Lecture Note

Microeconomic Theory 1

- Basic analytical framework of modern economics:
  - Economic environments: Number of agents, individuals' characteristics (preference, technology, endowment), information structures, institutional economic environments
  - Behavioral assumption: Selfish and rational agents
  - Economics institutional arrangements: Economic mechanism
  - Equilibrium
  - Evaluation: Welfare analysis (first-best, second-best)
- Mathematics as a tool for economists
  - To describe model and assumptions clearly and precisely
  - To make analysis rigorous
  - To obtain results that may not be available through verbal arguments
  - To reduce unnecessary debates

Chapter 1

Consumer Theory

4 Building blocks: consumption set, feasible set (Budget set), preference relation, and behavioral assumption

1.1 Consumption Set and Budget Constraint

Assume that there are L goods.

- Consumption set: set of all conceivable consumption bundles $=: X \ni \mathbf{x} = (x_1, \cdots, x_L)$
  - Usually $X = \mathbb{R}_+^L$ but more specific sometimes

    cf) $X =$ bundles that gives the consumer a subsistence existence
  - We assume that $X \subseteq \mathbb{R}_+^L$ is closed and convex
- Budget constraint: set of all affordable bundles given prices $\mathbf{p} = (p_1, \cdots, p_L)$ and income m, which is given by

$$B(\mathbf{p}, m) = \{\mathbf{x} \in X | \mathbf{p} \cdot \mathbf{x} \leq m\}$$

  cf) Budget constraint with 2 goods

1.2 Preferences and Utility

- Preference: binary relation on X to compare or order the bundles
  - $\mathbf{x} \succeq \mathbf{y}$: '$\mathbf{x}$ is at least as good as $\mathbf{y}$' $\rightarrow$ weak preference
  - $\mathbf{x} \succ \mathbf{y}$ iff $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \not\succeq \mathbf{x}$: '$\mathbf{x}$ is strictly preferred to $\mathbf{y}$' $\rightarrow$ strict preference
  - $\mathbf{x} \sim \mathbf{y}$ iff $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{x}$: '$\mathbf{x}$ is indifferent to $\mathbf{y}$' $\rightarrow$ indifference relation
- Subsets of X derived from the preference relation: for a given bundle $\mathbf{y} \in X$,
  - Upper contour set: $P(\mathbf{y}) = \{\mathbf{x} \in X | \mathbf{x} \succeq \mathbf{y}\}$
  - Strictly upper contour set: $P_s(\mathbf{y}) = \{\mathbf{x} \in X | \mathbf{x} \succ \mathbf{y}\}$
  - Lower contour and strictly lower contour sets, denoted $L(\mathbf{y})$ and $L_s(\mathbf{y})$
  - Indifference set (or curve): $I(\mathbf{y}) = \{x \in X | x \sim y\}$
- The preference relation $\succeq$ on X is called *rational* if it possesses the following properties:
  - Complete: $\forall \mathbf{x}, \mathbf{y} \in X$, either $\mathbf{x} \succeq \mathbf{y}$ or $\mathbf{y} \succeq \mathbf{x}$
  - Transitive: $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X$, if $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{z}$, then $\mathbf{x} \succeq \mathbf{z}$
- Other desirable properties
  - Monotonicity: $\forall \mathbf{x}, \mathbf{y} \in X$, if $\mathbf{x} \geq \mathbf{y}$, then $\mathbf{x} \succeq \mathbf{y}$ while if $\mathbf{x} \gg \mathbf{y}$, then $\mathbf{x} \succ \mathbf{y}$
  - Strict monotonicity: $\forall \mathbf{x}, \mathbf{y} \in X$, if $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$, then $\mathbf{x} \succ \mathbf{y}$
  - Continuity: $\forall \mathbf{y} \in X$, $P(\mathbf{y})$ and $L(\mathbf{y})$ are closed or $P_s(\mathbf{y})$ and $L_s(\mathbf{y})$ are open

    EXAMPLE 1.2.1. **(Lexicographic Preference)** Assume $L = 2$. Define $\succeq$ as follows: $\mathbf{x} \succeq \mathbf{y}$ iff either '$x_1 > y_1$' or '$x_1 = y_1$ and $x_2 \geq y_2$' $\rightarrow$ Neither continuous nor even upper semi-continuous (or the upper contour set is not closed)
  - Non-satiation: $\forall \mathbf{x} \in X$, $\exists \mathbf{y} \in X$ such that $\mathbf{y} \succ \mathbf{x}$
  - Local non-satiation: $\forall \mathbf{x} \in X$ and $\forall \epsilon > 0$, $\exists \mathbf{y} \in X$ with $|\mathbf{x} - \mathbf{y}| < \epsilon$ such that $\mathbf{y} \succ \mathbf{x}$
  - Convexity: If $\mathbf{x} \succeq \mathbf{y}$, then $t\mathbf{x} + (1-t)\mathbf{y} \succeq \mathbf{y}, \forall t \in [0, 1]$
  - Strict convexity: If $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$, then $t\mathbf{x} + (1-t)\mathbf{y} \succ \mathbf{y}, \forall t \in (0, 1)$

1.3 The Utility Function

Definition 1.3.1. *A real-valued function $u : X \to \mathbb{R}$ is called a utility function representing the preference relation $\succeq$ iff $\forall \mathbf{x}, \mathbf{y} \in X$, $u(\mathbf{x}) \geq u(\mathbf{y}) \Leftrightarrow \mathbf{x} \succeq \mathbf{y}$.*

Example 1.3.1. **(Some Utility Functions)**

(1) Cobb-Douglas: $u(\mathbf{x}) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_L^{\alpha_L}$ with $\alpha_\ell > 0, \forall \ell \to$ continuous, strictly monotone, and strictly convex in $\mathbb{R}^L_{++}$

(2) Linear: $u(\mathbf{x}) = \sum_{\ell=1}^{L} \alpha_\ell x_\ell \to$ continuous, strictly monotone, and convex in $\mathbb{R}^L_+$

(3) Leontief: $u(\mathbf{x}) = \min\{\alpha_1 x_1, \cdots, \alpha_L x_L\} \to$ continuous, monotone, and convex in $\mathbb{R}^L_+$

Proposition 1.3.1. *A preference relation $\succeq$ can be represented by a utility function only if it is rational.*

Proof. Straightforward. ▮

The converse, however, does not hold. That is, a rational preference in itself does *not* guarantee the existence of utility function representing it.

Theorem 1.3.1. **(Existence of a Utility Function)** *Suppose that preference relation $\succeq$ is complete, reflexive, transitive, continuous, and strictly monotonic. Then, there exists a continuous utility function $u : \mathbb{R}^L_+ \to R$ which represents $\succeq$.*

Proof. Let $\mathbf{e} = (1, 1, \cdots, 1) \in \mathbb{R}^L_+$. Then, given any vector $\mathbf{x} \in R^L_+$, let $u(\mathbf{x})$ be defined such that $\mathbf{x} \sim u(\mathbf{x})\mathbf{e}$. We first show that $u(\mathbf{x})$ exists and is unique.

Existence: Let $B = \{t \in \mathbb{R} | t\mathbf{e} \succeq x\}$ and $W = \{t \in \mathbb{R} | \mathbf{x} \succeq t\mathbf{e}\}$. Neither B nor W is empty while $B \cap W = \mathbb{R}_+$. Also, the continuity of $\succeq$ implies both sets are closed. Since the real line is connected, there exists $t_x \in \mathbb{R}$ such that $t_x\mathbf{e} \sim \mathbf{x}$.

Uniqueness: If $t_1\mathbf{e} \sim \mathbf{x}$ and $t_2\mathbf{e} \sim \mathbf{x}$, then $t_1\mathbf{e} \sim t_2\mathbf{e}$ by the transitivity of $\sim$. So, by strict monotonicity, $t_1 = t_2$.

Next, we show that $u(\cdot)$ represents $\succeq$, which results from the following: $\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^L_+$,

$$\begin{aligned}
&\mathbf{x}^1 \succeq \mathbf{x}^2 \\
\Leftrightarrow &u(\mathbf{x}^1)\mathbf{e} \sim \mathbf{x}^1 \succeq \mathbf{x}^2 \sim u(\mathbf{x}^2)\mathbf{e} \\
\Leftrightarrow &u(\mathbf{x}^1)\mathbf{e} \succeq u(\mathbf{x}^2)\mathbf{e} \\
\Leftrightarrow &u(\mathbf{x}^1) \geq u(\mathbf{x}^2).
\end{aligned}$$

Continuity of $u(\cdot)$: Consider $\{\mathbf{x}^m\}$ with $\mathbf{x}^m \to \mathbf{x}$. Suppose to the contrary that $u(\mathbf{x}^m) \nrightarrow u(\mathbf{x})$. Consider the case $u' := \lim_{k\to\infty} u(\mathbf{x}^m) > u(\mathbf{x})$. Then, by monotonicity, $u'\mathbf{e} \succ u(\mathbf{x})\mathbf{e}$. Let $\hat{u} := \frac{1}{2}[u' + u(\mathbf{x})]$. By monotonicity, $\hat{u}\mathbf{e} > u(\mathbf{x})\mathbf{e}$. Now, since $u(\mathbf{x}^m) \to u' > \hat{u}$, $\exists M$ such that for all $m > M$, $u(\mathbf{x}^m) > \hat{u}$. For all such m, $\mathbf{x}^m \sim u(\mathbf{x}^m)\mathbf{e} \succ \hat{u}\mathbf{e}$. By the continuity of $\succeq$, this would imply $\mathbf{x} \succeq \hat{u}\mathbf{e}$, which in turn implies $u(\mathbf{x})\mathbf{e} \sim \mathbf{x} \succeq \hat{u}\mathbf{e}$, a contradiction. The other case $u' < u(\mathbf{x})$ can be dealt with similarly. ▮

Example 1.3.2. **(Non-representation of Lexicographic Preference by a Utility Function)** *Lexicographic preference cannot be represented by any function whether continuous or not.*

The utility function is said to be unique up to the monotonic transformation in the following sense.

Theorem 1.3.2. **(Invariance of Utility Function to Positive Monotonic Transforms)** *If $u(\mathbf{x})$ represents some preference $\succeq$ and $f : R \to R$ is strictly increasing, then $v(\mathbf{x}) = f(u(\mathbf{x}))$ represents the same preference.*

Proof. This is because $f(u(\mathbf{x})) \geq f(u(\mathbf{y}))$ iff $u(\mathbf{x}) \geq u(\mathbf{y})$. ▮

The utility function inherits the properties of the preference relation that it represents.

Theorem 1.3.3. *Let $\succeq$ be represented by $u : \mathbb{R}^L_+ \to \mathbb{R}$. Then,*

1. *$u(x)$ is strictly increasing iff $\succeq$ is strictly monotonic.*
2. *$u(x)$ is quasiconcave iff $\succeq$ is convex.*
3. *$u(x)$ is strictly quasiconcave iff $\succeq$ is strictly convex.*

Remark 1.3.1. The strict quasiconcavity of $u(\mathbf{x})$ can be checked by verifying if the principal minors of bordered Hessian have determinants alternating in sign:

$$\begin{vmatrix} 0 & u_1 & u_2 \\ u_1 & u_{11} & u_{12} \\ u_2 & u_{21} & u_{22} \end{vmatrix} > 0,$$

$$\begin{vmatrix} 0 & u_1 & u_2 & u_3 \\ u_1 & u_{11} & u_{12} & u_{13} \\ u_2 & u_{21} & u_{22} & u_{23} \\ u_3 & u_{31} & u_{32} & u_{33} \end{vmatrix} < 0,$$

and so on, where $u_\ell = \frac{\partial u}{\partial x_\ell}$ and $u_{\ell k} = \frac{\partial^2 u}{\partial x_\ell \partial x_k}$.

- Marginal rate of substitution: given a bundle $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$,

$$MRS_{12} \text{ at } \mathbf{x} := \frac{\partial u(\mathbf{x})/\partial x_1}{\partial u(\mathbf{x})/\partial x_2}$$

  - This measures the (absolute value of) slope of indifference curve at bundle $\mathbf{y}$
  - Invariance of MRS to monotone transforms: if $v(\mathbf{x}) = f(u(\mathbf{x}))$, then

$$\frac{\partial v(\mathbf{x})/\partial x_1}{\partial v(\mathbf{x})/\partial x_2} = \frac{f'(u)\partial u(\mathbf{x})/\partial x_1}{f'(u)\partial u(\mathbf{x})/\partial x_2} = \frac{\partial u(\mathbf{x})/\partial x_1}{\partial u(\mathbf{x})/\partial x_2}$$

1.4 Utility Maximization and Optimal Choice

A fundamental hypothesis in the consumer theory is that a rational consumer will choose a most preferred bundle from the set of affordable alternatives.

- Utility maximization problem: for $\mathbf{p} \gg 0$, and $m > 0$,

$$\max_{\mathbf{x}\in\mathbb{R}^L_+} u(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{x} \in B(\mathbf{p}, m) \text{ or } \mathbf{p}\cdot\mathbf{x} \le m.$$

  - At least one solution exists since $B(\mathbf{p}, m)$ is compact and $u(\mathbf{x})$ is continuous (by the Weierstrass theorem).
  - If $u(\mathbf{x})$ is strictly quasiconcave, then the solution is unique.
  - If $u(\mathbf{x})$ is locally non-satiated, then the budget constraint is binding at the optimum.

- Marshallian demand correspondence or function:

$$\mathbf{x}(\mathbf{p}, m) = \arg\max_{\mathbf{x}\in\mathbb{R}^L_+} u(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{p}\cdot\mathbf{x} \le m$$

  - $\mathbf{x}(t\mathbf{p}, tm) = \mathbf{x}(\mathbf{p}, m)$: homogeneous of degree zero
  - $\mathbf{x}(\mathbf{p}, m)$ is continuous by the Berge's Maximum Theorem.

- Lagrangian method and first-order condition:

$$\mathcal{L}(\mathbf{x}, \lambda) = u(\mathbf{x}) + \lambda[m - \mathbf{p}\cdot\mathbf{x}],$$

  where $\lambda \ge 0$ is Lagrangian multiplier associated with the budget constraint.

  - If $\mathbf{x}^* \in \mathbf{x}(\mathbf{p}, m)$, then there exists a Lagrange multiplier $\lambda \ge 0$ such that for all $i = 1, \cdots, L$,

$$\frac{\partial u(\mathbf{x}^*)}{\partial x_\ell} \le \lambda p_\ell, \text{ with equality if } x^*_\ell > 0.$$

  - If $\mathbf{x}^* \gg 0$, then it is called interior solution: for all ℓ and k,

$$\frac{\partial u(\mathbf{x}^*)}{\partial x_\ell} = \lambda p_\ell,$$

$$\text{so } \frac{\partial u(\mathbf{x}^*)/\partial x_\ell}{\partial u(\mathbf{x}^*)/\partial x_k} = \frac{p_\ell}{p_k}$$

– This may not hold if the solution is not interior

$$MRS_{12}(\mathbf{x}^*) > \frac{p_1}{p_2}$$

– λ measures the change in utility from a marginal increase in m:

$$\sum_{\ell=1}^{L} \frac{\partial u(\mathbf{x}(\mathbf{p}, m))}{\partial x_\ell} \frac{\partial x_\ell(\mathbf{p}, m)}{\partial m} = \sum_{\ell=1}^{L} \lambda p_\ell \frac{\partial x_\ell(\mathbf{p}, m)}{\partial m} = \lambda,$$

$\rightarrow \lambda$: 'shadow price' of income.

Example 1.4.1. **(Demand function for the Cobb-Douglas Utility Function)** By log transform $f(u) = \ln u$, we have $v(\mathbf{x}) = f(u(\mathbf{x})) = \alpha \ln x_1 + (1-\alpha)\ln x_2$. Then, the consumer solves

$$\max_{x_1, x_2} \alpha \ln x_1 + (1-\alpha)\ln x_2 \quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 = m,$$

which results in the following F.O.C.'s

$$\frac{\alpha}{x_1} = \lambda p_1$$
$$\frac{1-\alpha}{x_2} = \lambda p_2.$$

So, at the optimum, $p_1 x_1 = \frac{\alpha}{1-\alpha} p_2 x_2$, which can be substituted into the budget constraint to yield

$$p_1 x_1 = \frac{\alpha}{1-\alpha}(m - p_1 x_1).$$

Thus, we have $x_1(\mathbf{p}, m) = \frac{\alpha m}{p_1}$ and $x_2(\mathbf{p}, m) = \frac{(1-\alpha)m}{p_2}$.

- Indirect utility function $v : \mathbb{R}_+^L \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is given by

$$v(\mathbf{p}, m) = \max_{\mathbf{x} \in \mathbb{R}_+^L} u(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{p} \cdot \mathbf{x} \leq m.$$

– Hence, $v(\mathbf{p}, m) = u(\mathbf{x}(\mathbf{p}, m))$

PROPOSITION 1.4.1. **(Properties of the Indirect Utility Function)** *If $u(\mathbf{x})$ is continuous and locally non-satiated on $\mathbb{R}^L_+$ and $(\mathbf{p}, m) \gg 0$, then the indirect utility function is*

(1) Homogeneous of degree zero

(2) Nonincreasing in $\mathbf{p}$ and strictly increasing in m

(3) Quasiconvex in $\mathbf{p}$ and m.

(4) Continuous in $\mathbf{p}$ and m.

Proof. (1) Follows from $v(t\mathbf{p}, tm) = u(\mathbf{x}(t\mathbf{p}, tm)) = u(\mathbf{x}(\mathbf{p}, m)) = v(\mathbf{p}, m)$.
(2) Note that $B(\mathbf{p}', m) \subseteq B(\mathbf{p}, m)$ for $\mathbf{p}' \geq \mathbf{p}$. Then,

$$v(\mathbf{p}, m) = \max_{\mathbf{x} \in B(\mathbf{p}, m)} u(\mathbf{x}) \geq \max_{\mathbf{x} \in B(\mathbf{p}', m)} u(\mathbf{x}) = v(\mathbf{p}', m).$$

The strict monotonicity regarding m follows from a similar argument and the fact that $u(\mathbf{x})$ is locally non-satiated.
(3) Suppose that $v(\mathbf{p}, m) \leq \bar{v}$ and $v(\mathbf{p}', m') \leq \bar{v}$. Let $(\mathbf{p}'', m'') = (t\mathbf{p} + (1-t)\mathbf{p}', tm + (1-t)m')$. We need to show that $v(\mathbf{p}'', m'') \leq \bar{v}$

Let us first show that $B(\mathbf{p}'', m'') \subseteq B(\mathbf{p}, m) \cup B(\mathbf{p}', m')$. Suppose not. There must exist an $\mathbf{x}$ such that $(t\mathbf{p} + (1-t)\mathbf{p}') \cdot \mathbf{x} \leq tm + (1-t)m'$ but $\mathbf{p} \cdot \mathbf{x} > m$ and $\mathbf{p}' \cdot \mathbf{x} > m'$. Two inequalities imply

$$t\mathbf{p} \cdot \mathbf{x} > tm$$
$$(1-t)\mathbf{p}' \cdot \mathbf{x} > (1-t)m',$$

which sum to

$$(t\mathbf{p} + (1-t)\mathbf{p}') \cdot \mathbf{x} > tm + (1-t)m',$$

a contradiction.

Now note that

$$\begin{aligned} v(\mathbf{p}'', m'') &= \max u(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{x} \in B(\mathbf{p}'', m'') \\ &\leq \max u(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{x} \in B(\mathbf{p}, m) \cup B(\mathbf{p}', m') \\ &\leq \bar{v}. \end{aligned}$$

(4) Follows from the Berge's Maximum Theorem. ▌

1.5 Expenditure and Hicksian Demand Functions

In this section, we study the expenditure minimization problem: To minimize the expenditure needed to achieve a given level of utility.

- Expenditure minimization problem: Given a price vector $\mathbf{p}$ and utility level $u \in \mathbb{R}$,

$$\min_{\mathbf{x}\in\mathbb{R}_L^+} \mathbf{p}\cdot\mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) \geq u$$

  - The Hicksian demand is the solution of this problem and denoted as $\mathbf{h}(\mathbf{p}, u)$.
  - The expenditure function is defined as $e(\mathbf{p}, u) := \mathbf{p}\cdot\mathbf{h}(\mathbf{p}, u)$.

Proposition 1.5.1. **(Properties of the Expenditure Function)** *If $u(\mathbf{x})$ is continuous and locally non-satiated on $\mathbb{R}_+^L$ and $\mathbf{p} \gg 0$, then $e(\mathbf{p}, u)$ is*

(1) Homogeneous of degree 1 in $\mathbf{p}$.

(2) Strictly increasing in u and nondecreasing in $\mathbf{p}$.

(3) Continuous in $\mathbf{p}$ and u.

(4) Concave in $\mathbf{p}$.

If, in addition, $u(\mathbf{x})$ is strictly quasiconcave, we have

(5) Shephard's lemma: $h_\ell(\mathbf{p}, u) = \dfrac{\partial e(\mathbf{p}, u)}{\partial p_\ell}, \forall \ell$

Proof. We only prove (1), (4), and (5).

(1) We first prove that $h(\mathbf{p}, u)$ is homogeneous of degree zero in $\mathbf{p}$, that is $h(t\mathbf{p}, u) = h(\mathbf{p}, u)$:

$$\begin{aligned}\mathbf{h}(\mathbf{p}, u) &= \arg\min_{\mathbf{x}\in\mathbb{R}_+^L} \mathbf{p}\cdot\mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) \geq u \\ &= \arg\min_{\mathbf{x}\in\mathbb{R}_+^L} t\mathbf{p}\cdot\mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) \geq u = \mathbf{h}(t\mathbf{p}, u).\end{aligned}$$

Thus,

$$e(t\mathbf{p}, u) = t\mathbf{p} \cdot \mathbf{h}(t\mathbf{p}, u) = t\mathbf{p} \cdot \mathbf{h}(\mathbf{p}, u) = te(\mathbf{p}, u).$$

(4) Fix utility level at u, and let $\mathbf{p}'' = t\mathbf{p} + (1-t)\mathbf{p}'$ and $\mathbf{x}'' = \mathbf{h}(\mathbf{p}'', u)$. Then, we have

$$\begin{aligned} e(\mathbf{p}'', u) &= \mathbf{p}'' \cdot \mathbf{x}'' \\ &= t\mathbf{p} \cdot \mathbf{x}'' + (1-t)\mathbf{p}' \cdot \mathbf{x}'' \\ &\geq te(\mathbf{p}, u) + (1-t)e(\mathbf{p}', u), \end{aligned}$$

where the last inequality follows since $\mathbf{p} \cdot \mathbf{x}'' \geq e(\mathbf{p}, u)$ and $\mathbf{p}' \cdot \mathbf{x}'' \geq e(\mathbf{p}', u)$.

(5) For any price vector $\mathbf{p}$, consider $\mathbf{p}'$ such that $p'_\ell > p_\ell$ and $p'_k = p_k, \forall k \neq \ell$. We have the following inequality:

$$\begin{aligned} e(\mathbf{p}', u) - \mathbf{p}' \cdot \mathbf{h}(\mathbf{p}, u) &\leq 0 = e(\mathbf{p}, u) - \mathbf{p} \cdot \mathbf{h}(\mathbf{p}, u) \\ \text{or } e(\mathbf{p}', u) - e(\mathbf{p}, u) &\leq (p'_\ell - p_\ell)h_\ell(\mathbf{p}, u). \end{aligned}$$

Similarly, we have

$$e(\mathbf{p}', u) - e(\mathbf{p}, u) \geq (p'_\ell - p_\ell)h_\ell(\mathbf{p}', u).$$

Combining two inequalities yields

$$h_\ell(\mathbf{p}, u) \leq \frac{e(\mathbf{p}', u) - e(\mathbf{p}, u)}{p'_\ell - p_\ell} \leq h_\ell(\mathbf{p}', u).$$

So, by the continuity of $h_\ell(\mathbf{p}, u)$, $\lim_{p'_\ell \searrow p_\ell} \frac{e(\mathbf{p}',u)-e(\mathbf{p},u)}{p'_\ell - p_\ell} = h_\ell(\mathbf{p}, u)$. A parallel argument shows $\lim_{p'_\ell \nearrow p_\ell} \frac{e(\mathbf{p}',u)-e(\mathbf{p},u)}{p'_\ell - p_\ell} = h_\ell(\mathbf{p}, u)$. Thus, we have $\frac{\partial e(\mathbf{p},u)}{\partial p_\ell} = h_\ell(\mathbf{p}, u)$. ▮

Remark 1.5.1. The property (5) can also be derived using the envelope theorem: for some $\lambda \geq 0$,

$$e(\mathbf{p}, u) = \min_{\mathbf{x} \in \mathbb{R}^L_+} \mathcal{L}(\mathbf{x}, \lambda) = \min_{\mathbf{x} \in \mathbb{R}^L_+} \mathbf{p} \cdot \mathbf{x} + \lambda(u - u(\mathbf{x})).$$

Thus, by the envelope theorem,

$$\frac{\partial e(\mathbf{p}, u)}{\partial p_\ell} = \left. \frac{\partial \mathcal{L}(\mathbf{x}, u)}{\partial p_\ell} \right|_{\mathbf{x} = \mathbf{h}(\mathbf{p}, u)} = h_\ell(\mathbf{p}, u).$$

1.6 Some Important Identities

Indeed, the utility maximization and the expenditure minimization problems are closely related, which can be seen through the relationships between the value and solution functions resulting from solving two problems.

Theorem 1.6.1. **(Relationship between Indirect Utility and Expenditure Functions)** *Suppose that the utility function is continuous and strictly increasing. Then, for all* $\mathbf{p} \gg 0$*,* m*, and* u*,*

(1) $e(\mathbf{p}, v(\mathbf{p}, m)) \equiv m$.

(2) $v(\mathbf{p}, e(\mathbf{p}, u)) \equiv u$.

Proof. Note that, by definition, we have (i) $e(\mathbf{p}, v(\mathbf{p}, m)) \leq m$ since m is large enough to achieve $v(\mathbf{p}, m)$ and (ii) $v(\mathbf{p}, e(\mathbf{p}, u)) \geq u$ since $e(\mathbf{p}, u)$ achieves at least u.

(1) If $e(\mathbf{p}, v(\mathbf{p}, m)) < m$, then m is more than enough to achieve $v(\mathbf{p}, m)$ so extra money could be used to buy some more bundle, which, by strict monotonicity, will result in a higher utility, a contradiction.

(2) If $v(\mathbf{p}, e(\mathbf{p}, u)) > u$, then for small $\epsilon > 0$, $v(\mathbf{p}, e(\mathbf{p}, u) - \epsilon) > u$ due to the continuity of the indirect utility function (which is due to the continuity of utility function). But this means that the money needed to achieve u is at most $e(\mathbf{p}, u) - \epsilon$, a contradiction. ▌

This leads to the duality between Marshallian and Hicksian demand functions.

Theorem 1.6.2. **(Duality between Marshallian and Hicksian Demand Functions)** *Suppose that the utility function is continuous and strictly increasing. Then, for all* $\mathbf{p} \gg 0$*,* m*, and* u*,*

(1) $\mathbf{x}(\mathbf{p}, m) \equiv \mathbf{h}(\mathbf{p}, v(\mathbf{p}, m))$

(2) $\mathbf{h}(\mathbf{p}, u) \equiv \mathbf{x}(\mathbf{p}, e(\mathbf{p}, u))$

Proof. (1) Note that $u(\mathbf{x}(\mathbf{p}, m)) = v(\mathbf{p}, m)$ and, by the above theorem, $e(\mathbf{p}, v(\mathbf{p}, m)) \equiv m = \mathbf{p} \cdot \mathbf{x}(\mathbf{p}, m)$. Thus, $\mathbf{x}(\mathbf{p}, m)$ solves

$$\min_{\mathbf{x} \in \mathbb{R}^L_+} \mathbf{p} \cdot \mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) \geq v(\mathbf{p}, m),$$

which implies that $\mathbf{x}(\mathbf{p}, m) \equiv \mathbf{h}(\mathbf{p}, v(\mathbf{p}, m))$.

(2) Note that $e(\mathbf{p}, m) = \mathbf{p} \cdot \mathbf{h}(\mathbf{p}, u)$ and, by the above theorem, $v(\mathbf{p}, e(\mathbf{p}, u)) \equiv u = u(\mathbf{h}(\mathbf{p}, u))$. Thus, $\mathbf{h}(\mathbf{p}, u)$ solves

$$\max_{\mathbf{x} \in \mathbb{R}^L_+} u(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{p} \cdot \mathbf{x} \leq e(\mathbf{p}, u),$$

which implies that $\mathbf{h}(\mathbf{p}, u) \equiv \mathbf{x}(\mathbf{p}, e(\mathbf{p}, u))$. ▮

One nice application of the above identities is the Roy's identity.

Theorem 1.6.3. **(Roy's Identity)** *If $\mathbf{x}(\mathbf{p}, m)$ is the Marshallian demand function, then*

$$x_\ell(\mathbf{p}, m) = -\frac{\frac{\partial v(\mathbf{p}, m)}{\partial p_\ell}}{\frac{\partial v(\mathbf{p}, m)}{\partial m}},$$

provided that the RHS is well defined and $p_\ell > 0$ and $m > 0$.

Proof. Remember the identity, $u \equiv v(\mathbf{p}, e(\mathbf{p}, u))$. Differentiate this with p_ℓ and evaluate it at $u = v(\mathbf{p}, m)$ to get

$$0 \equiv \frac{\partial v(\mathbf{p}, m)}{\partial p_\ell} + \frac{\partial v(\mathbf{p}, m)}{\partial m} \frac{\partial e(\mathbf{p}, v(\mathbf{p}, m))}{\partial p_\ell}.$$

Thus, we have

$$x_\ell(\mathbf{p}, m) \equiv h_\ell(\mathbf{p}, v(\mathbf{p}, m)) = \frac{\partial e(\mathbf{p}, v(\mathbf{p}, m))}{\partial p_\ell} \equiv -\frac{\partial v(\mathbf{p}, m)/\partial p_\ell}{\partial v(\mathbf{p}, m)/\partial m},$$

where the first identity is due to (1) of Theorem 1.6.2 while the first equality is due to the Shepard's Lemma. ▮

There is another proof of Roy's identity, which uses the envelope theorem applied to the indirect utility function. Try it for yourself.

1.7 Properties of Demand Functions

Proposition 1.7.1. **(Properties of Hicksian demand)** *The substitution matrix defined as $\left(\frac{\partial h_\ell}{\partial p_k}\right)_{\ell, k}$ satisfies*

(1) $\left(\frac{\partial h_\ell}{\partial p_k}\right)_{\ell,k} = \left(\frac{\partial^2 e}{\partial p_\ell \partial p_k}\right)_{\ell,k}$.

(2) $\left(\frac{\partial h_\ell}{\partial p_k}\right)_{\ell,k}$ *is negative semidefinite.*

(3) $\left(\frac{\partial h_\ell}{\partial p_k}\right)_{\ell,k}$ *is symmetric.*

(4) $\sum_{k=1}^{L} \frac{\partial h_\ell}{\partial p_k} p_k = 0, \forall \ell$.

Proof. (1) follows from

$$\left(\frac{\partial h_\ell}{\partial p_k}\right)_{\ell,k} = \left(\frac{\partial}{\partial p_k}\left(\frac{\partial e}{\partial p_\ell}\right)\right)_{\ell,k} = \left(\frac{\partial^2 e}{\partial p_\ell \partial p_k}\right)_{\ell,k}.$$

(2) and (3) follow from (1) and the fact that since $e(\mathbf{p}, u)$ is a twice continuously differentiable concave function, its Hessian is symmetric and negative semidefinite.

(4) Note that $h_\ell(t\mathbf{p}, u) = h_\ell(\mathbf{p}, u)$. Differentiating both sides with t yields the result. ∎

From this , we have the following corollary

Corollary 1.7.1. **(Nice Features of Hicksian Demand)** *For each good ℓ, it holds that*

(1) $\frac{\partial h_\ell}{\partial p_\ell} \leq 0$*: (compensated) own price effect is nonpositive.*

(2) $\frac{\partial h_\ell}{\partial p_k} = \frac{\partial h_k}{\partial p_\ell}$ *for all $k \neq \ell$: cross price effects are symmetric.*

(3) $\frac{\partial h_\ell}{\partial p_k} \geq 0$ *for some $k \neq \ell$: each good has at least one substitute.*

Theorem 1.7.1. **(Slutsky Equation)**

$$\frac{\partial x_\ell(\mathbf{p}, m)}{\partial p_k} = \frac{\partial h_\ell(\mathbf{p}, v(\mathbf{p}, m))}{\partial p_k} - \frac{\partial x_\ell(\mathbf{p}, m)}{\partial m} x_k(\mathbf{p}, m)$$

Proof. Let $\mathbf{x}^* := \mathbf{x}(\mathbf{p}, m)$ and $u^* := u(\mathbf{x}^*)$. By the above identity,

$$h_\ell(\mathbf{p}, u^*) \equiv x_\ell(\mathbf{p}, e(\mathbf{p}, u^*)).$$

Differentiating this with p_k yields

$$\frac{\partial h_\ell(\mathbf{p}, u^*)}{\partial p_k} = \frac{\partial x_\ell(\mathbf{p}, e(\mathbf{p}, u^*))}{\partial p_k} + \frac{\partial x_\ell(\mathbf{p}, e(\mathbf{p}, u^*))}{\partial m} \frac{\partial e(\mathbf{p}, u^*)}{\partial p_k}. \tag{1.7.1}$$

Now, by the Shephard's Lemma and the identity,

$$\frac{\partial e(\mathbf{p}, u^*)}{\partial p_k} = h_k(\mathbf{p}, u^*) = h_k(\mathbf{p}, v(\mathbf{p}, m)) = x_k(\mathbf{p}, m).$$

Plugging this into (1.7.1) yields

$$\frac{\partial h_\ell(\mathbf{p}, u^*)}{\partial p_k} = \frac{\partial x_\ell(\mathbf{p}, e(\mathbf{p}, u^*))}{\partial p_k} + \frac{\partial x_\ell(\mathbf{p}, e(\mathbf{p}, u^*))}{\partial m} x_k(\mathbf{p}, m), \tag{1.7.2}$$

which, by rearrangement, leads to the result. ▌

- According to this theorem, a price change involves two effects:
  - $\dfrac{\partial h_\ell(\mathbf{p}, v(\mathbf{p}, m))}{\partial p_k}$: *substitution effect* which measures the change in demand due to the change in relative prices
  - $\dfrac{\partial x_\ell(\mathbf{p}, m)}{\partial m} x_k(\mathbf{p}, m)$: *income effect* which measures the change in demand due to the change in the 'purchasing' power

Corollary 1.7.2. **(Symmetric and Negative Semidefinite Slustky Matrix)** *The Slutsky matrix defined as*

$$S(\mathbf{p}, m) := \left(\frac{\partial x_\ell(\mathbf{p}, m)}{\partial p_k} + x_k(\mathbf{p}, m) \frac{\partial x_\ell(\mathbf{p}, m)}{\partial m} \right)_{\ell, k}$$

is symmetric and negative semidefinite.

Proof. From the equation (1.7.2) above, letting $u^* := v(\mathbf{p}, m)$,

$$\left(\frac{\partial x_\ell(\mathbf{p}, m)}{\partial p_k} + x_k(\mathbf{p}, m) \frac{\partial x_\ell(\mathbf{p}, m)}{\partial m} \right)_{\ell, k} = \left(\frac{\partial h_\ell(\mathbf{p}, u^*)}{\partial p_k} \right)_{\ell, k},$$

which is symmetric and negative semidefinite by Proposition 1.7.1. ▌

1.8 Money Metric (Indirect) Utility Functions

From the expenditure function, we can construct some other functions which will prove useful for the welfare analysis.

- The money metric utility function is defined by

$$m(\mathbf{p},\mathbf{x}) := e(\mathbf{p},u(\mathbf{x})).$$

  - This measures how much money the consumer would need to achieve the same utility as he could with the bundle $\mathbf{x}$.
  - Note that $m(\mathbf{p},\mathbf{x})$ is monotonic, homogeneous of degree one, and concave in $\mathbf{p}$.
  - For a fixed $\mathbf{p}$, $m(\mathbf{p},\mathbf{x})$ is a monotonic transform of the utility function and is therefore itself a utility function.

- Money metric indirect utility function is defined by

$$\mu(\mathbf{p};\mathbf{q},m) := e(\mathbf{p},v(\mathbf{q},m)),$$

  - This measures the amount of money the consumer would need at price $\mathbf{p}$ to achieve the same utility as he could under the prices $\mathbf{q}$ and income m.
  - For a fixed $\mathbf{p}$, $\mu(\mathbf{p};\mathbf{q},m)$ is a monotone transform of the indirect utility function.
  - Both money metric and money metric indirect utility functions are not subject to the monotone transform of the underlying utility function.

Chapter 2

Topics in Consumer Theory

2.1 Integrability

- If a demand function $\mathbf{x}(\mathbf{p}, m) \in \mathcal{C}^1$ is generated by rational preference, then it satisfies

  (P.1) Homogeneity: $\mathbf{x}(t\mathbf{p}, tm) = \mathbf{x}(\mathbf{p}, m)$.

  (P.2) Budget Balancedness (or Walras' Law): $\mathbf{p} \cdot \mathbf{x}(\mathbf{p}, m) = m$.

  (P.3) Symmetry: $S(\mathbf{p}, m)$ is symmetric.

  (P.4) Negative Semidefinite: $S(\mathbf{p}, m)$ is negative semidefinite.

- Integrability problem: 'does the inverse hold?'

  – If we observe a demand function $\mathbf{x}(\mathbf{p}, m)$ that satisfies properties (P.1)$-$(P.4) above, then can we find preference that generated $\mathbf{x}(\mathbf{p}, m)$?

  – The answer is 'yes' according to Antonelli (1886) and Hurwicz and Uzawa (1971).

- How then can we find such preference? $\rightarrow$ 2 steps

  – Recover $e(\mathbf{p}, u)$ from $\mathbf{x}(\mathbf{p}, m)$

  – Recover preferences from $e(\mathbf{p}, u)$

  The following proposition tells us how to recover preference from $e(\mathbf{p}, u)$.

Proposition 2.1.1. *Given $e(\mathbf{p}, u)$, define*

$$V_u := \{\mathbf{x} \in \mathbb{R}^L_+ | \mathbf{p} \cdot \mathbf{x} \geq e(\mathbf{p}, u), \forall \mathbf{p} \gg 0\}.$$

Then, for each utility $u \in \mathbb{R}$, V_u is an upper contour set in the sense that

$$e(\mathbf{p}, u) = \min_{\mathbf{x} \in \mathbb{R}^L_+} \mathbf{p} \cdot \mathbf{x} \quad s.t. \quad \mathbf{x} \in V_u.$$

Proof. From the definition of V_u, $e(\mathbf{p}, u) \leq \min\{\mathbf{p} \cdot \mathbf{x} : \mathbf{x} \in V_u\}$. We need to prove that $e(\mathbf{p}, u) \geq \min\{\mathbf{p} \cdot \mathbf{x} | \mathbf{x} \in V_u\}$.

For any $\mathbf{p}$ and $\mathbf{p}'$, the concavity of $e(\mathbf{p}, u)$ in $\mathbf{p}$ implies that

$$e(\mathbf{p}', u) \leq e(\mathbf{p}, u) + \nabla_p e(\mathbf{p}, u) \cdot (\mathbf{p}' - \mathbf{p}),$$

where $\nabla_p e(\mathbf{p}, u) := (\frac{\partial e}{\partial p_1}, \cdots, \frac{\partial e}{\partial p_L})$. Since $e(\mathbf{p}, u)$ is homogeneous of degree one in $\mathbf{p}$, Euler's formula tells us that $e(\mathbf{p}, u) = \nabla_p e(\mathbf{p}, u) \cdot \mathbf{p}$. Thus, $e(\mathbf{p}', u) \leq \nabla_p e(\mathbf{p}, u) \cdot \mathbf{p}'$ for all $\mathbf{p}'$, which means that $\nabla_p e(\mathbf{p}, u) \in V_u$. It follows that $\min\{\mathbf{p} \cdot \mathbf{x} | \mathbf{x} \in V_u\} \leq \mathbf{p} \cdot \nabla_p e(\mathbf{p}, u) = e(\mathbf{p}, u)$. ∎

When preference is convex When preference is not convex

- How to recover $e(\mathbf{p}, u)$ from $\mathbf{x}(\mathbf{p}, m)$
  - For a fixed u^0 and $(\mathbf{p}^0, m^0)$, by Shephard's Lemma, we can consider the following system of partial differential equations:

$$\begin{aligned} \frac{\partial e(\mathbf{p}, u^0)}{\partial p_1} &= h_1(\mathbf{p}, u^0) = x_1(\mathbf{p}, e(\mathbf{p}, u^0)) \\ &\vdots \\ \frac{\partial e(\mathbf{p}, u^0)}{\partial p_L} &= h_L(\mathbf{p}, u^0) = x_L(\mathbf{p}, e(\mathbf{p}, u^0)), \end{aligned} \tag{2.1.1}$$

with initial condition, $e(\mathbf{p}^0, u^0) = m^0$.

– Frobenius' theorem tells us that a solution exists iff the $L \times L$ derivative matrix of the functions on the the right hand side is symmetric, that is

$$\left(\frac{\partial x_\ell(\mathbf{p}, e(\mathbf{p}, u^0))}{\partial p_k} + \frac{\partial x_\ell(\mathbf{p}, e(\mathbf{p}, u^0))}{\partial m} \frac{\partial e(\mathbf{p}, u^0)}{\partial p_k} \right)_{\ell,k}$$

is symmetric.

– Thus, the necessary and sufficient condition for the recovery of an underlying expenditure function is the symmetry and negative semidefiniteness of the Slutsky matrix.

Example 2.1.1. Suppose that $L = 3$ and a consumer's demand is summarized by

$$x_\ell(\mathbf{p}, m) = \alpha_\ell \frac{m}{p_\ell}, \ell = 1, 2, 3.$$

One can easily check that this demand function satisfies the integrability condition (P.1)−(P.4). Thus, we can be sure that there exists an underlying preference which generates this demand. In order to recover the expenditure function, one can apply (2.1.1) to have

$$\frac{\partial e(\mathbf{p}, u)}{\partial p_\ell} = \alpha_\ell \frac{e(\mathbf{p}, u)}{p_\ell}, \ell = 1, 2, 3.$$

This can be rewritten as

$$\frac{\partial \ln(e(\mathbf{p}, u))}{\partial p_\ell} = \frac{\alpha_\ell}{p_\ell}, \ell = 1, 2, 3,$$

which implies (how?) that for some function $c(\cdot)$,

$$\ln(e(\mathbf{p}, u)) = \alpha_1 \ln(p_1) + \alpha_2 \ln(p_2) + \alpha_3 \ln(p_3) + c(u).$$

Thus, we have

$$e(\mathbf{p}, u) = \exp[c(u)] p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}.$$

But there is no problem (why?) with replacing $\exp[c(u)]$ with u to finally obtain

$$e(\mathbf{p}, u) = u p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}.$$

Think for yourself about how to obtain the utility function from $e(\mathbf{p}, u)$.

2.2 Welfare Evaluation of Price Changes

Let us assume that a change from $(\mathbf{p}^0, m^0)$ to $(\mathbf{p}^1, m^1)$ is proposed.

- An easy way to measure the welfare change involved in moving from $(\mathbf{p}^0, m^0)$ to $(\mathbf{p}^1, m^1)$ is to calculate
$$v(\mathbf{p}^1, m^1) - v(\mathbf{p}^0, m^0).$$
  - Subject to the monotone transformation
  - Not good for the cost-benefit analysis
- Instead, we use
$$\mu(\mathbf{q}; \mathbf{p}^1, m^1) - \mu(\mathbf{q}; \mathbf{p}^0, m^0) = e(\mathbf{q}, u^1) - e(\mathbf{q}, u^0),$$
where $u^0 = v(\mathbf{p}^0, m^0)$ and $u^1 = v(\mathbf{p}^1, m^1)$.
  - This measures the difference between the utilities $v(\mathbf{p}^0, m^0)$ and $v(\mathbf{p}^1, m^1)$ in *monetary terms* using $\mathbf{q}$ as the base price.
  - *Equivalent Variation (EV)*: Setting $\mathbf{q} = \mathbf{p}^0$,
$$EV := \mu(\mathbf{p}^0; \mathbf{p}^1, m^1) - \mu(\mathbf{p}^0; \mathbf{p}^0, m^0) = \mu(\mathbf{p}^0; \mathbf{p}^1, m^1) - m^0,$$
which measures what income change at the current prices would be equivalent to the proposed change in terms of its impact on utility.
  - *Compensating Variation (CV)*: Setting $\mathbf{q} = \mathbf{p}^1$,
$$CV := \mu(\mathbf{p}^1; \mathbf{p}^1, m^1) - \mu(\mathbf{p}^1; \mathbf{p}^0, u^0) = m^1 - \mu(\mathbf{p}^1; \mathbf{p}^0, m^0),$$
which measures what income change at the new prices would be necessary to compensate the consumer for the proposed change.
- Suppose that $p_1^0 > p_1^1$ while $\mathbf{p}_{-1}^0 = \mathbf{p}_{-1}^1 = \mathbf{p}_{-1}$ and $m^0 = m^1 = m$. Then, EV and CV can be alternatively expressed as
$$EV = e(\mathbf{p}^0, u^1) - e(\mathbf{p}^1, u^1) = \int_{p_1^0}^{p_1^1} h_1(p_1, \mathbf{p}_{-1}, u^1) dp_1$$
$$CV = e(\mathbf{p}^0, u^0) - e(\mathbf{p}^1, u^0) = \int_{p_1^0}^{p_1^1} h_1(p_1, \mathbf{p}_{-1}, u^0) dp_1.$$

- Often, the change in consumer's surplus due to the price change is defined as

$$CS := \int_{p_1^0}^{p_1^1} x_1(p_1, \mathbf{p}_{-1}, m) dp_1,$$

  - This is the area to the left of the Marshallian demand curve between p_1^0 and p_1^1.
  - If the good is normal, then we have $EV > CS > CV$.

2.3 Revealed Preference

- Question: Can we derive the above predictions for a consumer's market behavior by imposing a few simple and sensible assumptions (or axioms) on the consumer's *observable choices* themselves, rather than on his *unobservable preferences*?
  - 'Yes' if the consumer's observable choices satisfy the certain axioms.
  - The basic idea is simple: If the consumer buys one bundle instead of another affordable bundle, then the first bundle is considered to be *revealed preferred* to the second.

- Let $(\mathbf{p}, \mathbf{x})$ denote a price-quantity data such that the bundle $\mathbf{x}$ is chosen under the price $\mathbf{p}$.
  - *Weak Axiom of Reveal Preference (WARP):* The observations satisfy WARP if $\mathbf{p} \cdot \mathbf{x}' \leq \mathbf{p} \cdot \mathbf{x}$ and $\mathbf{x} \neq \mathbf{x}'$, then we must have $\mathbf{p}' \cdot \mathbf{x} > \mathbf{p}' \cdot \mathbf{x}$.
  - *Strong Axiom of Revealed Preference (SARP):* The observations satisfy SARP if for any list $(\mathbf{p}^1, \mathbf{x}^1), \cdots, (\mathbf{p}^n, \mathbf{x}^n)$, with the property that $\mathbf{x}^m \neq \mathbf{x}^{m+1}$ for all $m \leq n-1$, and
$$\mathbf{p}^m \cdot \mathbf{x}^{m+1} \leq \mathbf{p}^m \cdot \mathbf{x}^m \text{ for all } m \leq n-1,$$
then we must have $\mathbf{p}^n \cdot \mathbf{x}^1 > \mathbf{p}^n \cdot \mathbf{x}^n$.
  - *Generalized Axiom of Revealed Preference (GARP):* The observations satisfy GARP if for any list $(\mathbf{p}^1, \mathbf{x}^1), \cdots, (\mathbf{p}^n, \mathbf{x}^n)$, with the property that $\mathbf{x}^m \neq \mathbf{x}^{m+1}$ for all $m \leq n-1$, and
$$\mathbf{p}^m \cdot \mathbf{x}^{m+1} \leq \mathbf{p}^m \cdot \mathbf{x}^m \text{ for all } m \leq n-1,$$
we must have $\mathbf{p}^n \cdot \mathbf{x}^1 \geq \mathbf{p}^n \cdot \mathbf{x}^n$.
  - Clearly, SARP is stronger than WARP while it is only slightly stronger than GARP.

Whether these axioms are satisfied is closely related to whether there exists a utility function (or preference) that generates (or rationalizes) the data.

PROPOSITION 2.3.1. *Let $\mathbf{x}(\mathbf{p}, m)$ denote the choice made by the consumer who faces prices $\mathbf{p}$ and income m. If $\mathbf{x}(\mathbf{p}, m)$ satisfies WARP and budget balancedness, then it must have homogeneity of degree zero and negative semidefinitenes of the Slutsky matrix.*

Proof. We only prove the negative semidefiniteness. Fix $\mathbf{p} \gg 0$, $m > 0$, and let $\mathbf{x} := \mathbf{x}(\mathbf{p}, m)$. Then, for any other price vector $\mathbf{p}'$and $\mathbf{x}' := \mathbf{x}(\mathbf{p}', \mathbf{p}' \cdot \mathbf{x})$, WARP implies that

$$\mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{x}' \tag{2.3.1}$$

with inequality being strict if $\mathbf{x}' \neq \mathbf{x}$. By budget balancedness,

$$\mathbf{p}' \cdot \mathbf{x} = \mathbf{p}' \cdot \mathbf{x}'. \tag{2.3.2}$$

Subtracting (2.3.1) from (2.3.2) yields[1]

$$(\mathbf{p}' - \mathbf{p}) \cdot \mathbf{x} \geq (\mathbf{p}' - \mathbf{p}) \cdot \mathbf{x}' = (\mathbf{p}' - \mathbf{p}) \cdot \mathbf{x}(\mathbf{p}', \mathbf{p}' \cdot \mathbf{x}). \tag{2.3.3}$$

Letting $\mathbf{p}' = \mathbf{p} + t\mathbf{z}$, where $t > 0$ and $\mathbf{z} \in R^L$ is arbitrary, (2.3.3) becomes

$$\mathbf{z} \cdot \mathbf{x} \geq \mathbf{z} \cdot \mathbf{x}(\mathbf{p} + t\mathbf{z}, (\mathbf{p} + t\mathbf{z}) \cdot \mathbf{x}). \tag{2.3.4}$$

This means that for small $\bar{t} > 0$ so that $\mathbf{p} + t\mathbf{z} \gg 0$, the function $f : [0, \bar{t}] \to \mathbb{R}$ defined by

$$f(t) := \mathbf{z} \cdot \mathbf{x}(\mathbf{p} + t\mathbf{z}, (\mathbf{p} + t\mathbf{z}) \cdot \mathbf{x})$$

is maximized at $t = 0$, which implies $f'(0) \leq 0$ or

$$f'(0) = \sum_{\ell} \sum_{k} z_\ell \left[\frac{\partial x_\ell(\mathbf{p}, m)}{\partial p_k} + x_k(\mathbf{p}, m) \frac{\partial x_\ell(\mathbf{p}, m)}{\partial m} \right] z_k = \mathbf{z} \cdot S(\mathbf{p}, m)\mathbf{z} \leq 0.$$

Thus, $S(\mathbf{p}, m)$ is negative semidefinite. ▌

- If $\mathbf{x}(\mathbf{p}, m)$ satisfies SARP, it must also have the symmetric Slutsky matrix $\to$ SARP is essentially equivalent to the existence of a utility function that rationalizes the data.
- For a finite data set, GARP is equivalent to the existence of a locally nonsatiated, continuous, increasing, and concave utility function that rationalizes the data.

[1] This is sometimes called 'law of compensated demand' since it implies

$$(\mathbf{p}' - \mathbf{p}) \cdot (\mathbf{x}' - \mathbf{x}) \leq 0,$$

which says that prices and compensated demands move in the opposite direction.

2.4 Aggregate Demand

Suppose that there are I consumers with Marshallian demand functions $\mathbf{x}^i(\mathbf{p}, m)$ for consumer $i = 1, \cdots, I$. In general, given prices $\mathbf{p} \in \mathbb{R}^L_+$ and wealths $(m^1, \cdots, m^I)$, aggregate demand is written as

$$\mathbf{x}(\mathbf{p}, m^1, \cdots, m^I) = \sum_{i=1}^{I} \mathbf{x}^i(\mathbf{p}, m^i).$$

When would the aggregate demand be as if it were generated by a single *representative* consumer?

- One such case is where the aggregate demand can be expressed as a function of prices and *aggregate wealth*, $\sum_{i=1}^I m^i$:

$$\mathbf{x}(\mathbf{p}, m^1, \cdots, m^I) = \mathbf{X}(\mathbf{p}, \sum_{i=1}^{I} m^i).$$

  - What is required here is that for all wealth levels $(m^1, \cdots, m^I)$ and its differential change $(dm^1, \cdots, dm^I)$ satisfying $\sum_{i=1}^I dm^i = 0$,

$$\sum_{i=1}^{I} \frac{\partial x^i_\ell(\mathbf{p}, m^i)}{\partial m^i} dm^i = 0, \text{ for every } \ell.$$

  - This will hold if

$$\frac{\partial x^i_\ell(\mathbf{p}, m^i)}{\partial m^i} = \frac{\partial x^j_\ell(\mathbf{p}, m^j)}{\partial m^j}, \text{ for all } i, j$$

    that is, the wealth effect must be the same across consumers.

PROPOSITION 2.4.1. **(Gorman Form Indirect Utility Function)** *A representative consumer in the above sense exists if and only if every consumer has the following form of indirect utility function:*

$$v^i(\mathbf{p}, m^i) = a^i(\mathbf{p}) + b(\mathbf{p})m^i.$$

Proof. We only prove the sufficiency while the proof of necessity can be found in Deaton and Muellbauer (1980). By Roy's identity, consumer i's demand for good ℓ takes the form

$$x^i_\ell(\mathbf{p}, m^i) = a^i_\ell(\mathbf{p}) + b_\ell(\mathbf{p})m^i,$$

where

$$a_\ell^i(\mathbf{p}) = -\frac{\frac{\partial a^i(\mathbf{p})}{\partial p_\ell}}{b(\mathbf{p})} \text{ and } b_\ell(\mathbf{p}) = -\frac{\frac{\partial b(\mathbf{p})}{\partial p_\ell}}{b(\mathbf{p})}.$$

Thus, the aggregate demand for good ℓ takes the form

$$X_\ell(\mathbf{p}, \sum_{i=1}^{I} m^i) = -\left[\sum_{i=1}^{I} a_\ell^i(\mathbf{p}) + b_\ell(\mathbf{p})\left(\sum_{i=1}^{I} m^i\right)\right]. \tag{2.4.1}$$

This can be generated by a representative consumer whose indirect utility function is given by

$$V(\mathbf{p}, m) = \sum_{i=1}^{I} a^i(\mathbf{p}) + b(\mathbf{p})m.$$

To verify this, apply Roy's identity to $V(\mathbf{p}, m)$ to obtain the demand demand function given in (2.4.1). ▮

Example 2.4.1. Two utility functions whose indirect utility function is of Gorman form:

(1) Homothetic utility function: $u^i(\mathbf{x}) = g(h(\mathbf{x}))$, where g is a strictly increasing function and h is a function which is homogeneous of degree 1 $\rightarrow \quad v^i(\mathbf{p}, m) = b(\mathbf{p})m$.

(2) Quasi-linear utility function: $u^i(\mathbf{x}) = x_1 + w^i(x_2, \cdots, x_L) \rightarrow \quad v^i(\mathbf{p}, m) = m + a^i(\mathbf{p})$.

Chapter 3

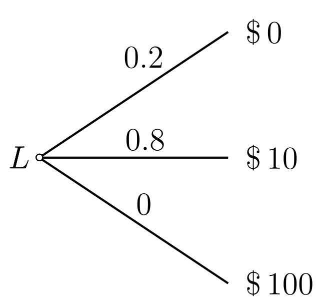
Choice under Uncertainty

3.1 Expected Utility Theory

Let us imagine a decision maker who faces a choice among a number of risky alternatives. Each risky alternative results in one of outcomes 1 to N, following some probability distribution over those outcomes. We call a risky alternative a *lottery.*

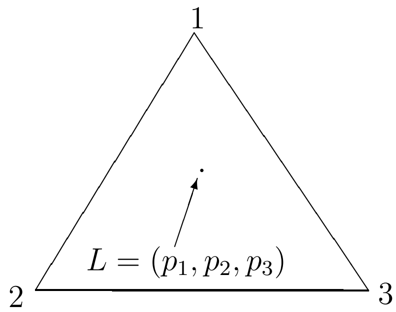
- A *simple lottery* L is a list $L = (p_1, \cdots, p_N)$ with $p_n \geq 0$ for all n and $\sum_n p_n = 1$, where p_n is the probability of outcome n occurring.

  Example 3.1.1. If the set of outcomes is $\{\$0, \$10, \$100\}$, then $L = (0.2, 0.8, 0)$ means that the decision maker can obtain $\$0, \$10$ and $\$100$ with probabilities 0.2, 0.8, and 0, respectively. Then, L can be expressed using the following diagram

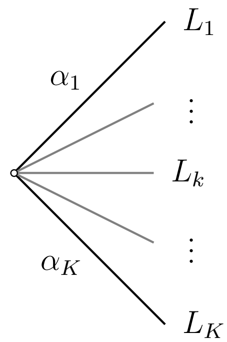


  – Let $\mathcal{L}$ denote the set of all simple lotteries.

  Example 3.1.2. In case of three outcomes, $\mathcal{L}$ can be represented by the simplex



– A *compound lottery* is a lottery whose outcomes are again lotteries: Given K simple lotteries $L_k = (p_1^k, \cdots, p_N^k), k = 1, \cdots, K$, the compound lottery is a list $(L_1, \cdots, L_K; \alpha_1, \cdots, \alpha_K)$, where α_k is the probability of lottery k occurring and $\sum_k \alpha_k = 1$.

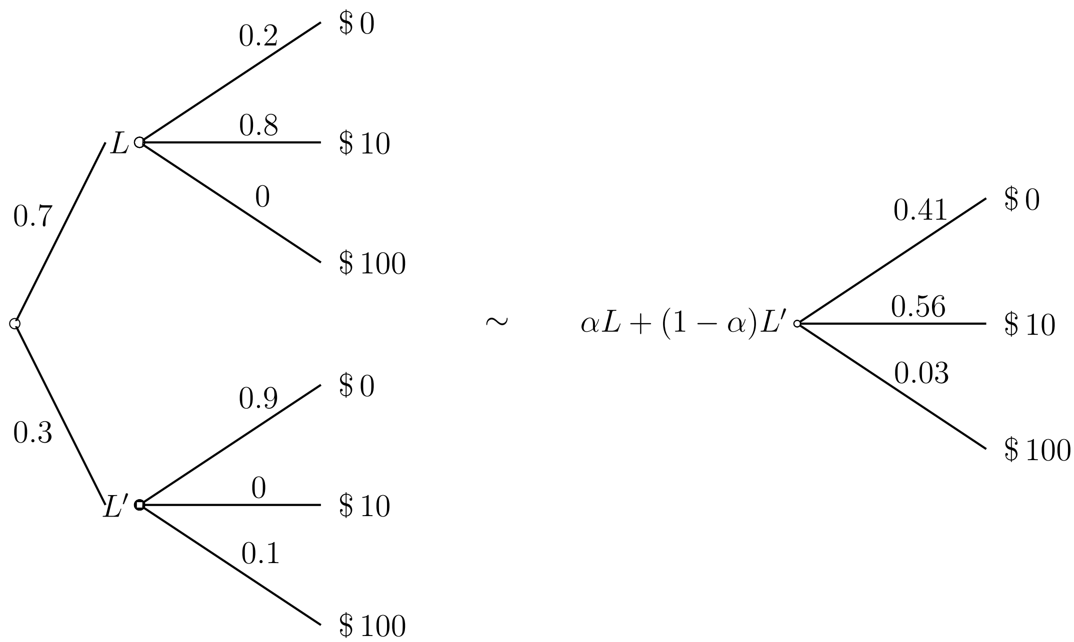


- Let $\succeq$ denote the decision maker's preference relation on $\mathcal{L}$. We assume that $\succeq$ is complete and transitive. In addition, we make the following assumptions:

– *Reduction of compound lotteries*: For any $L, L' \in \mathcal{L}$,

$$\begin{aligned}(L, L', \alpha, 1-\alpha) \sim & \alpha L + (1-\alpha)L' \\ & = (\alpha p_1 + (1-\alpha)p_1', \cdots, \alpha p_N + (1-\alpha)p_N').\end{aligned}$$

Example 3.1.3.



– *Continuity*: For any three lotteries $L, L', L'' \in \mathcal{L}$ satisfying $L \succ L' \succ L''$, there exists $\alpha \in (0,1)$ such that

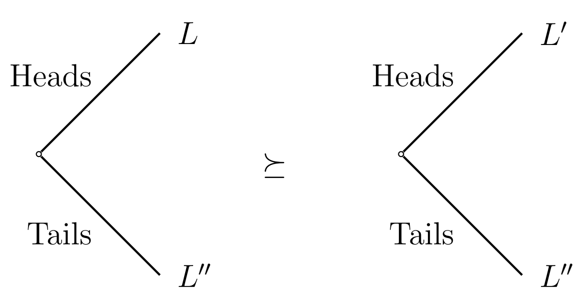
$$\alpha L + (1-\alpha)L'' \sim L'.$$

– *Independence*: For any $L, L', L'' \in \mathcal{L}$ and $\alpha \in (0,1)$, we have

$$L \succeq L' \text{ if and only if } \alpha L + (1-\alpha)L'' \succeq \alpha L' + (1-\alpha)L''.$$

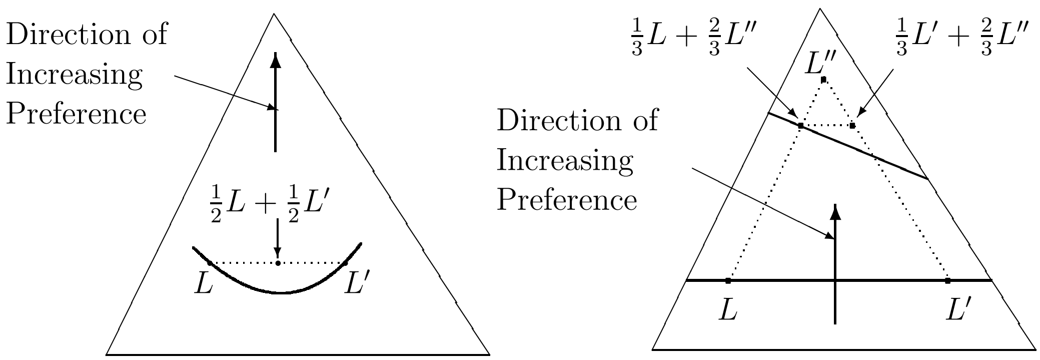
In words, if we mix each of two lotteries with a third one, then the order of two mixtures does not depend on the third lottery.

Example 3.1.4. Suppose that $\alpha = \frac{1}{2}$. Then, $\frac{1}{2}L + \frac{1}{2}L''$ can be considered as the compound lottery arising from a coin toss where L (L'') obtains if heads (tails) comes up. The independence axiom requires



if and only if $\quad L \succeq L'$

Example 3.1.5. In general, the independence axiom requires that the indifference curves are straight and parallel on the simplex corresponding to $\mathcal{L}$.



Violations of Independence Axiom

– *Best and worst lotteries*: There exist two lotteries $\bar{L}$ and $\underline{L}$ such that for all $L \in \mathcal{L}$, $\bar{L} \succeq L \succeq \underline{L}$.

- The utility function $u : \mathcal{L} \to R$ has the *expected utility form* if for every $L \in \mathcal{L}$,

$$U(L) = u_1 p_1 + \cdots u_N p_N, \tag{3.1.1}$$

where u_n is the utility assigned to the nth outcome.

– $U : \mathcal{L} \to R$ is called a *von Neumann-Morgenstern (v.N-M)* expected utility function while u_k is often called *Bernoulli* utility function in case each outcome is a monetary value.

Theorem 3.1.1. **(Expected Utility Theorem)** *The above assumptions are necessary and sufficient for the preference $\succeq$ on $\mathcal{L}$ to admit a utility representation of the expected utility form.*

Proof. The proof of the necessity is straightforward. Below, we prove the sufficiency part. First, it follows from the independence that for any $\alpha, \beta \in [0, 1]$,

$$\beta\bar{L} + (1-\beta)\underline{L} \succ \alpha\bar{L} + (1-\alpha)\underline{L} \text{ if and only if } \beta > \alpha. \tag{3.1.2}$$

By the continuity, for any $L \in \mathcal{L}$, there exists an $\alpha_L \in [0, 1]$ such that $\alpha_L\bar{L} + (1-\alpha_L)\underline{L} \sim L$. Also, such α_L is unique due to (3.1.2).

Consider the function $U : \mathcal{L} \to R$ that assigns α_L to each $L \in \mathcal{L}$. Then, $U(\cdot)$ represents $\succeq$ since for any two lotteries $L, L' \in \mathcal{L}$, we have

$$L \succeq L' \text{ if and only if } \alpha_L \bar{L} + (1 - \alpha_L)\underline{L} \succeq \alpha_{L'} \bar{L} + (1 - \alpha_{L'})\underline{L},$$

which, by (3.1.2), implies that $L \succeq L'$ if and only if $\alpha_L \geq \alpha_{L'}$.

Next, we show that $U(\cdot)$ is linear, that is for any $L, L' \in \mathcal{L}$ and any $\beta \in [0, 1]$,

$$U(\beta L + (1 - \beta)L') = \beta U(L) + (1 - \beta)U(L'). \tag{3.1.3}$$

First, $L \sim U(L)\bar{L} + (1 - U(L))\underline{L}$ and $L' \sim U(L')\bar{L} + (1 - U(L'))\underline{L}$. Then, applying the independence axiom (twice), we obtain

$$\begin{aligned}
&\beta L + (1 - \beta)L' \\
\sim &\beta[U(L)\bar{L} + (1 - U(L))\underline{L}] + (1 - \beta)L' \\
\sim &\beta[U(L)\bar{L} + (1 - U(L))\underline{L}] + (1 - \beta)[U(L')\bar{L} + (1 - U(L'))\underline{L}],
\end{aligned}$$

which, by the reduction of compound lottery, is equivalent to

$$[\beta U(L) + (1 - \beta)U(L')]\bar{L} + [1 - \beta U(L) - (1 - \beta)U(L')]\underline{L}.$$

Thus, (3.1.3) follows from the definition of $U(\cdot)$.

Finally, it is straightforward to obtain (3.1.1) from (3.1.3).(How?) ▮

The v.N-M utility function is *not* unique up to the monotone transformation but unique up to the *affine* transformation:

PROPOSITION 3.1.1. *The v.N-M utility functions U and $\tilde{U}$ represent the same preference if and only if $\tilde{U}(L) = \beta U(L) + \gamma$ for some scalars $\beta > 0$ and γ.*

Proof. We only prove the necessity. Consider any lottery $L \in \mathcal{L}$ and define $\lambda_L \in [0, 1]$ by

$$U(L) = \lambda_L U(\bar{L}) + (1 - \lambda_L)U(\underline{L}). \tag{3.1.4}$$

Thus,

$$\lambda_L = \frac{U(L) - U(\underline{L})}{U(\bar{L}) - U(\underline{L})}.$$

Since $\lambda_L U(\bar{L}) + (1-\lambda_L)U(\underline{L}) = U(\lambda_L \bar{L} + (1-\lambda_L)\underline{L})$, we must have $L \sim \lambda_L \bar{L} + (1-\lambda_L)\underline{L}$. Since $\tilde{U}$ is also linear and represents the same preference, we have

$$\begin{aligned} \tilde{U}(L) &= \tilde{U}(\lambda_L \bar{L} + (1-\lambda_L)\underline{L}) \\ &= \lambda_L \tilde{U}(\bar{L}) + (1-\lambda_L)\tilde{U}(\underline{L}) \\ &= \lambda_L(\tilde{U}(\bar{L}) - \tilde{U}(\underline{L})) + \tilde{U}(\underline{L}). \end{aligned} \tag{3.1.5}$$

Letting

$$\beta = \frac{\tilde{U}(\bar{L}) - \tilde{U}(\underline{L})}{U(\bar{L}) - U(\underline{L})}$$

and

$$\gamma = \tilde{U}(\underline{L}) - U(\underline{L})\frac{\tilde{U}(\bar{L}) - \tilde{U}(\underline{L})}{U(\bar{L}) - U(\underline{L})},$$

it is easy to show by comparing (3.1.4) and (3.1.5) that $\tilde{U}(L) = \beta U(L) + \gamma$. ▌

The following example casts doubt on the expected utility theory, especially the independence assumption.

EXAMPLE 3.1.6. **(Allais Paradox)** There are three monetary outcomes, \$2.5 million, \$0.5 million, and \$0. You are subjected to two choice tests. The first one is to choose between

$$L_1 = (0, 1, 0) \quad \text{and} \quad L_1' = (0.10, 0.89, 0.01).$$

The second one is to choose between

$$L_2 = (0, 0.11, 0.89) \quad \text{and} \quad L_2' = (0.10, 0, 0.90).$$

People often exhibit $L_1 \succ L_1'$ and $L_2' \succ L_2$. But if the expected utility theory holds true, then $L_1 \succ L_1'$ implies that

$$U(0.5M) > 0.10U(2.5M) + 0.89U(0.5M) + 0.01U(0).$$

Adding $0.89U(0) - 0.89U(0.5M)$ to both sides, we get

$$0.11U(0.5M) + 0.89U(0) > 0.10U(2.5M) + 0.90U(0).$$

The following example casts doubt on whether people always assign (objective or subjective) probabilities to the uncertain events.

Example 3.1.7. **(Ellsberg Paradox)** You are told that an urn contains 300 balls. One hundred of the balls are red and 200 are either blue or green. You are subjected to two choice tests. The first one is to choose between

Gamble A: You receive $ 1000 if the ball is red.

Gamble B: You receive $ 1000 if the ball is blue

The second one is to choose between

Gamble C: You receive $ 1000 if the ball is not red.

Gamble D: You receive $ 1000 if the ball is not blue.

People often exhibit $A \succ B$ and $C \succ D$. $A \succ B$ implies $p(R)u(1000) > p(B)u(1000)$ or $p(R) > p(B)$ while $C \succ D$ implies that $1 - p(R) > 1 - p(B)$, a contradiction ! This paradox suggests that people dislike *ambiguity*.

3.2 Risk Aversion

In case a lottery yields the monetary outcomes, the lottery can be represented by a cumulative distribution function $F : \mathbb{R} \to [0, 1]$. We can extend the expected utility theory to obtain the v.N-M utility function

$$U(F) = \int u(x) dF(x),$$

where $u(x)$ is a Bernoulli utility assigned to x amount of money.

- A decision maker is *risk averse* if for any lottery F,

$$\int u(x) dF(x) \leq u\left(\int x dF(x)\right).$$

  - One way to see the worth of lottery $F(\cdot)$ is to look at the *certainty equivalent*, denoted $c(F, u)$ and defined by

$$u\left(c(F, u)\right) = \int u(x) dF(x).$$

  - The followings are equivalent:

    (1) The decision maker is risk averse.

    (2) $u(\cdot)$ is concave.

(3) $c(F,u) \leq \int x dF(x)$.

Proof. (2) $\rightarrow$ (1): If $u(\cdot)$ is concave, then Jensen's inequality immediately implies

$$\int u(x)dF(x) \leq u\left(\int x dF(x)\right). \tag{3.2.1}$$

(1) $\rightarrow$ (3): By (3.2.1), we have

$$u(c(F,u)) = \int u(x)dF(x) \leq u\left(\int x dF(x)\right),$$

so $c(F,u) \leq \int x dF(x)$.

(3) $\rightarrow$ (2): Suppose that (2) does not hold. Then, there must exist x, y, and $\lambda \in (0,1)$ such that

$$u(\lambda x + (1-\lambda)y) < \lambda u(x) + (1-\lambda)u(y).$$

Now, consider a binary distribution $F(\cdot)$ according to which x is drawn with probability λ while y is drawn with $(1-\lambda)$. Then,

$$u(\lambda x + (1-\lambda)y) < \lambda u(x) + (1-\lambda)u(y) = u(c(F,u)).$$

Thus,

$$\int x dF(x) = \lambda x + (1-\lambda)y < c(F,u),$$

contradicting (3). ▌

EXAMPLE 3.2.1. **(Demand for a Risky Asset)** Suppose that there are two assets, a *safe* asset with 1 dollar return per dollar and a *risky* one with a random return of z dollars per dollar. Assume that $\int z dF(z) > 1$, meaning that a risk is actuarially favorable. Letting α and β denote the amounts invested in the risky and safe assets, the utility maximization problem of the investor with wealth w is

$$\max_{\alpha,\beta \geq 0} \int u(\alpha z + \beta)dF(z) \quad \text{subject to} \quad \alpha + \beta = w.$$

or

$$\max_{0 \leq \alpha \leq w} \int u(w + \alpha(z-1))dF(z).$$

The Kuhn-Tucker first-order condition at the optimum α^* is

$$\phi(\alpha^*) = \int u'(w + \alpha^*(z-1))(z-1)dF(z) \begin{cases} \leq 0 & \text{if } \alpha^* < w \\ \geq 0 & \text{if } \alpha^* > 0. \end{cases}$$

Since $\phi(0) > 0$, $\alpha^* = 0$ is not optimal, which implies that if a risk is actuarially favorable, then a risk averse investor will always accept at least a small amount of the risk.

- The degree of risk aversion is measured by the *Arrow-Pratt coefficient of absolute risk aversion*: At wealth level x,
$$r_A(x) = -u''(x)/u'(x).$$
  - Given two utility functions $u_1(\cdot)$ and $u_2(\cdot)$, the followings are equivalent:
    (1) $r_A(x, u_2) \geq r_A(x, u_1)$ for every x,
    (2) $u_2(x) = \psi(u_1(x))$ for some increasing concave function $\psi(\cdot)$.
    (3) $c(F, u_2) \leq c(F, u_1)$ for any $F(\cdot)$.

    So, we say that $u_2(\cdot)$ is more risk averse than $u_1(\cdot)$.
  - The Bernoulli utility function $u(\cdot)$ is said to exhibit
$$\left.\begin{array}{c}\text{decreasing absolute risk aversion (DARA)}\\ \text{constant absolute risk aversion (CARA)}\\ \text{increasing absolute risk aversion (IARA)}\end{array}\right\} \text{ if } r_A(x,u) \text{ is } \left\{\begin{array}{c}\text{decreasing in } x\\ \text{constant in } x\\ \text{increasing in } x\end{array}\right.$$
  - The (CARA) utility function takes the following form:
$$u(x) = -e^{-ax}, a > 0 \quad \rightarrow \quad r_A(x, u) = a, \forall x.$$

Example 3.2.2. **(Example 3.2.1 Continued)** Suppose that the risk averse investor has a DARA Bernoulli utility function. We ask whether he invests more as he holds more wealth. Consider two wealth levels w_1 and $w_2 > w_1$. Define $u_1(x) := u(w_1 + x)$ and $u_2(x) := u(w_2 + x)$. Because of the DARA property, $u_1(x) = \psi(u_2(x))$ for some concave function $\psi(\cdot)$. Then, the utility maximization problem of the investor with wealth w_i is
$$\max_{0 \leq \alpha \leq w_i} \int u(w_i + \alpha(z-1))dF(z) = \int u_i(\alpha(z-1))dF(z).$$
The first order condition for the investor with w_1 is
$$\phi_1(\alpha_1^*) = \int (z-1)u_1'(\alpha_1^*(z-1))dF(z) = 0.$$
For the investor with w_2, it is
$$\int (z-1)u_2'(\alpha_2^*(z-1))dF(z) = 0.$$

Since $\phi_1(\cdot)$ is deceasing due to the concavity of $u_1(\cdot)$, we will have $\alpha_2^* > \alpha_1^*$ if $\phi_1(\alpha_2^*) < 0$, which holds because

$$\begin{aligned}
\phi_1(\alpha_2^*) &= \int (z-1)\psi'(u_2(\alpha_2^*(z-1)))u_2'(\alpha_2^*(z-1))dF(z) \\
&< \psi'(u_2(0)) \int_{-\infty}^{1} (z-1)u_2'(\alpha_2^*(z-1))dF(z) \\
&\quad + \psi'(u_2(0)) \int_{1}^{\infty} (z-1)u_2'(\alpha_2^*(z-1))dF(z) \\
&= \psi'(u_2(0)) \int (z-1)u_2'(\alpha_2^*(z-1))dF(z) = 0.
\end{aligned}$$

Thus, the demand of risky asset is increasing in wealth, i.e., the risky asset is a normal good if the investor has a DARA utility function. Also, the risky asset is an inferior good if the investor has an IARA utility function.

- Another useful measure of risk aversion is *the coefficient of relative risk aversion at x* defined as $r_R(x,u) := -xu''(x)/u'(x)$.
  - *Constant relative risk aversion (CRRA)* utility function takes the following form:

$$u(x) = \beta x^{1-\rho} + \gamma, \beta > 0 \quad \rightarrow \quad r_R(x,u) = \beta, \forall x.$$

  - Since $r_R(x) = xr_A(x)$, a consumer who has DRRA utility function must exhibit DARA.

Example 3.2.3. **(Example 3.2.1 Continued)** The *proportion* α^*/w of wealth invested in the risky asset is decreasing (increasing) with w if the investor has an increasing relative risk aversion or IRRA (DRRA) utility function.

3.3 Comparison between Lotteries

Now that we have learned how to compare risk attitudes, let us learn how to compare between lotteries.

- The lottery $F(\cdot)$ *first-order stochastically dominates (FOSD)* the lottery $G(\cdot)$ if, for every nondecreasing function $u : R \rightarrow R$, we have

$$\int u(x)dF(x) \geq \int u(x)dG(x).$$

- The following holds:

$$F(\cdot)\ FOSD\ G(\cdot) \text{ if and only if } F(x) \leq G(x) \text{ for every } x,$$

  that is the probability of getting at least x is higher under $F(\cdot)$ than under $G(\cdot)$. Graphically,

- $FOSD$ does not mean that the payoff drawn from $F(\cdot)$ is always higher than the one from $G(\cdot)$.
- $FOSD$ is stronger than requiring that the mean is higher under $F(\cdot)$ than under $G(\cdot)$.

We can also compare two lotteries with the same mean in terms of their riskiness.

- For two lotteries $F(\cdot)$ and $G(\cdot)$ with the same mean, $F(\cdot)$ *second-order stochastically dominates (SOSD)* (or is less risky than) $G(\cdot)$ if for every nondecreasing concave function $u : \mathbb{R} \to \mathbb{R}$, we have

$$\int u(x)dF(x) \geq \int u(x)dG(x).$$

  - $FOSD$ implies $SOSD$.
  - Given a lottery $F(\cdot)$, its *mean-preserving spread* is another lottery $G(\cdot)$ whose payoff is $y = x + z$, where x is first drawn from $F(\cdot)$ and then z from a distribution $H_x(\cdot)$ with $\int z dH_x(z) = 0$.
  - The followings are equivalent:
    (1) $F(\cdot)\ SOSD\ G(\cdot)$
    (2) $G(\cdot)$ is a mean-preserving spread of $F(\cdot)$.
    (3) $\int_0^x G(t)dt \geq \int_0^x F(t)dt$ for all x.

Proof. We only prove (2) $\rightarrow$ (1). If $G(\cdot)$ is a mean-preserving spread of $F(\cdot)$, then for any concave function $u(\cdot)$,

$$\begin{aligned}\int u(y)dG(y) &= \int \left(\int u(x+z)dH_x(z) \right) dF(x) \\ &\leq \int u\left(\int (x+z)dH_x(z) \right) dF(x) \\ &= \int u(x)dF(x),\end{aligned}$$

where the inequality is due to the Jensen's inequality. ▮

Chapter 4

Producer Theory

4.1 Production

- Consider an economy with L commodities.
  - $y = (y_1, \cdots, y_L) \in \mathbb{R}^L$: A *production plan*, where if $y_\ell > 0$ ($y_\ell < 0$), then $|y_\ell|$ units of ℓth commodity are produced (used) as an output (input).
  - A given technology is described by the *production set* denoted $Y \subset \mathbb{R}^L$, which is the set of all feasible production plans.
- We assume that
  - Y is nonempty and closed.
  - Free disposal: If $\mathbf{y} \in Y$ and $\mathbf{y}' \leq \mathbf{y}$ (so that $\mathbf{y}'$ produces at most the same amount of outputs using at least the same amount of inputs), then $\mathbf{y}' \in Y$.
- With only one output and m inputs, the technology can also be described by the *input requirement set*:

  $$V(y) := \{\mathbf{x} \in \mathbb{R}^m_+ | (y, -\mathbf{x}) \in Y\},$$

  which is the set of all input bundles $\mathbf{x}$ that produce at least y units of output.
  - $f(\mathbf{x}) := \max\{y \in \mathbb{R}_+ : \mathbf{x} \in V(y)\}$: Production function.
  - $Q(y) := \{\mathbf{x} \in \mathbb{R}^m_+ | f(\mathbf{x}) = y\}$: Isoquant for output level y.

– $f_\ell(\mathbf{x}) := \dfrac{\partial f(\mathbf{x})}{\partial x_\ell}$: Marginal product of input ℓ.

– $MRTS_{\ell k}(\mathbf{x}) := \dfrac{\partial f(\mathbf{x})/\partial x_\ell}{\partial f(\mathbf{x})/\partial x_k}$: Marginal rate of technical substitution (MRTS) between inputs ℓ and k when the current input vector is $\mathbf{x}$.

EXAMPLE 4.1.1. **(Cobb-Douglas Technology)** Let there be one output and two inputs, and a be a parameter such that $0 < a < 1$. The Cobb-Douglas technology is defined as follows:

$$\begin{aligned}
Y &= \{(y, -x_1, -x_2) \in \mathbb{R}^3 | y \leq x_1^a x_2^{1-a}\} \\
V(y) &= \{(x_1, x_2) \in \mathbb{R}^2_+ | y \leq x_1^a x_2^{1-a}\} \\
f(\mathbf{x}) &= x_1^a x_2^{1-a} \\
Q(y) &= \{(x_1, x_2) \in \mathbb{R}^2_+ | y = x_1^a x_2^{1-a}\} \\
Y(\bar{x}_2) &= \{(y, -x_1, -x_2) | y \leq x_1^a x_2^{1-a}, x_2 = \bar{x}_2\}.
\end{aligned}$$

- Elasticity of substitution between input ℓ and k at the point $\mathbf{x}$ is defined as

$$\sigma_{\ell k} := \frac{d \ln(x_k/x_\ell)}{d \ln(f_\ell(\mathbf{x})/f_k(\mathbf{x}))} = \frac{d(x_k/x_\ell)}{x_k/x_\ell} \frac{f_\ell(\mathbf{x})/f_k(\mathbf{x})}{d(f_\ell(\mathbf{x})/f_k(\mathbf{x}))}.$$

– The larger $\sigma_{\ell k}$ is, the easier substitution between inputs ℓ and k.

– The elasticity of substitution is constant for the CES production function defined by

$$f(\mathbf{x}) = \left(\sum_{\ell=1}^m \alpha_\ell x_\ell^\rho\right)^{1/\rho}, \text{ where } \sum_{\ell=1}^m \alpha_\ell = 1.$$

Proof. Note first that $\frac{f_\ell(\mathbf{x})}{f_k(\mathbf{x})} = \frac{\alpha_\ell x_\ell^{\rho-1}}{\alpha_k x_k^{\rho-1}}$, which implies that

$$\ln\left(\frac{f_\ell(\mathbf{x})}{f_k(\mathbf{x})}\right) = \ln\left(\frac{\alpha_\ell}{\alpha_k}\right) + (1-\rho)\ln\left(\frac{x_k}{x_\ell}\right).$$

Thus,

$$\sigma_{\ell k} := \frac{d \ln(x_k/x_\ell)}{d \ln(f_\ell(\mathbf{x})/f_k(\mathbf{x}))} = \frac{1}{1-\rho},$$

which is immediate from the linear relationship between $\ln\left(\frac{x_k}{x_\ell}\right)$ and $\ln\left(\frac{f_\ell(\mathbf{x})}{f_k(\mathbf{x})}\right)$. ▮

– Some well-known production functions belong to the CES class:

(1) $\rho = 1 \rightarrow \quad f(\mathbf{x}) = \sum_{\ell=1}^{m} \alpha_\ell x_\ell$, which is *linear*.

(2) $\rho = 0 \rightarrow \quad f(\mathbf{x}) = \prod_{\ell=1}^{m} x_\ell^{\alpha_\ell}$, which is *Cobb-Douglas*.

(3) $\rho = -\infty \rightarrow \quad f(\mathbf{x}) = \min\{x_1, \cdots, x_m\}$, which is *Leontief*.

- Returns to scale: A production function $f(\mathbf{x})$ has the property of (globally)

  (1) **Constant returns to scale** if $f(t\mathbf{x}) = tf(\mathbf{x})$ for all $t > 0$ and all $\mathbf{x}$.

  (2) **Increasing returns to scale** if $f(t\mathbf{x}) > tf(\mathbf{x})$ for all $t > 0$ and all $\mathbf{x}$.

  (3) **Decreasing returns to scale** if $f(t\mathbf{x}) < tf(\mathbf{x})$ for all $t > 0$ and all $\mathbf{x}$.

  – A local measure of returns to scale is often useful:

$$e(\mathbf{x}) := \left.\frac{d\ln[f(t\mathbf{x})]}{d\ln(t)}\right|_{t=1} = \left.\frac{df(t\mathbf{x})}{dt}\frac{t}{f(t\mathbf{x})}\right|_{t=1} = \frac{\sum_{\ell=1}^{m} f_\ell(\mathbf{x})x_\ell}{f(\mathbf{x})},$$

  which is called the *elasticity of scale at the point* $\mathbf{x}$.

4.2 Profit Maximization

Let us consider a competitive firm that takes the prices as given.

- Given a price vector $\mathbf{p} \gg 0$, the firm's profit maximization problem (PMP) is

$$\max_{\mathbf{y}} \mathbf{p} \cdot \mathbf{y} \quad \text{s.t.} \quad \mathbf{y} \in Y.$$

  – $\mathbf{y}(\mathbf{p})$: The set of profit-maximizing production plans, which is singleton if Y is strictly convex.

  – $\pi(\mathbf{p}) := \mathbf{p} \cdot \mathbf{y}(\mathbf{p})$: Maximized profit, called *profit function*.

- In case of one output and m inputs, PMP can be alternatively expressed as

$$\max_{\mathbf{x}} pf(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x},$$

  where $\mathbf{w}$ is the input price vector.

  – Thus, $\mathbf{y}(\mathbf{p}) = (y(p, \mathbf{w}), -\mathbf{x}(p, \mathbf{w}))$, where $y(p, \mathbf{w})$ and $\mathbf{x}(p, \mathbf{w})$ are called the *output supply* and *factor demand* functions, respectively.

– Assuming that at the optimum, $\mathbf{x}^* \gg 0$, the first order condition requires

$$p\frac{\partial f(\mathbf{x}^*)}{\partial x_\ell} = w_\ell \text{ for every } \ell = 1, \cdots, m,$$

which states that the value of the marginal product of input ℓ must equal the cost per unit of input ℓ.

– This implies that

$$MRTS_{\ell k}(\mathbf{x}^*) = \frac{f_\ell(\mathbf{x}^*)}{f_k(\mathbf{x}^*)} = \frac{w_\ell}{w_k}, \forall k, \ell.$$

THEOREM 4.2.1. **(Properties of Profit Function)** *If Y is closed and satisfies the free disposal, then the profit function $\pi(\mathbf{p})$ satisfies*

(1) If $p'_\ell \geq p_\ell$ for all output ℓ and $p'_k \leq p_k$ for all input k, then $\pi(\mathbf{p}') \geq \pi(\mathbf{p})$.

(2) Homogeneous of degree one.

(3) Convex in $\mathbf{p}$.

(4) Hotelling's lemma: For all $\ell = 1, \cdots, L$,

$$\frac{\partial \pi(\mathbf{p})}{\partial p_\ell} = y_\ell(\mathbf{p}). \tag{4.2.1}$$

Proof. (1) Since $p'_\ell \geq p_\ell$ for all l for which $y_\ell(\mathbf{p}) \geq 0$, and $p'_k \leq p_k$ for all k for which $y_k(\mathbf{p}) \leq 0$, we have $\mathbf{p}' \cdot \mathbf{y}(\mathbf{p}) \geq \mathbf{p} \cdot \mathbf{y}(\mathbf{p})$. So, we have

$$\pi(\mathbf{p}') = \mathbf{p}' \cdot \mathbf{y}(\mathbf{p}') \geq \mathbf{p}' \cdot \mathbf{y}(\mathbf{p}) \geq \mathbf{p} \cdot \mathbf{y}(\mathbf{p}) = \pi(\mathbf{p}),$$

where the first inequality holds since $y(\mathbf{p}')$ is the profit-maximizing vector under $\mathbf{p}'$.

(2) follows easily from the fact that $\mathbf{y}(t\mathbf{p}) = \mathbf{y}(\mathbf{p})$, that is, $\mathbf{y}(\mathbf{p})$ is homogeneous of degree zero.

(3) Given two price vectors $\mathbf{p}$ and $\mathbf{p}'$, let $\mathbf{p}'' = t\mathbf{p} + (1-t)\mathbf{p}'$. Then, we have

$$\pi(\mathbf{p}'') = \mathbf{p}'' \cdot \mathbf{y}(\mathbf{p}'') = (t\mathbf{p} + (1-t)\mathbf{p}') \cdot \mathbf{y}(\mathbf{p}'') = t\mathbf{p} \cdot \mathbf{y}(\mathbf{p}'') + (1-t)\mathbf{p}' \cdot \mathbf{y}(\mathbf{p}'') \leq t\pi(\mathbf{p}) + (1-t)\pi(\mathbf{p}'),$$

where the inequality is due to the fact that $\mathbf{p} \cdot \mathbf{y}(\mathbf{p}'') \leq \pi(\mathbf{p})$ and $\mathbf{p}' \cdot \mathbf{y}(\mathbf{p}'') \leq \pi(\mathbf{p}')$.

(4) The proof is quite similar to that of Shephard's lemma. A simpler proof can be provided if $\pi(\mathbf{p})$ is assumed to be differentiable. Suppose that $\mathbf{y}^*$ is the profit-maximizing vector under $\mathbf{p}^*$. Then, define the function

$$g(\mathbf{p}) := \pi(\mathbf{p}) - \mathbf{p} \cdot \mathbf{y}^*.$$

Note that $g(\mathbf{p})$ reaches a minimum value of 0 at $\mathbf{p} = \mathbf{p}^*$. Thus, the first-order conditions for a minimum require that

$$\frac{\partial g(\mathbf{p}^*)}{\partial p_\ell} = \frac{\partial \pi(\mathbf{p}^*)}{\partial p_\ell} - y_\ell^* = 0, \forall \ell = 1, \cdots, L.$$

Since this is true for all choices of $\mathbf{p}^*$, the proof is done. ▌

REMARK 4.2.1. Hotelling's lemmas tells us that

$$\left(\frac{\partial^2 \pi}{\partial p_\ell \partial p_k} \right)_{\ell,k} = \left(\frac{\partial y_\ell}{\partial p_k} \right)_{\ell,k}.$$

Since the profit function is convex, we have

$$\frac{\partial y_\ell(\mathbf{p})}{\partial p_\ell} = \frac{\partial^2 \pi(\mathbf{p})}{\partial p_\ell^2} \geq 0.$$

That is, the own price effect is non-negative (non-positive) if the good is output (input). *But* there is an easier way to show this. Let us consider two price vectors $\mathbf{p}$ and $\mathbf{p}'$, and let $\mathbf{y} = \mathbf{y}(\mathbf{p})$ and $\mathbf{y}' = \mathbf{y}(\mathbf{p}')$. Then,

$$\begin{aligned} \mathbf{p} \cdot (\mathbf{y} - \mathbf{y}') &\geq 0 \\ \text{and} \quad \mathbf{p}' \cdot (\mathbf{y}' - \mathbf{y}) &\geq 0, \end{aligned}$$

which add up to

$$(\mathbf{p}' - \mathbf{p}) \cdot (\mathbf{y}' - \mathbf{y}) \geq 0.$$

Letting $p_k' = p_k$ for $k \neq \ell$, we have

$$(p_\ell' - p_\ell)(y_\ell' - y_\ell) \geq 0.$$

4.3 Cost Minimization

In this part, we study the cost-minimizing behavior of the firm. Note that to minimize the cost is necessary for maximizing the profit, whether the firm is competitive or not. From now on, we only consider the case where there are only one output and m inputs.

- Given input price vector $\mathbf{w} \gg 0$ and output level y, the firm's cost minimization problem is

$$\min_{\mathbf{x}} \mathbf{w} \cdot \mathbf{x} \quad \text{s.t.} \quad f(\mathbf{x}) = y.$$

  - $\mathbf{x}(\mathbf{w}, y)$: Cost-minimizing input vector, called *conditional factor demand function*
  - $c(\mathbf{w}, y) := \mathbf{w} \cdot \mathbf{x}(\mathbf{w}, y)$: Minimized cost, called *cost function*
  - $mc(\mathbf{w}, y) := \frac{\partial c(\mathbf{w},y)}{\partial y}$: Marginal cost.

THEOREM 4.3.1. *If f is continuous and strictly increasing, then $c(\mathbf{w}, y)$ is*

(1) Homogeneous of degree 1 in $\mathbf{w}$.

(2) Strictly increasing in y and nondecreasing in $\mathbf{w}$.

(3) Continuous in $\mathbf{w}$ and y.

(4) Concave in $\mathbf{w}$.

If, in addition, $f(\mathbf{x})$ is strictly quasiconcave, we have

(5) Shephard's lemma: $x_\ell(\mathbf{w}, y) = \dfrac{\partial c(\mathbf{w}, y)}{\partial w_\ell}, \forall \ell = 1, \cdots, m.$

Proof. Identical to that of Proposition 1.5.1. ∎

REMARK 4.3.1. The structural similarity between the firm's cost-minimization problem and the consumer's expenditure-minimization problem implies:

(1) the conditional factor demand function has the same properties as the Hicksian demand function does and

(2) the production function can be recovered from the cost function by applying the same integrability argument.

- Given the cost function, the problem of maximizing the profit can be expressed alternatively as

$$\max_{y \in \mathbb{R}_+} py - c(\mathbf{w}, y).$$

  - The (well-known) first-order condition is

$$p = mc(\mathbf{w}, y^*),$$

    that is to equate the marginal revenue to the marginal cost at the optimal level of output.

- Consider the short-run situation where the amounts of some inputs $\mathbf{x}_2$ are fixed at $\bar{\mathbf{x}}_2$. Then, the firm's short-run cost minimization problem is

$$\min_{\mathbf{x}_1} \mathbf{w}_1 \cdot \mathbf{x}_1 + \mathbf{w}_2 \cdot \bar{\mathbf{x}}_2 \quad \text{s.t.} \quad f(\mathbf{x}_1, \bar{\mathbf{x}}_2) = y.$$

  - $\mathbf{x}_1(\mathbf{w}, y, \bar{\mathbf{x}}_2)$: Short-run conditional factor demand function.
  - $c_s(\mathbf{w}, y, \bar{\mathbf{x}}_2) := \mathbf{w}_1 \cdot \mathbf{x}_1(\mathbf{w}, y, \bar{\mathbf{x}}_2) + \mathbf{w}_2 \cdot \bar{\mathbf{x}}_2$: Short-run cost function, where $\mathbf{w}_1 \cdot \mathbf{x}_1(\mathbf{w}, y, \bar{\mathbf{x}}_2)$ is *short-run variable cost* while $\mathbf{w}_2 \cdot \bar{\mathbf{x}}_2$ is *short-run fixed cost.*

- Long-run and short-run cost functions are related in the following way:

$$c(\mathbf{w}, y) = c_s(\mathbf{w}, y, \mathbf{x}_2(\mathbf{w}, y))$$

  since $\mathbf{x}_2(\mathbf{w}, y)$ is the long-run cost-minimizing level, given $\mathbf{w}$ and y.

  - The long-run (average) cost curve is the lower envelope of the short-run (average) cost curves.
  - The response of output to its price change is greater in the long-run than in the short-run. To see this, observe that given $\bar{\mathbf{x}}_2 = \mathbf{x}_2(\mathbf{w}, \bar{y})$ for some $\bar{y}$, since $g(y) := c(\mathbf{w}, y) - c_s(\mathbf{w}, y, \bar{\mathbf{x}}_2)$ is maximized at $y = \bar{y}$, the first- and second-order necessary conditions imply

$$\left.\frac{dg(y)}{dy}\right|_{y=\bar{y}} = \left.\frac{d}{dy}[c(\mathbf{w}, y) - c_s(\mathbf{w}, y, \bar{\mathbf{x}}_2)]\right|_{y=\bar{y}} = mc(\mathbf{w}, \bar{y}) - mc_s(\mathbf{w}, \bar{y}, \bar{\mathbf{x}}_2) = 0$$

    and

$$\left.\frac{d^2g(y)}{dy^2}\right|_{y=\bar{y}} = \left.\frac{d^2}{dy^2}[c(\mathbf{w}, y) - c_s(\mathbf{w}, y, \bar{\mathbf{x}}_2)]\right|_{y=\bar{y}} = \frac{d}{dy}[mc(\mathbf{w}, \bar{y}) - mc_s(\mathbf{w}, \bar{y}, \bar{\mathbf{x}}_2)] \leq 0.$$