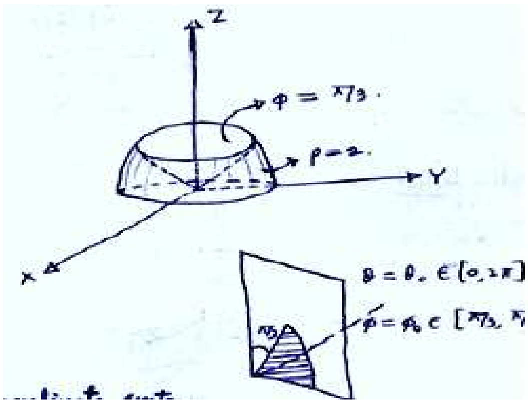


$dv = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$

<u>Example 1:</u> Find the iterated integral in the order of $d\rho \, d\phi \, d\theta$ to evaluate volume of the solid bounded below by the $xy-$ plane, on the sides by the $sph \; \rho = 2$ and above by the cone $\phi = \frac{\pi}{3}$.

(Spherical Coordinate)

$$\int_{\theta=0}^{2\pi} \int_{\emptyset=\frac{\pi}{3}}^{\frac{\pi}{2}} \int_{\rho=0}^{\rho=2} \rho^2 \sin\emptyset \;\; d\rho \; d\emptyset \; d\theta$$



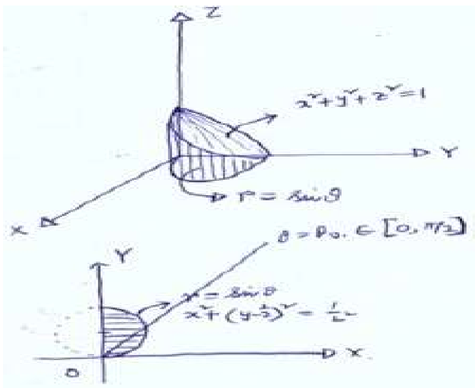
<u>Example 2:</u> Find the integral in the cylindrical coordinate system to evaluate volume of the solid, in the 1st hyper octant that is bounded on the side by $r = \sin\theta$, above by the sphere $x^2 + y^2 + z^2 = 1$

$$\int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{Sin\,\theta} \int_{z=0}^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta$$

$r = \sin\emptyset => r^2 = r\sin\theta$

$= x^2 + y^2 = y$

$= x^2 + (y - \frac{1}{2})^2 = \frac{1}{2^2}$

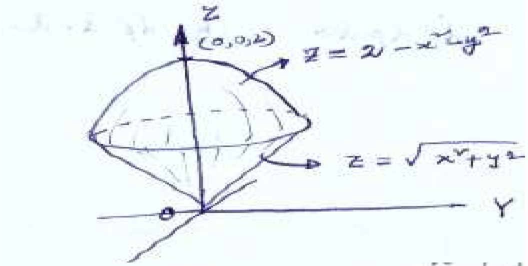


<u>Example 3:</u> (changing order of integration in cylindrical coordinate)

Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. Set up the triple integral in cylindrical coordinates that give the volume of D using the following order of integration.

(a) $dz\ dr\ d\theta$ (b) $dr\ dz\ d\theta$ (c) $d\theta\ dz\ dr$

Solution. (a) $dz\ dr\ d\theta$

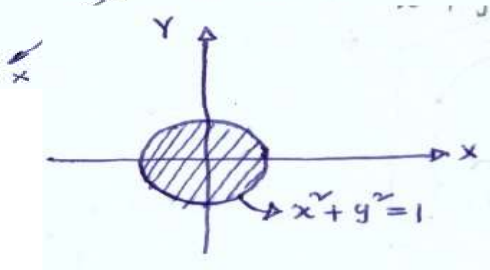


Eliminate; $z = \sqrt{x^2+y^2} = z = 2 - x^2 - y^2$.

⇨ $\left(\sqrt{x^2+y^2}\right)^2 + \sqrt{x^2+y^2}$ - 2= 0

⇨ $\left(\sqrt{x^2+y^2} - 1\right)\left(\sqrt{x^2+y^2} + 2\right) = 0$

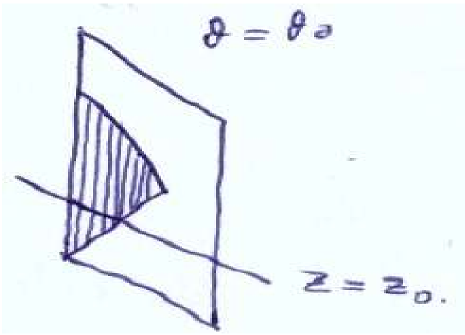
⇨ $x^2 + y^2 = 1$.



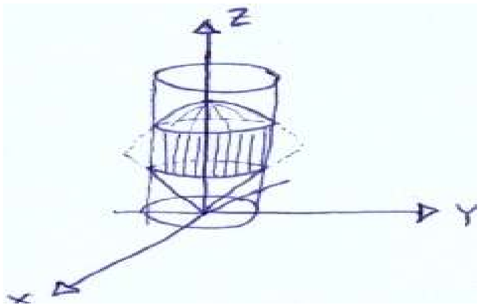
$$\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=r}^{2-r^2} r\ dz\ dr\ d\theta$$

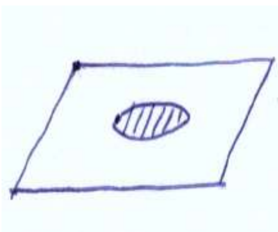
(b) $$\int_{\theta=0}^{2\pi}\int_{z=0}^{1}\int_{r=0}^{z} r\ dr\ dz\ d\theta$$

$$+\int_{\theta=0}^{2\pi}\int_{z=1}^{2}\int_{r=0}^{\sqrt{2-z}} r\ dr\ dz\ d\theta$$



(c) $$\int_{r=0}^{1}\int_{z=r}^{2-r^2}\int_{\theta=0}^{2\pi} r\ d\theta\ dz\ dr$$



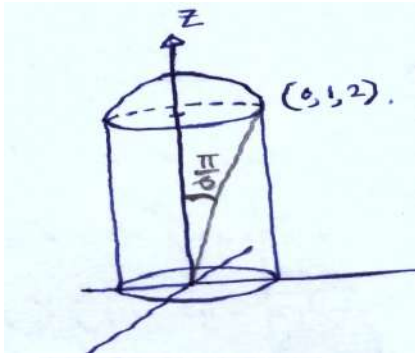


$r = r_0$, $z = z_0$

<u>Example 4:</u> (Changing order of integration in spherical coordinates)

Let D be the region bounded below by the plane $z = 0$, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration:

(a) $d\rho\, d\emptyset\, d\theta$ (b) $d\emptyset\, dp\, d\theta$



Solution: (a): $\int_{\theta=0}^{2\pi}\int_{\emptyset=0}^{\frac{\pi}{6}}\int_{\rho=0}^{2}\rho^2 \sin\emptyset\; d\rho\, d\emptyset\, d\theta$

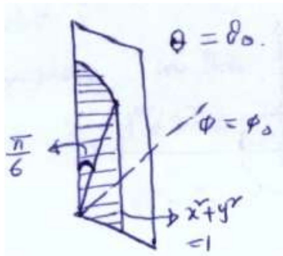
$$+\int_{\theta=0}^{2\pi}\int_{\emptyset=\frac{\pi}{6}}^{\frac{\pi}{2}}\int_{\rho=0}^{cosec\,\emptyset}\rho^2 \sin\emptyset\; d\rho\, d\emptyset\, d\theta$$

$$\left.\begin{aligned} x^2+y^2+z^2 &= 4 \\ x^2+y^2 &= 1 \end{aligned}\right\} z = \sqrt{3} \qquad \therefore z \geq 0$$

$$\begin{cases} x^2+y^2 = 1 \\ z = \sqrt{3} \end{cases}$$

$$\emptyset = \angle\left((0,0,1),(0,1,\sqrt{3})\right)$$

$$= \cos^{-1}\frac{\sqrt{3}}{\sqrt{1}\sqrt{4}} = \frac{\pi}{6}$$



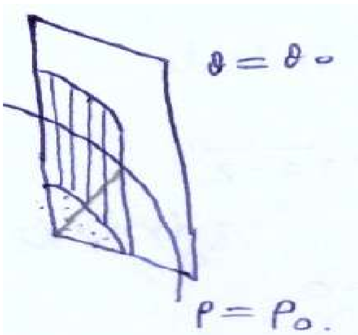
$x^2 + y^2 = 1$

$$\rho^2 \sin^2\emptyset = 1$$

$$\rho = cosec\,\emptyset$$

(b). $\int_{\theta=0}^{2\pi}\int_{\rho=1}^{2}\int_{\rho=0}^{\sin^{-1}\frac{1}{\rho}}\rho^2 \sin\emptyset\; d\emptyset\, d\rho\, d\theta$

$$+\int_{\theta=0}^{2\pi}\int_{\rho=0}^{1}\int_{\emptyset=0}^{\frac{\pi}{2}} d\emptyset\, d\rho\, d\theta$$



(1) Double & iterated integral / repeated integral $\int\int f\ dA$, $\int_x \int_y f\ dy\ dx$, $\int_y \int_x f\ dx\ dy$ (Fubini's Theorem).

(2) Rules in integrations, properties.

Assignment 1

(1) Find Volume of the region bounded above by the surface $z = y - 2x^2$ and below by the square $|x| + |y| = 1$.

Sol.: Vol $= \int_1^0 \int_{-x-1}^{x+1} (y - 2x^2)\ dy\ dx + \int_1^0 \int_{x-1}^{1-x} (y - 2x^2)\ dy\ dx = -\frac{2}{3}$

(2) Evaluate $\iint_R xy\ \ dA$ where r is the region bounded by the lines $y = x,\ y = 2x,\ \&\ x + y = 2$.

Sol.: $\iint_R xy\ \ dA = \int_0^{2/3} \int_x^{2x} xy\ dy\ dx + \int_{2/3}^1 \int_x^{2-x} xy\ dy\ dx = \frac{13}{81}$

(3) Find the volume of the solid cut from the square col. $|x| + |y| \leq 1$ by the planes $z = 0\ \&\ \ 3x + z = 3$.

Sol.: Vol $= \int_{-1}^0 \int_{-x-1}^{x+1} (3 - 3x)\ dy\ dx + \int_0^1 \int_{x-1}^{1-x} (3 - 3x)\ dy\ dx = 6$

(4) Find $\iint_R \sqrt{4 - x^2}\ \ dA$ where R is the sector cut from the disk $x^2 + y^2 \leq 4$ by the rays $\theta = {}^{\pi}/_{6}\ \&\ \theta = {}^{\pi}/_{2}$.

Sol.: $= \frac{\pi}{6}$, $\quad y = \frac{x}{\sqrt{3}}$; $\qquad \iint_R \sqrt{4 - x^2}\ \ dA = \int_0^{\sqrt{3}} \int_{x/\sqrt{3}}^{\sqrt{4-x^2}} \sqrt{4 - x^2}\ \ dy\ dx = \frac{20\sqrt{3}}{9}$.

(5) Evaluate the integral $\int_0^2 (\tan^{-1} \pi x - \tan^{-1} x)\ dx$ by converting it to a double integral**.**

Sol: $\int_0^2 \int_x^{\pi x} \frac{1}{1+y^2}\ dy\ dx = \ \int_0^2 \int_{y/\pi}^{y} \frac{1}{1+y^2}\ dx\ dy + \ \int_2^{2\pi} \int_{y/\pi}^{2} \frac{1}{1+y^2}\ dx\ dy.$

$= 2\tan^{-1} 2\pi - 2\tan^{-1} 2 - \frac{1}{2\pi}\quad \ln{(1 + 4\pi^2)} + \frac{ln5}{2}$

(6) A solid right (noncircular) cylinder has its base R in the $xy -$ plane and is bounded above by the parabolid $z = x^2 + y^2$. The cylinder's Volume is

$$\int_0^1 \int_0^y (x^2+y^2)dx\,dy + \int_1^2 \int_0^{2-y} (x^2+y^2)dx\,dy$$

Sketch the base and region R and express the cylinder's Volume as a single iterated integral with order of integral reversed. Then evaluate the integral.

Sol.: $\int_0^1 \int_0^{2-x} (x^2+y^2)dy\,dx = 4/3$

(7) Change the order of integration and hence evaluate $\int_0^2 \int_0^{4-x^2} \frac{x^{2y}}{4-y} dx\,dy$ & $\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dy\,dx}{y^4+1}$

Sol.: $\int_0^8 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y} dx\,dy = \frac{e^8-1}{8}$; $\int_0^2 \int_0^{y^3} \frac{1}{1+y^4} dx\,dy = \frac{ln17}{4}$

(8) How would you evaluate the double integration of constant function $f(x,\ y)$ over the region R in the $x, y -$ plane enclosed by the hincye with vertices $(0,1), (2,0), \& (1,2)$;

Sol: $\int_0^1 \int_{2-2y}^{2-y/2} f(x\,,y)dx\,dy + \int_1^2 \int_{y-1}^{2-y/2} f(x\,,y)dx\,dy$

Gradient, Divergence and Curl:

We will start with the concept of Point function:

A variable quantity whose value at any point in the region of the space deptnds upon the positon of the point, is called a point function.

Types of point functions are

1)- **scaler point function**:if to the each point P(x,y,z) of region Rin the space there corresponding a unique scaler f(P), then f is called a scaler point function. We also call it a scalar field. Hence, a scalar field is a function points in space to points real numbers. For example Temperature distribution in a heated body, density of a body and the potiential due to gravity are scaler point function or scalar field.

2)-**Vector point function**: if to each point P(x,y,z) of the region R in the space there corresponds a unique vector f(P), then f is called a vector point function. This is also called as vector field. Hence, vector field is a function from points in space to vectors in space. The velocity of a moving fluid, gravitational force are the examples of the vector point function.

Gradient of a Scaler function:

Definition of gradient: If φ(x,y,z) be a scaler point function then $\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$ is called the gradient of the scaler function φ.

It is denoted by grad φ or ∇ φ.

Thus, $$\text{grad } \varphi = \hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$$

$$\text{grad } \varphi =(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z})\ \varphi(x,y,z)$$

$$\text{grad } \varphi = \nabla\ \varphi$$

Hence, We recognize the gradient as the generalization of the derivative.

Remark: From Vector Calculus we know that the partial derivatives $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}$ give the rates of change of f(x, y, z) in the directions of $\vec{i},\vec{j},\vec{k},$ respectively. Hence, if the gradient of t at a point *P* is not zero, i.e., $\vec{\nabla} f(P) = Grad\ f(P) \neq 0,$ then it is a vector in the direction of maximum increase of *f* at *P*.

Properties of gradient

(i) Gradient of a scalar quantity is a Vector quantity.

(ii) Magnitude of that vector quantity is equal to the Maximum rate of change of that scalar quantity.

(iii) Change of scalar quantity does not depend only on the coordinate of the point, but also on the direction along which the change is shown.

Geometrical meaning of the Gradient

As we know that a surface is all points in $(x, y, z) \in \mathbb{R}^3$ that verifies $f(x, y, z) = c,$ for some smooth scalar field f and constant c.

Example: Consider the unit sphere in $\mathbb{R}^3$, S^2 :

$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$

From the following, we can understand the geometrical meaning of gradient:

Definition of level surface: If a surface φ(x,y,z) =c passes through a point and the value of the function at each point of the surface is the same as at P, then such a surface is called a level surface through P. For example, if φ(x,y,z) is represents potiential at the point P,then equipotential surface φ(x,y,z)=c is a *level surface.* Two level surface cannot intersect.

Let a level surfce pass through the point P at which the value of the function is φ. Consider another level surface passing through Q,where the value of the function is φ+dφ.

Let $\bar{r}$ and $\bar{r}+\delta\bar{r}$ be the position vector of P an Q then $\overrightarrow{PQ} = \delta\bar{r}$

$$\nabla \varphi.d\bar{r} = (\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}).(\hat{\imath}\, dx + \hat{\jmath} dy + \hat{k}\, dz)$$

$$= \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{yx}dy + \frac{\partial\varphi}{\partial z}dz = d\varphi \qquad \text{….. ………………..(1)}$$

If Q lies on the level surface of P, then dφ=0

Equation (1) beecomes $\nabla \varphi.d\bar{r}=0$. Then $\bar{\nabla}\varphi$ is $\perp$ to $d\bar{r}$ (tangent).

Hence, ∇ φ is normal to the surface φ(x,y,z)=c.

Let $\nabla \varphi = |\nabla \varphi|\hat{N}$,where $\hat{N}$ is a unit normal veator. Let δn be the perpendicular distance between two level surface through P and R. then the rate of change of φ in the direction of the normal to the surface through P is $\frac{\partial\varphi}{\partial n}$.

$$\frac{d\varphi}{dn} = \lim_{\delta n \to 0}\frac{\partial\varphi}{\partial n} = \lim_{\delta n \to 0}\frac{\nabla\varphi.d\bar{r}}{\partial n}$$

$$= \lim_{\delta n \to 0}\frac{|\nabla\varphi|\hat{N}.d\bar{r}}{\partial n}$$

$$=\lim_{\delta n\to 0}\frac{|\nabla\varphi|\delta n}{\partial n} = |\nabla\varphi|$$

$$|\nabla\ \varphi| = \frac{\partial\varphi}{\partial n}.$$

Hence, gradient φ is a vector normat to the surface φ =c and has a magnitude equal to the rate of change of φ along this normal. Hence, we have the following theorem as geometrical interpritation of gradient:

Theorem: Let *f* be differentiable and S = $S=\{(x,y,z)\in \mathbb{R}^3 : f(x,y,z)=c\}$ be a surface. Then the gradient of *f*, ∇f at a point P of the surface S is a normal vector to S at P (provided $\nabla f(P)\neq 0$).

Engineers use the gradient vector in many physical laws such as:

1. Electric Field (E) and Electric Potential (V): E=-Grad.V
2. Heat Flow (q) and Temperature (T): q(x,y,z)=-kGradT(x,y,z), k is constant
3. Force Field (F) and Potential Energy (U): F(x,y,z)=-GRadU(x,y,z)

Divergence of a Vector function:

The divergence of a vector point function $\bar{F}$ is denoted by div F and is defined as below.

Let $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$

Div $=\vec{F} = \vec{\nabla}.\vec{F}$

$$= (\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}).(F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k})$$

$$=\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

It is evident that div $\vec{F}$ is scaler function.

Physical Intuition

- Divergence (div) is “flux density”—the amount of flux entering or leaving a point. We can think of it as the rate of flux expansion (positive divergence) or flux contraction (negative divergence).
- So, divergence is just the net flux per unit volume, or “flux density”, just like regular density is mass per unit volume (of course, we don’t know about “negative” density). Imagine a tiny cube—flux can be coming in on some sides, leaving on others, and we combine all effects to figure out if the total flux is entering or leaving.
- The bigger the flux density (positive or negative), the stronger the flux source or sink. A div of zero means there’s no net flux change in side the region. i.e.,

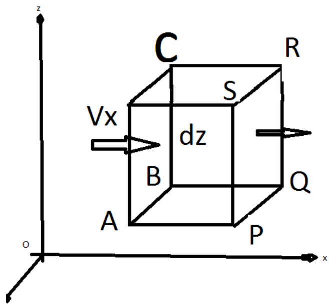
$$Divergence = \frac{Flux}{Volume}$$

- Imagine a fluid, with the vector field representing the velocity of the fluid at each point in space. Divergence measures the net flow of fluid out of (i.e., diverging from) a given point. If fluid is instead flowing into that point, the divergence will be negative.

  A point or region with positive divergence is often referred to as a "source" (of fluid, or whatever the field is describing), while a point or region with negative divergence is a "sink".

Geometrical Interpretation of Divergence:

Let us consider the case of a fluid flow. Consider aSmall rectangle pallellopiped of dimension dx,dy,dz parallel to x,y and z axes resprctively.



Let $\vec{V} = V_x\hat{\imath} + V_y\hat{\jmath} + V_z\hat{k}$ be the velocity of the fluid at P(x,y,z).

Mass of the fluid flowing in through the face ABCD in unit time

=Velocity ×Aear of the face

=V_x(dy dz)

Mass of the fluid flowing out across the face PQRS in unit time

= V_x(x+dx)(dy dz)

=(V_x+ $\frac{\partial V_x}{\partial x}$dx)(dy dz)

Net decrease in mass of fluid in the parallelopiped corresponding to the flow along x-axis per unit time

= V_x(dy dz)- (V_x+ $\frac{\partial V_x}{\partial x}$dx)(dy dz)

=- $\frac{\partial V_x}{\partial x}$ dxdy dz (minus sign shows decrease)

Similarly, the decrease in mass of fluid to the flow y-axis = $\frac{\partial V_y}{\partial y}$ dx dy dz

And the decrease in mass of fluid to the flow z-axis = $\frac{\partial V_z}{\partial z}$ dx dy dz

Total decrease of the amount of the fluid per unit time =$(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z})$ dx dy dz

Thus the rate of the loss of fluid per unit volume = $\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

$$=(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}).(V_x\hat{\imath} + V_y\hat{\jmath} + V_z\hat{k})$$

$$= \overline{\nabla}.\overline{V} = \text{div } \overline{V}\ .$$

Remark: If the fluid is incompressible, there can be no gain or no loss in the fluid in the volume element.

Hence

$$\text{div } \overline{V} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and V is called a *solenoidal vector function*.

Equation (1) is also called the equation of the **continuity or conservation of mass**.

CURL of a Vector

The curl of a vector point function s definrd as below

$$\text{Curl} \times \vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k})$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \hat{\imath}(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}) - \hat{\jmath}(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}) + \hat{k}(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})$$

Curl $\vec{F}$ is the vector quantity.

Physical Meaning of the CURL:

As we know that $\vec{V} = \vec{\omega} \times \vec{r}$, where ω is the angular velocity, $\vec{V}$ is the linear velocity and $\vec{r}$ is the position vector of a point on the rotating body.

Curl $\vec{V} = \vec{\nabla} \times \vec{V}$

$= \vec{\nabla} \times (\vec{\omega} \times \vec{r}) = \vec{\nabla} \times [(\omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}) \times (x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k})]$

$= \vec{\nabla} \times \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \vec{\nabla} \times [(\omega_2 z - \omega_3 y)\,\hat{\imath} + (\omega_1 z - \omega_3 x)\,\hat{\jmath} + (\omega_1 y - \omega_2 x)\,\hat{k}]$

$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times [(\omega_2 z - \omega_3 y)\,\hat{\imath} + (\omega_1 z - \omega_3 x)\,\hat{\jmath} + (\omega_1 y - \omega_2 x)\,\hat{k}]$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (\omega_1 z - \omega_3 x) & (\omega_1 y - \omega_2 x) \end{vmatrix}$$

$=(\omega_1+\omega_2)\hat{\imath} - (-\omega_2 - \omega_2)\hat{\jmath} + (\omega_3 + \omega_3)\hat{k}$

$= 2(\omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}) = 2\,\omega.$

Remark: Curl $\vec{V} = 2\,\omega$ which shows the curl of a vector field is connected with the rotational properties of the vector field and the justifies the name rotation used for curl.

If Curl $\vec{V}$=0, the field is termed as **irrotational.**

Lemma -1: Gradient fields are irrotationals. That is, if $F = \nabla f$, for some smoothscalar field f, then curl F = 0.

Similarly, some identites related to curl, divergence and Gradient:

for vectors functions , a and b, and scaler functions U are given below:

i)- div curl = $\nabla \cdot \nabla \times$a =0

ii)- $\nabla \cdot \boldsymbol{U}$a = $(\nabla \boldsymbol{U})$a + U($\nabla \cdot$ a)

= (grad U)a +U(div a)

iii)- $\nabla \times$Ua= U$\nabla \times$a +$(\nabla \boldsymbol{U}) \times$ a

iv)- div a $\times$ b = (curl a).b - a.(curl b)

v)- curl a $\times$ b =$\nabla \times$(a $\times$ b)

=(∇. b)a –(∇. a)b + [b .∇] a - [a .∇] b

Where [a .∇] = $\left(\boldsymbol{a_x}\frac{\partial}{\partial x} + \boldsymbol{a_y}\frac{\partial}{\partial y} + \boldsymbol{a_z}\frac{\partial}{\partial z}\right)$

vi)- curl (curl a) = grad (div a) - ∇^2 a where ∇^2 a = $\nabla^2 \boldsymbol{a_x}$+ $\nabla^2 \boldsymbol{a_y}$+ $\nabla^2 \boldsymbol{a_z}$

Assignment 2

1)- If $\bar{r}$ =(x $\hat{\imath}$ + y $\hat{\jmath}$ + z $\hat{k}$) then show that

i)- grad r =$\frac{\bar{r}}{r}$ ii)- grad($\frac{1}{r}$) = $\frac{\bar{r}}{\mathbf{r^3}}$

2) if u= x+y+z, v=x^2+ y^2 +z^2, w=yz +zx+xy prove that the grad u, grad v, grad w are coplanar vectors.

3) if u= x^2+ y^2 +z^2, and $\vec{r}$ =(x $\hat{\imath}$ + y $\hat{\jmath}$ + z $\hat{k}$), then find div (u $\vec{r}$) in terms of u.

4) find the value of n for which the vector $r^n\vec{r}$ is solenoidal, where $\vec{r}$ =(x $\hat{\imath}$ + y $\hat{\jmath}$ + z $\hat{k}$).

5) show that div (grad r^n) =n (n+1) r^{n-2}, where $\vec{r}$ =(x $\hat{\imath}$ + y $\hat{\jmath}$ + z $\hat{k}$).

Hence show that, $\Delta^2(\frac{1}{r})$ = 0.

6) find the divergence and curl of $\bar{v}$= (x y z) $\hat{\imath}$ + (3 x^2 y) $\hat{\jmath}$ + (x z^2 –y^2 z) $\hat{k}$ at (2,-1,1).

7) Prove that (y^2 - z^2 +3xyz -2x) $\hat{\imath}$ + (3xz +2xy) $\hat{\jmath}$ + (3xy –2xz +2z) $\hat{k}$ is both solenoidal and irrotational.

8) $\vec{F}$=(x^2 – y^2 + x) $\hat{\imath}$ - (2xy + y)$\hat{\jmath}$. Is this field is irrotational ? if so, find its scaler potiential.

9) P**rove that**, the vactor field $\vec{F}$=$\frac{\bar{r}}{|\mathbf{r^3}|}$ is irrotational as well as solenoidal. Find the scaler potential.

10) For a solenoidal vector $\vec{F}$, show that

curl (curl (curl $\vec{F}$))) = $\nabla^4\vec{F}$.

11. The gravitational field $\vec{p}$ is the force between two particles at points $P_0 = (x_0, y_0, z_0)$ and $P = (x, y, z)$. It is defined by

$$\vec{p} = -\frac{c}{r^3}\vec{r} = -\frac{c}{r^3}[x - x_0, y - y_0, z - z_0],$$

where $\vec{r} = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$ and it is irrotational field as the scalar field $f(x, y, z) = \frac{c}{r}$ is a potential for it. Verify it.

Surface Area
of a closed and bounded region on the surface
of $f(x, y, z) = c$ with the help of projection on a plane

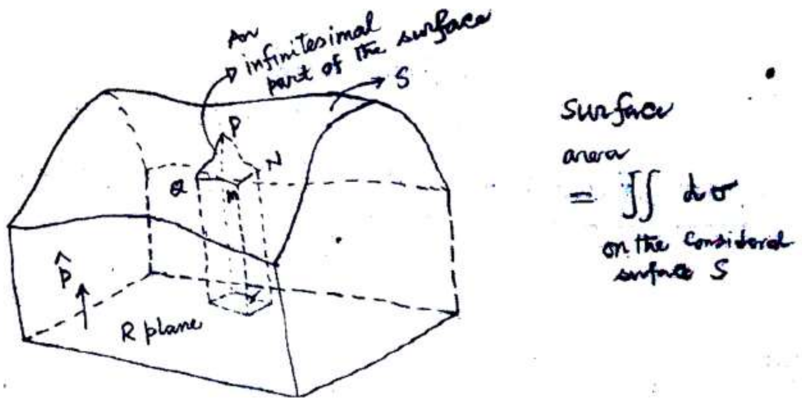
We often come across to deal with some surfaces that are presented by $(x, y, z) = c$, a constant, for instance, the equipotential surfaces in the gravitational or electrical or magnetic fields.

In this lecture we intend to evaluate the surface area of a closed and bounded region on the surface of $f(x, y, z) = c$. In due course we shall project the surface under consideration on a flat region or plane.

Let this plane be *R* and $\hat{p}$ be the unit normal on *R.* We assume that

(i) The surface is smooth; if not entirely, at least the patch of the surface that we consider to evaluate the area is smooth. Let this patch be *S*. Mathematically, this condition presumes that on *S,* the function *f* is differentiable and $\vec{\nabla} f$ is nonnull and continuous.

(ii) When we take perpendicular projection of *S* on *R*, we never observe that the surface folds back over itself. Mathematically, $\vec{\nabla} f . \hat{p} \neq 0$ on *S.*



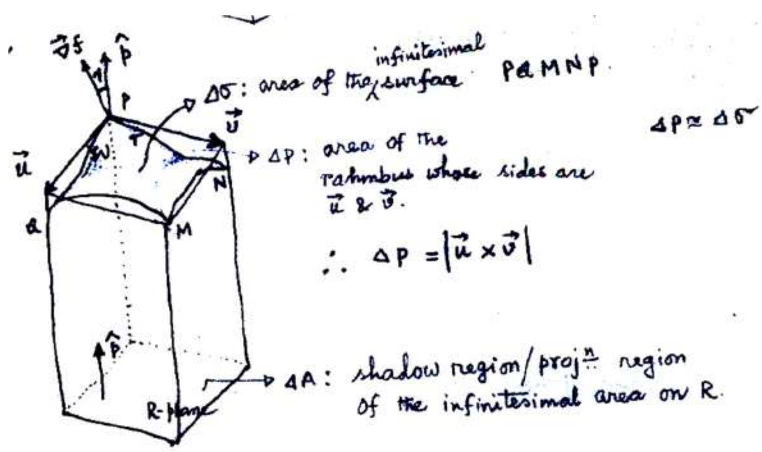


Fig : A surface and its projection on R-plane.

We approximate the area of the infinitesimal surface by the tangent surface at *p*.

As $\vec{\nabla} f$ is perpendicular to the tangent plane $\vec{\nabla} f$ *is parallel to* $\vec{u} \times \vec{v}$,

when $\vec{u}$ is tangent to the curve PWQ and $\vec{v}$ is tangent to the curve PTN.

The value $|\vec{u}\ x\ \vec{v}|$ measures area of the rhombus whose sides are $\vec{u}$ & $\vec{v}$. Here we let $\Delta\sigma = |\vec{u} \times \vec{v}|$.

As $\vec{u} \times \vec{v}$ has the direction perpendicular to the rhombus, the dot product $|\vec{u} \times \vec{v}|.\hat{p}$ measures area of the region perpendicular to $\hat{p}$.

Hence $\Delta A = \left||\vec{u} \times \vec{v}.\hat{p}|\right|$ (absolute value of mod of $|\vec{u} \times \vec{v}|.\hat{p}$)

$= \left||\vec{u} \times \vec{v}|\ |\hat{p}|\ cos\ r\right|$, $\quad r = <(\hat{p}, \vec{\nabla} f)$

$= \Delta P\ |cos\ r|$

$\Rightarrow\ \Delta P = \frac{\Delta A}{|cos\ r|}$ $\quad$ (*cos r ≠0,* due to assumption (ii))

As we approximate $\Delta\sigma$ by the value of ΔP,

$$\Delta\sigma \approx \Delta P = \frac{\Delta A}{|Cos\ r|}$$

Since $r = <(\hat{p}, \vec{\nabla} f)$, above $\Rightarrow\ |\vec{\nabla} f.\hat{p}| = |\vec{\nabla} f|\ Cos\ r$

$$\Rightarrow\ cos\ r = \frac{|\vec{\nabla} f\ .\ \hat{p}|}{|\vec{\nabla} f|}$$

Therefore $\quad \Delta\sigma = \frac{\Delta A|\vec{\nabla} f|}{|\vec{\nabla} f\ .\ \hat{p}|}$

Hence the required surface area $= \iint\limits_{\substack{projection\ on\\ R\ plane\ region}} \frac{|\vec{\nabla} f| dA}{|\vec{\nabla} f\ .\ \hat{p}|}$

Surface integrals

If S be the shadow region of a smooth surface Ω, defined by $f(x, y, z) = c$ on the plane R whose unit normal is $\hat{p}$, and *g* be a continuous function over the surface Ω, then the integral of *g* over Ω, is the integral

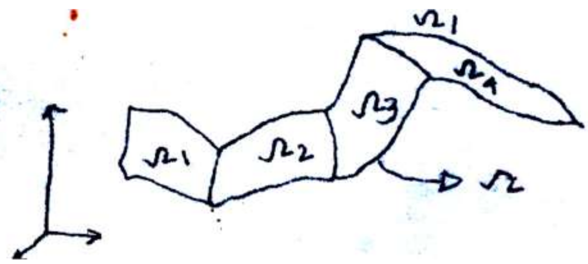
$$\iint_{\Omega} g d\,\sigma = \iint_{S} g(x, y, z) \frac{|\vec{\nabla} f|}{|\vec{\nabla} f\ .\ \hat{p}|} dA$$

Here we assume that *R* be such a plane that $\vec{\nabla} f.\hat{p} \neq 0$.

PROPERTY

If Ω being partitioned by smooth curves into a finite number of nonoverlapping patches $\Omega_1, \Omega_2, \ldots .\Omega_n$

then $\iint_{\Omega} g d\sigma = \iint_{\Omega_1} g d\sigma + \iint_{\Omega_2} g d\sigma + \cdots \ldots \ldots \ldots . \iint_{\Omega_n} g d\sigma$

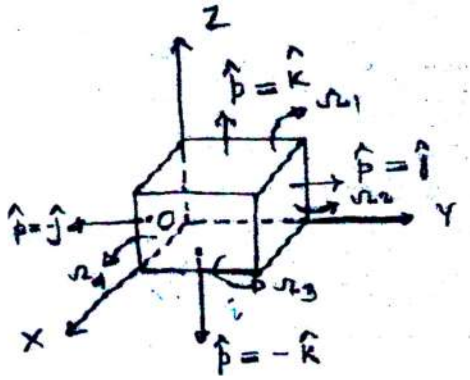


Example 1 Integrate $g(x,y,z) = x^2y^2z$ over the surface of the cube out from the first octant by the plane $x=1$, $y=1$ and $z=1$.

Solution Ω = six faces of the unit cube

$= \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 + \Omega_5 + \Omega_6$

For



Ω_1 : $\hat{p} = \hat{k}$, equation : $z=1$

For Ω_2 : $y=1$, $\hat{p} = \hat{j}$

For Ω_3 : $z=0$, $\hat{p} = -\hat{k}$

For Ω_4 : $y=0$, $\hat{p} = -\hat{j}$

For Ω_5 : $x=0$, $\hat{p} = -\hat{\imath}$

For Ω_6 : $x=1$, $\hat{p} = \hat{\imath}$

$$\therefore \iint_{\Omega} g d\sigma = \iint_{\Omega_1} g d\sigma + \iint_{\Omega_2} g d\sigma + \iint_{\Omega_3} g d\sigma + \iint_{\Omega_4} g d\sigma + \iint_{\Omega_5} g d\sigma + \iint_{\Omega_6} g d\sigma$$

Here $\iint_{\Omega_1} g d\sigma = \iint_{\Omega_1} g(x,y,z) \frac{|\vec{\nabla} f|}{|\vec{\nabla} f . \hat{p}|} d_A$, where $f(x,y,z) \equiv z - 1 = 0$

$$= \int_{x=0}^{1} \int_{y=0}^{1} x^2 y^2 1^2 \frac{|\hat{k}|}{|\hat{k} . \hat{k}|} dx\, dy = \frac{1}{9}$$

Similarly $\iint_{\Omega_i} g d\sigma = \frac{1}{9}$ for i = 2,3,4,5,6

$$\therefore \iint_{\Omega} g d\sigma = \frac{6}{9} = \frac{2}{3}.$$

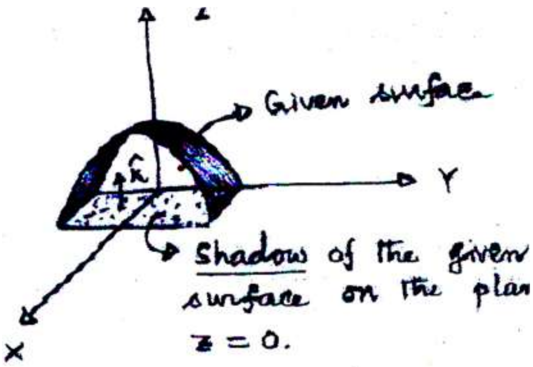
Example 2 Find the surface area of the surface cut from the cylinder $y^2 + z^2 = 1$, $z \geq 0$ by the planes $x = 0$ and $x = 1$.

Solution The required surface area

$$= \iint_{\Omega} d\sigma, \qquad \Omega: \begin{cases} y^2 + z^2 = 1, \ z \geq 0 \\ 0 \leq x \leq 1 \end{cases}$$

$$= \iint_{S} \frac{|\vec{\nabla} f|}{|\vec{\nabla} f.\ \hat{p}|} d_A,$$

where $f(x, y, z) = y^2 + z^2 - 1 = 0$, $z \geq 0$ & $\hat{p} = \hat{k}$



$$= \int_{x=0}^{1} \int_{y=-1}^{1} \frac{|2y\hat{j} + 2z\hat{k}|}{|(2y\hat{j} + 2z\hat{k}).\hat{k}|} dxdy$$

$$= \int_{x=0}^{1} \int_{y=-1}^{1} \frac{z\sqrt{y^2 + z^2}}{z|z|} dxdy$$

$$= \int_{x=0}^{1} \left(\int_{y=-1}^{1} \frac{1}{\sqrt{1 - y^2}} dy \right) dx$$

$$= \int_{x=0}^{1} [sin^{-1} y]_{-1}^{1} dx$$

$$= 2\ sin^{-1}\ 1 \qquad \text{sq. unit}$$

Mathematically this shadow region is

$$S: \begin{cases} 0 \leq x \leq 1 \\ -1 \leq y \leq 1 \end{cases}$$

References

1. Apostol, T., Mathematical Analysis, Addison-Wesley, 2nd Edition, 1974
2. Thomas, George Brinton, Maurice D. Weir, Joel Hass, and Frank R. Giordano. Thomas' calculus. Addison-Wesley, 2005.
3. Kreyszig, Erwin, Advanced engineering mathematics, 12th Edition, Wiley.

Module-7

Trigonometry

Lectures required -02

1. Generation of Angles:

In plane geometry, an angle is usually said to be formed by two rays (half-lines), radiating from the same point called **vertex**. All angles – acute, obtuse or reflex, are positive and less than 360°.

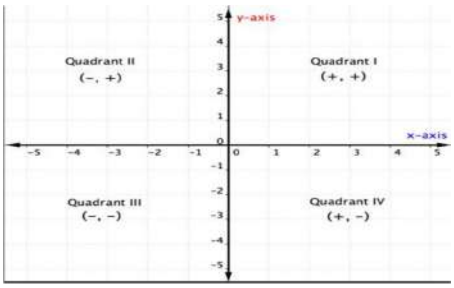
An angle, in trigonometry, is formed by the rotation of a ray in a plane, around the end point.

The revolving ray is called **generating line** and its initial and final position are respectively called **initial side (arm)** and **terminal side (arm)**. The end point is called **vertex**.

Counter clockwise movement of generating line generates **positive angle** while clockwise generates **negative angle**.

2. Useful Terminology:

1. Quadrants: Two perpendicular lines divide the plane into four different Quadrants as shown in figure.



2. Quadrantal and Co-terminal angles:

An angle is said to be in the quadrant, in which the terminal side of the angle, when placed in standard position, is located. If the terminal side coincides with one of the axes, then the angle is called a **quadrantal angle**. Any multiple of 90° are all quadrantal angles.

If the initial and terminal sides coincide the angles are said to be **co-terminal**. If any integral multiple of 360° is added or subtracted for an angle ϴ, then all such angles are co-terminal.

If $0° < \theta < 90°$, ϴ is in first quadrant
If $90° < \theta < 180°$, ϴ is in second quadrant
If $180° < \theta < 270°$, ϴ is in third quadrant
If $270° < \theta < 360°$, ϴ is in fourth quadrant

3. Measurement of angles: different systems°

There are three principal systems for measuring angles:

4. Sexagesimal or English Systems or Degree System:

Most commonly used system of measuring angle.

5. Circular System or Radian Measure (or system):

In advanced mathematics, the most convenient system for the measurement of angles is the circular or radian measures.

1. Definition of π:

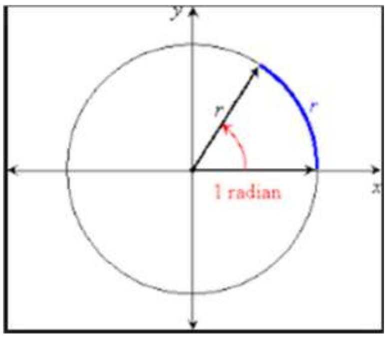
For any circle, the ratio of the circumference to its diameter is the same, i.e., this ratio is independent of the size of the circle.

$$\pi = \frac{\text{circumference of circle}}{\text{diameter of circle}}$$

2. Numerical value of π:

Approximate value is 22/7. This is correct to 2 decimal places. The fraction 355/113 is correct to 6 decimal places. Approximate value of π, correct upto 10 decimal places, is 3.1415926536.

A ***radian*** is the measure of an angle at the center of a circle subtended by an arc equal in length to the radius of the circle.



Radian is a constant angle. One radian = $\frac{180°}{\pi}$

One radian is denoted by 1^c. Thus π^c stands for π radian. Generally, π radian is just denoted by π.

$180° = \pi$, $90° = \frac{\pi}{2}$, $45° = \frac{\pi}{4}$, $60° = \frac{\pi}{3}$, etc.

1 radian = $\frac{180°}{\pi}$ = 180° x 0.31831 = 57.2958° = 57° 17' 44.81'' approx. = 206265 seconds approx.

3. Relation between three systems:

For the same angle let the measure be D in degree, G in grades and C in radian.

180° = π radians.

D degrees = $\frac{\pi D}{180}$ radian …. (i)

100 grades = 1 right angle = 90° = $\frac{\pi}{2}$

1 grade = $\frac{\pi}{200}$radian, G grades = $\frac{\pi G}{200}$ ….(ii)

From equation (i) and (ii),
$\frac{\pi D}{180} = \frac{\pi G}{200} = C$

Hence, $\frac{\mathbf{D}}{\mathbf{90}} = \frac{\mathbf{G}}{\mathbf{100}} = \frac{\mathbf{2C}}{\boldsymbol{\pi}}$

$$\text{Magnitude of angle in circular measure} = \frac{\textbf{Length of corresponding arc}}{\textbf{radius}}$$

$$\Theta = \frac{\mathbf{L}}{\mathbf{R}}$$

<u>Examples</u>:

Q1) Express 73°25'30'' in centesimal measure.
Solution: 25'30'' = $\frac{25.5°}{\mathbf{60}}$ = 0.425°,
Hence, 73° 25'30'' = 73.425° = $\frac{73.425°}{\mathbf{90}}$ rt angles = 81.5833^g = 81^g 58'33''

Q2) The angles of a triangle are in AP and the ratio of number of radian in the greatest angle is to the number of degree in the least one as π : 60. Find the angles in degree.
Solution: Let the number of degree in the angle be A-B, A, A+B, then A= 60°
Again the greatest angle = (A+B)° = (A+B) * $\frac{\pi}{\mathbf{180}}$
The least angle = (A-B)°

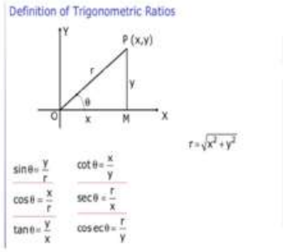
Hence, $\frac{\frac{(A+B)\pi}{180}}{A-B} = \frac{\pi}{60}$

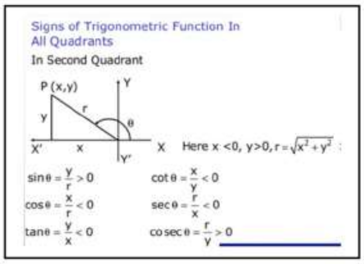
A+B = 3(A-B), A= 2B

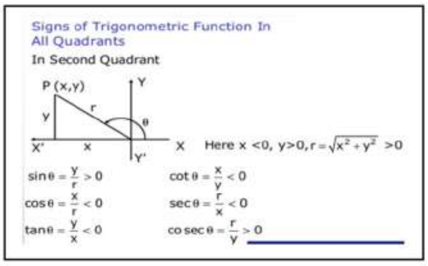
B= 30°. Hence, 30°, 60° and 90°.

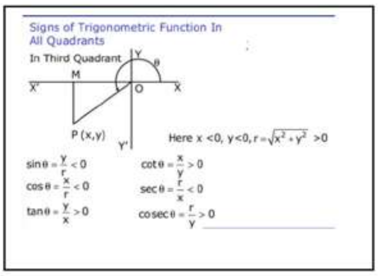
<u>TRIGNOMETRICAL RATIO and FUNCTION</u>

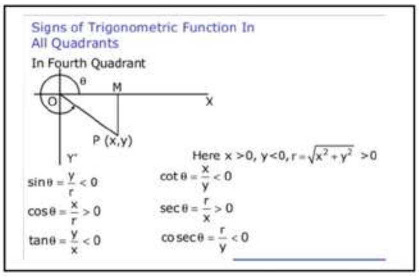
A class of real functions defined in terms of ratios of sides of a right angled triangle are known as trigonometrical function. Sign of trigonometrical ratios of an angle with reference to a right angled triangle will depend on the quadrant in which the terminal side is located.







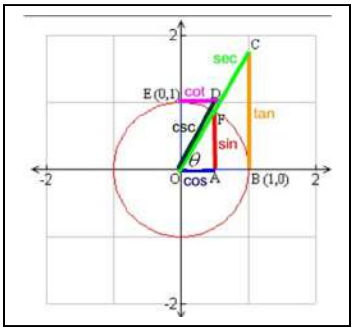




<u>Some Basic Identities:</u>

- $\sin\theta\times\mathrm{cosec}\,\theta=1$; $\theta\neq n\pi, n\in 1$
- $\cos\theta\times\sec\theta=1$; $\theta\neq(2n+1)\frac{\pi}{2}, n\in I$
- $\tan\theta\times\cot\theta=1$; $\theta\neq(2n+1)\frac{\pi}{2}$; $\theta\neq n\pi, n\in I$
- $\sin^2\theta+\cos^2\theta=1$ • $\sec^2\theta-\tan^2\theta=1$
- $\mathrm{cosec}^2\theta-\cot^2$
- $\tan\theta=\frac{\sin\theta}{\cos\theta}$; $\theta\neq(2n+1)\frac{\pi}{2}, n\in I$
- $\cot\theta=\frac{\cos\theta}{\sin\theta}$; $\theta\neq n\pi, n\in I$

1. <u>Definition of Trigonometrical Function unit Circle</u>:



The trigonometrical ratio can be represented through a unit circle.

Take any point on the circle at a arc length of Θ.

$\llcorner$DOB = = $\frac{l}{r}$ = $\frac{\theta}{1}$ = Θ(in radian)

OD = 1, OA = x , AD = y.

$\sin\Theta = \frac{DA}{OA} = \frac{y}{1} = y$

$\cos\Theta = \frac{OA}{OD} = \frac{x}{1} = x$

$\tan\Theta = \frac{y}{x}$, $\cot\Theta = \frac{x}{y}$, $\mathrm{cosec}\Theta = \frac{1}{y}$ and $\sec\Theta = \frac{1}{x}$

From the diagram, $x^2 + y^2 = 1$, all equation in section 1 can be derived from this.

For example, $\sin^2\Theta + \cos^2\Theta = 1$

$$1 + \tan^2\Theta = \frac{x^2+y^2}{y^2} = \sec^2\Theta$$

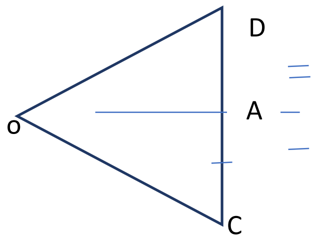
$$1 + \cot^2\Theta = \frac{x^2+y^2}{x^2} = \text{cosec}^2\Theta$$

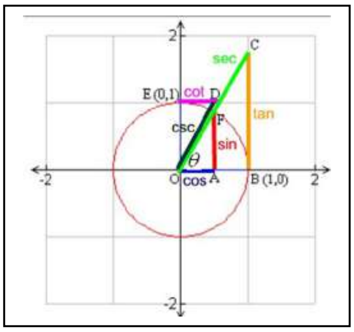
The formulae, mentioned above, which are true for all (admissible) values of Θ, are called **trigonometrical identities.**

Values of Circular functions for some standard angles:

To find the value of trigonometry ratios for $\Theta = \frac{\pi}{6}$ and $\frac{\pi}{3}$

$\llcorner$DOB = 30°, $\llcorner$ODA = 60°





The triangle DOC is equilateral with each side of 1unit. $\llcorner$DOA = 60°/2 = 30°

Hence, DA= 1/2. $\text{Sin}(\frac{\pi}{6}) = DA = y = ½$

$OA = \sqrt{1-(½)^2} = \sqrt{3}/2 = \cos(\frac{\pi}{6}) = \cos 30°$

Similarly, $\sin 60° = \text{Sin}(\frac{\pi}{3}) = \sqrt{3}/2$, $\cos 60° = ½$

Examples:

Q1) Prove that $\frac{\tan A+\sec A-1}{\tan A-\sec A+1} = \frac{1+\sin A}{\cos A}$

Solution: Left side $= \frac{\tan A+\sec A-1}{\tan A-\sec A+1} = \frac{\tan A+\sec A-(sec^2-tan^2)}{\tan A-\sec A+1} = (\sec A + \tan A)\frac{1-\sec A+\tan A}{\tan A-\sec A+1} = \sec A + \tan A$
$= \frac{1+\sin A}{\cos A}$

Q2) if $\cos A + \sin A = \sqrt{2}\cos A$, prove that $\cos A - \sin A = \sqrt{2}\sin A$

Solution: Squaring both sides

$$\cos^2 A + \sin^2 A + 2\cos A \sin A = 2\cos^2 A$$

$$\cos A - \sin A = \frac{2\sin A\cos A}{\cos A+\sin A} = \sqrt{2}\sin A$$

2. Conversion of circular functions of -Θ(negative angle) in terms of circular functions of Θ

$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$ $\cot(-\theta) = -\cot\theta$

$\sec(-\theta) = \sec\theta$ $\text{cosec}(-\theta) = -\text{cosec}\,\theta$

3. Cosine Formula

1. $\cos(A - B) = \cos A \cos B + \sin A \sin B$
2. $\cos(A + B) = \cos A \cos B - \sin A \sin B$

Consider the unit circle with center O and radius OP=1

Let OP rotate about O in the anti-clockwise sense and reached OP_1 such that, are $PP_1 = B$. Then the angle generated is $\angle POP_1 = B$.

Instead if it rotates about O in the anticlock wise sense to reach OP_2 such that are $PP_2 = A$ and then rotates about O in the clockwise sense to reach OP_3 such that are $P_2P_3 = B$.

Then the angle generated is

$\angle P_3OP = \angle P_2OP - \angle P_2OP_3 = A - B$

Co-ordinate of $P_1 = (x_1, y_1) = (\text{Cos } B, \text{Sin } B)$
Co-ordinate of $P_2 = (x_2, y_2) = (\text{Cos } A, \text{Sin } A)$
Co-ordinate of $P_2 = (x_3, y_3) = (\text{Cos } A, \text{Sin } A)$

Also $P(1,0)$, as are $PP_1 = P_2P_3 = B$, we get cord PP_1 = cord P_2P_3.

arc PP_1 + arc P_1P_3 = arc P_2P_3 + arc P_1P_3, arc PP_3 = arc P_1P_2

$PP_3 = P_1P_2$, ie, $PP_3^2 = P_1P_2^2$ or $(1 - x_3)^2 + (0 - y_3)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

or $[1 - \text{Cos}(A - B)]^2 + [(0 - \text{Sin}(A - B))]^2 = (\text{Cos } A - \text{Cos } B)^2 + (\text{Sin } A - \text{Sin } B)^2$

or $1 - 2\text{Cos}(A - B) + \text{Cos}^2(A-B) + \text{Sin}^2(A-B) = \text{Cos}^2A + \text{Cos}^2B + \text{Sin}^2A + \text{Sin}^2A - 2\text{Cos } A \text{ Cos } B - 2\text{Sin } A \text{ Sin} B$

or $2 - 2\text{Cos}(A - B) = 2 - 2(\text{Cos } A \text{ Cos } B + \text{Sin } A \text{ Sin } B)$

$\text{Cos}(A-B) = \text{Cos } A \text{ Cos } B + \text{Sin } A \text{ Sin } B$

Cos (A + B) = Cos (A – (-B)) = Cos A Cos (-B) + Sin A Sin (-B) = Cos A Cos B – Sin A Sin B

3. <u>Sine Formula:</u>

Sin (A + B) = Cos [$\frac{\pi}{2}$ - (A +B)] = Cos [($\frac{\pi}{2} - A$) -B] = Cos ($\frac{\pi}{2} - A$) Cos B + Sin ($\frac{\pi}{2} - A$) Sin B = Sin A Cos B +Cos A Sin B

Sin (A+B) = Sin A Cos B +Cos A Sin B

Similarly, Sin (A-B) = Sin A Cos B – Cos A Sin B

1. tan (A+B) = $\frac{\tan A+\tan B}{1-\tan A\tan B}$
2. tan (A+B) = $\frac{\tan A-\tan B}{1+\tan A\tan B}$

Sin (A+B) = Sin A Cos B +Cos A Sin B …(i)

Cos (A + B) = Cos A Cos B – Sin A Sin B …(ii)

Dividing (i) and (ii) :

Tan (A+B) = $\frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$ … (iii)

Dividing (iii) by Cos A Cos B

Tan (A+B) = $\frac{\tan A+\tan B}{1-\tan A\tan B}$

3. <u>To transform sum or difference into product:</u>

Adding two sine formula:

2sinAcosB = Sin (A+B) + Sin (A-B)

2cosAsinB = Sin (A+B) – Sin (A-B)

2cosAcosB = Cos (A+B) + Cos (A-B)

2sinAsinB = Cos (A-B) – Cos (A+B)

To transform sum or difference into product

Let A+B = C and A-B = D, then A= C+D/2 , B = C-D/2

Replacing these values in 1, 2, 3 and 4 above

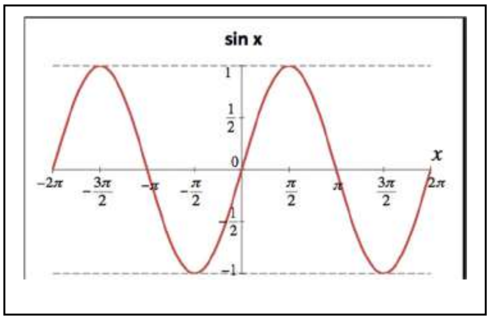
Sin C + Sin D = 2 Sin $(\frac{C+D}{2})$ Cos $(\frac{C-D}{2})$

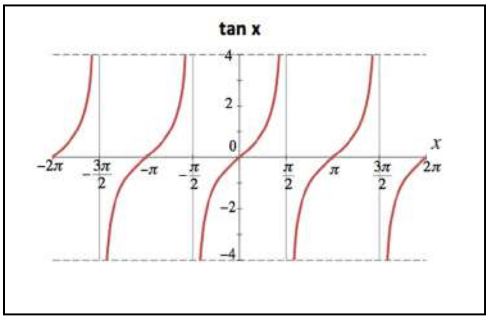
Sin C – Sin D = 2 Cos $(\frac{C+D}{2})$ Sin $(\frac{C-D}{2})$

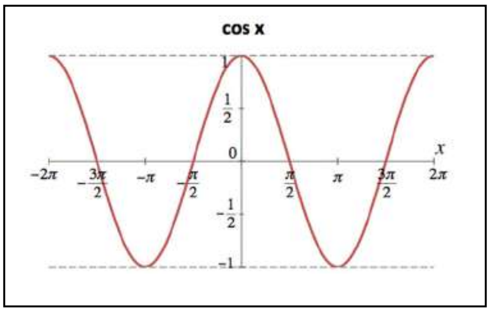
Cos C + Cos D = 2 Cos $(\frac{C+D}{2})$ Cos $(\frac{C-D}{2})$

Cos C – Cos D = 2 Sin $(\frac{C+D}{2})$ Sin $(\frac{D-C}{2})$

4. Graph of trigonometry ratio:







Examples:

Q1) Prove that, $\frac{\tan(A+B)+\tan(A-B)}{1-\tan(A-B)\tan(A+B)}$ = tan2A

Solution: Let A+B = C and A-B = D

Left side $\frac{\tan C+\tan D}{1-\tan C\tan D}$ = tan(C+D) = tan (A+B +A -B) = tan2A

Q2) If A+B+C = π and cos A = cos B.cosC, Prove that
tanA = tanB+tanC

Solution: Right side = tan B + tanC $=\frac{\sin B}{\cos B}+\frac{\sin C}{\cos C} = \frac{\sin B\cos C+\sin C\cos B}{\cos B\cos C} = \frac{\sin(B+C)}{\cos A} = \frac{\sin(\pi-B)}{\cos A} =$ tanA

Formulae on multiple angles :

Question 1 $\sin(A + B) = \sin A \cos B + \cos A \sin B$

Taking B = A, we get

$\sin 2A = \boxed{2 \sin A \cos A}$

$= 2 \frac{\sin A}{\cos A} \cos^2 A$

$= \frac{2\tan A}{\sec^2 A} = \boxed{\frac{2\tan A}{1+\tan^2 A}}$

Question 2 $\cos(A + B) = \cos A \cos B - \sin A \sin B$

taking B = A, we get

$\cos 2A = \boxed{\cos^2 A - \sin^2 A}$

$= \cos^2 A - (1 - \cos^2 A)$

$= \boxed{2\cos^2 A - 1}$

$= \cos^2 A\left(1 - \frac{\sin^2 A}{\cos^2 A}\right)$

$= \boxed{\frac{1-\tan^2 A}{1+\tan^2 A}}$

$= (1 - \sin^2 A) - \sin^2 A$

$= \boxed{1 - 2\sin^2 A}$

Question 3 $\tan 2A = \frac{\sin 2A}{\cos 2A} = \boxed{\frac{2\tan A}{1-\tan^2 A}}$

Formulae on submultiple angles:

Again considering 2A = θ, we get the following formulae for submultiple angles

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$= \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

$= 2\cos^2\frac{\theta}{2} - 1$

$= 1 - 2\sin^2\frac{\theta}{2}$

$= \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

Inverse Trigonometric Functions

The equation $\sin\theta = x$, where θ is an angle whose sine measure is x, can be expressed as $\theta = \sin^{-1}x$. Therefore $\sin^{-1}x$ is an angle where $\sin\theta$ is the number. Both the relations are identical. If one is given, the other one follows.

The general value of $\sin^{-1}x$ is a multiple valued function as it gives an infinite number of values as

$$\sin\theta = x = \sin\alpha \qquad (-1 \le x \le 1)$$

$$\text{or, } \theta = n\pi + (-1)^n\alpha, \qquad n = 0, \pm 1, \pm 2 \ldots;$$

$$\text{or, } \sin^{-1}x = n\pi + (-1)^n\alpha.$$

Similarly the general value of $\cos^{-1}x$ and $\tan^{-1}x$ can be written as

$$\cos^{-1}x = 2n\pi \pm \cos^{-1}x:$$

$$\tan^{-1}x = n\pi + \tan^{-1}x;$$

The smallest numerical value, either positive or negative obtained by putting n=0 is called principal value. The principal value of $\sin^{-1}\frac{1}{2}$ is $\frac{\pi}{6}$. For the case of two equal numerical value, one positive and one negative, it is customary to take the positive one as the principal value. For this reason the principal value of $\cos^{-1}\frac{1}{2}$ is $\frac{\pi}{3}$ though $\left(\cos(-\frac{\pi}{3})\right) = \frac{1}{2}$.

one can easily prove

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2};$$

$$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2};$$

$$\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2};$$

The following formulae important for inverse circular functions.

$$\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right\}$$

$$\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left\{xy \pm \sqrt{(1-x^2)(1-y^2)}\right\}$$

$$\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\left\{\frac{x \pm y}{1 \mp xy}\right\}$$

ASSIGNMENT

Q1) Express in terms of radians(i) 75° and (ii) 225°? [ANS: 5/12 π and 5/4 π]

Q2) Reduce to degrees (i) -5π/6 (ii) 1.1radian? [ANS: -150° and 63°]

Q3) The angles of the triangle are as 1:3:5; find them in radian? [ANS: $\frac{\pi}{9}, \frac{\pi}{3}, \frac{5\pi}{9}$]

Q4) The angles of a triangle are in AP. The greatest angle is three times the least. Find them in (i) degrees and (ii) radians? [ANS: (i) 30°, 60°, 90° (ii) $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$]

Q5) A train is travelling on a circular path of radius $\frac{2}{3}Km.$ @ 33Km/hr. Through what angle in degrees has it turned in 15 seconds? [Ans: 11° 48' 45'']

Q6) If (1 – sinA)(1-sinB)(1-sinC) = (1+sinA)(1+sinB)(1+sinC) then prove that the value of each = ±cosAcosBcosC.

Q7) If $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$ and $x\sin\theta - y\cos\theta = 0$, prove that $x^2 + y^2 = 1$.

Q8) (a) if sin α + cos α =1, prove that sin α - cos α = ±1

(b) If acosA – bsinA = C then prove that

$a\sin A + b\cos A = \pm \sqrt{(a^2 + b^2 - c^2)}$

Q9) If u = secA +tanA, prove that tanA $=\frac{u^2-1}{2u}$ and sinA $=\frac{u^2-1}{u^2+1}$

Q10) If tanA = $\frac{m}{m+1}$ and tanB = $\frac{1}{2m+1}$, prove that A+B = $\frac{\pi}{4}$

Q11) If A+B =45°, prove that (1+tanA)(1+tanB) =2

Q12) Prove that tan3A – tan2A – tanA = tan3A.tan2A.tanA

Q13) Prove that $\frac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A} = \tan 4A$

Q14) If $\frac{\sin(\theta+\alpha)}{\cos(\theta-\alpha)} = \frac{1-m}{1+m}$, prove that $\tan(\frac{\pi}{4} - \theta)\tan(\frac{\pi}{4} - \alpha) = m$

Q16) Prove that,

1. $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$
2. $\tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\frac{3x-x^2}{1-3x^2}$, where $x^2 < \frac{1}{3}$
3. $\tan[\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2} + \frac{1}{2}\cos^{-1}\frac{1-x^2}{1+x^2}] = \frac{2x}{1-x^2}$

Q17) Prove that,

$$\cos^{-1}\frac{\cos\theta+\cos\alpha}{1+\cos\theta\cos\alpha} = 2\tan^{-1}[\tan\frac{\theta}{2}\tan\frac{\alpha}{2}]$$

Reference Books:

1. Plane Trigonometry, by S.L. Loney Part 1
2. Modern Approach to Intermediate Trigonometry, by Das Gupta and Prasad.

Module-8

Probability

Lectures required -02

Probability theory deals with the rules governing the chances of occurrence of phenomena, which are random in nature. To introduce probability, certain basic terms are needed, we first discuss those terms.

Random experiment:

A **random experiment** is an experiment which can be repeated under identical conditions, the set of all possible outcomes is known but before any particular performance of the experiment, we cannot predict which outcome will occur. For example, tossing of a coin, rolling of a die etc. Any particular performance of a random experiment is called a trial.

Sample space:

The set S of all possible outcomes of a random experiment is called the **sample space** associated with that random experiment. For example, in tossing of a coin, $S = \{H,\ T\}$ where H and T resp. denote getting a head and tail. In rolling of a die, $S = \{s_1,\ s_2,\ s_3,\ s_4,\ s_5,\ s_6\}$ where s_i denotes the outcome in which i appears at the uppermost face, $i = 1, 2, 3, 4, 5, 6$.

Event:

An **event** in a random experiment is defined as any subset E of the sample space S. Singleton subsets are called **elementary events**, $\emptyset$ is called the **impossible event** and S is called the **certain** event, e.g., in rolling of a die $\{s_1,\ s_3,\ s_5\ \}$ is an event which is precisely 'getting an odd number'.

If for any two events A and B, $s \in A \cup B$ we say that at least one of the events A or B occurs. If $s \in A \cap B$, then we say that both A and B have occurred. If $s \notin A$, then we say that the **complementary event** A' has occurred.

Favourable outcomes to an event:

An outcome is said to be **favourable** to an event A if it entails the happening of that event. The outcomes favourable to an event A are precisely the outcomes which belong to A.

Equally likely outcomes:

Two outcomes are said to be **equally likely** if there is no reason to expect any one of them more than the other. For example, if a coin in unbiased, then getting a head or tail will be equally likely but if the coin is biased then getting a head or tail will not be equally likely.

Mutually exclusive events:

Two events A and B are said to be **mutually exclusive** if $A \cap B = \emptyset$, i.e, there is no outcome favourable to both the events simultaneously.

Classical definition of probability:

If in a random experiment, the sample space consists of a finite number n of equally likely outcomes and m of them are favourable to an event A, then probability of an event A, denoted by $P(A)$, is defined as $\frac{m}{n}$

It is easy to see that

I. $0 \leq P(A) \leq 1$
II. $P(\emptyset) = 0, P(S) = 1$
III. $P(A') = 1 - P(A)$

Example 1: An urn contains 8 white balls and 3 red balls. If two balls are drawn at random, find the probability that (i), both are white, (ii) both are red, (iii) one is white and one is red.

Solution. These are 11_{C_2} mutually exclusive, equally likely and exhaustive ways to draw two balls out of 11 balls.

(i) The no. of ways in which two while balls can be drawn from 8 while balls, is 8_{C_2}. Hence the required probability is $\frac{8_{C_2}}{11_{C_2}} = \frac{28}{55}$

(ii) The probability of getting 2 red balls $= \frac{3_{C_2}}{11_{C_2}} = \frac{3}{55}$

(iii) The probability of getting one white and one red ball$= \frac{8\times3}{11_{C_2}} = \frac{24}{55}$

The classical definition of probability has some defects:

(i) It involves the term "equally likely" outcomes. It cannot be applied if the outcomes are not equally likely.
(ii) It is assumed that the number of possible outcomes is finite. If the number of possible outcomes is infinite, the classical definition is not applicable.

To overcome these shortcomings, the following definition was introduced.

Statistical (or Empirical or Frequency) definition of probability:

Let a random experiment E be repeated N times under uniform conditions in which an event A occurs $N\ (A)$ times, then $N\ (A)$ is called the **absolute frequency** of A. The ratio $\frac{N\ (A)}{N}$ is called the **relative frequency** or the frequency ratio. The probability of the event A, denoted by $P\ (A)$, is defined as $\lim_{n\to\infty} \frac{N\ (A)}{N}$, assuming that this limit is finite.

Later on, another approach to introduce probability was developed which is called the **axiomatic** development of the probability. In this approach, probability is defined as a function *'P'* which associates a real number *P (A)* to each event *A,* which satisfies the following three axioms:

(i) $P(A) \geq 0$, for each A
(ii) $P(S) = 1$
(iii) If $A_1,\ A_2,\ A_3, \ldots$ are a countable number of events such that

$A_i \cap A_j = \emptyset$ for $i \neq j$, then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

Now we mention two basic laws of probability which are stated in the form of two theorems as follows:

Additive Law of probability (Theorem of total probability):

For any two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

In particular if A and B are mutually exclusive then $P(A \cup B) = P(A) + P(B)$, since in this case $A \cup B = \emptyset => P(A \cap B) = 0$

If there are N mutually exclusive events $A_1,\ A_2,\ A_3, \ldots \ldots \ldots \ldots \ldots \ldots A_n$ then

$$P(A_1 \cup A_2 \ldots \ldots \ldots \ldots \ldots \ldots .. \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)\ ,$$

Next we define conditional probability.

Definition:

The probability of occurrence of an event B on the hypothesis that the event A has already occurred, is called the **conditional probability** of B, given A, denoted by $P(B|A)$, is defined as follows:

$P(B|A) = \frac{P(A \cup B)}{P(B)}$ provided $P(B) \neq 0$

Similarly, $P(A|B) = \frac{P(A \cap B)}{P(A)}$, provided $P(A) \neq 0$

Thus if $P(A) \neq 0,\ P(B) \neq 0$ then

$P(A \cap B) = P(A).P((B|A)) = P(B).P((A|B))$

Definition:

Two events A and B are said to be **independent** if $P(A \cap B) = P(A).P(B)$

For any two independent events A and B, the following statements are equivalent:

(i) $P(A \cap B) = P(A).P(B)$
(ii) $P(A|B) = P(A)$ if $P(B) > 0$
(iii) $P(B|A) = P(B)$ if $P(A) > 0$

Example 2: An urn contains 10 white and 8 black balls. Two balls are drawn at random. Find the probability that they are of the same colour.

Solution: The total no. of ways of drawing two balls is18_{C_2}. The balls drawn will be of the same colour if either both are while or both are black.

$$\text{The required probability} = \frac{10_{C_2}}{18_{C_2}} + \frac{8_{C_2}}{18_{C_2}} = \frac{45+28}{153}$$

$$= \frac{73}{153}$$

Example 3. A can solve 75% of the problems of a book and B can solve 70%. What is the probability that either A or B can solve a problem chosen at random from the book?

Solution. The required probability

$$=P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A).P(B)$$

$$= \frac{3}{4} + \frac{7}{10} - \frac{3}{4}.\frac{7}{10} = \frac{37}{40}$$

Example 4. A bag contains 4 red balls and 3 black balls. Two drawings of two balls are made. Find the chance that the first drawing gives 2 red balls and the second drawing gives two blue balls:

(i) If the balls are returned to the bag after the first draw,
(ii) If the balls are not returned.

Solution. (i) The total no. of ways in which two balls are drawn out of 7 balls, is 7_{C_2}. The total no. of ways of drawing 2 red balls out of 4, is 4_{C_2}. Hence in the first draw the probability of drawing two red balls is $\frac{^4C_2}{^7C_2}$. In the second draw, the probability of drawing 2 blue balls is $\frac{^3C_2}{^7C_2}$. The required probability is

$$\frac{^4C_2}{^7C_2} \times \frac{^3C_2}{^7C_2} = \frac{2}{7} \times \frac{1}{7} = \frac{2}{49}$$

(ii) In this case the probability of drawing 2 red balls in the first draw is $\frac{^4C_2}{^7C_2}$ in the second draw, the probability of drawing 2 blue balls is $\frac{^3C_2}{^5C_2}$.

Hence the required probability $\frac{^4C_2}{^7C_2} \times \frac{^3C_2}{^5C_2} = \frac{2}{7} \times \frac{3}{10} = \frac{3}{35}$

In the following theorem we state **Bayes' formula.**

Theorem: If $A_1, A_2, A_3, \ldots\ldots\ldots\ldots\ldots\ldots A_n$ are given set of mutually exclusive and exhaustive events such that

$A = \bigcup_{i=1}^{n} A_i$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, then for any event A,

(i) $P(A) = \sum_{i=1}^{n} P(A_i).P(A|A_i)$

(ii) **Bayes' Formula:** $\mathrm{P}(A_k|A) = \frac{\mathrm{P}(A_k)\,\mathrm{P}(A_k|A)}{\mathrm{P(A)}}$, provided $P(A) \neq 0$.

$$= \frac{\mathrm{P}(A_k)\,\mathrm{P}(A_k|A)}{\sum_{i=1}^{n} P(A_i).P((A|A_i)}$$

Example. 5 The contents of three boxes are as follows:

Box 1: 1 white and 2 black balls

Box 2: 2 white and 1 black balls

Box 3: 2 white and 2 black balls

One of these boxes are selected at random and one ball is drawn at random from it. What is the probability that the third box was chosen?

Solution: Let A_i denote the event that the ball is drawn from the i^{th} box, i=1, 2, 3. Then the events $A_1,\ A_2,\ A_3$ are mutually exclusive and exhaustive and

$$\mathrm{P}(A_1) = \mathrm{P}(A_2) = \mathrm{P}(A_3) = \frac{1}{3}$$

Let A denote the event of drawing a white ball then

$$\mathrm{P}(A|A_1) = \frac{1}{3},\ \mathrm{P}(A|A_2) = \frac{2}{3},\ \mathrm{P}(A|A_3) = \frac{1}{2}$$

Using Bayes′formula, we get

$$\mathrm{P}(A_3|A) = \frac{P(A_3).P(A|A_3)}{\sum_{i=1}^{3} P(A_i).P(A|A_i)} = \frac{1/3\ \times 1/2}{\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{2}}$$

$$= \frac{1/3\ \times 1/2}{\frac{1}{3}(\frac{1}{3}+\frac{2}{3}+\frac{1}{2})} = \frac{1}{3}$$

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Random Variables and Probability Distributions

Random Variable:

A real valued function *X* defined on a sample space is called a **random variable.** The range of the function *X* is called the **spectrum** of the random variable. The random variable *X* is called **discrete** or **continuous** according as the spectrum is discrete or continuous.

Discrete Probability Distribution:

Let x be a discrete random variable and suppose its possible values are x_1, x_2, $x_3, \ldots\ldots, x_n \ldots\ldots$, then the function f defined as follows:

$$f(x) = P(x = x_k), \qquad \text{when } x = x_k$$

$$= 0 \qquad of\ x \neq x_k.$$

is called the **probability function** or the **probability mass function** of *x*. The probability function always satisfies:

(i) $f(x) \geq 0$ for each x

(ii) $\sum_{i=1}^{\infty} f(x_k) = 1$

Mean and variance of a discrete probability distribution are defined as follows.

Mean (*x*) $= \bar{x} = \sum_{k=1}^{\infty} x_k f(x_k)$

Var (*x*) $= \sigma^2 = \sum_{k=1}^{\infty} (x_k - \bar{x})^2 f(x_k)$

$= \sum_{k=1}^{\infty} x_k^2 f(x_k) - \bar{x}^2$

Positive square root of the variance is called the standard deviation of x.

Example1. Find the mean and variance of a random variable X where X denotes the number appeared at the uppermost face in a throw of an unbiased die.

Solution.

X:	1	2	3	4	5	6
$P(X)$:	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Mean(X): $\sum_{X=1}^{6} XP(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$

$= \frac{1}{6}(1+2+3+4+5+6)$

$= \frac{21}{6} = \frac{7}{3}$

Var(X)= $\sum_{x=1}^{6}(X - \bar{X})^2 P(X) = \sum_{x=1}^{6} X^2 P(X) - \bar{X}^2$

$= \frac{1}{6}\{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2\} - \frac{49}{9}$

$= \frac{175}{18}$

Distribution function:

The distribution function (also called cumulative distribution function) 'F' of a random variable X is a function of real variable x defined as follows:

$$F(x) = P(-\infty < X \leq x)$$

The distribution function $F(x)$ satisfies the following properties:

(i) $F(x)$ is non $-$ decreasing, $i.e$, for $x \leq y$, $F(x) \leq F(y)$

(ii) $F(-\infty) = 0, \quad F(\infty) = 1$

(iii) F is continuous from the right, i.e. , $\lim_{h \to 0+} F(x+h) = F(x)$ for each x.

For a discrete probability distribution $F(x) = \sum_{x_k \leq x} f(x_k)$

Example 2. Suppose an unbiased coin is tossed two times so that the sample space is $S = \{HH, HT, TH, TT\}$. Let *X* represent the number of heads obtained. Then for each sample point, we associate a number *X* as follows:

Sample point:	HH	HT	TH	TT
X:	2	1	1	0

Then *X* is a discrete random variable with the probabilities given by:

$$P(X = 0) = P(TT) = P(T).P(T) = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$$

$$P(X = 1) = P(HT \cup TH) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P(X = 2) = P(HH) = P(H).P(H) = \frac{1}{4}$$

The corresponding cumulative distribution function is gives as follows:

$$\begin{aligned} F(x) &= 0, && -\infty < x < 0 \\ &= \frac{1}{4}, && 0 \le x < 1 \\ &= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}, && 1 \le x < 2 \\ &= 1, && 2 \le x < \infty \end{aligned}$$

Now we consider probability distribution of a continuous random variable.

If *X* is a continuous random variable, it can assume values on a continuous scale. In this case, the probability that *X* takes a value on the interval $a \le x \le b$ is defined as :

$$P(a \le x \le b) = \int_a^b f(x)dx$$

where f is an integrable function defined for all values of the random variable with which we are concerned, satisfying the following conditions:

(1) $f(x) \geq 0$ for all x within the domain of f.

(2) $\int_{-\infty}^{\infty} f(x)dx = 1$

Here f is called the probability density function of X.

The cumulative distribution function F of a continuous random variable X, is given by:

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

It follows that

$$P(a \leq x \leq b) = F(-\infty < X \leq b) - F(-\infty < X \leq a)$$

$$= F(b) - f(a) = \int_{a}^{b} f(x)dx$$

According to the fundamental theorem of integral calculus:

$$\frac{dF(x)}{dx} = f(x)$$

where the derivative exists.

In continuous probability distribution mean and variance are defined as follows:

$$\text{Mean }(x) = \int_{-\infty}^{\infty} x\, f(x)dx = \bar{x}$$

$$\text{Var }(x) = \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \bar{x}^2$$

Example 2: Find the constant k such that the following function becomes a probability density function:

$$f(x) = 1/k\ , \qquad a \leq x \leq b$$

$$= 0 \text{ otherwise.}$$

Determine the mean, variance of the distribution and the cumulative distribution function.

Solution: For $f(x)$ to be a probability density function:

(i) $f(x) \geq 0$, for all x and (ii) $\int_{-\infty}^{\infty} f(x)dx \ = 1$

which gives $\int_a^b \frac{1}{k} dx = 1 \quad => \ k = (b-a)$

Mean $(x) = \int_{-\infty}^{\infty} x\, f(x)dx = \int_a^b \frac{x}{(b-a)} dx = \frac{1}{2} \frac{(b^2-a^2)}{(b-a)} = \frac{1}{2}\ (a+b)$

$$\text{Var}\ (x) = \int_{-\infty}^{\infty} (x-\overline{x})^2 f(x)dx = \int_a^b \frac{x^2}{(b-a)} dx - \frac{1}{4}(a+b)^2$$

$$= \frac{1}{3}\frac{(b^3-a^3)}{(b-a)} - \frac{1}{4}(a+b)^2$$

$$= \frac{1}{3}(b-a).\frac{(b^2+ab+a^2)}{(b-a)} - \frac{1}{4}(a+b)^2$$

$$= \frac{1}{3}(b^2+ab+a^2) - \frac{1}{4}\ (a^2+b^2+2ab)$$

$$= \frac{a^2+b^2}{12} - ab\ (\frac{1}{2}-\frac{1}{3})$$

$$= \frac{a^2+b^2}{12} - \frac{ab}{6} = \frac{a^2+b^2-2ab}{12}$$

$$= \frac{1}{12}(a-b)^2$$

Cumulative distribution function:

$F(x) = \int_{-\infty}^{x} f(x)dx = 0, \qquad if = x\ \leq a$

$= \int_{-\infty}^{a} f(x)dx \ + \int_a^x f(x)dx = \frac{x-a}{b-a}$ if $a < x < b$

$= \int_{-\infty}^{a} f(x)dx \ + \int_a^b f(x)dx + \ \int_b^x f(x)dx = 1$ if $\ x \ \geq b$

Binomial Probability Distribution:

Suppose a trial is repeated n times. Let us call the occurrence of an event a 'success' and its non-occurrence', a 'failure' we assume that 'p' is the probability of a success and 'q' is the probability of a failure. So $p+q=1$. There are two assumptions: (i) all the trials are independent and (ii) The probability p of a success remains the same in each trial. Let x denote the number of successes obtained in a series of n trials. Then the probability of getting $x=r$ successes is given by

$$P(x = r) = nc_r\, p^r\, q^{n-r}, \qquad r = 0, 1, 2, \ldots\ldots\ldots\ldots\ldots n \qquad (1)$$

We have $\sum_{r=0}^{n} P(x = r) = \sum_{r=0}^{n} nc_r\, p^r\, q^{n-r} = 1$

Hence (i) defines a discrete probability distribution, called as a binomial probability distribution.

Mean and variance of this distribution are given by:

Mean $(x) = n\, p$ and variance $(x) = n\, p\, q$.

Example 3:

The incidence of occupational disease in an industry is such that the workers have a 20% chance of suffering from it. What is the probability that out of six workers, 4 or more will catch the disease?

Solution.

Here $n = 6, p = 0.2, q = 0.8$.

The required probability = $6c_4(0.2)^2(0.8)^2 + 6c_5(0.2)(0.8)^5 + (0.8)^6$

Example 4.

Suppose the probability of a new born baby being a boy, is 0.51. In a family of 8 children, calculate the probability that there are 4 or 5 boys.

Solution.

Here $n = 8$, $p = 0.51$, $q = 0.49$

The required probability is $8c_4(0.51)^4\ (0.49)^4 + 8c_5(0.51)^5\,(0.49)^3$

ASSIGNMENT

1. If A and B are two independent events, show that (i) A and B' are independent, (ii) A' and $B'$are independent. (Here A', B' denote respectively the complementary events of A and B).
2. When two perfect dice are thrown, find the probability that the sum of the numbers obtained is 6 and 7. (Ans. $\frac{11}{36}$)
3. A and B alternately throw a pair of dice, A, starting the game. A wins if he throws six before B throws 7 and B wins if he throw 7 before A throws 6. What is the probability of A's winning? (Ans. $\frac{30}{61}$)
4. There are three persons aged 50 years, 60 years and 70 years respectively. The probability to live 10 years more is $\frac{4}{5}$ for a 50 years old, $\frac{1}{2}$ for a 60 years old and $\frac{1}{5}$ for a 70 years old person. Find the probability that at least two of them will survive 10 years more. (Ans. $\frac{1}{2}$)
5. Two urns contain respectively 2 white and 1 black ball, and 1 white and 5 black balls. One ball is transferred from the first urn to the second urn, and then a ball is drawn from the second urn. What is the probability that the ball drawn is white? . (Ans. $\frac{5}{21}$)
6. In a factory manufacturing bulbs, machines 1, 2, 3 manufacture respectively 20, 45, and 35% of the total output, of this 3, 5 and 4% respectively are defective. A bulb is drawn at random from the total output and found to be defective. Find the probability that it was manufactured by machine number 1. (Ans.0.14)
7. If 10 % bolts produced by a machine are defective, determine the probability that out of 10 bolts chosen at random, (i) one, (ii) none, (iii) at most two bolts will be defective.

   (Ans. (i) 0.3874, (ii) 0.3487, (iii) 0.9298)

8. Compute the mean, median and made for the following frequency distribution.

   Table: Frequency distribution of I.Q. for 309 six-years-old children.

I.Q.	160-169	150-159	140-149	130-139	120-129	110-119	100-109
frequency	2	3	7	19	37	79	69

I.Q	90-99	80-89	70-79	60-69	50-59	40-49	**Total**
frequency	65	17	5	3	2	1	**309**

Ans.108.48, 108.41, 111.42

9. For a set of 250 observations on a certain variable x, the mean and s. d. are respectively 65.7 and 4.4. However, on scrutining the data it is found that two observations, which should be correctly read as 71 and 83, had been wrongly recorded as 91 and 80. Obtain the correct value of mean and s. d. (Ans. $\bar{x}$= 65.6, $s = 4.2$

References:

(i) Mathematical Statistics, J. N. Kapur & H. C. Saxena.

(ii) Probability and Statistics for Engineers, Irwin Miller & John E. Freund.

(iii) Elements of Probability and Statistics, A. P. Baisnab & M. Jas.

(iv) Statistics, Schaum's Outlines, M. R. Spiegel & L. J. Stephens.

(v) Probability and Statistics, Schaum's Outlines, M. R. Spiegel, J. J. Schiller, R. A. Srinivasan.

Module-9 Statistics

Measures of Central Tendency

The primary purpose of statistical methods is to summarize the information contained in any set of collected data. The purpose is served by classifying the data in form of a frequency distribution and using various graphs, viz., line diagrams, bar diagrams, pictorial diagrams, representation of percentages, statistical maps. When the data related to a variable, the process of summarization can be taken a long step further by using certain descriptive measures. The aim is to focus on certain features of the data which will describe the general nature of the data. The two most important features are **Central tendency** and **Dispersion**.

Central Tendency:

Let us consider the following table.

Table 1: Yield per plant for 12 tomato plants of a particular verifies:

Plant No.	Yield (gm.)	Plant No.	Yield (gm.)
1.	1,216	7.	1,202
2.	1,374	8.	1,372
3.	1,167	9.	1,278
4.	1,232	10.	1,141
5.	1,407	11.	1,221
6.	1,453	12.	1,329

From Table 1, it is clearly evident that the figures seem to cluster around some point between 1,200 gm. and 1,300 gm. However, we need a single value, the central value, to represent the whole set of figures. Such a representative or typical value of a variable is called the measure of central tendency or an average.

Three commonly used measures of central tendency are

I. Arithmetic mean
II. Median
III. Mode

Arithmetic Mean:

Let us denote the variable by x, and the corresponding values of the variable x by $x_1, x_2, \ldots \ldots . x_n$.

For example, let x represents the height of n students and the corresponding heights are represented by $x_1, x_2, \ldots \ldots . x_n$.

Then the arithmetic mean $(A.M)$ of x is given by,

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad (1.1)$$

<u>Example 1</u>. For the data given in Table 1. $A.M$ is **…..**

$$= \frac{1{,}216 + 1{,}374 + \cdots + 1{,}329}{12}$$

$$= \frac{15{,}392}{12}$$

= 1,282.67 gm.

<u>Example 2.</u>

Let us consider the following table

Table 2. Frequency distribution of number of peas per pod for 198 pods.

No. of Peas	**Frequency (f_i)**
1.	**4**
2.	**33**
3.	**76**
4.	**50**
5.	**26**
6.	**8**
7.	**1**
Total	**198 =$\sum_{i=1}^{7} f_i$**

In the above table, the value 1 occurs 4 times, value 2 occurs 33 times, and so on.

Therefore, if in the above example x represents the no. of peas per pod and the corresponding value of x, i.e., x_i (i=1, 2, …. . …198) represents the no. of peas in it n pod, then…

$$\sum_{i=1}^{198} x_i = 11 \times 4 + 2 \times 33 + 3 \times 76 + 4 \times 50 + 5 \times 26 + 6 \times 8 + 7 \times 1$$

= **683.**

Hence, $A.M$ of x is given by….

$$\bar{x} = \frac{11 \times 4 + 2 \times 33 + 3 \times 76 + 4 \times 50 + 5 \times 26 + 6 \times 8 + 7 \times 1}{198}$$

$= \frac{683}{98}$ =20.697

Therefore, if the numbers occur $f\_1, f\_2, \ldots \ldots \ldots \ldots .. f_n$ times, respectively (i.e., occur with frequencies $f_1, f_2, \ldots \ldots \ldots \ldots . f_n$ the arithmetic mean is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i f_i}{\sum_{i=1}^{n} f_i} = \frac{1}{N} \sum_{i=1}^{n} x_i f_i \qquad (1.2)$$

Where N= $\sum_{i=1}^{n} f_i$ is the total frequency.

When we have a frequency table which represents the frequencies in the different classes, then also we will use the formula (1.2) for calculating the arithmetic mean. However, in this case x_i will represent the mid value of width class interval. But in this case (1.2) will give only an approximate value of the mean. The error of approximation will be negligible provided the range of x is very large compared to the width of the clan-intervals.

Example 3.

(a.) Find the arithmetic mean of the following frequency distribution

x_i :	1	2	3	4	5	6	7
f_i :	5	9	12	17	14	10	6

(b.) Calculate the arithmetic mean of the marks from the following table:

Marks:	0-10	10-20	20-30	30-40	40-50	50-60
No. of students.	12	18	27	20	17	6

Solution:

(A)

x_i	f_i	$x_i f_i$
1	5	5
2	9	18
3	12	36
4	17	68
5	14	70
6	10	60
7	6	42
Total	**73**	**299**

$$\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{299}{73} = 4.0959$$

(b) :

Marks	**Mid Points** x_i	**No. of Students** f_i	$x_i f_i$
0-10	5	12	60
10-20	15	18	270
20-30	25	27	675
30-40	35	20	700
40-50	45	17	765
50-60	55	6	330
		100	**2800**

Arithmetic mean (A.M) $= \dfrac{\sum x_i f_i}{\sum f_i}$

$$= \frac{2800}{100} = 28.$$

Average marks of the students are 28.

It may be noted here that if the values of $x_i's$ (and) or $f_i's$ are large, the calculation of mean by formula (1.2) is quite time-consuming and tedious.

Let $d_i = x_i - A, i = 1, 2, \dots\dots\dots n$

$$f_i\, d_i = \sum f_i\,(x_i - A) = f_i\, x_i - Af_i, \qquad i = 1, 2, \dots.n$$

$$\frac{1}{N}\sum f_i\, d_i = \frac{1}{N}\sum_{i=1}^{n} x_i\, f_i - A \qquad \therefore N = \sum_{i=1}^{n} f_i$$

$$= \bar{x} - A.$$

$$\boxed{\bar{x} = \mathrm{A} + \frac{1}{N}\sum_{i=1}^{n} f_i\, d_i} \qquad (1.3)$$

where A is any arbitrary point.

Let us verify frequency distribution.

Let $d_i = \frac{x_i - A}{h}$, $\quad i = 1,2, \dots \dots \dots \dots \dots n$

A is an arbitrary point

h is the common magnitude of class interval

Now $h\, d_i = x_i - A$

- $\sum_{i=1}^{n} h\, d_i\, f_i = \sum_{i=1}^{n} x_i\, f_i - A \sum_{i=1}^{n} f_i$
- $\frac{h}{N} \sum_{i=1}^{n} d_i\, f_i = \bar{x} - A$

$$\boxed{\bar{x} = \mathrm{A} + \frac{h}{N} \sum_{i=1}^{n} d_i\, f_i} \qquad (1.4)$$

Example 4.

Calculate the simple mean/arithmetic mean/ mean for the following frequency distribution.

Class interval:	0-8	8-16	16-24	24-32	32-40	40-48
Frequency:	8	7	16	24	15	7

Solution:

Class interval	Mid Value(x_i)	Frequency (f_i)	$(d_i) = \frac{x_i - A}{h}$	$f_i d_i$
0-8	4	8	-3	-24
8-16	12	7	-2	-14
16-24	20	16	-1	-16
24-32	28(=A)	24	0	0
32-40	36	15	1	15
40-48	44	7	2	14
Total		**77(=N)**		**-25**

Let us consider A=28 & h=8

$\bar{x} = \mathrm{A} + \frac{h}{N} \sum f_i \, d_i \; = 28 + \frac{8}{77}\,(-25)$

=25.404

The Weighted Arithmetic Mean:

In calculating arithmetic mean we assume that all the items in the distribution have equal importance. But in practice this may not be so. If same items in a distribution are more important than others, then this point must be considered in calculating the average. In such cases, proper weights must be given to various items. The weights attached to each item being proportional to the importance of the item in the distribution.

For example, let w_i be the weight attached to the item

x_i, i= 1, 2, . . . ………….n then we define:

Weighted arithmetic mean (or weighted mean)

$$=\sum_{i=1}^{n} w_i \, x_i \; / \; \sum_{i=1}^{n} w_i \qquad (1.5)$$

Example 5:

If a final examination in a course is weighted 3 times as much as a quiz and a student has a final examination grade of 85 and quiz grades of and 90, then the mean grade is.

$$\overline{x} = \frac{1\times70+1\times90+3\times85}{1+1+3} = 415/5 = 83.$$

Median:

Median of a distribution is the value of the variable which divides the entire set of values into two equal parts. The median is thus a positional average.

In case of ungrouped data, if the number of observations is odd then median is the middle value after the values have been arranged in ascending or descending order of magnitude. In case of even number of observations, there are two middle terms and median is obtained by taking the arithmetic mean of the middle terms.

For example, the median of the values 25, 20, 15, 10, 5, 21, 7, i.e. 5, 7, 10, 15, 20, 21, 25 is 15. And the median of 8, 16, 12, 1, 2, 9, 15, 30, 25, 4 i.e. 1, 2, 4, 8, 9, 12, 15, 16, 25, 30 is

$$=\frac{1}{2}(9+12)=10.5$$

Remark: In case of even number of observation, in fact any value lying between the two middle values can be taken as median but conventionally we take it to be the mean of the middle term.

In case of discrete frequency distribution median is obtained by considering the cumulative frequencies.

The steps for calculating median are given below:

I. Find $\frac{1}{2}N$, where $N=\sum_{i=1}^{n} f_i$
II. see the (less then) cumulative frequency(c.f.) just greater than $\frac{1}{2}N$.
III. The corresponding value of x is median.

Example 6.

Obtain the median for the following frequency distribution:

x_i :	1	2	3	4	5	6	7	8	9
f_i :	8	10	11	16	20	25	15	9	6

Solution:

x_i	f_i	$c.f.$
1	8	8
2	10	18
3	11	29
4	16	45
5	20	**65**
6	25	90
7	15	105
8	9	114
9	6	120 (=N)

$\therefore \quad \frac{1}{2}N=\frac{120}{2}=60$

$\therefore$ The cumulative frequency just greater then $N/2$ is 65 and the corresponding value of x_i is 5.

∴ Median is 5.

Median for continuous frequency distribution:

In case of continuous frequency distribution, the class corresponding to the (less than) cumulative frequency just greater than $\frac{1}{2}N$ is called the median class and the value of median is obtained by the following formula.

$$\text{Median} = l+\frac{h}{f}\left(\frac{N}{2} - c\right) \qquad (1.6)$$

Where,

l is the lower limit of the median class.

f is the frequency of the median class.

h is the length of the median class.

c is the c. f. of the class preceding the median class.

Example. 7

Find the median wage of the following distribution

Wages (in Rs.):	2000-3000	3000-4000	4000-5000	5000-6000	6000-7000
No. of workers:	3	5	20	10	5

Solution:

$\boldsymbol{Wages(in\ Rs.)}$	$\boldsymbol{No. of\ workers}$	$\boldsymbol{c.f}$
2000-3000	3	3
3000-4000	5	8
4000-5000	20	28
5000-6000	10	38
6000-7000	5	43

N=43, => $\frac{N}{2} = 21.5$

Cumulative frequency just greater than 21.5 is 28 and the corresponding class is 4000-5000.

Thus the median class is 4000-5000.

These median wages is Rs 4,675

Mode:

Made is the value which occurs most frequently in a set of observations. In other words, made is the value of the variable which is predominant in the series. For example, in the following frequency distribution

x :	1	2	3	4	5	6	7	8
f :	4	9	16	25	22	15	7	3

Value of x corresponding to the maximum frequency, viz, 25 are 4. Hence, mode is 4.

Mode for continuous frequency distribution:

In case of continuous frequency distribution, made is given by the formula:

$$\text{Mode} = l + \frac{h(f_1 - f_0)}{(f_1 - f_0) - (f_2 - f_1)}$$

$$= l + \frac{h(f_1 - f_0)}{2f_1 - f_0 - f_2} \qquad (1.7)$$

Where, l is lower limit of the modal class.

h is length/magnitude of the modal class.

f_1 is frequency of the class preceding the modal class.

f_2 is Frequency of the class succeeding the modal class.

In modal class is the class with maximum frequency.

Example 8.

Find the mode for the following distribution

Class-interval:	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Frequency:	5	8	7	12	28	20	10	10

Solution: Maximum frequency is 28

∴ The modal class is 40-50

∴Mode = $10 + \frac{10(28-12)}{2\times 28-12-20}$ = 40+6.666=46.67

DISPERSION

Average or measures of central tendency give us an idea of the concentration of the observations about the central part of the distribution.

Let us consider the following three set of data

I.	x :	7,	8,	9,	10,	11	=> $\sum x_i = 45$ & $\bar{x} = 9$
II.	x :	3,	6,	9,	12,	15	=> $\sum x_i = 45$ & $\bar{x} = 9$
III.	x :	1,	5,	9,	13,	17	=> $\sum x_i = 45$ & $\bar{x} = 9$

In all the above cases we have 5 observations with mean 9. If we have given that the mean of 5 observations is 9 we cannot form an idea as to whether it is the average of 1st set of data or 3rd set of data or some other set of data. Whose sum is 45.

Thus we see that the measures of central tendency are inadequate to give us a complete idea of the distribution. They must be supported and supplemented by some other measures. One such measure is dispersion.

Literal meaning of dispersion is "scatteredness" Dispersion gives us an idea about the homogeneity or heterogeneity of the distribution.

Measures of Dispersion:

Various measures of dispersion are as follows:

I. Range.
II. Mean deviation.
III. Standard deviation.

Range:

The simplest measure of the dispersion of a variable is its range, which is defined as the difference between the highest (maximum) and the lowest (minimum) Values of the observation/variable.

Let us consider an example here. Suppose two students, A and B of a college received the following marks in eight monthly examinations in a particular subject:

Marks obtained by A	Marks obtained by B
63	61
47	54
56	56
44	57
66	60
65	59
80	55
43	62

In this example, average score of both the student A & B is same, i. e., 58. In this example, the range of the marks obtained by A is 80-43=37 and that of B is 62-54=8.

Mean deviation:

Let x_A be the chosen average value of the variable x, then $x_i - x_A$ is the deviation of the i^{th} given value of x from the average. Clearly the higher the deviations

$x_1 - x_A, x_2 - x_{A,}$ .... $x_n - x_A$

In magnitude, the higher is the dispersion of x. One may therefore, consider some way of combining the deviations to get a measure of dispersion. It is headily seen that the simple arithmetic mean of the deviations, viz. $\frac{1}{n}\sum_i(x_i - x_A)$, cannot serve this purpose, as the sum of the deviation and proportionally the arithmetic mean may be quite small even when the individual deviations are large, positive and negative deviations almost cancelling each other. In fact, if x_A is considered to the arithmetic mean of x, then the sum of the deviation vanishes, whatever the deviations are individually. This difficulty may be overcome by considering, their absolute values instead of the deviations in which the magnitude of the deviations (and not their sign) will be considered. The arithmetic mean of the absolute deviations of x_i from x_A is the required measure of dispersion and is referred to as the mean deviation of x about x_A, denoted by MD_A and given by-

$$MD_A = \frac{1}{n}\sum_{i=1}^{n} f_i|x_i - x_A| \tag{2.1}$$

It can be shown that MD_A is least when measure about median. Let us consider the same example as discussed above.

Marks obtained By $A(x^1)$	$x_i^{(1)} - x^{-(1)}$	**Marks obtained by $A(x^{(2)})$**	$x_i^{(2)} - x^{-(2)}$
63	5	61	3
47	-11	54	-4
56	-2	56	-2
44	-14	57	-1
66	8	60	2
65	7	59	1
80	22	55	-3
43	-15	62	4

Let

$\boldsymbol{x^{-(1)}} = \boldsymbol{x^{-(2)}} = \bar{x} = 58 = x_A$ (Arithmetic Mean)

$\therefore \sum_{i=1}^{8}\left|x_i^{(1)} - x_A\right|$=5+11+2+14+8+7+22+15=84

$\therefore \frac{1}{8}\sum_{i=1}^{8} |x_i - x_A| = \frac{84}{8} = 10.5$

$\sum_{i=1}^{8} \left|x_i^{(2)} - x_A\right|$=3+4+2+1+2+1+3+4=20

$\therefore \frac{1}{8}\sum_{i=1}^{8} \left|x_i^{(2)} - x_A\right| = \frac{20}{8} = 2.5$

∴ Mean deviation of the marks obtained by the students A and B about the arithmetic mean 58 are 10.5 and 2.5, respectively.

If (x_i, f_i), i=1, 2, ………………n be the frequency distribution of a variable x, the mean deviation of x about the average x_A (may be mean, median or mode) is given by…

$MD_A = \frac{1}{N}\sum_{i=1}^{n} f_i|x_i - x_A|$; $\sum_{i=1}^{n} f_i = N$ (2.2)

Example 9.

Calculate the mean deviation from mean (A.M) for the following data:

Marks:	0-10	10-20	20-30	30-40	40-50	50-60	60-70
No. of Students:	6	5	8	15	7	6	3

Marks	**Mid Points** ($\boldsymbol{x_i}$)	**No. of students** ($\boldsymbol{f_i}$)	$\boldsymbol{d_i = \frac{x_i - A}{h}}$	$\boldsymbol{d_i f_i}$	$\boldsymbol{\lvert x_i - \bar{x}\rvert}$ $\boldsymbol{\lvert x_i - 33.4\rvert}$
0-10	5	6	-3	-18	28.4
10-20	15	5	-2	-10	18.4
20-30	25	8	-1	--8	8.4
30-40	35(=A)	15	0	0	1.6
40-50	45	7	1	7	11.6
50-60	55	6	2	12	21.6
60-70	65	3	3	9	31.6
		50 (=N)		**8**	

$$N = \sum_{i=1}^{n} f_i = 50$$

$Let\ A = 35\ \&\ h = 10$

$A.m.=\bar{x} = A + \frac{h}{n}\sum_{i=1}^{7} f_i d_i = 35 + \frac{10}{50}(-8)$

=35-1.6 = **33.4 =** $\boldsymbol{\bar{x}}$

$$MD_{\bar{x}} = \frac{1}{N}\sum_{i=1}^{7} f_i|x_i - \bar{x}|$$

$= \frac{1}{50}[6 \times 28.4 + 5 \times 18.4 + 8 \times 8.4 + 15 \times 1.6 + 7 \times 11.6 + 6 \times 21.6 + 3 \times 31.6]$

$= \quad \boldsymbol{\frac{659.2}{50}}$

=13.184

Standard deviation and Root mean square deviation:

Standard deviation, usually denoted by the Greek letter small sigma (σ), is the positive square root of the arithmetic mean of the squares of the deviations of the given values from their arithmetic mean. For the frequency distribution $x_i \mid f_i \ (i = 1, 2, \ldots \ldots \ldots \ldots \ldots . n)$

For the arithmetic mean $\bar{x}$ of distribution

$$\sigma = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2} \tag{2.3}$$

For the frequency distribution $(\boldsymbol{x_i}, \boldsymbol{f_i})$ **,** i =1, 2, … … … … … . . , k

s. d. of the variable x is given by

$$\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^{k} f_i (x_i - \bar{x})^2} \quad , \quad N = \sum_{i=1}^{k} f_i \tag{2.4}$$

The square of the standard deviation (s. d.) is called the variance and is given by

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 \tag{2.5}$$

Or, $$\sigma^2 = \frac{1}{N} \sum_{i=1}^{k} (x_i - \bar{x})^2 \tag{2.6}$$

Root mean square deviation, denoted by "s" is given by:

$$s = \sqrt{\frac{1}{n} \sum_{i=1}^{k} f_i (x_i - \bar{A})^2} , \qquad N = \sum_{i=1}^{k} f_i \tag{2.7}$$

or, $$s = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{A})^2} , \tag{2.8}$$

where A is an arbitrary number s^2 is called the mean square deviation.

Relation between $\boldsymbol{\sigma}$ and s:

By definition

$$s^2 = \frac{1}{N} \sum_{i=1}^{k} f_i (x_i - A)^2$$

$$= \frac{1}{N} \sum_{i=1}^{k} f_i (x_i - \bar{x} + \bar{x} - A)^2$$

$$= \frac{1}{N} \sum_{i=1}^{k} f_i \{(x_i - \bar{x})^2 + (\bar{x} - A)^2 + 2(\bar{x} - A)(x_i - \bar{x})\}$$

$$= \frac{1}{N} \sum_{i=1}^{k} f_i (x_i - \bar{x})^2 + f_i (\bar{x} - A)^2 + 2(\bar{x} - A) \frac{1}{N} \sum_{i=1}^{k} f_i (x_i - \bar{x})$$

$= \frac{1}{N} \sum_{i=1}^{k} f_i \, (x_i - \bar{x})^2 + f_i(\bar{x} - A)^2 + 0 \qquad \therefore \sum_{i=1}^{k} f_i \, (x_i - \bar{x}) = 0$

$\therefore s^2 = \sigma^2 + d^2$, Where $d = \bar{x} - A$.

$\therefore s^2 \geq \sigma^2$

s^2 will be least when $d = 0$, i.e. $\bar{x} = A$, hence, we can conclude that mean square deviation and consequently root mean square deviation is least when the deviation are taken about A= $\bar{x}$, i.e., standard deviation is the least value of root mean square deviation.

Result1.

$$\sigma_x^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i^2 - 2x_i\bar{x} + \bar{x}^{\,2})$$

$$= \frac{1}{n} \sum_{i=1}^{n} x_i^2 - 2\bar{x}^{\,2} + \bar{x}^{\,2}$$

$$\boxed{\boldsymbol{\sigma_x^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \bar{x}^{\,2}}} \tag{2.9}$$

Result.2

$$\sigma_x^2 = \frac{1}{N} \sum_{i=1}^{k} f_i (x_i - \bar{x})^2 = \frac{1}{N} \sum_{i=1}^{k} f_i \, (x_i^2 - 2x_i\bar{x} + \bar{x}^{\,2})$$

$$= \frac{1}{N} \sum_{i=1}^{k} f_i \, x_i^2 - 2\bar{x} \, \frac{1}{N} \sum_{i=1}^{k} f_i \, x_i + f_i \, \bar{x}^{\,2})$$

$$= \frac{1}{N} \sum_{i=1}^{k} f_i \, x_i^2 - 2\bar{x}^{\,2} + \bar{x}^{\,2}$$

$$= \frac{1}{N} \sum_{i=1}^{k} f_i \, x_i^2 - \bar{x}^{\,2} \tag{2.10}$$

Variance of the combined series:

Let us suppose that we have two series of data of sizes x_1 and x_2, with mean $\overline{x_1}$ and $\overline{x_2}$, and standard deviations $\sigma_1$and σ_2, respectively. Then the standard deviation σ of the combined series of data of size $x_1 + x_2$, is given by

$$\sigma^2 = \frac{1}{n_{1+n_2}} \left[n_1 \left(\sigma_1{}^2 + d_1{}^2 \right) + n_2 \left(\sigma_2{}^2 + d_2{}^2 \right) \right] \tag{2.11}$$

where, $d_1 = \overline{x_1} - \bar{x}$, $d_2 = \overline{x_2} - \bar{x}$, $and \; \bar{x} = \frac{n_1\overline{x_1} + n_2\overline{x_2}}{n_{1+n_2}}, is \; the \; mean \; of \; the \; combined \; series.$

Proof: Let x_{1i} ; $i = 1, 2, \dots \dots \dots \dots, n_1$ and x_{2j} ; $j = 1, 2, \dots \dots \dots \dots n_2$

be two series of data, then n_1

$$\left.\begin{array}{l}\overline{x_1} = \frac{1}{n_1}\sum_{i=1}^{n_1} x_{1i} \\ \overline{x_2} = \frac{1}{n_2}\sum_{j=1}^{n_2} x_{2j}\end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l}\sigma_1{}^2 = \frac{1}{n_1}\sum_{i=1}^{n_1}(x_{1i} - \overline{x_1}) \\ \sigma_2{}^2 = \frac{1}{n_2}\sum_{i=1}^{n_2}(x_{2i} - \overline{x_1})^2\end{array}\right\}$$

Then the mean $\bar{x}$ of the combined series is given by

$$\bar{x} = \frac{1}{n_{1+n_2}}\left(\sum_{i=1}^{n_1} x_{1i} + \sum_{j=1}^{n_2} x_{2j}\right) = \frac{n_1\overline{x_1} + n_2\overline{x_2}}{n_{1+n_2}},$$

The variance σ^2 of the combined series of data is given by

$$\sigma^2 = \frac{1}{n_{1+n_2}}\left[\sum_{i=1}^{n_1}(x_{1i-}\bar{x})^2 + \sum_{j=1}^{n_2}(x_{2j} - \bar{x})^2\right]$$

Now

$$\sum_{i=1}^{n_1}(x_{1i-}\bar{x})^2 = \sum_{i=1}^{n_1}(x_{1i-}\overline{x_1} + \overline{x_1} - \bar{x})^2$$

$$= \sum_{i=1}^{n_1}(x_{1i-}\bar{x})^2 + n_1(\overline{x_1} - \bar{x})^2 + 2(\overline{x_1} - \bar{x})\sum_{i=1}^{n_1}(x_{1i-}\overline{x_1})$$

$$= n_1\sigma_1{}^2 + n_1(\overline{x_1} - \bar{x})^2 + 0 \qquad : \sum_{i=1}^{n_1}(x_{1i-}\overline{x_1}) = 0$$

$$= n_1\sigma_1{}^2 + n_1 d_1{}^2, \quad \text{Where } d_1 = \overline{x_1} - \bar{x}$$

Similarly, we get

$$\sum_{j=1}^{n_2}(x_{2j} - \bar{x})^2 = n_2\sigma_2{}^2 + n_2 d_{2,}{}^2, \text{ where } d_2 = \overline{x_2} - \bar{x}$$

$$\textit{Hence, } \sigma^2 = \frac{1}{n_{1+n_2}}\left[n_1\left(\sigma_1{}^2 + d_1{}^2\right) + n_2\left(\sigma_2{}^2 + d_2{}^2\right)\right]$$

Variance and consequently standard deviation is independent of change of origin

Let $d_i = x_i - A$, A *is an arbitray value*

$$\frac{1}{n}\sum_{i=1}^{k} f_i d_i = \frac{1}{n}\sum_{i=1}^{k} f_i(x_i - A) = \frac{1}{n}\sum_{i=1}^{k} f_i x_i - A$$

⇨ $\bar{d} = \bar{x} - A$

$$\therefore d_i - \bar{d} = (x_i - A) - (\bar{x} - A) = x_i - \bar{x}$$

$$\sigma_x^2 = \frac{1}{N}\sum_{i=1}^{k} f_i(x_i - \bar{x})^2 = \frac{1}{N}\sum_{i=1}^{k} f_i(d_i - \bar{d})^2$$

$$\boldsymbol{\sigma_x^2 = \sigma_d^2} \tag{2.12}$$

Or, $\sigma_x^2 = \frac{1}{n}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n}(d_i - \bar{d})^2 = \boldsymbol{\sigma_d^2}$

Variance is independent of change of origin.

Let $d_i = \frac{(x_i - A)}{h}$ => $x_i = A + h\, d_i$

$\therefore\ \bar{x} = A + \frac{h}{n} \sum_{i=1}^{k} f_i\, d_i = A + h\, \bar{d}$

$\therefore\ x_i - \bar{x} = h\,(d_i - \bar{d})$

$\therefore\ \sigma_x^2 = \frac{1}{n}\sum_{i=1}^{n} f_i (x_i - \bar{x})^2 = \frac{h^2}{n}\sum_{i=1}^{n}(d_i - \bar{d})^2 = h^2\, \sigma_d^2$

or

$$\boxed{\sigma_x^2 = h^2\, \sigma_d^2} \tag{2.13}$$

=> Variance is independent of change of change of origin but not of scale.

Example 10: Calculate the mean and standard deviation for the following table giving the age distribution of 542 members.

Age (in years):	20-30	30-40	40-50	50-60	60-70	70-80	80-90
No. of members:	3	61	132	153	140	51	2

Age Group	**Mid Values** (x_i)	**No. of Members** (f_i)	$d_i = \frac{x_i - A}{h}$	$f_i d_i$	$f_i {d_i}^2$
20-30	25	3	-3	-9	27
30-40	35	61	-2	-122	244
40-50	45	132	-1	-132	132
50-60	55(=A)	153	0	0	0
60-70	65	140	1	140	140
70-80	75	51	2	102	204
80-90	85	2	3	6	18
Total		**542**	**0**	**-15**	**765**

$\bar{x} = \quad A + \frac{h}{N} \sum_i f_i d_i = \frac{55+10\times(-15)}{542} = 55 - 0.28$

=54.72 (years). $\cong 55$

$\therefore \sigma_x^2 = h^2 \sigma_d^2 = h^2 \left[\frac{1}{N}\sum_i f_i d_i^2 - (\frac{1}{N} f_i d_i^2)\right]$

$=100\left[\frac{765}{542} - (\frac{-15}{542})^2\right]$= 141.07.

$\therefore$ Standard deviation (σ_x) = $\sqrt{141.07}$ years = 11.80 years

Example 11.

Calculate the s. d. of the marks obtained by two students A and B of a college in eight monthly examinations in a particular subject.

Marks obtained by A:	63	47	56	44	66	65	80	43
Marks obtained by B:	61	54	56	57	60	59	55	62

Solution: Let us consider series I ($x^{(1)}$) as marks obtained by A and series II ($x^{(2)}$) as the marks obtained by B.

x_i^1	$d_i^{(1)} = x_i^{(1)} - \bar{x}$	$(d_i^1)^2$	$x_i^{(2)}$	$d_i^{(2)} = x_i^{(2)} - \bar{x}$	$(d_i^{(2)})^2$
63	5	25	61	3	9
47	-11	121	54	-4	16
56	-2	4	56	-2	4
44	-14	196	57	-1	1
66	8	64	60	2	4
65	7	49	59	1	1
80	22	484	55	-3	9
43	-15	225	62	4	16
Total	**0**	**1168**	**464**	**0**	**6**

$\therefore \bar{x}^1 = 58 = \bar{x}^2 = \bar{x}$

$\therefore$ Arithmetic mean of the marks obtained by both the students A and B are same, i.e., 58.

$$\sigma^2_{x^{(1)}} = \sigma^2_{d^{(1)}} = \frac{1}{n}\sum_i (d_i^{(1)})^2 - (\frac{1}{n}\sum_i d_i^{(1)})^2$$

$$=\frac{1}{8} \times 1168 - \left(\frac{1}{8} \times 0\right) = 146$$

$$= \sigma_{x^{(1)}} = \sqrt{146} = 12.08$$

$$\sigma^2_{x^{(2)}} = \sigma^2_{d^{(2)}} = \frac{1}{n}\sum_i (d_i^{(2)})^2 - (\frac{1}{n}\sum_i d_i^{(2)})^2$$

$$=\frac{1}{8} \times 60 - \left(\frac{1}{8} \times 0\right) = 7.5$$

$$\therefore \sigma_2 = 2.739$$

Hence, it is observed that though $\bar{x}^{(1)} = 58 = \bar{x}^{(2)}$, but $\sigma^2_{x^{(1)}} > \sigma^2_{x^{(2)}}$.

- ⇨ Marks obtained by the student B is consistent and remain near about 58 throughout, whereas A received as high score as so and as low score as 43.
- ⇨ Thus arithmetic mean of the marks obtained by A and B gives an overall idea about the nature of the marks obtained by B is all eight examination where as it fails to give an idea about the nature of the marks obtained by A as $S.D.$ of the marks obtained by B is very- very less in compare to that of A. Hence, $A.M.$ can be a good approximation of the marks obtained by B whereas it will not be a good approximation for the less of marks obtained by A.

References:

(i) Mathematical Statistics, J. N. Kapur & H. C. Saxena.

(ii) Probability and Statistics for Engineers, Irwin Miller & John E. Freund.

(iii) Elements of Probability and Statistics, A. P. Baisnab & M. Jas.

(iv) Statistics, Schaum's Outlines, M. R. Spiegel & L. J. Stephens.

(v) Probability and Statistics, Schaum's Outlines, M. R. Spiegel, J. J. Schiller, R. A. Srinivasan.