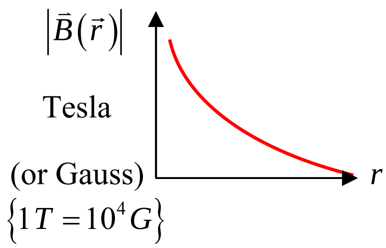
LECTURE NOTES 15

The Divergence & Curl of $\vec{B}$
<u>Ampere's Law</u>

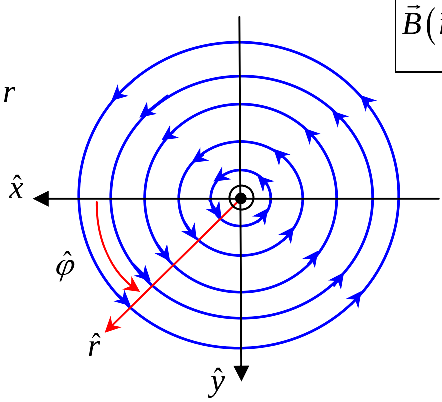
As we have discussed in the previous P435 Lecture Notes, for the case of an infinitely long straight wire carrying a steady (constant) line current $\vec{I} = I\hat{z}$, the macroscopic magnetic field associated with this system is given by:

$$\boxed{\vec{B}(\vec{r}) = \left(\frac{\mu_0}{2\pi}\right)\frac{I}{r}\hat{\varphi}} \quad \text{for} \quad \boxed{\vec{I} = I\hat{z}}$$



<u>Lines of $\vec{B}$ for $\vec{I} = I\hat{z}$ out of page:</u>

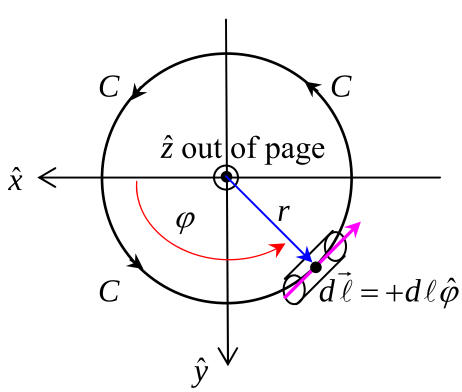
$$\boxed{\vec{B}(\vec{r}) = \left(\frac{\mu_0}{2\pi}\right)\frac{I}{r}\hat{\varphi}}$$



Clearly this vector $\vec{B}$-field has *<u>circulation</u>* (*a.k.a* "*<u>vorticity</u>*") associated with it !!!

$\Rightarrow$ $\underline{\underline{\vec{\nabla}\times\vec{B}\neq 0!!!}}$ (*cf* w/ $\vec{\nabla}\times\vec{E} = 0$ <u>always</u>, for electrostatics)

Let's take the line integral $\oint_C \vec{B}\cdot d\vec{\ell}$ where $d\vec{\ell} = +d\ell\hat{\varphi}$ at a radius, *r* (arbitrary) along a contour *C* as shown in the figure below, for this $\vec{B}$-field:

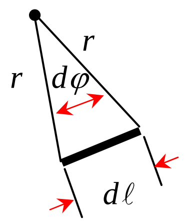


$$\therefore \left(\vec{B} = \left(\frac{\mu_0}{2\pi}\right)\frac{I}{r}\hat{\varphi}\right) \parallel \left(d\vec{\ell} = d\ell\hat{\varphi}\right)$$

$$\therefore \quad \vec{B}\cdot d\vec{\ell} = \left(\frac{\mu_0}{2\pi}\right)\frac{I}{r}\hat{\varphi}\cdot d\ell\hat{\varphi} = \left(\frac{\mu_0}{2\pi}\right)\frac{I}{r}d\ell \quad \text{(here)}$$

<u>n.b. Important Convention</u>:
The contour integrals' path is always taken such that the "outside" of the enclosing contour *C* is on the <u>right-hand</u> side of the infinitesimal vector element $d\vec{\ell}$.

By the arc length formula $s = r\theta$, we see that $d\ell = rd\varphi$:

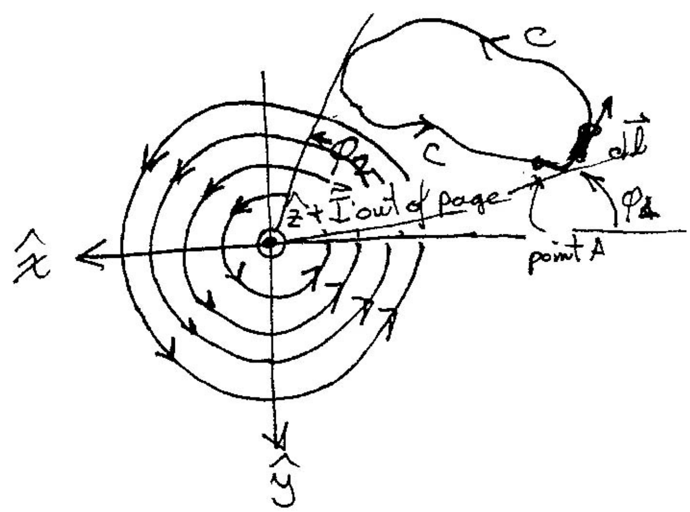


Thus, if $\boxed{\vec{B}(r) = \left(\frac{\mu_0}{2\pi}\right)\frac{I}{r}\hat{\varphi}}$ for $\boxed{\vec{I} = I\hat{z}}$ and $\boxed{d\vec{\ell} = d\ell\hat{\varphi} = rd\varphi\hat{\varphi}}$ (here), then:

$$\oint_C \vec{B}(r)\cdot d\vec{\ell} = \int_{\varphi=0}^{\varphi=2\pi}\left[\left(\frac{\mu_0}{2\pi}\right)\frac{I}{\cancel{r}}\hat{\varphi}\right]\cdot\left[\cancel{r}d\varphi\hat{\varphi}\right] = \left(\frac{\mu_0}{2\pi}\right)I\int_{\varphi=0}^{\varphi=2\pi}d\varphi = \left(\frac{\mu_0}{\cancel{2\pi}}\right)I * \cancel{2\pi} = \mu_0 I$$

<u>Thus</u>: $\boxed{\oint_C \vec{B}(\vec{r})\cdot d\vec{\ell} = \mu_0 I}$

If we had <u>instead</u> chosen a loop/contour/path which did not enclose (circle) the current, but rather, e.g. the contour shown in the figure below:



<u>Then here</u>: $\oint_C \vec{B}(\vec{r})\cdot d\vec{\ell} = \oint_a^a \vec{B}(\vec{r})\cdot d\vec{\ell} = \frac{\mu_0}{2\pi}I\int_{\substack{start\\ \varphi=\varphi_1}}^{\substack{end\\ \varphi=\varphi_1}}\left(\frac{1}{r(\varphi)}\right)d\varphi = 0$!!!

Thus, we have obtained <u>Ampere's Circuital Law (in integral form)</u>: $\boxed{\oint_C \vec{B}(\vec{r})\cdot d\vec{\ell} = \mu_0 I_{enclosed}}$

where $I_{enclosed}$ is the current <u>enclosed</u> by the bounding contour, C.

If we have N straight line/filamentary wires, each carrying current $\vec{I} = I\hat{z}$ and all of which are enclosed by the bounding contour, C then (by the principal of linear superposition) $I_{enclosed} = NI$, and thus:

$$\oint_C \vec{B}(\vec{r}) \cdot d\vec{\ell} = \mu_o I_{enclosed} = \underbrace{\mu_o NI}_{\substack{\text{n.b. has } no \\ \text{spatial} \\ \text{dependence} \\ = \text{constant.}}} \quad \text{for the } N \text{ currents } I \text{ enclosed by the bounding contour, } C$$

If electric charge flow is via a <u>free</u> volume electric current density $\vec{J}_{free}(\vec{r}\,')$ (Amps/m^2), then:

$$\boxed{I_{enclosed} = \int_{S'} \vec{J}_{free}(\vec{r}\,') \cdot d\vec{A}'}$$

n.b. the surface integral is taken over cross sectional area S' bounded by the corresponding closed contour, C

<u>Then</u>: $\boxed{\oint_{C'} \vec{B}(\vec{r}\,') \cdot d\vec{\ell}' = \mu_0 I_{enclosed} = \mu_0 \int_{S'} \vec{J}_{free}(\vec{r}\,') \cdot d\vec{A}'} \Leftarrow$ n.b. primes denote integration with respect to *source* variables.

<u>Now use Stokes' Theorem</u>:

$$\oint_C \vec{B}(\vec{r}\,') \cdot d\vec{\ell}' = \int_{S'} \left(\vec{\nabla} \times \vec{B}(\vec{r}\,')\right) \cdot d\vec{A}' = \mu_0 \int_{S'} \vec{J}_{free}(\vec{r}\,') \cdot d\vec{A}'$$

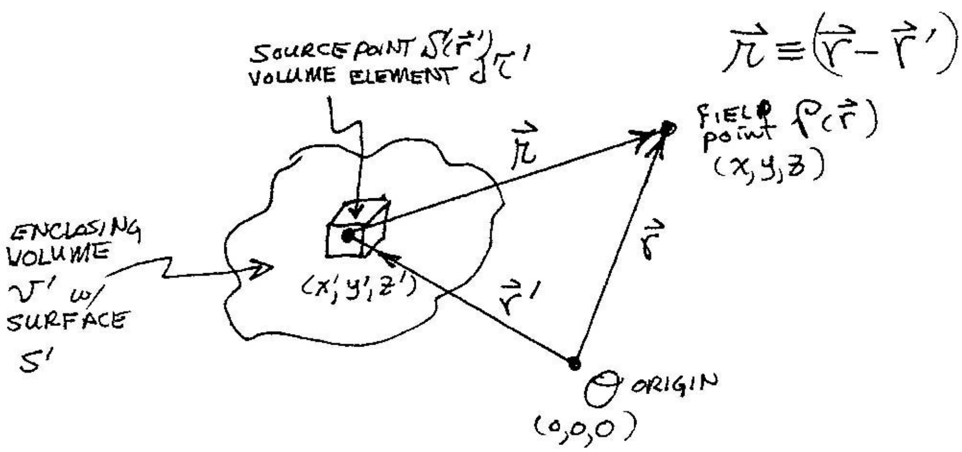
<u>Thus</u>: $\int_{S'} \left(\vec{\nabla} \times \vec{B}(\vec{r}\,')\right) \cdot d\vec{A}' - \mu_0 \int_{S'} \vec{J}_{free}(\vec{r}\,') \cdot d\vec{A}' = 0 \;\Rightarrow\; \boxed{\int_{S'} \left(\vec{\nabla} \times \vec{B}(\vec{r}\,') - \mu_0 \vec{J}_{free}(\vec{r}\,')\right) \cdot d\vec{A}' = 0}$

However, the enclosing surface S' is <u>*arbitrary*</u> (as long as $I_{enclosed}$ <u>remains</u> <u>the</u> <u>same</u>).
The only way this relation can hold for arbitrary enclosing surfaces S' is if the <u>integrand</u> = 0 at <u>every</u> point $\vec{r}\,'$ within S' and thus:

$\boxed{\vec{\nabla} \times \vec{B}(\vec{r}\,') - \mu_0 \vec{J}_{free}(\vec{r}\,') = 0}$ or: $\boxed{\vec{\nabla} \times \vec{B}(\vec{r}\,') = \mu_0 \vec{J}_{free}(\vec{r}\,')} \Leftarrow$ Ampere's Law (Differential Form)

A more rigorous way to prove this relation for <u>arbitrary</u> volume current density distributions $\vec{J}_{free}(\vec{r}\,')$ is as follows: We start with the formula for $\vec{B}(\vec{r})$ that we obtained earlier (see P435 Lect. Notes 14, p. 10 and/or p. 17):

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi} \int_{v'} \frac{\vec{J}(\vec{r}\,') \times \hat{\mathbf{r}}}{\mathbf{r}^2} d\tau'}$$



Note that:

$\vec{B}(\vec{r})$ is a function only of the field/observation point $P(\vec{r})$ located at $\vec{r} = (x\hat{x}, y\hat{y}, z\hat{z})$

$\vec{J}(\vec{r}\,')$ is a function only of the source point $S(\vec{r}\,')$ located at $\vec{r}\,' = (x'\hat{x}, y'\hat{y}, z'\hat{z})$

The field point-source point separation distance vector $\boldsymbol{\mathfrak{r}} \equiv (\mathbf{r} - \mathbf{r}')$ in Cartesian coordinates is:

$$\boxed{\boldsymbol{\mathfrak{r}} \equiv (\mathbf{r} - \mathbf{r}') = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}} \text{ and } \boxed{d\tau' = dx'dy'dz'}$$

The integration over the source volume v' is only over the source coordinates:

$$\boxed{\int_{v'} d\tau' = \int_{x'_{Lo}}^{x'_{Hi}} dx' \int_{y'_{Lo}}^{y'_{Hi}} dy' \int_{z'_{Lo}}^{z'_{Hi}} dz'}$$

Next, the divergence and curl of $\vec{B}(\vec{r})$ are to be taken with respect to the field coordinates (i.e. the unprimed coordinates), $\vec{r} = (x\hat{x}, y\hat{y}, z\hat{z})$ and *not* the source coordinates, $\vec{r}\,' = (x'\hat{x}, y'\hat{y}, z'\hat{z})$.

Thus:
$$\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r}) = \vec{\nabla}\cdot\left[\int_{v'} \frac{\vec{J}(\vec{r}\,')\times\hat{\boldsymbol{\mathfrak{r}}}}{\boldsymbol{\mathfrak{r}}^2} d\tau'\right] = \frac{\mu_o}{4\pi}\int_{v'} \vec{\nabla}\cdot\left(\frac{\vec{J}(\vec{r}\,')\times\hat{\boldsymbol{\mathfrak{r}}}}{\boldsymbol{\mathfrak{r}}^2}\right) d\tau'}$$

And:
$$\boxed{\vec{\nabla}\times\vec{B}(\vec{r}) = \vec{\nabla}\times\left[\int_{v'} \frac{\vec{J}(\vec{r}\,')\times\hat{\boldsymbol{\mathfrak{r}}}}{\boldsymbol{\mathfrak{r}}^2} d\tau'\right] = \frac{\mu_o}{4\pi}\int_{v'} \vec{\nabla}\times\left(\frac{\vec{J}(\vec{r}\,')\times\hat{\boldsymbol{\mathfrak{r}}}}{\boldsymbol{\mathfrak{r}}^2}\right) d\tau'}$$

Now $\boxed{\vec{\boldsymbol{\mathfrak{r}}} \equiv \vec{r} - \vec{r}\,'}$ and $\boxed{\boldsymbol{\mathfrak{r}} = |\vec{\boldsymbol{\mathfrak{r}}}| = |\vec{r} - \vec{r}\,'|}$ and also $\boxed{\hat{\boldsymbol{\mathfrak{r}}} = \vec{\boldsymbol{\mathfrak{r}}}/|\vec{\boldsymbol{\mathfrak{r}}}| = \vec{r} - \vec{r}\,'/|\vec{r} - \vec{r}\,'|}$.

For the divergence of $\vec{B}(\vec{r})$ we use the product rule on the RHS:

$$\vec{\nabla}\cdot\left(\vec{J}(\vec{r}\,')\times\frac{\hat{\boldsymbol{\mathfrak{r}}}}{\boldsymbol{\mathfrak{r}}^2}\right) = \frac{\hat{\boldsymbol{\mathfrak{r}}}}{\boldsymbol{\mathfrak{r}}^2}\cdot\left(\vec{\nabla}\times\vec{J}(\vec{r}\,')\right) - \vec{J}(\vec{r}\,')\cdot\left(\vec{\nabla}\times\frac{\hat{\boldsymbol{\mathfrak{r}}}}{\boldsymbol{\mathfrak{r}}^2}\right)$$

But: $\vec{\nabla}\times\vec{J}(\vec{r}\,') = 0$ (obvious) and (amazingly) $\vec{\nabla}\times\left(\frac{\hat{\boldsymbol{\mathfrak{r}}}}{\boldsymbol{\mathfrak{r}}^2}\right) = 0$ (see Griffiths problem 1.62 p. 57)

Thus: $\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r}) = 0}$ always ($\Rightarrow$ no *free* magnetic charges/no magnetic monopoles!!!)
no "bare" or "isolated" *N* or *S* *free* magnetic charges

For the curl of $\vec{B}(\vec{r})$ we use *another* product rule on the RHS

$$\vec{\nabla}\times\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi}\int_{v'} \vec{\nabla}\times\left(\frac{\vec{J}(\vec{r}\,')\times\hat{\boldsymbol{\mathfrak{r}}}}{\boldsymbol{\mathfrak{r}}^2}\right) d\tau' = \frac{\mu_o}{4\pi}\int_{v'}\left(\vec{J}(\vec{r}\,')\left(\vec{\nabla}\cdot\frac{\hat{\boldsymbol{\mathfrak{r}}}}{\boldsymbol{\mathfrak{r}}^2}\right) - \left(\vec{J}(\vec{r}\,')\cdot\vec{\nabla}\right)\frac{\hat{\boldsymbol{\mathfrak{r}}}}{\boldsymbol{\mathfrak{r}}^2}\right) d\tau'$$

Recall that $\vec{\nabla}\cdot\left(\frac{\hat{\boldsymbol{\mathfrak{r}}}}{\boldsymbol{\mathfrak{r}}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathfrak{r}})$; $\delta^3(\boldsymbol{\mathfrak{r}})$ is the 3-D Dirac δ-function (see Griffiths 1.100, p. 50)

Note also that $\vec{\nabla}\cdot\vec{J}(\vec{r}\,') = 0$ and $\vec{\nabla}\times\vec{J}(\vec{r}\,') = 0$ because $\vec{\nabla}$ depends only on the field point variable $\vec{r} = (\vec{x}, \vec{y}, \vec{z})$ whereas $\vec{J}(\vec{r}\,')$ depends only on the source point variable $\vec{r}\,' = (\vec{x}\,', \vec{y}\,', \vec{z}\,')$.

<u>Therefore the term</u>: $\boxed{\frac{\mu_o}{4\pi}\int_{v'}\vec{J}(\vec{r}')\left(\vec{\nabla}\bullet\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}\right)d\tau' = \frac{\mu_0}{\cancel{4\pi}}\int_{v'}\vec{J}(\vec{r}')\,\cancel{4\pi}\,\delta^3(\vec{r}-\vec{r}\,')\,d\tau' = \mu_0\vec{J}(\vec{r})}$!!!

Next, we focus on figuring out the details associated with the term: $-\vec{J}(\vec{r}')\bullet\vec{\nabla}\left(\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}\right)$

<u>We know that</u>: $-\vec{J}(\vec{r}')\bullet\vec{\nabla}\left(\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}\right) = +\vec{J}(\vec{r}')\bullet\vec{\nabla}'\left(\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}\right)$ because $\boxed{\vec{\nabla}\left(\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}\right) = -\vec{\nabla}'\left(\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}\right)}$

$\Rightarrow$ Please prove this relation to <u>yourself</u>, using $\boxed{\vec{\mathbf{r}} \equiv \vec{r}-\vec{r}'}$, $\boxed{\mathbf{r} = |\vec{\mathbf{r}}| = |\vec{r}-\vec{r}'|}$ and also

$\boxed{\hat{\mathbf{r}} = \vec{\mathbf{r}}/|\vec{\mathbf{r}}| = \vec{r}-\vec{r}'/|\vec{r}-\vec{r}'|}$, then work out: $\boxed{\vec{\nabla}\left(\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}\right) = \vec{\nabla}\left(\frac{\hat{r}-\hat{r}'}{|\vec{r}-\vec{r}'|^2}\right) = -\vec{\nabla}'\left(\frac{\hat{r}-\hat{r}'}{|\vec{r}-\vec{r}'|^2}\right) = -\vec{\nabla}'\left(\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}\right)}$

n.b. Can also show: $\boxed{\vec{\nabla}\left(\frac{1}{\mathbf{r}}\right) = \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -\vec{\nabla}'\left(\frac{1}{\mathbf{r}}\right)}$

$$\therefore\quad -\int_{v'}\vec{J}(\vec{r}')\bullet\vec{\nabla}\left(\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}\right)d\tau' = +\int_{v'}\vec{J}(\vec{r}')\bullet\vec{\nabla}'\left(\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}\right)d\tau'$$

<u>Now since</u>: $\boxed{\vec{\nabla}\bullet\left(f\vec{A}\right) = f\left(\vec{\nabla}\bullet\vec{A}\right) + \vec{A}\bullet\left(\vec{\nabla}f\right)}$ <u>then</u>: $\boxed{\vec{A}\bullet\left(\vec{\nabla}f\right) = \vec{\nabla}\bullet\left(f\vec{A}\right) - f\left(\vec{\nabla}\bullet\vec{A}\right)}$

<u>Thus</u>:
$$\vec{J}(\vec{r}')\bullet\vec{\nabla}'\frac{\hat{\mathbf{r}}}{\mathbf{r}^2} = \vec{\nabla}'\bullet\left(\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}\vec{J}(\vec{r}')\right) - \frac{\hat{\mathbf{r}}}{\mathbf{r}^2}\underbrace{\left(\vec{\nabla}'\bullet\vec{J}(\vec{r}')\right)}_{\substack{=0 \text{ for} \\ \text{steady state currents} \\ \left(=-\frac{\partial\rho_{free}}{\partial\tau}=0!!!\right)}}$$

$$\therefore\quad -\int_{v'}\vec{J}(\vec{r}')\bullet\vec{\nabla}\left(\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}\right)d\tau' = +\int_{v'}\vec{J}(\vec{r}')\bullet\vec{\nabla}'\left(\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}\right)d\tau' = \int_{v'}\vec{\nabla}'\bullet\left(\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}\vec{J}(\vec{r}')\,d\tau'\right) = \oint_{S'}\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}\vec{J}(\vec{r}')\bullet d\vec{A}'$$

{n.b. the last step used the divergence theorem}

Now let volume v' and corresponding enclosing surface $S' \to \infty$, then $\exists$ no currents on surface!

$$\Rightarrow\ -\int_{v'}\vec{J}(\vec{r}')\bullet\vec{\nabla}\left(\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}\right)d\tau' = \oint_{S'}\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}\vec{J}(\vec{r}')\bullet d\vec{A}' = 0\ !!!$$

$\therefore$ $\boxed{\vec{\nabla}\times\vec{B}(\vec{r}) = \mu_0\vec{J}(\vec{r})}$ holds/is valid $\forall$ $\vec{r}$ in volume v' !!!

Thus, finally, we obtain <u>Ampere's Law (in differential form)</u>: $\boxed{\vec{\nabla}\times\vec{B}(\vec{r}) = \mu_0\vec{J}(\vec{r})}$

APPLICATIONS OF AMPERE'S LAW

Ampere's Law in Differential Form: $\boxed{\vec{\nabla}\times\vec{B}(\vec{r})=\mu_0\vec{J}(\vec{r})}$

Ampere's Law in Integral Form: $\boxed{\oint_C \vec{B}(\vec{r})\bullet d\vec{\ell}=\mu_0 I_{enclosed}}$

However, by the Curl Theorem (see inside front cover of Griffith's book):

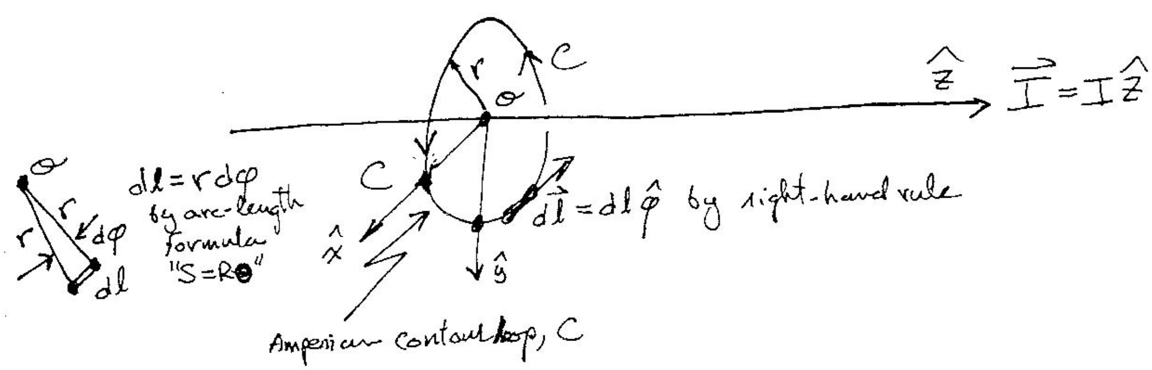
$$\boxed{\int_S \vec{\nabla}\times\vec{B}(\vec{r})\bullet d\vec{A}=\oint_C \vec{B}(\vec{r})\bullet d\vec{\ell}=\mu_0\int_S \vec{J}(\vec{r})\bullet d\vec{A}=\mu_0 I_{enclosed}}$$

Griffiths Example 5.7:

Use Ampere's Law to determine the macroscopic magnetic field $\vec{B}(\vec{r})$ a perpendicular distance *r* away from a (infinitely) long, straight filamentary wire carrying steady current, *I*.

We already know that (here) $\vec{B}\parallel\hat{\varphi}$ (i.e. solenoidal/phi field).

Use the integral form of Ampere's Law, take an "Amperian" loop contour *C*, enclosing the filamentary line current *I* as shown in the figure below:



$$\oint_C \vec{B}(\vec{r})\bullet d\vec{\ell}=\mu_0 I_{enclosed}=\mu_0 I$$

Now: $\vec{B}(\vec{r})\parallel\hat{\varphi},\ d\vec{\ell}\parallel\hat{\varphi}\ \Rightarrow\ \vec{B}(\vec{r})\parallel d\vec{\ell},\ \therefore\ \vec{B}(\vec{r})\bullet d\vec{\ell}=B(\vec{r})\,d\ell$

And: $d\ell=rd\varphi$ by the arc length formula, $S=R\theta$

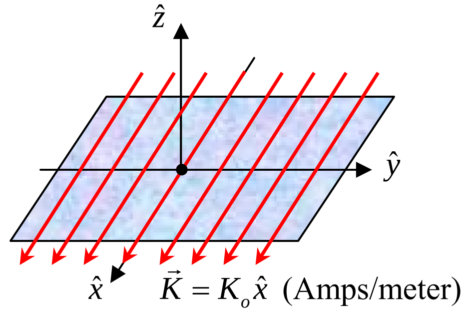
Thus: $\int_{\varphi=0}^{\varphi=2\pi}B(\vec{r})\,d\ell=\int_{\varphi=0}^{\varphi=2\pi}B(\vec{r})\,rd\varphi=2\pi rB(\vec{r})=\mu_o I$

$\boxed{\vec{B}(\vec{r})=\frac{\mu_0}{2\pi}\left(\frac{I}{r}\right)\hat{\varphi}}$ $\Leftarrow$ Same/identical result as that which we had previously obtained - but far less effort/work involved using Ampere's Law!!!

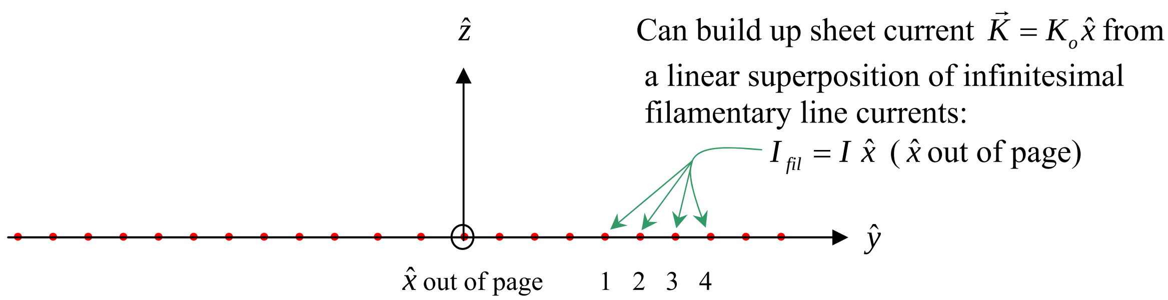
Griffiths Example 5.8:

Determine the magnetic field $\vec{B}(\vec{r})$ associated with an infinite planar sheet of uniform surface current, $\vec{K} = K_o\hat{x}$ Amperes/meter flowing over the *x-y* plane (K_o = constant).

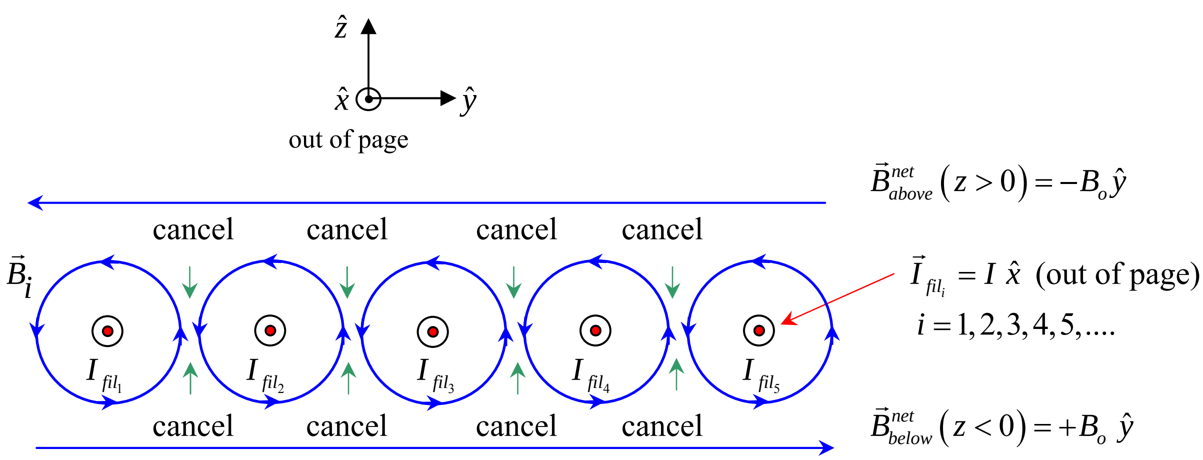
3-D View:



End-View of Uniform Surface/ "Sheet" Current:



Blow-Up/Expanded/Microscopic View of Filamentary/Infinitesimal Line Currents:

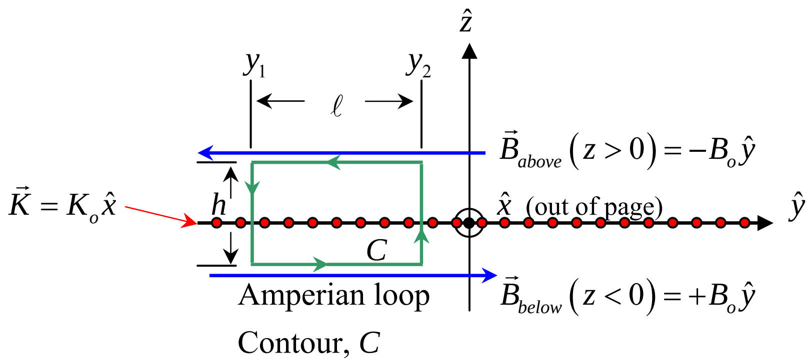


Use Ampere's Circuital Law – in integral form to determine $\vec{B}(\vec{r})$ associated with surface / sheet current $\vec{K} = K_o \hat{x}$:

$\oint_C \vec{B}(\vec{r}) \cdot d\vec{\ell} = \mu_0 I_{enclosed}$ - take the contour shown in the figure below:

Can either a.) do the line integral <u>explicitly</u> – i.e. break it up into 4 integrals - one for each side (will discover that the <u>vertical</u> sides will not contribute to the line integral), or b.) shrink height *h* of contour loop *C* to infinitesimally above/below current sheet:

$\vec{K} = K_o \hat{x}.$



What is $I_{enclosed}$? $\boxed{\vec{I}_{enclosed} = \int_{\ell_\perp = y_1}^{\ell_\perp = y_2} \vec{K}\, d\ell_\perp = \int_{\ell_\perp = y_1}^{\ell_\perp = y_2} K_o \hat{x}\, d\ell_\perp = K_o \hat{x} \int_{\ell_\perp = y_1}^{\ell_\perp = y_2} d\ell_\perp = K_o \ell \hat{x}}$

If the height *h* of the contour loop *C* is infinitesimally small, then <u>only</u> the <u>horizontal</u> portions of the contour loop *C* contribute to contour integral:

$$\oint \vec{B}(\vec{r}) \cdot d\vec{\ell} = \overbrace{\int_{y_2}^{y_1} \vec{B}_{above}(z>0) \cdot \underbrace{d\vec{\ell}}_{=-d\ell\,\hat{y}}}^{\text{n.b. } \vec{B}_{above}(z>0) \parallel d\vec{\ell} \text{ here}} + \overbrace{\int_{y_1}^{y_2} \vec{B}_{below}(z<0) \cdot \underbrace{d\vec{\ell}}_{=+d\ell\,\hat{y}}}^{\text{n.b. } \vec{B}_{below}(z<0) \parallel d\vec{\ell} \text{ here}} = \mu_o I_{enclosed} = \mu_o K_o \ell$$

$$= (-B_o \hat{y}) \cdot (-\ell \hat{y}) \quad + \quad (+B_o \hat{y}) \cdot (+\ell \hat{y}) = \mu_o K_o \ell$$

$$= +B_o \ell + B_o \ell = 2 B_o \ell = \mu_o K_o \ell$$

<u>Thus</u>: $\boxed{B_o = \frac{1}{2} \mu_o K_o}$ And hence finally:

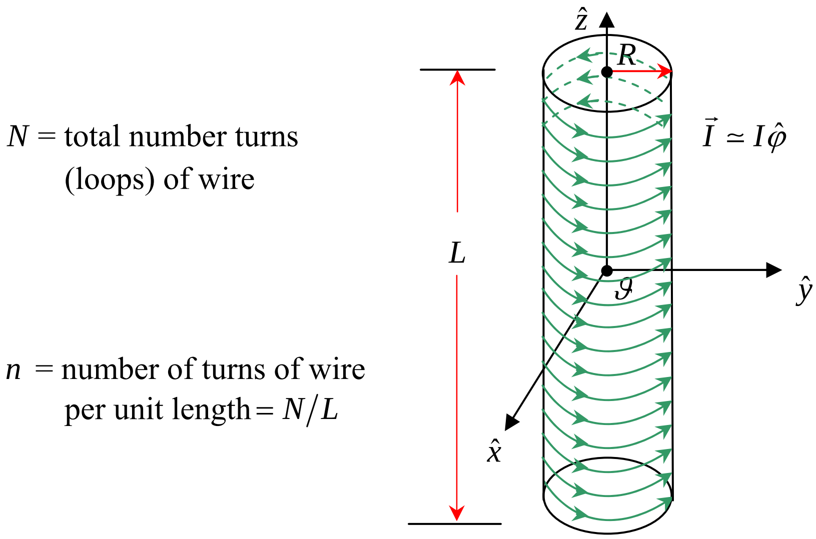
$$\boxed{\begin{aligned} \vec{B}_{above}(z>0) &= -\frac{1}{2}\mu_o K_o \hat{y} \\ \vec{B}_{below}(z<0) &= +\frac{1}{2}\mu_o K_o \hat{y} \end{aligned}}$$

Note that for the infinite planar sheet current, that the $\vec{B}$-field is <u>*independent*</u> of the height *z* of the observer above (or below) the sheet current – completely analogous to what we found for the $\vec{E}$-field associated with an infinite planar electrically charged conducting sheet of uniform surface charge density σ_o Coulombs/m^2!!!

Griffiths Example 5.9:

Determine the magnetic field associated with an infinitely long solenoid consisting of n turns (loops) per unit length, i.e. $n = N/L$ (where N = total number of turns or wire wound around a cylinder of radius R and carrying a steady current I).

n.b. There is a very small pitch angle, α associated with physically winding real, finite-diameter wire around a cylindrical form to make a real solenoid coil. The steady current flowing in the solenoid is then $\vec{I} = I\cos\alpha\,\hat{\varphi} + I\sin\alpha\,\hat{z}$, which for $\alpha \approx 0$ is nearly all in the phi-direction, i.e. $\vec{I} \simeq I\hat{\varphi}$. {n.b. If we neglect this finite-small pitch angle α, then $\vec{I} \equiv I\hat{\varphi}$.}



We will take several Amperian-loop contours in order to "explore" the nature of $\vec{B}_{solenoid}$:

Amperian Loop/Closed Contour #1:

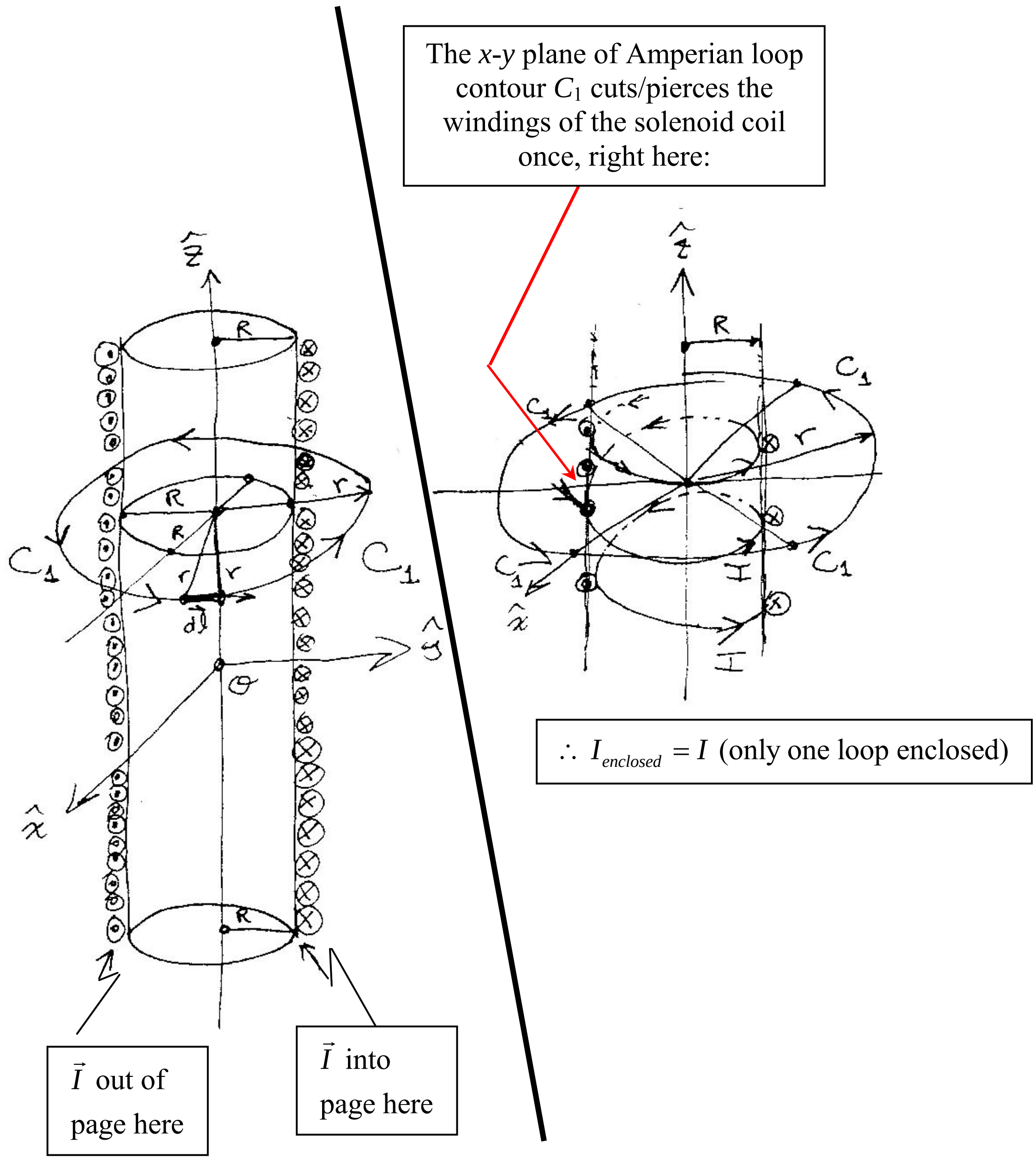
Circular loop contour C_1 (in x-y plane) with radius $r > R$ (i.e. outside solenoid) - see figure(s) on following page - encloses a single winding of the solenoid.

Thus: $I_{enclosed} = 1I = I$

$$\therefore \oint_{C_1} B_{outside}(\vec{r}) \bullet d\vec{\ell} = \mu_o I_{enclosed} = \mu_o I \quad \text{with} \quad d\vec{\ell} = d\ell\hat{\varphi} = rd\varphi\hat{\varphi} \quad \text{and} \quad \vec{B}_{outside}(\vec{r}) = B_{outside}(\vec{r})\hat{\varphi}$$

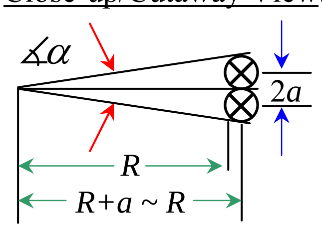
Thus: $\oint_{C_1} \vec{B}_{outside}(\vec{r}) \bullet d\vec{\ell} = 2\pi r B_{outside}(\vec{r}) = \mu_0 I$ or: $B_{outside}(\vec{r}) = \dfrac{\mu_0 I}{2\pi r}$ i.e: $\boxed{\vec{B}_{outside}(\vec{r}) = \dfrac{\mu_0 I}{2\pi r}\hat{\varphi}}$

The $\vec{B}$-field outside of a long solenoid, $\vec{B}_{outside}(r > R) =$ same as that for infinitely long wire carrying steady current $\vec{I} = I\hat{z}$!!! It arises solely due to finite pitch angle, α associated with physically winding a real solenoid. If we neglect the finite pitch angle of the solenoid windings, i.e. $\alpha = 0$, then $\vec{B}_{outside}(r > R) = 0$.



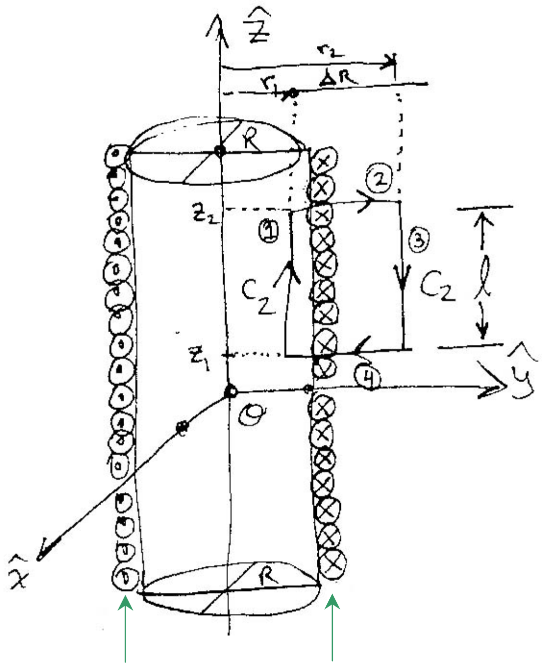
n.b. The finite pitch angle α associated with the windings of a solenoid coil arises due to winding the coil with wire of finite thickness (e.g. wire radius = a).
For $a \ll R$, then: $\tan\alpha \approx \alpha = 2a/R$.

Close-up/Cutaway View:



<u>Amperian Loop/Closed Contour #2</u>:

Take closed contour C_2 e.g. in the *y-z* plane, as shown below in a cross-sectional view of the long solenoid:



$\vec{I}$ out of page $\vec{I}$ into page

Ampere's Circuital Law for closed contour C_2: $\oint_{C_2} \vec{B}(\vec{r}) \cdot d\vec{\ell} = \mu_0 I_{enclosed}$

What is $I_{enclosed}$ here?

The vertical sections of contour C_2 have height ℓ, and if the long solenoid has $\boxed{n = N/L}$ turns per unit length, then the total number of turns of wire (each carrying current I) enclosed by the contour C_2 is $\boxed{N_{enclosed} = n\ell}$, and thus: $\boxed{I_{enclosed} = N_{enclosed} I = n\ell I}$.

Here, we will initially explicitly carry out this closed contour integral by breaking it up into the four segments (1), (2), (3) and (4):

$$\oint_{C_2} \vec{B}(\vec{r}) \cdot d\vec{\ell} = \int_{(1)} \vec{B}_1(\vec{r}) \cdot d\vec{\ell}_1 + \int_{(2)} \vec{B}_2(\vec{r}) \cdot d\vec{\ell}_2 + \int_{(3)} \vec{B}_3(\vec{r}) \cdot d\vec{\ell}_3 + \int_{(4)} \vec{B}_4(\vec{r}) \cdot d\vec{\ell}_4 = \mu_0 I_{enclosed}$$

<u>where</u>:

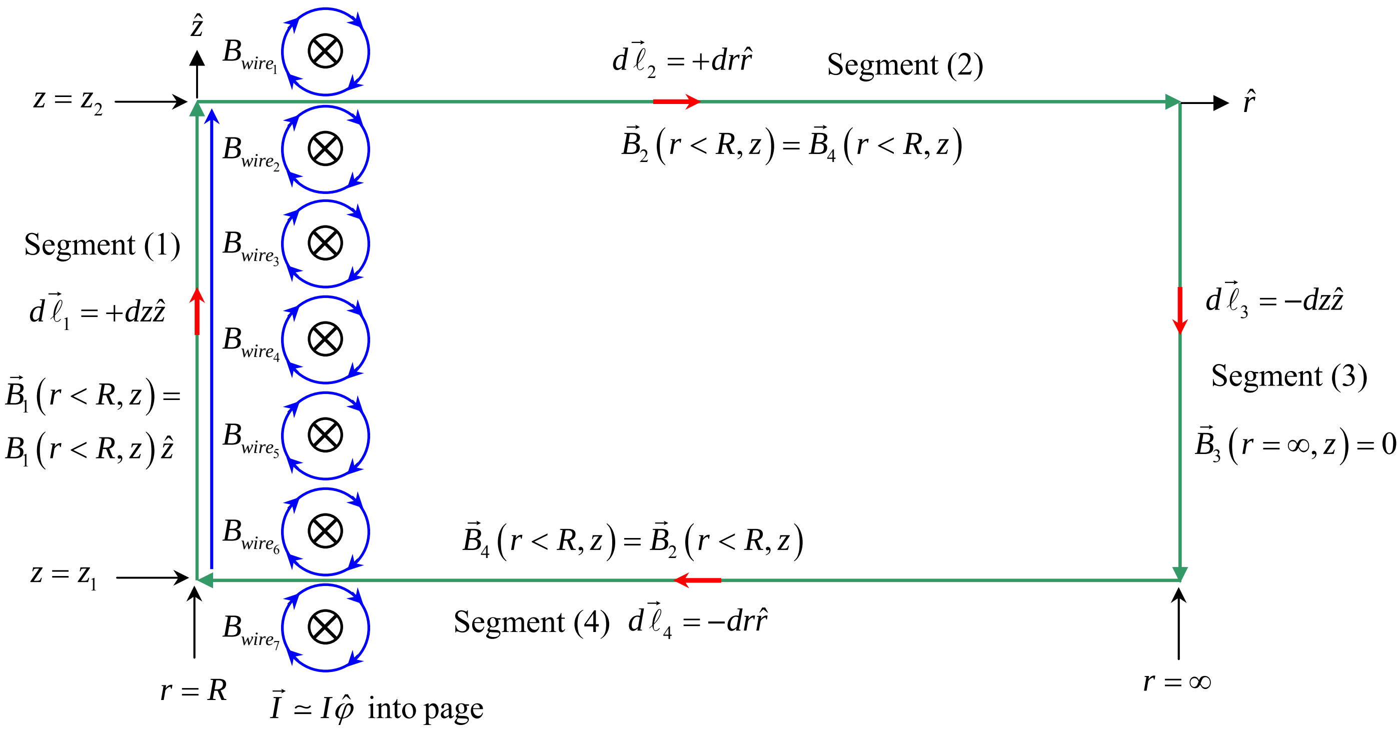
$d\vec{\ell}_1 = +dz\hat{z}$

$d\vec{\ell}_2 = +dy\hat{y} = +dr\hat{r} \leftarrow$ in cylindrical coordinates

$d\vec{\ell}_3 = dz(-\hat{z}) = -dz\hat{z}$

$d\vec{\ell}_4 = dy(-\hat{y}) = -dy\hat{y} = -dr\hat{r}$

We will extend the RHS of contour C_2 out to infinity – i.e. in the radial direction (in cylindrical coordinates) such that segment (3) of contour C_2 is located at $r = \infty$, as shown in a close-up view of the solenoid & contour integral's path, below:



For an infinitely long solenoid carrying a steady current *I*, there can be no *z*-dependence of the magnetic field (anywhere) (i.e. $\vec{B}(\vec{r}) \neq fcn(z)$), just as we saw for the case of the magnetic field associated with an infinitely long filamentary wire carrying steady current *I*.

One can see that on segment (1), with $r \leq R$ and $z_1 \leq z \leq z_2$, that the macroscopic/net $\vec{B}$-field $\vec{B}_1(r<R,z)$ will be non-zero and also point in the $+\hat{z}$ direction, because the individual $\vec{B}$-field contributions from each wire add together (analogous to what we saw for the macroscopic $\vec{B}$-field associated with superposing infinitesimal filamentary currents for the planar current sheet!). Thus $\vec{B}_1(r<R,z) = B_z(r<R,z)\hat{z} \parallel d\vec{\ell}_1 = +dz\hat{z}$. Note also that $\ell = z_2 - z_1$

For segment (3), which is located at $r = \infty$, note that $\vec{B}_3(r=\infty,z) = 0$.

If the solenoid is infinitely long, then the $\vec{B}$-field along segments (2) and (4), can have no *z*-dependence, thus we know that $\vec{B}_2(r,z_2) \equiv \vec{B}_4(r,z_1)$. Furthermore, for an infinitely long solenoid, the macroscopic/net $\vec{B}$-field cannot have any radial component, i.e. $B_r(\vec{r})\hat{r} = 0$. However, along segments (2) and (4), $d\vec{\ell}_2 = +dr\hat{r}$ and $d\vec{\ell}_4 = +dr\hat{r}$, thus $\vec{B}_2(\vec{r}) \cdot d\vec{\ell}_2 = 0$ and $\vec{B}_4(\vec{r}) \cdot d\vec{\ell}_4 = 0$

Then:

$$\oint_{C_2} \vec{B}(\vec{r})\bullet d\vec{\ell} = \int_{(1)} \vec{B}_1(\vec{r})\bullet d\vec{\ell}_1 + \int_{(2)} \vec{B}_2(\vec{r})\bullet d\vec{\ell}_2 + \int_{(3)} \vec{B}_3(\vec{r})\bullet d\vec{\ell}_3 + \int_{(4)} \vec{B}_4(\vec{r})\bullet d\vec{\ell}_4 = \mu_0 I_{enclosed} = \mu_0 n\ell I$$

$$= \int_{z=z_1}^{z=z_2} B_z(r \le R)\,dz + \underbrace{\int_{r=R}^{r=\infty} \cancel{\vec{B}_2(\vec{r})\bullet dr\hat{r}}}_{=0} - \underbrace{\int_{z=z_2}^{z=z_1} \cancel{\vec{B}_3(r=\infty)}\bullet dz\hat{z}}_{=0} - \underbrace{\int_{r=\infty}^{r=R} \cancel{\vec{B}_4(\vec{r})\bullet dr\hat{r}}}_{=0}$$

$$= B_z(r \le R)\cancel{\ell} = \mu_0 n \cancel{\ell} I$$

Thus we learn that the magnetic field is only non-zero ***inside*** ($r < R$) the infinitely long solenoid and that it points in the $\hat{z}$-direction:

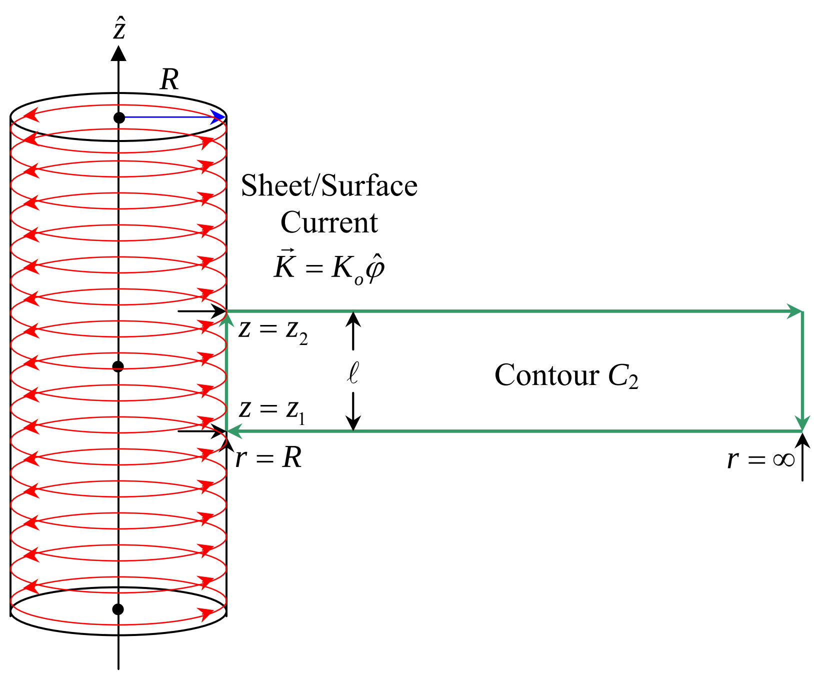
$$\boxed{\vec{B}^{infinite}_{solenoid}(r \le R) = \mu_0 nI\,\hat{z}}$$

We can also calculate the magnetic flux passing through the bore of the solenoid:

$$\Phi_m = \int_S \vec{B}^{infinite}_{solenoid}(r \le R)\bullet d\vec{A} \quad \text{where} \quad d\vec{A} = dA\hat{n} = dA\hat{z}\,,$$

thus: $\boxed{\Phi_m = \int_S \vec{B}^{infinite}_{solenoid}(r \le R)\bullet d\vec{A} = \int_S (\mu_0 nI\,\hat{z})\bullet dA\hat{z} = \mu_0 nI\pi R^2}$ (SI Units: Webers = Tesla-m^2).

If we now imagine the thickness of the wire (of radius a) used to wind the coil to become smaller and smaller, then each loop/each turn of wire, in the limit $a \to 0$ becomes an infinitesimal filamentary "line" current wrapped around the cylindrical surface of the solenoid, the pitch angle $\alpha \to 0$, and in this limit we can equivalently view the infinitely long solenoid as having a sheet/surface current $\vec{K} = K_o\hat{\varphi}$ flowing azimuthally around the surface of the cylinder of radius R, as shown in the figure below:



For the same closed contour C_2 as used above, Ampere's Circuital Law is:

$\oint_{C_2} \vec{B}(\vec{r})\bullet d\vec{\ell} = \mu_0 I_{enclosed}$ with $\vec{I}_{enclosed} = \int \vec{K}\,d\ell_\perp = \int_{z_1}^{z_2} (K_o\hat{\varphi})\,dz = K_o\ell\hat{\varphi}$. Thus: $\boxed{I_{enclosed} = K_o\ell}$.

We again break the contour C_2 up into its four segments (as before), and the result is the same as before – the only non-zero contribution to this contour integral is that from segment (1)

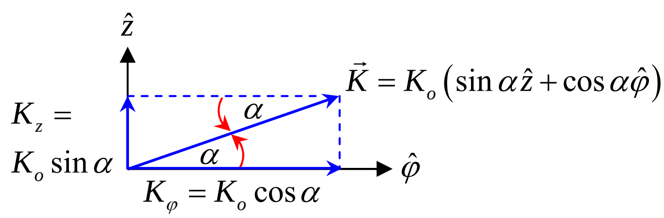
$$\oint_{C_2} \vec{B}(\vec{r})\bullet d\vec{\ell} = \int_{(1)} \vec{B}_1(\vec{r})\bullet d\vec{\ell}_1 + \int_{(2)} \vec{B}_2(\vec{r})\bullet d\vec{\ell}_2 + \int_{(3)} \vec{B}_3(\vec{r})\bullet d\vec{\ell}_3 + \int_{(4)} \vec{B}_4(\vec{r})\bullet d\vec{\ell}_4 = \mu_0 I_{enclosed} = \mu_0 K_o \ell$$

$$= \int_{z=z_1}^{z=z_2} B_z(r \le R)\, dz + \int_{r=R}^{r=\infty} \underbrace{\cancel{\vec{B}_2(\vec{r})\bullet dr\hat{r}}}_{=0} - \int_{z=z_2}^{z=z_1} \underbrace{\cancel{\vec{B}_3(r=\infty)}\bullet dz\hat{z}}_{=0} - \int_{r=\infty}^{r=R} \underbrace{\cancel{\vec{B}_4(\vec{r})\bullet dr\hat{r}}}_{=0}$$

$$= B_z(r \le R)\cancel{\ell} = \mu_0 K_o \cancel{\ell}$$

Thus we see that $\boxed{\vec{B}^{K-sheet}_{cylinder}(r \le R) = \mu_0 K_o \hat{z}}$ is equivalent to the $\vec{B}$-field associated with an infinitely long solenoid $\boxed{\vec{B}^{infinite}_{solenoid}(r \le R) = \mu_0 nI\, \hat{z}}$ wound with $\boxed{n = N/L}$ turns per unit length of wire, both in magnitude and direction, provided the equivalent azimuthal sheet/surface current $\boxed{K_o = nI = NI/L}$ (Amperes/meter).

We can now even "jazz up"/improve our azimuthal current-sheet model of an infinitely long solenoid in order to mimic the finite-small pitch angle α associated with physically winding finite-diameter wire (of radius a) around a cylindrical tube to make a real solenoid, by giving the sheet/surface current a small z-component, i.e. $\vec{K} = K_z\hat{z} + K_\varphi\hat{\varphi} = K_o(\sin\alpha\hat{z} + \cos\alpha\hat{\varphi})$:



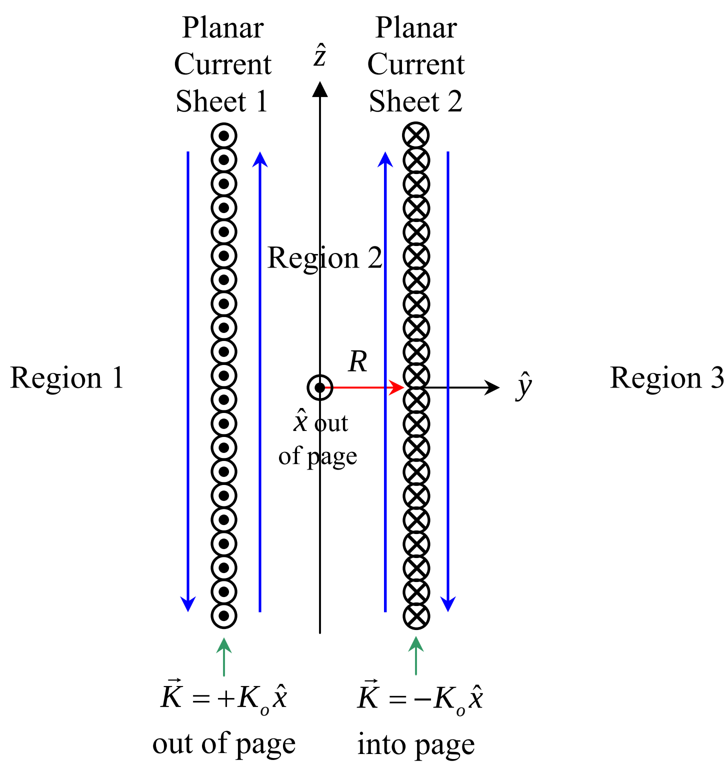
This sheet current $\vec{K} = K_o(\sin\alpha\hat{z} + \cos\alpha\hat{\varphi})$ flows down the surface of the cylinder, spiraling slowly (because α is small) around it - in a helical fashion - analogous to that of a strand of DNA, or e.g. a red & white barber-shop pole. The resulting $\vec{B}$-field is still primarily inside the bore of the solenoid, $\boxed{\vec{B}^{K-sheet}_{cylinder-z}(r \le R) = \mu_0 K_o \cos\alpha\hat{z}}$ however with a finite-small pitch angle α, there is also a weak φ-component of the magnetic field which can be obtained using Ampere's Circuital Law, and taking contour C_1: $\boxed{\vec{B}^{K-sheet}_{\substack{cylinder-\varphi \\ outside}}(\vec{r}) = \frac{\mu_0 K_o \ell \sin\alpha}{2\pi r}\hat{\varphi}}$ which is equivalent to the φ-component of the weak external (i.e. $r > R$) magnetic field that we obtained for the infinite solenoid with finite pitch angle α: $\boxed{\vec{B}_{outside}(\vec{r}) = \frac{\mu_0 I}{2\pi r}\hat{\varphi}}$.

We can learn some additional interesting things about the nature of the magnetic field associated with the infinitely long solenoid. For the simplest model in which the pitch angle $\alpha = 0$, the magnetic field is only non-zero (and constant/uniform) inside the solenoid:

$$\boxed{\vec{B}^{infinite}_{solenoid}\left(r \le R\right) = \mu_0 n I\ \hat{z}}$$

This result may initially seem rather strange – why is the magnetic field outside of this solenoid identically zero?

First, let's investigate the situation where we have two parallel, infinite-extent planar current sheets that have opposing sheet current flows, and which are laterally spaced apart from each other by a distance $d = 2R$ as shown in the figure below.



In each of the three Regions 1, 2 and 3, the net magnetic field there is the linear superposition of the magnetic field contributions in that region associated with each of the two current sheets:

$$\vec{B}_{Region\,1}\left(y<-R\right) = \vec{B}^{planar}_{\substack{current\\ sheet\,1}}\left(y<-R\right) + \vec{B}^{planar}_{\substack{current\\ sheet\,2}}\left(y<-R\right) = -\frac{1}{2}\mu_o K_o\hat{z} + \frac{1}{2}\mu_o K_o\hat{z} = 0!!!$$

$$\vec{B}_{Region\,2}\left(|y|<R\right) = \vec{B}^{planar}_{\substack{current\\ sheet\,1}}\left(|y|<R\right) + \vec{B}^{planar}_{\substack{current\\ sheet\,2}}\left(|y|<R\right) = +\frac{1}{2}\mu_o K_o\hat{z} + \frac{1}{2}\mu_o K_o\hat{z} = \mu_o K_o\hat{z}$$

$$\vec{B}_{Region\,3}\left(y>R\right) = \vec{B}^{planar}_{\substack{current\\ sheet\,1}}\left(y>R\right) + \vec{B}^{planar}_{\substack{current\\ sheet\,2}}\left(y>R\right) = +\frac{1}{2}\mu_o K_o\hat{z} - \frac{1}{2}\mu_o K_o\hat{z} = 0!!!$$

Thus, we see that for this situation, the net magnetic field is only non-zero in the region between the two opposing, infinite-planar current sheets!

The above picture is also the cross-sectional view of an infinitely-long solenoid! In fact, if we simply take a single infinitely-long planar current sheet and topologically deform it into an infinitely long cylinder of radius R, we have precisely the same situation as that of an azimuthal sheet/surface current flowing on the surface of an infinitely long cylinder.

However, the cross sectional view shows that outside the cylinder, the magnetic field associated with the "current sheet" on one side of the cylinder is cancelled by the magnetic field associated with the "current sheet" on the opposing side of the cylinder, whereas inside the cylinder, these two magnetic field contributions add together!

Thus, it can be seen that the magnetic field in the exterior region of an infinitely long solenoid is zero due to this cancellation of fields associated with the two opposing "current sheets" of the solenoid in the cross-sectional view!

Yet another equivalent way to view the infinitely long solenoid is as an infinite linear superposition of individual, planar current loops each carrying steady current I. Inside the bore of the solenoid, the B-fields add, whereas outside they cancel! One can even slant the planar current loops by a small pitch angle α to mimic the winding of turns on a real solenoid!

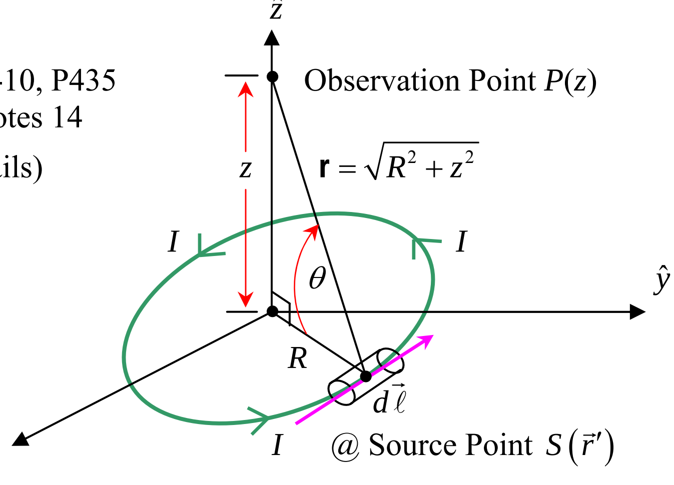
The Magnetic Field of a Short Solenoid

If the length L of the solenoid is not $L >> R$ where R = radius of solenoid, then end effects are not negligible – then the magnetic field along the solenoid is no longer axial, especially at the ends of the solenoid – it develops a radial component. Ampere's Law isn't very useful here either, in this situation, because detailed information gets "integrated over" (i.e. lost…) using $\oint_C \vec{B}\bullet d\vec{\ell} = \mu_o I_{enclosed}$. Ampere's law (in integral form) actually tells us very little about the detailed geometry of the current-carrying structure.

On the other hand, we have seen that, on the symmetry axis of a single current-carrying loop of radius R with steady current I, i.e. using the formula: $\vec{B}_{loop}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right) I \oint_C \frac{d\vec{\ell}' \times \hat{\boldsymbol{r}}}{\boldsymbol{r}^2}$ that:

$$\vec{B}_{loop}(z) = \left(\frac{\mu_o}{\cancel{4\pi}}\right)\frac{\cancel{2\pi} RI}{\boldsymbol{r}^2}\cos\theta\hat{z} = \left(\frac{\mu_o}{2}\right)\frac{RI}{\boldsymbol{r}^2}\cos\theta\hat{z} = \left(\frac{\mu_o}{2}\right)\frac{R^2 I}{\left(R^2+z^2\right)^{3/2}}\hat{z}, \quad \cos\theta = \frac{R}{\boldsymbol{r}}, \quad \boldsymbol{r} = \sqrt{R^2+z^2}$$

(See p. 7-10, P435 Lect. Notes 14 for details)



We can "stack up" N such current loops along the z-axis to create a short solenoid of length L.

Each such current loop contributes its own infinitesimal $d\vec{B}_{loop}(z) = \left(\frac{\mu_o}{2}\right)\frac{R^2 I}{\left(R^2+z^2\right)^{3/2}}\hat{z}$

Noting that $n = N/L$ and $I \to nIdz$, then summing all of these individual contributions to get the total / net result (i.e.integrating this expression (appropriately)) gives the (net) $\vec{B}$-field on the symmetry axis of a short solenoid:

$$\vec{B}_{\substack{short\\solenoid}}(z) = \int d\vec{B}_{loop} = \left(\frac{\mu_o}{2}\right)\int_{z_{LO}}^{z_{HI}}\frac{R^2 nIdz}{\left(R^2+z^2\right)^{3/2}}\hat{z} \quad \text{or:} \quad \boxed{\vec{B}_{\substack{short\\solenoid}}(z) = \left(\frac{\mu_o}{2}\right)R^2 nI\int_{z_{LO}}^{z_{HI}}\frac{dz}{\left(R^2+z^2\right)^{3/2}}\hat{z}}$$

We can then use this formula to calculate:

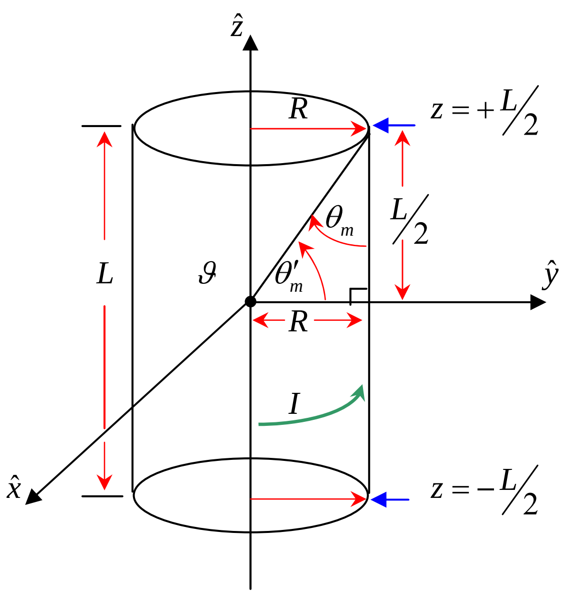
A.) The $\vec{B}$-field at the Center of a Short Solenoid:

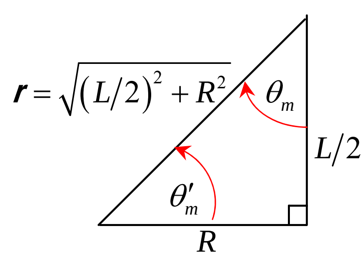
$$\vec{B}_{\substack{short\\solenoid}}(z=0) = \left(\frac{\mu_o}{2}\right)R^2 nI\int_{-L/2}^{L/2}\frac{dz}{\left(z^2+R^2\right)^{3/2}}\hat{z} \quad \text{gives:} \quad \boxed{\vec{B}_{\substack{short\\solenoid}}(z=0) = \mu_o nI\sin\theta'_m\hat{z}}$$

Where: $\boxed{\sin\theta'_m = \frac{L/2}{\sqrt{(L/2)^2+R^2}}}$ and $\boxed{\theta'_m = 90^o - \theta_m}$ and $n = N/L = \#$ turns per unit length.

Thus: $\boxed{\sin\theta'_m = \sin\left(90^o-\theta_m\right) = \sin 90^o\cos\theta_m - \cancel{\cos 90^o}^{=0}\sin\theta_m = \cos\theta_m}$

Thus, equivalently: $\boxed{\vec{B}_{\substack{short\\solenoid}}(z=0) = \mu_o nI\cos\theta_m}$ where: $\boxed{\cos\theta_m = \sin\theta'_m = \frac{L/2}{\sqrt{(L/2)^2+R^2}}}$





Note that if the short solenoid → infinitely long solenoid that:

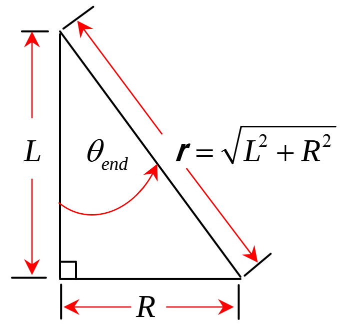
$\theta'_m \to 90^o \left(=\dfrac{\pi}{2}\right)$ and $\sin\theta'_m \to 1$, $\cos\theta_m \to 1$ $\left(i.e.\ \theta_m \to 0^o\right)$, then $\boxed{\vec{B}_{\substack{long\\solenoid}}(z=0) = \mu_o nI\hat{z}}$!!!

B.) The $\vec{B}$-field at the End of a Short Solenoid:

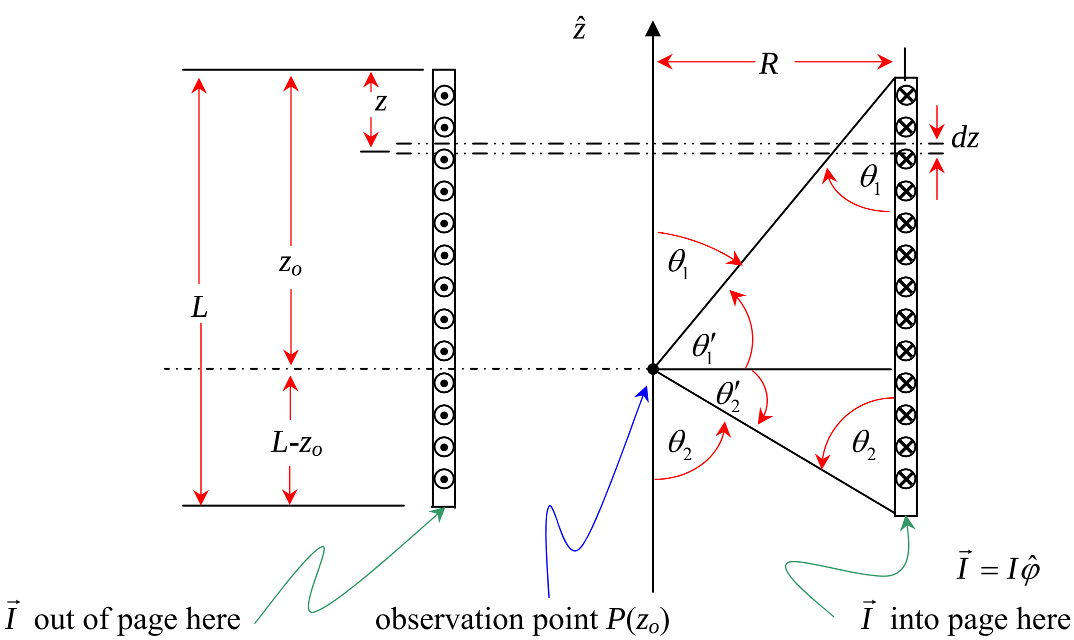
At one end of the short solenoid, e.g. $z = +L/2$:

$\vec{B}_{\substack{short\\solenoid}}\left(z=+\dfrac{L}{2}\right) = \left(\dfrac{\mu_o}{2}\right)R^2 nI\int_{-L}^{0}\dfrac{dz}{\left(z^2+R^2\right)^{3/2}}\hat{z}$ or: $\boxed{\vec{B}_{\substack{short\\solenoid}}\left(z=+\dfrac{L}{2}\right) = \left(\dfrac{\mu_o}{2}\right)nI\sin\theta_{end}\hat{z}}$

where: $\boxed{\sin\theta_{end} = \dfrac{R}{\sqrt{R^2+L^2}}}$



B.) The $\vec{B}$-field at an Arbitrary Point on Along the Symmetry Axis of a Short Solenoid:



$$\boxed{B_z(z_o) = \frac{1}{2}\mu_o nIR^2\int_0^L \frac{dz}{\left[\left(z_o-z\right)^2+R^2\right]^{3/2}}}$$

$$\boxed{\sin\theta' = \frac{\left(z-z_o\right)}{\sqrt{\left(z-z_o\right)^2+R^2}}}\qquad \boxed{\cos\theta' = \frac{R}{\sqrt{\left(z-z_o\right)^2+R^2}}}$$

Let: $(z-z_o)=R\tan\theta'$ then: $\tan\theta_1' = -\dfrac{z_o}{R}$ and: $\theta_1' = -\tan^{-1}\left(\dfrac{z_o}{R}\right)$

Then: $dz = \dfrac{-R}{\sin^2\theta'}d\theta'$ and: $\tan\theta_2' = \dfrac{(L-z_o)}{R}$ and: $\theta_2' = \tan\left(\dfrac{L-z_o}{R}\right)$

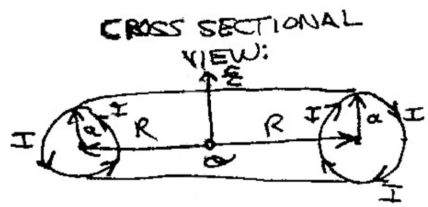
Then: $B_z(z_o) = \left(\dfrac{\mu_o}{2}\right)nI\int_{\theta_1'}^{\theta_2'}\cos\theta' d\theta'$ Or: $\boxed{B_z(z_o) = \left(\dfrac{\mu_o}{2}\right)nI\left(\sin\theta_2' - \sin\theta_1'\right)}$

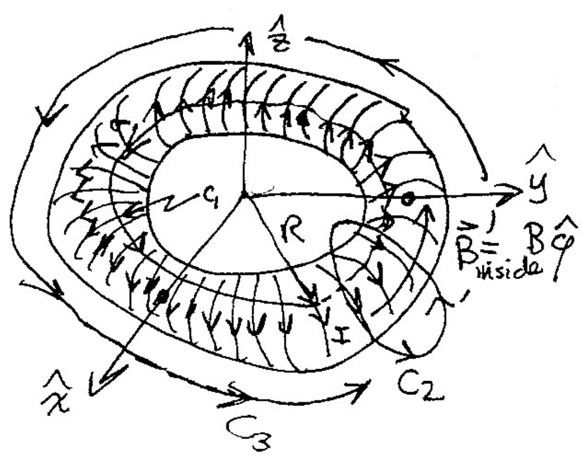
But: $\sin\theta' = \sin(90-\theta) = \cos\theta$ Thus: $\boxed{B_z(z_o) = \left(\dfrac{\mu_o}{2}\right)nI\left(\cos\theta_2 - \cos\theta_1\right)}$

Another Application of Ampere's Circuital Law

Griffiths Example 5.10: The *B*-field Associated with a Toroid Coil

A toroid is a long solenoid bent into a circle = doughnut!!!
$\Rightarrow \vec{B}$-field inside the toroid is in the $\hat{\varphi}$ direction.





Take Amperian Loop Contour C_1:
Take contour C_1 at mean radius of toroid R:

$$\oint_{C_1}\vec{B}(\vec{r})\cdot d\vec{\ell} = \mu_o I_{enclosed} \quad \text{where} \quad I_{enclosed} = NI$$

N total turns in toroid.

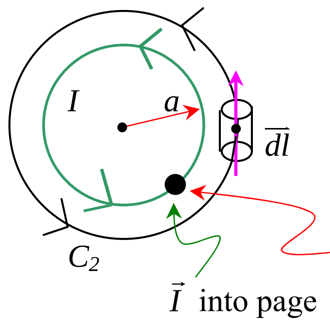
$d\vec{\ell} = Rd\varphi\hat{\varphi}$ (by arc length formula "$S = R\theta$")
$\vec{B} \parallel d\vec{\ell}$ and $\vec{B} = B\hat{\varphi}$ by the right-hand rule

Thus: $\int_0^{\varphi=2\pi} BRd\varphi = \mu_o NI$ or: $2\pi RB(R) = \mu_o NI$ or: $\boxed{\vec{B}_{toroid}(r=R) = \dfrac{\mu_o NI}{2\pi R}\hat{\varphi}}$

The (mean) circumference of the toroid is $C = 2\pi R$ and $\dfrac{N}{C} = \dfrac{N}{2\pi R} = n = \#$ turns per unit length

Thus: $\boxed{\vec{B}_{toroid}(r=R) = \mu_o nI\hat{\varphi}}$ and note that: $\boxed{\left|\vec{B}_{toroid}(r=R)\right| = \left|\vec{B}^{inside}_{\substack{long\\solenoid}}(r\le R)\right| = \mu_o nI}$

Amperian Loop Contour C_2: Take contour C_2 around "waist" of doughnut – see above figure.



Contour C_2 encloses single current I right here (e.g.)

$\oint_{C_2} \vec{B} \bullet d\vec{\ell} = \mu_o I_{enclosed} = \mu_o I \qquad \vec{B}_{here} \parallel d\vec{\ell}_{here} \qquad$ (n.b. $\vec{B}_{here} \simeq \vec{B}$ from long straight wire!)

Bend wire into a circular loop of radius R

= magnetic dipole loop!!!

$\Rightarrow \vec{B}$-field outside toroid wound with N total turns and finite pitch angle, α = $\vec{B}$-field of single circular loop carrying steady current I !!! If pitch angle $\alpha = 0$, $\Rightarrow \vec{B}$-field outside toroid = 0.

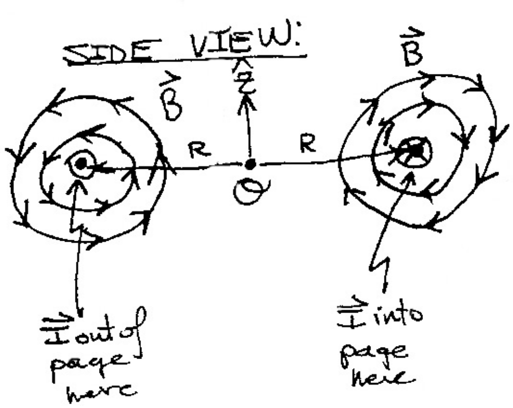
Amperian Loop Contour C_3:
Take contour C_3 at radius r larger than toroid $R+a$:

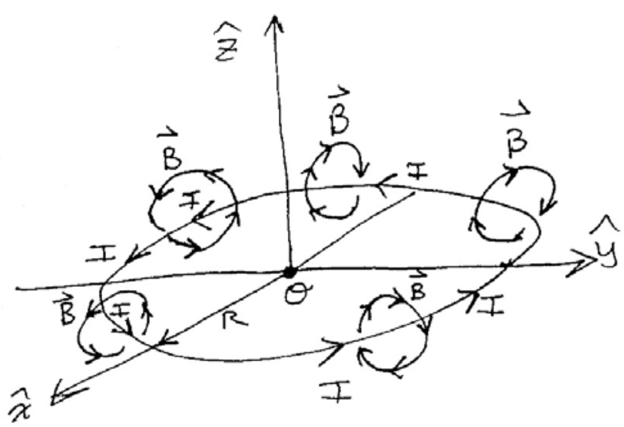
$$\int \vec{B}_{outside}\left(r > R+a\right) \bullet d\vec{\ell} = \mu_o I_{enclosed} = 0$$

= 0 !! Net $I = 0$

$$\Rightarrow \vec{B}_{outside}\left(r > R+a\right) = 0\hat{\varphi}$$

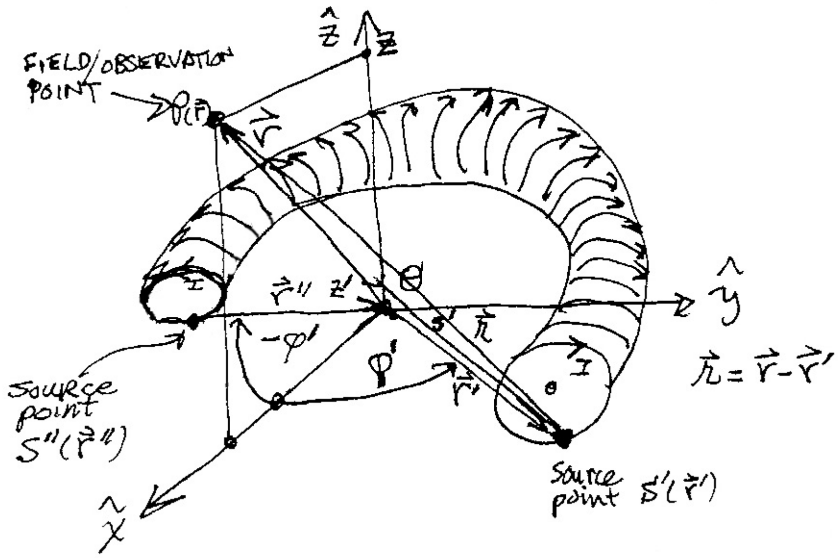
The $\vec{B}$-field "detected" by contour C_3 is equivalent to that of single circular loop of radius R carrying steady current I – it's the $\vec{B}$-field one gets from a long straight wire carrying current I after it is bent into circle of radius R.





One can also use $d\vec{B}(\vec{r}) = \left(\dfrac{\mu_o}{4\pi}\right)\dfrac{\vec{I}\times\boldsymbol{r}}{\boldsymbol{r}^3}d\ell'$ to obtain $\vec{B}(\vec{r})$ at the observation point $P(\vec{r})$

Locate the observation point $P(\vec{r})$ in the x–z plane $\vec{r} = (x,0,z)$ as shown in the figure below:



Field & source point <u>coordinates</u> are located at: $\vec{r} = (x,0,z)$, $\vec{r}' = (s'\cos\varphi', s'\sin\varphi', z')$

<u>Thus</u>: $\boldsymbol{r} = \vec{r} - \vec{r}' = (x - s'\cos\varphi', s'\sin\varphi', z - z')$

<u>and</u>: $\boldsymbol{r} = |\boldsymbol{r}| = |\vec{r} - \vec{r}'| = \sqrt{(x - s'\cos\varphi')^2 + (-s'\sin\varphi')^2 + (z - z')^2}$

If we neglect the pitch angle α of the windings of the toroid, this is (again) equivalent to neglecting a/the (small) $\hat{\varphi}$-component of the current I, then: $\vec{I} = (I_s\cos\varphi', I_s\sin\varphi', I_z)$

Note that: $\hat{s} = \cos\varphi\hat{x} + \sin\varphi\hat{y}$ and that: I_s is in the radial direction $\hat{s}$ whereas I_z is in the $\hat{z}$ direction. Then:

$$\vec{I}\times\boldsymbol{r} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ I_s\cos\varphi' & I_s\sin\varphi' & I_z \\ (x - s'\cos\varphi') & (-s'\sin\varphi') & (z - z') \end{pmatrix}$$

$$\vec{I}\times\boldsymbol{r} = \left[\sin\varphi'\left(I_s(z - z') + s'I_z\right)\right]\hat{x} - \left[I_z(x - s'\cos\varphi') - I_s\cos\varphi'(z - z')\right]\hat{y} + \left[-I_s x\sin\varphi'\right]\hat{z}$$

However, $\exists$ a symmetrically-situated current element at $\vec{r}''$ that has the same value of s', same value of $\boldsymbol{r} = |\vec{r} - \vec{r}''| = |\vec{r} - \vec{r}'|$ and the same $d\vec{\ell}'$, same I_s and same I_z as at $\vec{r}'$, except that it has negative φ' (see figure). Then because $\sin(-\varphi') = -\sin\varphi'$, the $\hat{x}$ and $\hat{z}$ contributions from the two source points $\vec{r}'$ and $\vec{r}''$ <u>cancel</u> each other, and only the $\hat{y}$ term in $\vec{I}\times\boldsymbol{r} = \{\}$ above survives. From the figure, one concludes that the $\vec{B}$-field at the observation point $P(\vec{r})$, which is in the x-z plane, points in the $\hat{y}$ direction and more generally, the $\vec{B}$-field points in the $\hat{\varphi}$ direction, which is what we already knew from Ampere's circuital law.

Note that one can also bend a "solenoid" made up of tube of azimuthally-circulating surface/sheet current $\vec{K} = K_o\hat{\varphi}$ into a circular tube of (mean) radius R – thus making a toroid comprised of a tube of surface/sheet current – now circulating around the waist of the toroid – the toroid's $\vec{B}$-fields (N turns of wire vs. surface/sheet current) can thus be seen to be the same/equivalent to each other!!!

Maxwell's Equations for Electrostatics (Differential Form):

: $\boxed{\vec{\nabla}\bullet\vec{E}(\vec{r}) = \rho_{TOT}(\vec{r})/\varepsilon_o}$ (Gauss' Law) with $\boxed{\rho_{TOT}(\vec{r}) = \rho_{free}(\vec{r}) + \rho_{bound}(\vec{r})}$

$\boxed{\vec{\nabla}\times\vec{E}(\vec{r}) = 0}$ (always) $\leftarrow$ $\vec{E}(\vec{r})$ associated with conservative electrostatic force

Maxwell's Equations for Magnetostatics (Differential Form):

$\boxed{\vec{\nabla}\bullet\vec{B}(\vec{r}) = 0}$ $\leftarrow$ No magnetic charges / no magnetic monopoles

$\boxed{\vec{\nabla}\times\vec{B}(\vec{r}) = \mu_o\vec{J}(\vec{r})}$ (Ampere's Law)

Force Law for Electric Charges:

$$\boxed{\vec{F}_e(\vec{r}) = Q\left(\underbrace{\vec{E}(\vec{r})}_{\text{Coulomb's Law}} + \underbrace{\vec{v}(\vec{r})\times\vec{B}(\vec{r})}_{\text{Lorentz Force Law}}\right)}$$

Since (apparently) no magnetic charges/no magnetic monopoles exist in nature (despite many physicists searching hard to discover them!), in the absence of magnetic charges/magnetic monopoles, the lines of $\vec{B}$ are always closed – i.e. they start/end on themselves.

If magnetic charges $\pm g_m$ did exist in nature, then lines of $\vec{B}$ would terminate on/emanate from magnetic charges $\pm g_m$, exactly analogous to lines of $\vec{E}$ terminating on/emanating from electric charges $\pm e$.

Then, note also that $\boxed{\vec{\nabla}\bullet\vec{B}(\vec{r}) = \mu_o\rho_m(\vec{r}) \neq 0}$ where ρ_m = volume magnetic charge density (SI Units of ρ_m: Amperes/m^2), which would be the magnetic analog/equivalent of that for electrostatics: $\boxed{\vec{\nabla}\bullet\vec{E}(\vec{r}) = \rho_e(\vec{r})/\varepsilon_o \neq 0}$.

There would then also be a magnetic force law of the form:

Force Law for Magnetic Charges:

$$\boxed{\vec{F}_m(\vec{r}) = g_m\left(\underbrace{\vec{B}(\vec{r})}_{\text{Coulomb's Law}} - \underbrace{\frac{1}{c^2}\vec{v}(\vec{r})\times\vec{E}(\vec{r})}_{\text{Lorentz Force Law}}\right)}$$

