# Lecture Notes

# EE301 Signals and Systems I

**Department of Electrical and Electronics Engineering**

**Middle East Technical University (METU)**

# Preface

These lecture notes were prepared with the purpose of helping the students to follow the lectures more easily and efficiently. This course is a fast-paced course with a significant amount of material, and to cover all of this material at a reasonable pace in the lectures, we intend to benefit from these partially-complete lecture notes. In particular, we included important results, properties, comments and examples, but left out most of the mathematics, derivations and solutions of examples, which we do on the board and expect the students to write into the provided empty spaces in the notes. We hope that this approach will reduce the note-taking burden on the students and will enable more time to stress important concepts and discuss more examples.

These lecture notes were prepared using mainly our textbook titled "Signals and Systems" by Alan V. Oppenheim, Alan S. Willsky and S. Hamid Nawab, but also from handwritten notes of Fatih Kamisli and A. Ozgur Yilmaz. Most figures and tables in the notes are also taken from the textbook.

This is the first version of the notes. Therefore the notes may contain errors and we also believe there is room for improving the notes in many aspects. In this regard, we are open to feedback and comments, especially from the students taking the course.

Fatih Kamisli
December $2^{nd}, 2016$.

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# Chapter 1

# Fundamental Concepts

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What is signal processing ? Watch the following videos for a great description of signal processing and some great examples of its applications :

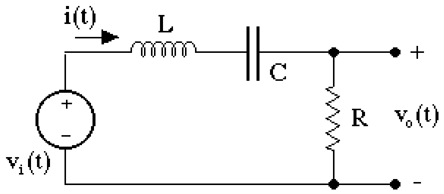
- `https://www.youtube.com/watch?v=EErkgr1MWw0`
  (Search youtube for : What is Signal Processing?)
- `https://www.youtube.com/watch?v=mexN6d8QF9o`
  (Search youtube for : Signal Processing and Machine Learning)

This chapter introduces basic signals, systems and their properties.

## 1.1 Signals

**Definition 1** *A signal is the variation of a physical, or non-physical, quantity with respect to one or more independent variable(s). Signals typically carry information that is somehow relevant for some purpose.*

**Ex:** Electrical signals : voltage as a function of time




**Ex:** Acoustic signals : acoustic pressure as a function of time

Speech is produced by creating fluctuations in acoustic pressure, which can be sensed by a microphone and converted into an electrical signal.

**Ex:** Picture : brightness as a function of two spatial variables

A camera senses the incoming light and records the light reflectivity as a function of space onto a magnetic film.

**Ex:** Other examples : sequence of bases in a gene (biological signal), sequence of daily stock prices in the financial market, ...

We will mostly refer to the **independent variable** as *time* ($t$), although it can be other things (such as space) depending on application.

We consider two types of signals : **continuous-time** (CT) signals and **discrete-time** (DT) signals.

- In continuous-time (CT) signals, the independent variable is continuous.
- In discrete-time (DT) signals, the independent variable is discrete.

**Ex:** Examples of CT and DT signals

**Note** : DT signals are undefined at values other than the specified discrete values.

Independent variables can be 1-D, 2-D, 3-D,...
**Ex:**
Speech signal :
Image signal :
Video signal :

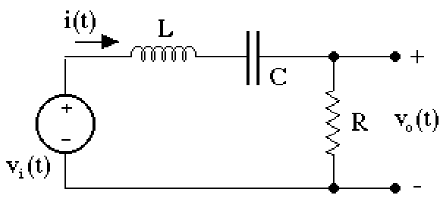
Throughout the course, following notation is used to represent CT and DT signals :
CT signals :
DT signals :

Most physical signals are CT, but not all. A DT signal may be obtained from phenomena

- that is inherently discrete (as in the 'people attending lecture' example)
- that is obtained by taking samples from a CT signal (as in the 'blood pressure' example).

**Definition 2** *System is defined as any process in which input signals are transformed to output signals.*

**Ex:** Electrical circuit with an input signal ($v_i(t)$) and an output signal ($v_o(t)$)




We will discuss systems in Section 1.2 in more detail.

### 1.1.1 Transformations of the independent variable of signals

We sometimes consider signals after modifying the independent variable.

- **Time Shift** : $x(t) \to y(t) = x(t - t_0)$ or $x[n] \to y[n] = x[n - n_0]$
  **Ex:**

  In general :

- **Time Reversal (Reflection)** : $x(t) \rightarrow y(t) = x(-t)$ or $x[n] \rightarrow y[n] = x[-n]$
  **Ex:**

- **Time Scaling** : $x(t) \rightarrow y(t) = x(at)$, $a \in \mathbb{R}$ or $x[n] \rightarrow y[n] = x[bn]$, $b \in \mathbb{Z}$
  **Ex:**

  In general :

**Note** : $x(at)$ is always defined for $a \epsilon \mathbb{R}$. The same is not true for $x[bn]$, unless $b \in \mathbb{Z}$, i.e. $b$ is an integer.
**Ex:** For $x[n]$ : $x[2n]$ $x[\sqrt{2}n]$ $x[\frac{n}{2}]$

**Ex:** Find and plot $y(t) = x(-t+1)$ for

**Ex:** Find and plot $y[n] = x[-2n+1]$ for

### 1.1.2 Periodic signals

A periodic CT signal $x(t)$ has the property that there is a **period** $T \in \mathbb{R}^+$ for which

$$x(t) =$$

It is said that $x(t)$ is periodic with $T$.
Periodicity is defined similarly for DT signals :
**Ex:**

**Note** : If $x(t)$ is periodic with $T$ then it is also periodic with $2T, 3T, 4T, ...$, i.e. $k \cdot T, k \in \mathbb{Z}^+$.(Same holds for DT signals.)
**Fundamental period** $T_0$ of $x(t)$ ($N_0$ of $x[n]$) is the smallest positive $T$ ($N$) for which $x(t)$ ($x[n]$) is periodic, i.e. the above equalities hold.

### 1.1.3 Even and Odd Signals

A CT signal is **even** if $\quad x(t) = x(-t) \quad \forall t. \qquad$ (In DT: $x[-n] = x[n] \quad \forall n$)
A CT signal is **odd** if $\quad x(t) = -x(-t) \quad \forall t. \qquad$ (In DT: $x[n] = -x[-n] \quad \forall n$)
**Ex:**

Any signal $x(t)$ ($x[n]$) can be written uniquely as a sum of its even and odd part :

In the next subsections, we will discuss some basic CT and DT signals, in particular,

- DT and CT unit impulse and step signals and
- CT and DT complex exponential signals.

### 1.1.4 DT Unit Impulse and Unit Step Sequences

**DT Unit Impulse signal**

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

**DT Unit Step signal**

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0. \end{cases}$$

**Relations between $\delta[n]$ and $u[n]$ and some properties**

- $\delta[n] =$
- $u[n] =$
- $\sum_{k=k_1}^{k_2} \delta[k] =$
- $x[n]\delta[n - n_0] =$
- $\sum_{k=k_1}^{k_2} x[k]\delta[k - n_0] =$

**Ex:** : Compute the following expressions
$\sum_{i=k_1}^{k_2} x[i]\delta[i]$
$u[n] = \sum_{i=-\infty}^{n} \delta[i]$ (show it)

### 1.1.5 CT Unit Impulse and Unit Step Signals

To study the CT unit step ($u(t)$) and impulse ($\delta(t)$) signals, let us first examine their approximations $u_\Delta(t)$ and $\delta_\Delta(t)$ :

Note that $u_\Delta(t)$ and $\delta_\Delta(t)$ are related by :

As $\Delta \to 0$, $u(t)$ and $\delta(t)$ are obtained, which still satisfy the above relations :

**CT Unit Step signal** $u(t)$

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0. \end{cases}$$

**CT Unit Impulse signal** $\delta(t)$

$\delta(t)$ is not defined directly as in many other functions, but by its properties :

- $\delta(t) = \frac{d}{dt}u(t)$

- $\delta(t) = 0,\ t \neq 0$ and

- $\int_{\tau=t_1}^{t_2} \delta(\tau)d\tau =$

- $u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$ (Running Sum Interpretation)

- $x(t)\delta(t - t_0) =$

- $\int_{\tau=t_1}^{t_2} x(\tau)\delta(\tau - t_0)d\tau =$

**Ex:** : Compute the following expressions
$\int_{\sigma=0}^{\infty} \delta(t - \sigma)d\sigma =$
$\int_{\tau=-\infty}^{t} x(\tau)\delta(\tau)d\tau =$

**Ex:** (Square type signal)

### 1.1.6 Brief review of complex algebra and arithmetic

Rectangular (cartesian) form $z = x + j \cdot y$

Euler's relation : $e^{j\theta} =$

Polar form : $z = r \cdot e^{j\theta}$

Components of rectangular and polar form are related :

Complex conjugate of a complex number $z$ is represented with $z^*$ :

Complex arithmetic :

### 1.1.7 CT Complex Exponential Signals

The CT complex exponential signal is of the form

$$x(t) = Ce^{at}$$

where $C$ and $a$ are in general complex numbers.

#### Real Exponential Signals

If $C$ and $a$ are real numbers, real exponential signals are obtained.

- $a > 0$, growing exponential:

- $a < 0$, decaying exponential:

#### Periodic Complex Exponential and Sinusoidal Signals

If $a$ is purely imaginary, <u>periodic</u> complex exponentials are obtained.

Notes:

- Fundamental period of $x(t) = e^{j\omega_0 t}$ is $T_0 = \dfrac{2\pi}{\omega_0}$
- $e^{jw_0 t}$ and $e^{-jw_0 t}$ have the same fundamental period $\rightarrow T_0 = \dfrac{2\pi}{|\omega_0|}$
- The sinusoidal signals $x_a(t) = A\cos(\omega_0 t)$ and $x_b(t) = A\sin(\omega_0 t)$ are closely related to $e^{j\omega_0 t}$ :

- Let's define fundamental frequency as $|\omega_0| = \dfrac{2\pi}{T_0}$ ($|\omega_0|$ represents rate of oscillation)

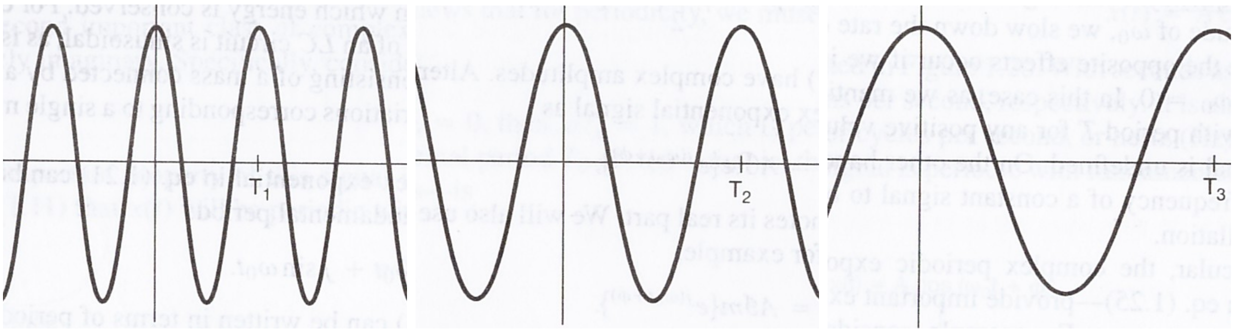



Figure 1.1: Sinusoidal signals $x_i(t) = \cos(\omega_i t)$ for $i = 1, 2, 3$ with varying fundamental frequencies and periods.

- Units of $T_0$, $\omega_0$ :
- Set of harmonically related complex exponentials : $x_k(t) = e^{jk\omega_0 t}, \qquad k = ... -2, -1, 0, 1, 2...,$
  - Fundamental period of $x_k(t)$ is $T_{0,k} =$
  - A common period for all $x_k(t)$ is
  - Plot of $Re\{x_k(t)\}$ for $k = 1, 2, 3$

**Ex:** Sum of two complex exponentials.

**General Complex Exponential Signals**

Both $C$ and $a$ are complex. Let us represent $C$ in polar form as $C = |C|e^{j\theta}$ and $a$ in rectangular form as $a = \alpha + j\omega_0$. Then
$x(t) = Ce^{at} =$

- $\alpha = 0$ : both real and imaginary parts are
- $\alpha > 0$ : both real and imaginary parts are

- $\alpha < 0$ : both real and imaginary parts are

**Ex:** $x(t) = 2e^{(2+j\omega_0)t}$

**Ex:** $x(t) = 2e^{(-1+j\omega_0)t}$

### 1.1.8 DT Complex Exponential Signals

The DT complex exponential signal is of the form

$$x[n] = C\alpha^n \qquad \text{or} \qquad x[n] = Ce^{\beta n} \quad (\text{where } \alpha = e^{\beta})$$

where $C$ and $\alpha$ are in general complex numbers.

#### Real Exponential Signals

If $C$ and $a$ are real numbers, real exponential signals are obtained with various behavior.
**Ex:**

#### Sinusoidal Signals

If $\beta$ is purely imaginary (i.e. $|\alpha| = 1$), DT sinusoidals are obtained. Consider $x[n] = Ce^{j\Omega_0 n}$.

**General Complex Exponential Signals**

If $\beta = \alpha + j \cdot \Omega$, then damped sinusoidals (decaying or growing) are obtained (see textbook for details.)

**Periodicity Properties of DT Complex Exponential Signals**

Although there are many similarities between CT and DT signals, there are also important differences. One of these is the different properties of complex exponential signals $x(t) = e^{j\omega_0 t}$ and $x[n] = e^{j\Omega_0 n}$.

We discussed $x(t) = e^{j\omega_0 t}$ previously and we can identify two important properties of it :

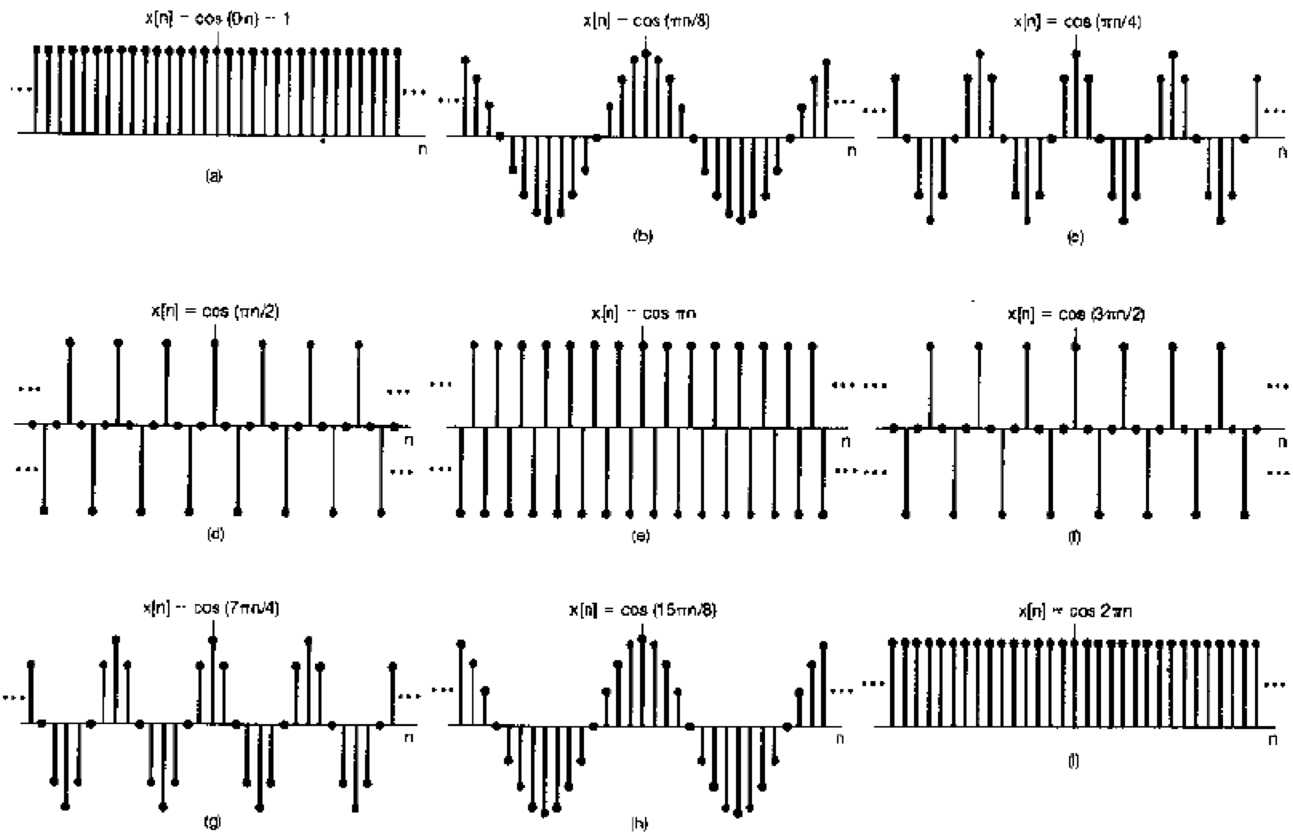
- it is periodic for any value of $\omega_0$ and its fundamental period is $T_0 = \dfrac{2\pi}{\omega_0}$
- the larger the magnitude of $\omega_0$, the higher the rate of oscillation (i.e. frequency) in the signals.

Both of the above properties are different for $x[n] = e^{j\Omega_0 n}$:

- $x[n] = e^{j\omega_0 n}$ is periodic only if $\Omega_0$ can be written in the form $\Omega_0 = 2\pi\frac{m}{N}$ for some integers $N > 0$, and $m$.

- $x[n] = e^{j\Omega_0 n}$ does not have a continually increasing rate of oscillation as we increase the magnitude of $\Omega_0$. In particular, $x_1[n] = e^{j\Omega_0 n}$ is equal to $x_2[n] = e^{j(\Omega_0 + k2\pi)n}$, $k \in \mathbb{Z}$.

**Ex:** Which of the following are periodic ? For each, find fundamental period if periodic.
$x(t) = e^{j2t}$ $\qquad x[n] = e^{j2n}$ $\qquad x[n] = e^{j\pi n}$ $\qquad x[n] = \cos(\pi n)$




(a) DT sinusoidal sequences $x[n] = cos(\Omega_0 n)$ for several frequencies (taken from texbook pg. 27)

**TABLE 1.1** Comparison of the signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$.

| $e^{j\omega_0 t}$ | $e^{j\omega_0 n}$ |
|---|---|
| Distinct signals for distinct values of $\omega_0$ | Identical signals for values of $\omega_0$ separated by multiples of $2\pi$ |
| Periodic for any choice of $\omega_0$ | Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and $m$ |
| Fundamental frequency $\omega_0$ | Fundamental frequency* $\omega_0/m$ |
| Fundamental period<br>$\omega_0 = 0$: undefined<br>$\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$ | Fundamental period*<br>$\omega_0 = 0$: undefined<br>$\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$ |

*Assumes that $m$ and $N$ do not have any factors in common.

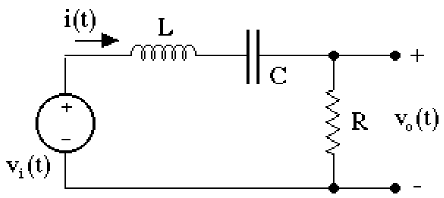
(b) Comparison of properties of $e^{j\omega_0 t}$ and $e^{j\Omega_0 n}$(pg 28 in texbook)

Figure 1.2: DT sinusoidal signals for several frequencies and comparison of $e^{j\omega_0 t}$ and $e^{j\Omega_0 n}$

## 1.2 Systems and Basic System Properties

**Definition 3** *System is defined as any process in which input signals are transformed to output signals.*

**Ex:** Electrical circuit with an input signal ($v_i(t)$) and an output signal ($v_o(t)$)




CT and DT system notations :

Interconnection of systems :

### 1.2.1 Memory Property

A system is **memoryless** (instantaneous) if the output at any time instant depends only on the input at that same instant.
**Ex:**

**Ex:**

**Ex:**

### 1.2.2 Causality Property

A system is **causal** if its output at any time instant depends only on the input at the same time instant and/or the past instants.
**Ex:**

**Ex:**

**Ex:**

### 1.2.3 Invertibility

A system is **invertible** if distinct inputs lead to distinct outputs. If a system is invertible, then a corresponding inverse system exists such that

**Ex:**

**Ex:**

(You are not responsible from this property as other sections do not discuss it, but see textbook for details and examples if interested.)

### 1.2.4 Stability

A system is **stable** if it produces bounded output signal for any bounded input signal.
*Bounded signal* : Its value at any time is bounded by two finite values.

**Ex:**

**Ex:**

**Ex:**

**Ex:**

### 1.2.5 Time Invariance

A system is **time-invariant** if a time shift of the input causes a time shift at the output by the same amount for any input. In other words,

**Ex:**

**Ex:**

**Ex:**

**Ex:**

### 1.2.6 Linearity

A linear system must satisfy the **superposition property**, which is as follows :

**Ex:**

**Ex:**

**Ex:**

**Ex:** (Previous midterm question) The system

$$y[n] = n \cdot x[n-1] + x^2[n] + 3x[n] + \sum_{k=n-10}^{n} x[k] + \sum_{k=-\infty}^{n} x[k] + x[4n] + 2\sin(\pi n)x[n] + 5$$

is not memoryless, not causal, not stable, not linear not time-invariant. Drop minimum number of terms from right-side of equation to make system **linear/ stable/causal/time-invariant/memoryless**.

**Ex:** Another example ?

# Chapter 2

# Linear Time-Invariant Systems

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Two of the important system properties discussed in the previous chapter are *linearity* and *time-invariance*. Systems possessing these two properties are called Linear Time-Invariant (LTI) systems and LTI systems are very important for system and signal analysis for two reasons:

- many physical systems are LTI, and
- powerful mathematical tools have been developed to study/analyze/design LTI systems.

This chapter discusses LTI systems in detail.

## 2.1 DT LTI Systems : The convolution sum

### 2.1.1 Representation of DT Signals in terms of Impulses

Any DT signal can be written as a sum of weighted and shifted DT impulses :

$$\boxed{x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]}$$

### 2.1.2 DT Unit Impulse Response and the Convolution Sum

Let us derive the famous convolution sum, which is very important since it allows us to compute the output $y[n]$ of a DT LTI system for any DT input signal $x[n]$.
Let us begin by calling the output of the system as the signal $h[n]$ when the input signal is $\delta[n]$ :

By the time-invariance property of the system, we can find the output when the input is any shifted impulse :

If we also use the linearity property of the system, we can find the output to any arbitrary input signal as follows :

In summary, using both the time-invariance and linearity properties of an LTI system, we can write its output $y[n]$ to an arbitrary input $x[n]$ via the famous convolution sum :

- Symbolic representation for convolution sum : $\boxed{y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]}$
- $h[n]$ is called the <u>*impulse response*</u> of the LTI system and is required to compute the convolution. ($h[n]$ is the output of the LTI system when the input is just the impulse $\delta[n]$)
- Output $y[n]$ can be interpreted as sum of weighted and shifted impulse responses where the weights are given by input sequence $x[n]$.
- The convolution sum can be used to compute the output $y[n]$ of a DT LTI system with impulse response $h[n]$ for an arbitrary input signal $x[n]$.

**Ex:** For an LTI system with impulse response $h[n] = 2\delta[n+1] + \delta[n-1]$, and input signal $x[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$, find and plot output of the system $y[n]$.(First plot $x[n]$ and $h[n]$.)

Solution 1 (uses linearity and time-invariance property of LTI system)

Solution 2 (uses definition of convolution sum)

.

**Ex:** Let $x[n] = \alpha^n u[n]$ $(0 < \alpha < 1)$ and $h[n] = u[n]$. Find $y[n] = x[n] * h[n]$.

## 2.2 CT LTI Systems : The convolution integral

### 2.2.1 Representation of CT Signals in terms of Impulses

Consider a staircase approximation to a CT signal $x(t)$ :

Overall, an arbitrary CT signal $x(t)$ can be written as an integral of shifted and weighted CT impulses :

$$\boxed{x(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau}$$

Notice the similarity of this result and the one we obtained for DT signals $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$.

### 2.2.2 CT Unit Impulse Response and the Convolution Integral

Let us derive the famous convolution integral, which is very important since it allows us to compute the output $y(t)$ of a CT LTI system for any CT input signal $x(t)$.
Let us begin by calling the output of the system as the signal $h_\Delta(t)$ when the input signal is $\delta_\Delta(t)$ :

By the time-invariance property of the system, we can find the output when the input is a shifted $\delta_\Delta(t)$, i.e. $\delta_\Delta(t-k\Delta)$ :

If we also use the linearity property of the system, we can find the output to any arbitrary input signal as follows :

In summary, using both the time-invariance and linearity properties of an LTI system, we can write its output $y(t)$ to an arbitrary input $x(t)$ via the famous convolution integral :

- Symbolic representation for convolution integral : $\boxed{y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$
- $h(t)$ is called the *impulse response* of the LTI system and is required to compute the convolution. ($h(t)$ is the output of the LTI system when the input is just the impulse $\delta(t)$)

- Output $y(t)$ can be interpreted as the integral of weighted and shifted impulse responses where the weights are given by input sequence $x(t)$.
- The convolution integral can be used to compute the output $y(t)$ of a CT LTI system with impulse response $h(t)$ for an arbitrary CT input signal $x(t)$.

**Ex:** $x(t) = u(t) - u(t-2) \quad \text{and} \quad h(t) = u(t)$. Find $y(t) = x(t) * h(t)$

**Ex:** $x(t) = 2u(t) - 2u(t-2) \quad \text{and} \quad h(t) = u(t+2) - u(t-1)$. Find $y(t) = x(t) * h(t)$ (Exercise for you.)

**Ex:** $x(t) = u(t) - u(t-1) \quad \text{and} \quad h(t) = (-2t+2)(u(t) - u(t-1))$. Find $y(t) = x(t) * h(t)$

**Ex:** $x(t) = 3\delta(t - t_0)$ and $h(t) = u(t) - u(t-1)$. Find $y(t) = x(t) * h(t)$

## 2.3 Properties of Convolution and LTI Systems

An **LTI** system is completely determined by its **impulse response** (or step response).

Note that the above statement is not true for systems that are not LTI, i.e. for non-LTI systems the response of the system to an impulse does not completely determine the system.
**Ex:** Consider the following two non-LTI systems :
$y_1[n] = (x[n] + x[n-1])^2$
$y_2[n] = \max(x[n], x[n-1])$.

### 2.3.1 Commutative property of convolution

$x[n] * h[n] = h[n] * x[n]$ (same for CT convolution)

### 2.3.2 Associative property of convolution

$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$ (same for CT convolution)

### 2.3.3 Distributive property of convolution

$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$ (same for CT convolution)

**Ex:** $x(t) = u(t) - u(t-1)$ and $g(t) = (-2t+2)(u(t) - u(t-1)) + 3\delta(t-2)$. Find $y(t) = x(t) * g(t)$.
(Hint : Remember the two final examples of Section 2.2.2 and use distributive property.)

### 2.3.4 Memory property in LTI systems

An LTI system is memoryless if and only if $h[n] = A \cdot \delta[n]$ $\quad ( h(t) = A \cdot \delta(t).\ )$

### 2.3.5 Causality property in LTI systems

An LTI system is causal if and only if $h[n] = 0,\ n < 0$ $\quad ( h(t) = 0,\ t < 0.\ )$

### 2.3.6 Stability property in LTI systems

An LTI system is stable if and only if its impulse response is absolutely summable (integrable) $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ $\quad ( \int_{t=-\infty}^{\infty} |h(t)| dt < \infty.\ )$

Note that we have only shown that absolute summability of $h[n]$ is sufficient for stability. Absolute summability is also a *necessary condition*, but we will not show it. You can see Problem 2.49 in the textbook for the necessary condition.

**Ex:** Time Shift $y[n] = x[n-2]$

**Ex:** Integrator $y(t) = \int_{\tau=-\infty}^{t} x(t) d\tau$

### 2.3.7 Invertibility property in LTI systems

If an LTI system is invertible then it has an LTI inverse system, which if connected in cascade with the first system produces an overall identity system.

**Note :** Since LTI systems are completely determined by their impulse responses $h[n]$ or $h(t)$, properties of LTI system can be inferred from the impulse responses, as can be seen from the above discussed properties.

### 2.3.8 Unit Step Response of LTI systems

Impulse response :

Step response :

We have seen that its impulse response $h[n]$ completely determines an LTI system. Same is true for step response $s[n]$ since $h[n]$ can be obtained from $s[n]$ or vice versa.

## 2.4 Systems Described by Differential and Difference Equations and Determining Their Impulse Responses

Many systems are described by differential or difference equations.
We will consider only Linear Constant Coefficient Differential (Difference) equations (LCCDE).
**Ex:** $y(t) = x(t) + 3 \cdot \frac{d}{dt}y(t)$ $\quad\rightarrow$ Differential equation
$y[n] = x[n] + 5x[n-1] + 3y[n-1]$ $\quad\rightarrow$ Difference equation

Consider a simple differential equation : $\frac{d}{dt}y(t) + a \cdot y(t) = x(t)$

- The solution contains a particular and a homogeneous part :

- The differential equation by itself does not specify a unique solution.
- An auxiliary (boundary) condition is required to find a unique solution.
- Different auxiliary conditions can lead to different solutions.
- Not all auxiliary conditions lead to LTI systems.
- In this course,
  - we mostly focus on differential/difference equations that describe causal and LTI systems
  - and finding their impulse responses $h(t)$ or $h[n]$,
  - because given the impulse response $h(t)$ or $h[n]$, output $y(t)$ or $y[n]$ can be computed for any input $x(t)$ or $x[n]$ via convolution : $y(t) = x(t) * h(t)$.
- In this course, to obtain causal and LTI systems from differential/difference equitations, the initial rest conditions will be used. (Using the initial rest conditions, auxiliary conditions that lead to causal and LTI systems can be obtained.)

### 2.4.1 Determining The Impulse Response Using Initial Rest Conditions

We first discuss the approach using an example. Then we give general methods for differential and difference equations.

**Ex:** Find impulse response $h(t)$ using initial rest conditions for the system described by the following differential equation. $\frac{d^2}{dt^2}y(t) + 3 \cdot \frac{d}{dt}y(t) + 2 \cdot y(t) = x(t)$.

### 2.4.2 A Method for Differential Equations

To find impulse response $h(t)$ using initial rest conditions for a LCCDE of the form

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = x(t)$$

1. Determine general homogeneous solution :
2. Use following auxiliary conditions for $h(t)$ :
   (Note that these auxiliary conditions are obtained using the initial rest conditions and integrating the above LCCDE from $t = 0^-$ to $t = 0^+$.)

$$h(0^+) = \frac{d}{dt}h(0^+) = \frac{d^2}{dt^2}h(0^+) = ... = \frac{d^{N-2}}{dt^{N-2}}h(0^+) = 0 \qquad \text{and} \qquad \frac{d^{N-1}}{dt^{N-1}}h(0^+) = \frac{1}{a_N}$$

3. Solve for $h(t)$ using Steps 1 & 2.

To find impulse response $h(t)$ using initial rest conditions for a general LCCDE of the form

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

1. Assume right-side of equation is only $x(t)$ and apply Steps 1-3 above.

2. Use linearity of the system to find impulse response :

**Ex:** Find $h(t)$ using initial rest conditions for
Part 1 :

Part 2 :

### 2.4.3 A Method for Difference Equations

To find impulse response $h[n]$ using initial rest conditions for a LCCDE of the form

$$\sum_{k=0}^{N} a_k y[n-k] = x[n]$$

1. Determine general homogeneous solution :

2. Determine auxiliary conditions for $h[n]$ (i.e. values of $h[0]$, $h[1]$, ...$h[N-1]$) using initial rest conditions (ie. $h[-1] = h[-2] = ... = 0$) and recursively applying the above LCCDE :

$$h[n] = \frac{1}{a_0}\{-\sum_{k=1}^{N} a_k h[n-k] + \delta[n]\}$$

3. Solve for $h[n]$ using Steps 1 & 2.

To find impulse response $h[n]$ using initial rest conditions for a <u>general</u> LCCDE of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

1. Assume right-side of equation is only $x[n]$ and apply Steps 1-3 above.

2. Use linearity and time-invariance of the system to find impulse response :

**Ex:** Find $h[n]$ using initial rest conditions for
Part 1 :

Part 2 :

Important exercises that you should go through from the textbook on differential/difference equations : Problems 2.30, 2.55, 2.56.

**Note :** In the upcoming chapters, we will learn transform methods, in particular **Laplace and z transforms**, which provide **more convenient and powerful** methods to obtain impulse responses of LTI systems described by differential or difference equations.

### 2.4.4 Block Diagram Representations of First-Order Systems Described By Differential and Difference Equations

Block diagram representations are pictorial representations which can be useful to understand behavior and properties of such system, or implementing such systems.
**Ex:** $y[n] + a \cdot y[n-1] = b \cdot x[n]$
Let's start by rearranging the equation into a form as follows : $y[n] =$

**Ex:** Similar block diagrams are possible for differential equations. See textbook for details.

# Chapter 3

# Continuous-time Fourier Series

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The convolution sum/integral developed in the previous chapter for LTI systems is based on representing signals as linear combinations of shifted impulses.

In this chapter and the following chapters, our discussions are based again on a representation of signals as a linear combination of a set of basic signals, complex exponentials. The resulting representations are known as Fourier series or transform representations, which have very useful properties in signal and system analysis and design.

In this chapter, we discuss the CT Fourier series representation of periodic CT signals. The next chapter extends the representation to aperiodic CT signals, as the Fourier transform. The following chapter discusses similar representations for DT signals, known as the DT Fourier series and transform representations.

## 3.1 Response of LTI Systems to Complex Exponentials

In the study of LTI systems, it is very useful to represent signals as a linear combination of basic signals that have following two properties:

1. The response of an LTI systems to the basic signals should be simple.

2. The set of signals can be used to construct a broad and useful class of signals.

Both of these properties are satisfied by complex exponential signals, $e^{st}$ in CT and $z^n$ in DT, where $s$ and $z$ are general complex numbers :

1. The response of an LTI systems to a complex exponential is itself with only a change in amplitude :

2. Linear combinations of sets of complex exponentials can provide broad and useful classes of signals (more on this later.)

### 3.1.1 Eigenfunctions of LTI system

In general, an **eigenfunction of a system** is a function (signal) such that the response of the system to such a function (signal) is itself multiplied by a constant :

For **LTI** systems, **complex exponential** signals are **eigenfunctions** :

For LTI systems, representation of signals as a linear combination of complex exponentials is very useful since the response of the LTI system to a signal with such representation can be determined easily :

**Ex:** Let a system be described by

$$y(t) = x(t-3).$$

Is this system LTI ? If so, determine first its impulse response $h(t)$. Then, determine the output of this system to inputs $x_1(t) = e^{2jt}$, and $x_2(t) = cos(4t) + cos(7t)$.

## 3.2 Fourier series : Linear Combinations of Harmonically Related Complex Exponentials

Remember properties of a **periodic CT signal** $x(t)$ periodic with T, i.e. $\quad x(t) = x(t+T) \ \ \forall t.$

- Fundamental period $T_0$ (seconds) :
- Fundamental frequency $\omega_0$ (radians/sec):
- Two basic periodic CT signals periodic with $\omega_0$ are :

Consider now the set of **harmonically related complex exponentials** :

$$\phi_k(t) = e^{jk\omega_0 t}, \quad k = 0, \mp 1, \mp 2, ...$$

- The fundamental frequency and period of $\phi_k(t)$ are $\quad \omega_{0,k}= \qquad T_{0,k} =$
- The **common period** of this set of signals is
- Then a linear combination of $\phi_k(t)$ as in $x(t) = \sum_k a_k e^{jk\omega_0 t}$ is periodic with

Let us now introduce the **CT Fourier series** representation :

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

- A representation of this form is called the **Fourier series** representation of periodic CT signal $x(t)$ with fundamental period $T_0 = \frac{2\pi}{\omega_0}$.
- Coefficients $a_k$ are called the Fourier series (FS) coefficients (or spectral coefficients).
- Fundamental component of FS representation :

- $N^{th}$ harmonic component of FS representation :
- DC (constant) component FS representation :

**Remarks :**

- Note that we did not show that any CT periodic signal can be written in the form above.
- We only argued that if a periodic CT signal can be written in the form above, then it is periodic with $T_0 = \frac{2\pi}{|\omega_0|}$ and this representation is called the FS representation.
- Later we will discuss whether all CT periodic signals can be written in this form. (It turns out that almost all practical signals of interest to engineers can be actually written in the FS representation. More on that in Section 3.4.)

**Ex:** Consider $x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$ where $a_0 = 1$, $a_1 = a_{-1} = \frac{1}{4}$, $a_2 = a_{-2} = \frac{1}{2}$ and $a_3 = a_{-3} = \frac{1}{3}$.

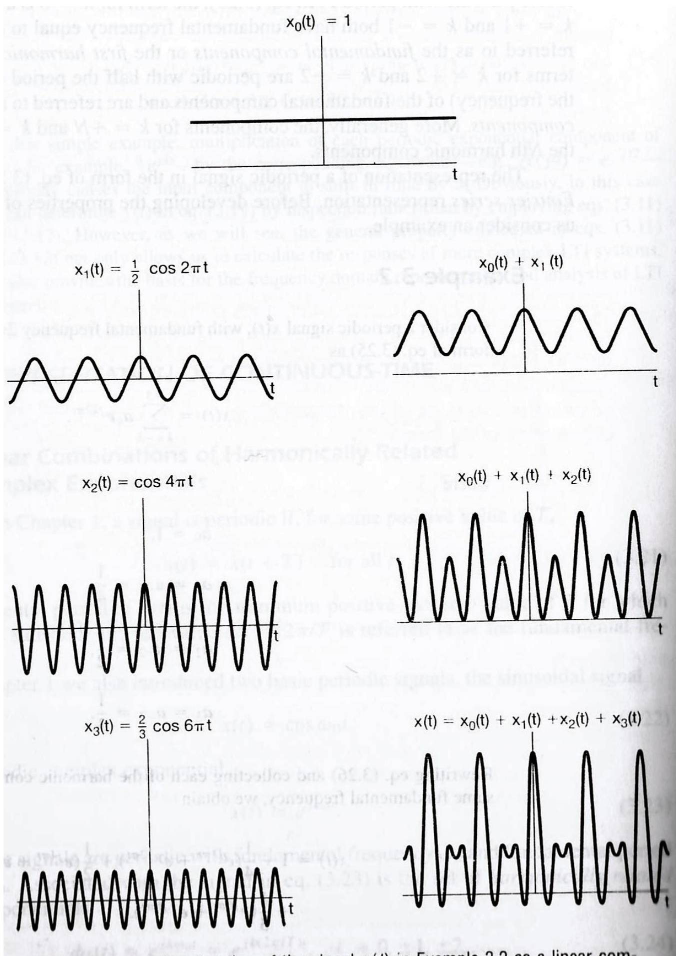



**Figure 3.4** Construction of the signal $x(t)$ in Example 3.2 as a linear combination of harmonically related sinusoidal signals.

## 3.3 Determination of CT Fourier Series Representation

Assuming that a given periodic CT signal can be represented with the FS representation given above, procedures for determining the FS coefficients $a_k$ are as follows.

### 3.3.1 Coefficient matching approach

Some signals are inherently expressed as a linear combination of complex exponentials. For such signals, the FS coefficients $a_k$ can be identified by inspection.
**Ex:** $x(t) = \sin(\omega_0 t)$

**Ex:** $x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos(2\omega_0 t + \frac{\pi}{4})$

### 3.3.2 General approach

The general approach is given by this integral : $a_k = \dfrac{1}{T_0} \int_0^T x(t) e^{-jk\omega_0 t} dt$.
To derive this result, consider a periodic signal with FS representation : $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ (fund. period is $T_0 = \frac{2\pi}{|\omega_0|}$)

Hence, this pair of **equations** defines the **Fourier series** of a **periodic CT signal** :

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}} \qquad \boxed{a_k = \frac{1}{T_0}\int_{<T_0>} x(t)e^{-jk\omega_0 t}dt}$$

**Note :** The boundaries of the integral can be over an arbitrary interval of one period $T_0$, i.e. from an arbitrary starting point $t_0$ to $t_0+T_0$, and a typical notation for that is $\int_{<T_0>}$. The interval should be chosen according to the signal considered. Typical choices are 0 to $T$, or $-\frac{T}{2}$ to $\frac{T}{2}$.

**Ex: Periodic square wave** defined over one period ($T$) as follows $x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < \frac{T}{2} \end{cases}$

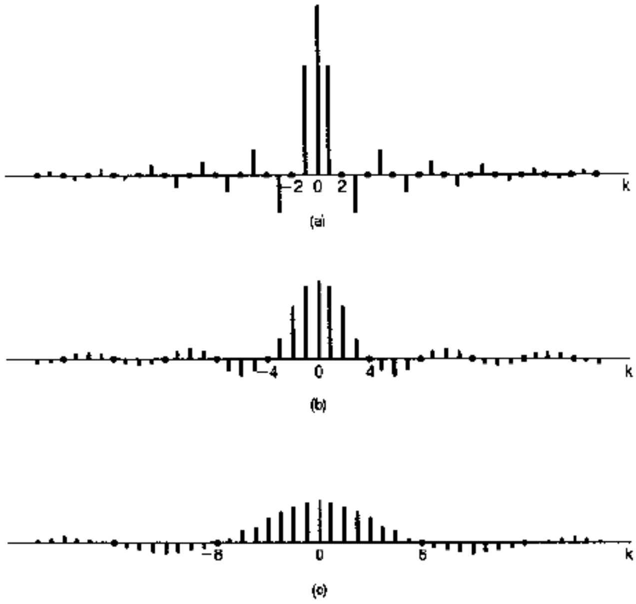


**Figure 3.7** Plots of the scaled Fourier series coefficients $Ta_k$ for the periodic square wave with $T_1$ fixed and for several values of $T$. (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2\sin\omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as $T$ increases.

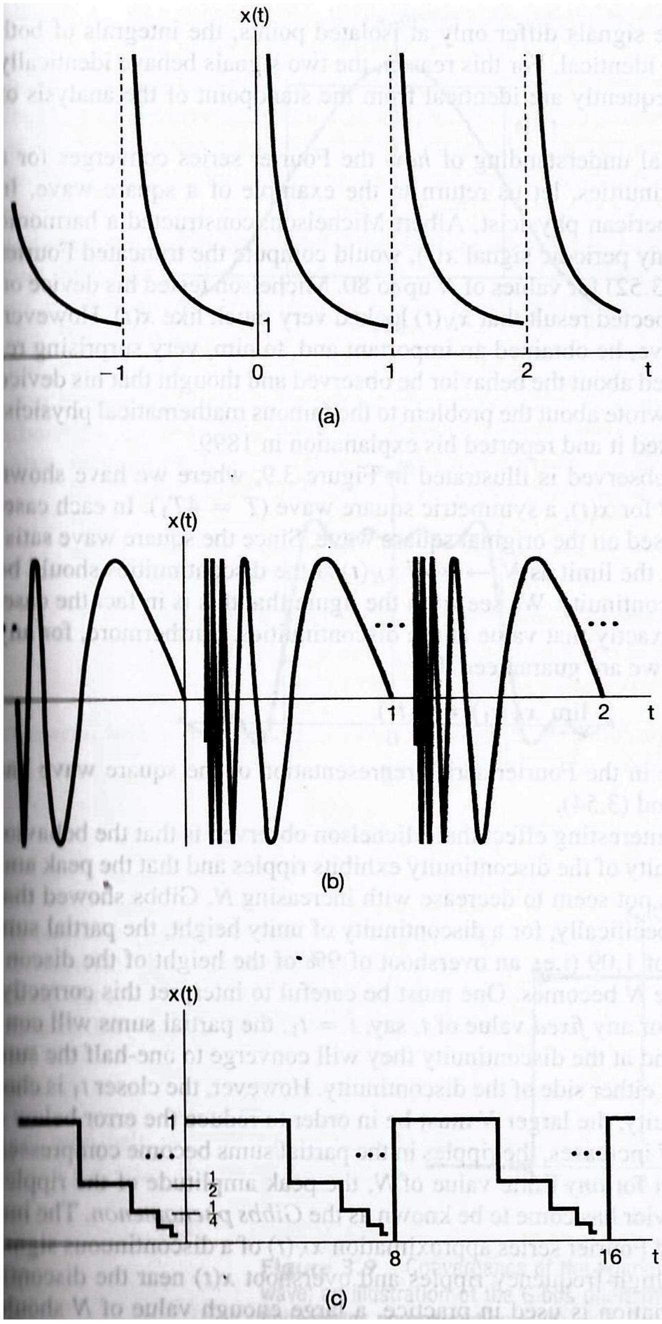
## 3.4 Existence and convergence of Fourier series

If a signal $x(t)$ equals $\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, then it is periodic and the coefficients $a_k$ are given by $a_k = \dfrac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$. However, this does not prove that any periodic signal can be expanded into $\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ (i.e. FS representation may not exist.)

In fact, the FS representation exists (i.e. $x(t)$ equals $\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$) if **either**

- $x(t)$ has **finite energy** over one period, i.e. $\int_T |x(t)|^2 dt < \infty$ **or**
- $x(t)$ satisfies the **Dirichlet conditions** :
  - $x(t)$ is absolutely integrable over any period $\int_0^T |x(t)| dt < \infty$
  - $x(t)$ has a finite number of maxima and minima within a period
  - $x(t)$ has a finite number of discontinuities within a period.

Many signals of interest in engineering (or in this course) have finite energy in one period and/or satisfy the Dirichlet conditions. In the figure are some functions that do not satisfy the Dirichlet conditions. Since such pathological signals do not arise in practical applications, convergence of FS does not play an important role in the remainder of this course.




Note however that the conditions discussed above do not guarantee that $x(t)$ equals $\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ for all $t$. In particular, under these conditions, for a periodic signal $x(t)$ **without discontinuities**, the FS representation $\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ converges and equals $x(t)$ for all $t$. But, for a periodic signal with **finite number of discontinuities** in a period (e.g. square wave), the FS representation $\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ converges and equals $x(t)$ for all $t$ except at the isolated discontinuity points, at which the FS expansion converges to the average value of either side of the discontinuity (i.e. if $x(t)$ has a discontinuity at $t = t_0$, then $\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t_0} = \frac{1}{2}(x(t_0^-) + x(t_0^+))$ ). In this case, the difference between signal $x(t)$ and its FS representation contains no energy and the two signals can be considered same for all practical purposes.

We will not go into any further detail on FS convergence, but if you are interested, please read the relevant section from the textbook.

## 3.5 Properties of Fourier Series

### 3.5.1 Linearity property

If $x(t)$ and $y(t)$ are periodic with the same period and have FS coefficients $a_k$ and $b_k$, respectively, then a linear combination of the signals, $\underline{Ax(t) + By(t)}$ has FS coefficients $\underline{Aa_k + Bb_k}$.

What if the signals have different periods, such as $T_0$ and $2T_0$, then what is the period and the FS coefficients of $Ax(t) + By(t)$?

### 3.5.2 Symmetry with real signals

If $x(t)$ is a real periodic signal (i.e. $\underline{x(t) = x^*(t)}$ ), then its FS coefficients must satisfy $\underline{a_k = a^*_{-k}}$.

### 3.5.3 Alternative forms of FS representation for real signals

If $x(t)$ is a real periodic signal, then two alternative forms for its FS representation can be derived :

### 3.5.4 Even and odd signals

If $x(t)$ is even (i.e. $\underline{x(t) = x(-t)}$ ), then its FS coefficients must satisfy $\underline{a_k = a_{-k}}$.

If $x(t)$ is odd, (i.e. $\underline{x(t) = -x(-t)}$ ), then its FS coefficients must satisfy $\underline{a_k = -a_{-k}}$.

### 3.5.5 FS coefficients of manipulated CT periodic signals

We introduce the following shorthand notation to indicate a periodic signal and its FS coefficients.

$$\text{signal} \longleftrightarrow \text{FS coefficients.}$$

**Time shift :** If $\underline{x(t) \longleftrightarrow a_k}$, then $\underline{x(t - t_0) \longleftrightarrow a_k e^{-jk\omega_0 t_0}}$

**Time reversal :** If $\underline{x(t) \longleftrightarrow a_k}$, then $\underline{x(-t) \longleftrightarrow a_{-k}}$

**Differentiation :** If $\underline{x(t) \longleftrightarrow a_k}$, then $\underline{\frac{d}{dt}x(t) \longleftrightarrow (jk\omega_0)a_k}$

### 3.5.6 Response of LTI systems to signals with FS representation

In an LTI system, if the input's FS representation is known, then the output's FS representation can be easily obtained from it. (Note that the output will also be periodic with the same period.)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

Pf:

1. Remember that $e^{st}$ is an eigenfunction of LTI systems and the eigenvalue was given by $H(s) = \int_{-\infty}^{\infty} e^{-st} h(t) dt$, where $s$ is a general complex number :

2. The system is linear :

**Ex:** Let input to an LTI system with impulse response $h(t) = u(t + \frac{T_1}{2}) - u(t - \frac{T_1}{2})$ be the periodic impulse train signal $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$. Find FS coefficients of $x(t)$, and the output $y(t)$. Also plot $y(t)$. (Assume $T > T_1$.)

**Ex:** One can find the FS coefficients of a periodic square wave from the FS coefficients of a periodic impulse train using FS properties. Let us start by finding a relationship between the periodic square wave and the impulse train signal.

### 3.5.7 Other properties of CTFS representation

There are many other properties of CTFS, but we will cover them in the next chapter together with the properties of the Fourier transform. The table below taken from the textbook lists all properties of the CTFS.

**TABLE 3.1** PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

| Property | Section | Periodic Signal | Fourier Series Coefficients |
|---|---|---|---|
| | | $\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$ | $a_k$ <br> $b_k$ |
| Linearity | 3.5.1 | $Ax(t) + By(t)$ | $Aa_k + Bb_k$ |
| Time Shifting | 3.5.2 | $x(t - t_0)$ | $a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$ |
| Frequency Shifting | | $y(t)\cdot e^{jM\omega_0 t} = e^{jM(2\pi/T)t}x(t)$ | $a_{k-M}$ |
| Conjugation | 3.5.6 | $x^*(t)$ | $a^*_{-k}$ |
| Time Reversal | 3.5.3 | $x(-t)$ | $a_{-k}$ |
| Time Scaling | 3.5.4 | $x(\alpha t), \alpha > 0$ (periodic with period $T/\alpha$) | $a_k$ |
| Periodic Convolution | | $\int_T x(\tau)y(t-\tau)d\tau$ | $Ta_k b_k$ |
| Multiplication | 3.5.5 | $x(t)y(t)$ | $\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$ |
| Differentiation | | $\frac{dx(t)}{dt}$ | $jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$ |
| Integration | | $\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$) | $\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$ |
| Conjugate Symmetry for Real Signals | 3.5.6 | $x(t)$ real | $\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$ |
| Real and Even Signals | 3.5.6 | $x(t)$ real and even | $a_k$ real and even |
| Real and Odd Signals | 3.5.6 | $x(t)$ real and odd | $a_k$ purely imaginary and odd |
| Even-Odd Decomposition of Real Signals | | $\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$ | $\Re e\{a_k\}$ <br> $j\Im m\{a_k\}$ |

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Figure 3.1: Properties of CTFS.

# Chapter 4

# Continuous-time Fourier Transform

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In the previous chapter, a representation of CT periodic signals as linear combinations of harmonically related complex exponentials, i.e. Fourier series, was developed. This representation was also used to analyze effects of LTI systems on signals with such representation. This chapter extends these concepts to aperiodic signals.

## 4.1 The Fourier Transform Representation of CT Aperiodic Signals

### 4.1.1 Intuition behind Fourier transform

Recall the periodic ($T_0$) square wave example from the previous chapter : $x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < \frac{T}{2} \end{cases}$

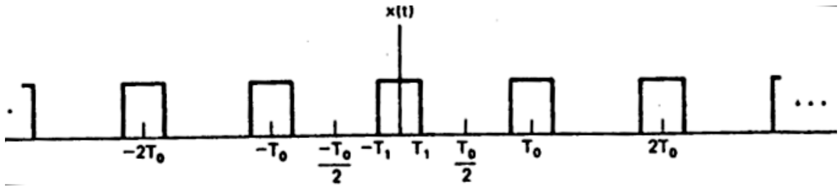



Figure 4.1: Periodic square wave.

Its Fourier Series coefficients $a_k$ were determined as : $a_k = \dfrac{2\sin(k\omega_0 T_1)}{k\omega_0 T_0}$ where $\omega_0 = \frac{2\pi}{T_0}$.

Consider now $T_0 a_k$ as **samples of a continuous envelope** :

$$T_0 a_k = \frac{2sin(k\omega_0 T_1)}{k\omega_0} = \frac{2sin(\omega T_1)}{\omega}|_{\omega = k\omega_0} \quad k = 0, \mp 1, \mp 2, ...$$

- Envelope $\dfrac{2sin(\omega T_1)}{\omega}$ is independent of period $T_0$.
- As period $T_0$ increases ($\omega_0 = \frac{2\pi}{T_0}$ decreases), the envelope is sampled denser.
- As period $T_0 \to \infty$,
  - the periodic square wave becomes a single pulse (i.e. aperiodic)
  - FS coefficients $a_k$ become infinitely-close samples of the envelope, approaching the continuous envelope
  - FS summation approaches an integral, i.e. Fourier transform
- This example illustrates the basic idea behind the Fourier transform

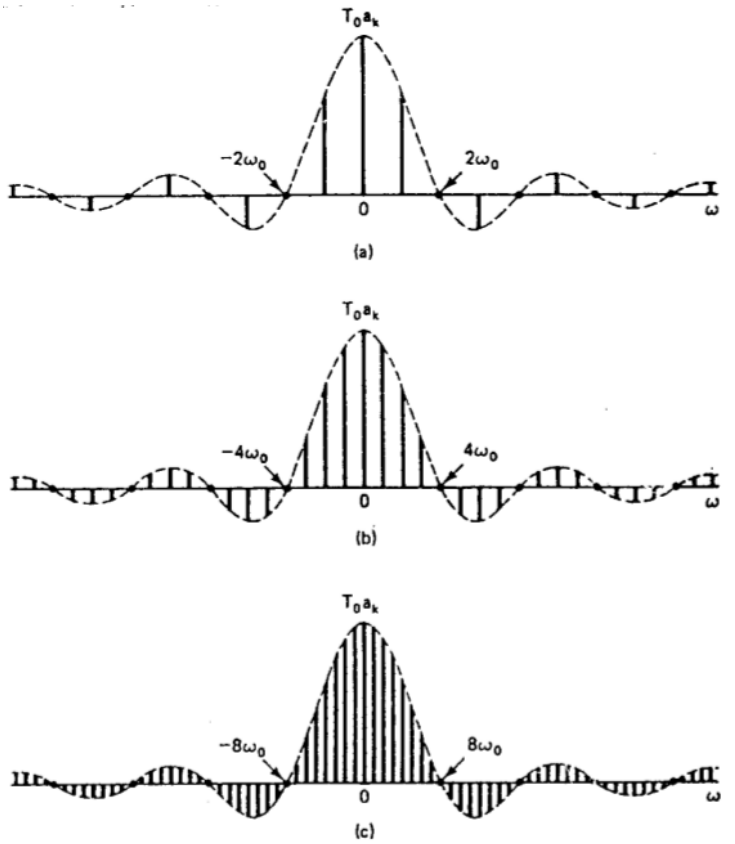



**Figure 4.11** Fourier coefficients and their envelope for the periodic square wave: (a) $T_0 = 4T_1$; (b) $T_0 = 8T_1$; (c) $T_0 = 16T_1$.

### 4.1.2 Formal development of Fourier transform

Consider an aperiodic signal $x(t)$ and its periodic version $\tilde{x}(t)$, as shown in the figure below.

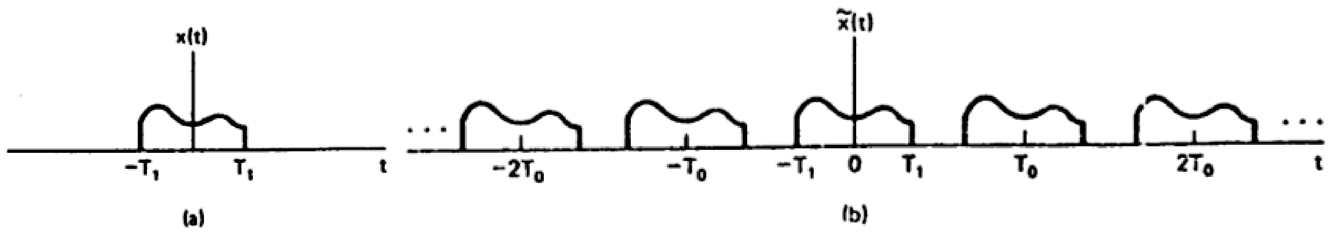



Figure 4.2: Aperiodic $x(t)$ and its periodic version $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t - kT_0)$.

Let us start by remembering the FS analysis and synthesis equations for $\tilde{x}(t)$ :

Hence, this pair of **equations** defines the **Fourier transform** or integral :

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega} \qquad \boxed{X(\omega) = \int_{-\infty}^{-\infty} x(t)e^{-j\omega t}dt}$$

**Notes :**

- $X(\omega)$ is called the Fourier transform (FT) of $x(t)$ (or spectrum of $x(t)$)
- These are also called **synthesis** and **analysis equations** of the FT.
- If FS and FT representations are compared :

- Both represent signals as a **linear combination of complex exponentials**.
- FS : complex exponentials occur at a **discrete set of harmonically related frequencies** $k\omega_0$ and have amplitudes $a_k$
- FT : complex exponentials occur at a **continuum of frequencies** and have amplitudes $\dfrac{X(\omega)}{2\pi}$
- From the development of FT, the FS coefficients $a_k$ of $\tilde{x}(t)$ are equally spaced samples of the FT of $x(t)$ : $a_k = \frac{1}{T_0} X(\omega)|_{\omega=k\omega_0}$

## 4.2 Convergence of Fourier Transform

Similar to convergence of FS representation, FT representation of a signal $x(t)$ exists (i.e. $x(t)$ equals $\dfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$) if **either**

- $x(t)$ has **finite energy**, i.e. $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$ **or**
- $x(t)$ satisfies the **Dirichlet conditions** :
  - $x(t)$ is absolutely integrable , i.e. $\int_{-\infty}^{\infty} |x(t)| dt < \infty$
  - $x(t)$ has a finite number of maxima and minima within any finite interval
  - $x(t)$ has a finite number of discontinuities within any finite interval, and each of these discontinuities are finite.

Although the above conditions are sufficient to guarantee that a signal has FT, we will see in the next section that **periodic signals**, which **neither have finite energy nor are absolutely integrable**, can be considered to **have FT** if **impulse functions** are permitted in the transform. This has the advantage that FS and FT representations can be combined into a single framework, which can be very convenient.

## 4.3 Examples of CT Fourier transforms

**Ex:** $x(t) = e^{-at}u(t) \quad a > 0.$

**Ex:** $x(t) = e^{-a|t|} \qquad a > 0.$

**Ex:** $x(t) = \delta(t).$

Let us introduce and plot two signals which are convenient to in the following examples and throughout the course :

1. $rect(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$
2. $sinc(t) = \dfrac{\sin(\pi t)}{\pi t}$

**Ex:** Rectangular pulse $x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$

**Ex:** $X(\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$

Notes :

- $rect(t)$ has FT $sinc(\frac{\omega}{2\pi})$ and $sinc(t)$ has FT $rect(\frac{\omega}{2\pi})$
- A $rect()$ in time domain corresponds to a $sinc()$ in frequency domain and vice versa. (Duality property of FT)
- As $rect()$ has a shorter (longer) duration, $sinc()$ has a longer (shorter) side lobe. (Scaling prop. of FT)

## 4.4 Response of LTI systems to complex exponentials (revisited)

Remember that $e^{st}$ is an **eigenfunction** of LTI systems and the eigenvalue was given by $H(s) = \int_{-\infty}^{\infty} e^{-st}h(t)dt$, where $s$ is a general complex number of the form $s = \alpha + j\omega$ :

Note that if $s$ is purely imaginary (i.e. $s = j\omega$), then the complex exponential $e^{st} = e^{j\omega t}$ and its eigenvalue reduces to the FT of $h(t)$, i.e. we have $H(s)|_{s=j\omega} = \mathscr{F}\{h(t)\} = H(\omega)$.

**Ex:** Let $x_1(t) = \cos(\frac{\pi}{4}t)$ and $x_2(t) = \cos(\frac{3\pi}{4}t)$ be inputs to an LTI system with $\mathscr{F}\{h(t)\} = H(\omega) = rect(\frac{\omega}{\pi})$. Find outputs $y_1(t)$ and $y_2(t)$.

## 4.5 Fourier transform of periodic signals

We developed the Fourier transform representation for aperiodic signals. But the FT representation can also be extended to periodic signals by allowing impulses in the transform $X(\omega)$. Hence, the FT representation can provide a unified framework for representing both periodic and aperiodic signals, which can be very convenient.

Consider a signal $x(t)$ with Fourier transform $X(\omega)$ that is a single impulse of strength $2\pi$ at $\omega = \omega_0$:

More generally, consider a linear combination of impulses equally spaced in frequency :

In summary, to determine the **FT of a periodic signal**,

1. determine its FS coefficients $a_k$
2. then write its FT as $X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$.

**Ex:** $x(t) = \cos(\omega_0 t)$

**Ex:** $x(t) = \sin(\omega_0 t)$

**Ex:** The periodic square wave (again!)

**Ex:** Periodic impulse train $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$

## 4.6 Properties of the Fourier Transform

To discuss Fourier transform properties, it is convenient to use the following shorthand notation to indicate the pairing of a signal and its transform :

- signal $\longleftrightarrow$ FT ( e.g. $x(t) \longleftrightarrow X(\omega)$ )

We sometimes also refer to a FT and inverse FT with the following notation :

- $\mathscr{F}\{\text{signal}\}$ and $\mathscr{F}^{-1}\{\text{FT}\}$ ( e.g. $X(\omega) = \mathscr{F}\{x(t)\}$ and $x(t) = \mathscr{F}^{-1}\{X(\omega)\}$)

For the following properties, let us assume that we have two signals $x(t)$ and $y(t)$ with corresponding FT $X(\omega)$ and $Y(\omega)$, i.e.

$$x(t) \longleftrightarrow X(\omega) \qquad \text{and} \qquad y(t) \longleftrightarrow Y(\omega).$$

We will also discuss some FS properties along with the corresponding FT properties since we omitted many FS properties in the previous chapter. Hence, we will use the same notation to indicate the pairing of a periodic signal and its FS coefficients. We will also use a tilde on top of periodic signals to differentiate them from aperiodic signals. To discuss FS properties, let us again assume that we have two periodic signals $\tilde{x}(t)$ and $\tilde{y}(t)$ with the same period $T_0$ and have corresponding FS coefficients $a_k$ and $b_k$, i.e.

$$\tilde{x}(t) \longleftrightarrow a_k \qquad \text{and} \qquad \tilde{y}(t) \longleftrightarrow b_k.$$

### 4.6.1 Linearity

$ax(t) + by(t) \longleftrightarrow aX(\omega) + bY(\omega)$

(FS: $A\tilde{x}(t) + B\tilde{y}(t) \longleftrightarrow Aa_k + Bb_k$ What if $\tilde{x}(t)$ and $\tilde{y}(t)$ do not have the same period ?)

### 4.6.2 Time Shift

$x(t - t_0) \longleftrightarrow e^{-j\omega t_0}X(\omega)$

(FS: $\tilde{x}(t - t_0) \longleftrightarrow e^{-jk\omega_0 t_0}a_k$ )

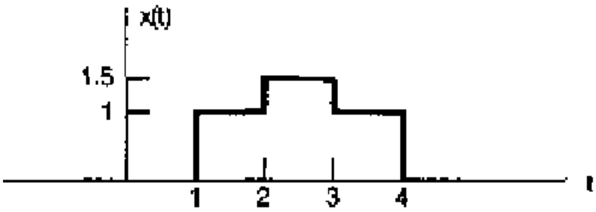
### 4.6.3 Time and Frequency Scaling

$x(at) \longleftrightarrow \frac{1}{|a|}X(\frac{\omega}{a})$

**Note** that this property indicates that time and frequency parameters scale inversely proportional. When one expands, the other contracts.

(FS: $\tilde{x}(at)$ becomes periodic with and $\tilde{x}(at) \longleftrightarrow a_k$ )
Let us write down and compare the FS expansions of $\tilde{x}(t)$ and $\tilde{x}(at)$:

**Ex:** $x(-t) \longleftrightarrow$
**Ex:** Find FT of $x(t)$ shown below by using FT of $rect(t)$ and some FT properties.




### 4.6.4 Conjugation and Conjugate Symmetry

$x^*(t) \longleftrightarrow X^*(-\omega)$

If $x(t)$ is <u>real</u> (i.e. $\underline{x(t) = x^*(t)}$), then :

- $X(\omega) =$ or $X^*(\omega) =$
- $Re\{X(\omega)\} = Re\{X(-\omega)\}$ and $Im\{X(\omega)\} = -Im\{X(-\omega)\}$

- $|X(\omega)| = |X(-\omega)|$ and $\angle X(\omega) = -\angle X(-\omega)$

If $x(t)$ is both real and even (i.e. $x(t) = x^*(t) = x(-t)$), then

- $X(\omega)$ is also both real and even.

If $x(t)$ is both real and odd (i.e. $x(t) = x^*(t) = -x(-t)$), then

- $X(\omega)$ is purely imaginary and odd.

Remember that any real signal $x(t)$ can be uniquely written as a sum of its even and odd parts: $x(t) = x_e(t) + x_o(t)$. Using some of the above properties it can be shown that :

- $x_e(t) \longleftrightarrow Re\{X(\omega)\}$
- $x_o(t) \longleftrightarrow j \cdot Im\{X(\omega)\}$

Showing these properties is left as an exercise.

(FS : $\tilde{x}^*(t) \longleftrightarrow a^*_{-k}$ ) Similar properties exist in case $\tilde{x}(t)$ is real, real and even, etc. See the CTFS properties table in the previous chapter in Section 3.5.7
**Ex:** $x(t) = e^{-|a|t} \quad , a > 0.$

### 4.6.5 Differentiation and Integration

$\frac{d}{dt}x(t) \longleftrightarrow j\omega \cdot X(\omega)$ and $\int_{-\infty}^{t} x(\tau)d\tau \longleftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$

(FS: $\frac{d}{dt}\tilde{x}(t) \longleftrightarrow jk\omega_0 \cdot a_k$ and $\int_{=\infty}^{t} \tilde{x}(\tau)d\tau \longleftrightarrow \frac{1}{jk\omega_0}a_k$)
Note that $\int_{=\infty}^{t} \tilde{x}(\tau)d\tau$ is finite and periodic only if $a_0 = 0$.

**Ex:** Find impulse response of LTI system described by $\frac{d}{dt}y(t) + \alpha y(t) = x(t)$

**Ex:** Find FT of linear ramp signal

**Ex:** Find FT of unit step signal $u(t)$

### 4.6.6 Duality

$X(t) \longleftrightarrow 2\pi x(-\omega)$ ( Or alternatively if $g(t) \longleftrightarrow f(\omega)$ then, $f(t) \longleftrightarrow 2\pi g(-\omega)$)

**Ex:** $x(t) = \dfrac{1}{1+t^2}$

**Ex:** Rectangular pulse and sinc functions. $rect(t) \longleftrightarrow sinc(\frac{\omega}{2\pi})$ implies

Duality principle can be applied to derive additional properties :

- Frequency shift :
- Frequency differentiation :
- Frequency integration :

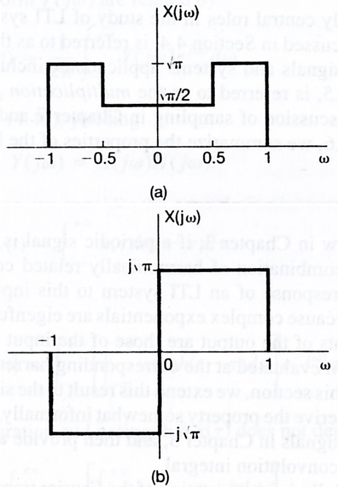
### 4.6.7 Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

- The term on left side is the **total energy in the signal** $x(t)$.
- Parseval's relation indicates that the total signal energy can be computed either
  - from $x(t)$ by integrating its energy per unit time ($|x(t)|^2$) over all time
  - or from $X(\omega)$ by integrating its energy per unit frequency ($\dfrac{|X(\omega)|^2}{2\pi}$) over all frequencies.
- $|X(\omega)|^2$ is called the *energy-density spectrum* of signal $x(t)$.

(FS: $\frac{1}{T_0}\int_{T_0} |\tilde{x}(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$ Left side of equation is the **average power of the periodic signal** $\tilde{x}(t)$, and equals to the sum of squared magnitudes of FS coefficients.)

**Ex:** Compute $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$ and $D = \frac{d}{dt}x(t)|_{t=0}$ for two signals of which the FT are given in the figure below.




### 4.6.8 Convolution Property

$x(t) * y(t) \longleftrightarrow X(\omega)Y(\omega)$

- The convolution property implies that we can analyze LTI systems also in frequency domain.

- $\mathscr{F}\{h(t)\} = H(\omega)$ is called the **frequency response** of the LTI system.
  - $H(\omega)$ may not be defined for all LTI systems. For $H(\omega)$ to be defined, $h(t)$ must have FT (i.e. have finite energy or satisfy Dirichlet conditions.)
  - If an LTI system is stable (i.e. $\int_{-\infty}^{\infty} |h(t)|dt < \infty$), then we typically say that it has a frequency response $H(\omega)$. (Why?)
  - To analyze unstable LTI systems, we will learn the Laplace transform in upcoming chapters.

(FS: $\int_{T_0} \tilde{x}(\tau)\tilde{y}(t-\tau)d\tau \longleftrightarrow T_0 \cdot a_k \cdot b_k$ Convolution over one period corresponds to multiplication of FS coefficients.)

**Ex:** An LTI system with $h(t) = \delta(t - t_0)$. Find its frequency response $H(\omega)$ and the outputs FT in terms of the input's FT.

**Ex:** The differentiator is an LTI system. Find its frequency response.

**Ex:** Consider an LTI system with $h(t) = e^{-at}u(t)$ and an input $x(t) = e^{-bt}u(t)$ where $a > 0$ and $b > 0$. While the output can be computed in time domain via convolution, let us find the output in frequency domain first and then in time-domain.

**Ex:** Frequency selective filtering is accomplished with an LTI system whose frequency response $H(\omega)$ passes desired range of frequencies and significantly attenuates frequencies outside that range. Consider the ideal low-pass filter with $H(\omega) = \begin{cases} 1, |\omega| < \omega_c \\ 0, \text{otherwise.} \end{cases}$

**Ex:** Consider an ideal LPF with cut-off frequency $w_c$ and an input signal $x(t) = \dfrac{\sin(\omega_i t)}{\pi t}$. Calculate the output.

### 4.6.9 Modulation (Multiplication) property

$x(t) \cdot y(t) \longleftrightarrow \dfrac{1}{2\pi} X(\omega) * Y(\omega)$

- Multiplication of one signal by another can be thought of using one signal to scale or **modulate** the amplitude of the other.
- Considering also the convolution property, it can be stated that **convolution in one domain** corresponds to **multiplication in the other domain** (Duality property.)

(FS: $\tilde{x}(t) \cdot \tilde{y}(t) \longleftrightarrow a_k * b_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$)

**Ex:** $x(t) = m(t)e^{j\omega_0 t}$. Find $X(\omega)$ in terms of spectrum of $m(t)$.

**Ex:** Let signal $s(t)$ have spectrum $S(\omega)$ as below and also consider signal $p(t) = \cos(\omega_0 t)$. Determine the spectrum of $r(t) = s(t) \cdot p(t)$.

**Ex:** Find FT of $x(t) = \dfrac{\sin(t)\sin(\frac{t}{2})}{\pi t^2}$.

### 4.6.10 Table of properties of CT FT

The following table from the textbook summarizes all properties.

**TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM**

| Section | Property | Aperiodic signal | Fourier transform |
|---|---|---|---|
| | | $x(t)$ <br> $y(t)$ | $X(j\omega)$ <br> $Y(j\omega)$ |
| 4.3.1 | Linearity | $ax(t)+by(t)$ | $aX(j\omega)+bY(j\omega)$ |
| 4.3.2 | Time Shifting | $x(t-t_0)$ | $e^{-j\omega t_0}X(j\omega)$ |
| 4.3.6 | Frequency Shifting | $e^{j\omega_0 t}x(t)$ | $X(j(\omega-\omega_0))$ |
| 4.3.3 | Conjugation | $x^*(t)$ | $X^*(-j\omega)$ |
| 4.3.5 | Time Reversal | $x(-t)$ | $X(-j\omega)$ |
| 4.3.5 | Time and Frequency Scaling | $x(at)$ | $\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$ |
| 4.4 | Convolution | $x(t)*y(t)$ | $X(j\omega)Y(j\omega)$ |
| 4.5 | Multiplication | $x(t)y(t)$ | $\frac{1}{2\pi}X(j\omega)*Y(j\omega)$ |
| 4.3.4 | Differentiation in Time | $\frac{d}{dt}x(t)$ | $j\omega X(j\omega)$ |
| 4.3.4 | Integration | $\int_{-\infty}^{t}x(t)dt$ | $\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$ |
| 4.3.6 | Differentiation in Frequency | $tx(t)$ | $j\frac{d}{d\omega}X(j\omega)$ |
| 4.3.3 | Conjugate Symmetry for Real Signals | $x(t)$ real | $\begin{cases} X(j\omega)=X^*(-j\omega) \\ \mathcal{Re}\{X(j\omega)\}=\mathcal{Re}\{X(-j\omega)\} \\ \mathcal{Im}\{X(j\omega)\}=-\mathcal{Im}\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert=\lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega)=-\sphericalangle X(-j\omega) \end{cases}$ |
| 4.3.3 | Symmetry for Real and Even Signals | $x(t)$ real and even | $X(j\omega)$ real and even |
| 4.3.3 | Symmetry for Real and Odd Signals | $x(t)$ real and odd | $X(j\omega)$ purely imaginary and odd |
| 4.3.3 | Even-Odd Decomposition for Real Signals | $x_e(t)=\mathcal{Ev}\{x(t)\}$ [$x(t)$ real] <br> $x_o(t)=\mathcal{Od}\{x(t)\}$ [$x(t)$ real] | $\mathcal{Re}\{X(j\omega)\}$ <br> $j\mathcal{Im}\{X(j\omega)\}$ |
| 4.3.7 | Parseval's Relation for Aperiodic Signals | $\int_{-\infty}^{+\infty}\lvert x(t)\rvert^2dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\lvert X(j\omega)\rvert^2d\omega$ | |
| | Duality | $X(t)$ | $2\pi\, x(-\omega)$ |

Figure 4.3: Properties of CT FT.

### 4.6.11 Table of basic signals and their CT FT and FS

The following table from the textbook summarizes the CT FT of some basic signals (and their CT FS if the signal is periodic) .

**TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS**

| Signal | Fourier transform | Fourier series coefficients (if periodic) |
|---|---|---|
| $\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$ | $a_k$ |
| $e^{j\omega_0 t}$ | $2\pi\delta(\omega - \omega_0)$ | $a_1 = 1$<br>$a_k = 0$, otherwise |
| $\cos \omega_0 t$ | $\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ | $a_1 = a_{-1} = \frac{1}{2}$<br>$a_k = 0$, otherwise |
| $\sin \omega_0 t$ | $\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$ | $a_1 = -a_{-1} = \frac{1}{2j}$<br>$a_k = 0$, otherwise |
| $x(t) = 1$ | $2\pi\delta(\omega)$ | $a_0 = 1,\ a_k = 0,\ k \neq 0$<br>(this is the Fourier series representation for any choice of $T > 0$) |
| Periodic square wave<br>$x(t) = \begin{cases} 1, & \|t\| < T_1 \\ 0, & T_1 < \|t\| \le \frac{T}{2} \end{cases}$<br>and<br>$x(t+T) = x(t)$ | $\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$ | $\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$ |
| $\sum_{n=-\infty}^{+\infty} \delta(t - nT)$ | $\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$ | $a_k = \frac{1}{T}$ for all $k$ |
| $x(t) \begin{cases} 1, & \|t\| < T_1 \\ 0, & \|t\| > T_1 \end{cases}$ | $\frac{2\sin \omega T_1}{\omega} = 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)$ | — |
| $\frac{\sin Wt}{\pi t}$ | $X(j\omega) = \begin{cases} 1, & \|\omega\| < W \\ 0, & \|\omega\| > W \end{cases}$ | — |
| $\delta(t)$ | $1$ | — |
| $u(t)$ | $\frac{1}{j\omega} + \pi\delta(\omega)$ | — |
| $\delta(t - t_0)$ | $e^{-j\omega t_0}$ | — |
| $e^{-at}u(t),\ \mathcal{R}e\{a\} > 0$ | $\frac{1}{a + j\omega}$ | — |
| $te^{-at}u(t),\ \mathcal{R}e\{a\} > 0$ | $\frac{1}{(a + j\omega)^2}$ | — |
| $\frac{t^{n-1}}{(n-1)!} e^{-at}u(t),$<br>$\mathcal{R}e\{a\} > 0$ | $\frac{1}{(a + j\omega)^n}$ | — |

Figure 4.4: Basic CT FT pairs.

## 4.7 Some applications of Fourier transform

### 4.7.1 Amplitude Modulation (AM)

**Modulation :** An information bearing signal $x(t)$ is **modulated** by multiplying it with a sinusoidal carrier signal $c(t) = \cos(\omega_c t)$. This will produce a signal with a spectrum centered around the carrier's frequency which is more convenient than the original signal $x(t)$ since

- antenna size is inversely proportional with $f_c = \frac{\omega_c}{2\pi}$ and
- multiple communications can be achieved with different carriers frequencies.

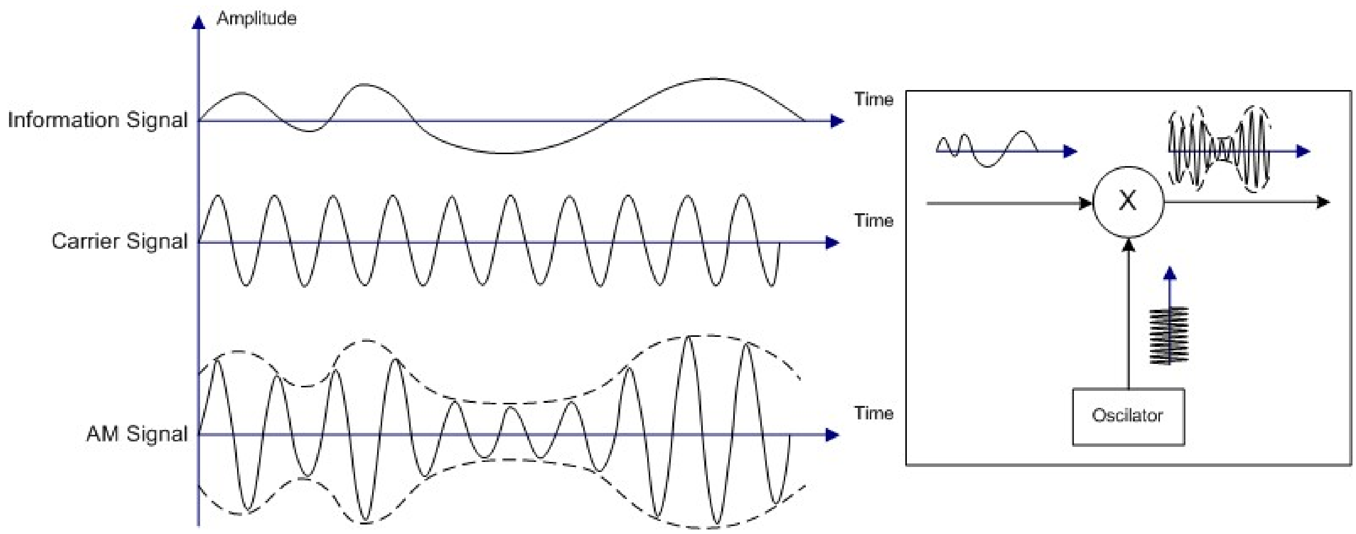



Figure 4.5: Basic CTFT pairs.

**Demodulation : Recovery** of the original information bearing signal $x(t)$ from the modulated signal. We will discuss **synchronous demodulation**, where the frequency and phase of the carrier is assumed to be known perfectly at the receiver.

Synchronous demodulation is achieved in two steps :

1. modulate with the same carrier and
2. low-pass filter

In summary,

- Modulation :
- Demodulation :

### 4.7.2 Frequency Division Multiplexing (FDM)

If **multiple** information bearing signals need to be transmitted at the same time, each can be modulated to different carrier frequencies.

How about demodulation in FDM ?

### 4.7.3 Single Sideband Modulation (SSB)

Consider a **real signal** $x(t)$.

# Chapter 5

# Discrete-time Fourier Series and Transform

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This chapter discusses Fourier series and transform representations for discrete-time signals. Similar to the CT representations, the DT Fourier series represents DT periodic signals as a linear combination of harmonically related DT complex exponentials and the DT Fourier transform representation extends this approach to aperiodic DT signals.

## 5.1 DT Fourier Series

### 5.1.1 Response of DT LTI Systems to Complex Exponentials

Similar to the CT case, DT complex exponentials $x[n] = z^n = (re^{j\Omega_0})^n$ are eigenfunctions of DT LTI systems.

Again, representation of DT signals as a linear combination of DT complex exponentials is very useful since

- such a representation can form a broad and useful class of signals and
- the response of the LTI system to a signal with such representation is easily determined:

**Ex:** Let an LTI system have impulse response $h[n] = \frac{1}{2}\delta[n+1] + \delta[n] + \frac{1}{2}\delta[n-1]$. Find outputs of system for inputs $x_1[n] = \cos(\frac{\pi}{2}n)$ and $x_2[n] = \sin(\pi n)$.

### 5.1.2 DT Fourier series representation of periodic DT signals

Remember that for CT complex exponentials,

- $x(t) = e^{j\omega_0 t}$ is always periodic with period $T_0 = \frac{2\pi}{|\omega_0|}$
- the set of all complex exponentials that have a common period $T_0 = \frac{2\pi}{|\omega_0|}$ are given by

$$\phi_k(t) = e^{jk\frac{2\pi}{T_0}t}, \quad k = 0, \pm1, \pm2, ...$$

For DT complex exponentials,

- $x[n] = e^{j\Omega_0 n}$ is periodic only if $\Omega_0$ is in the form
- the period of $x[n] = e^{j\frac{2\pi}{N}n}$ is $N$

- the set of all complex exponentials that have a common period $N$ are given by

$$\phi_k[n] = e^{jk\frac{2\pi}{N}n}, \quad k = 0, \pm1, \pm2, ...$$

Another important difference between CT and DT complex exponentials is that while

- $e^{jk\omega_0 t}$ are always distinct for different $k$,
- there are only $N$ distinct $e^{jk\frac{2\pi}{N}n}$, in particular, $e^{jk\frac{2\pi}{N}n} = e^{j(k+rN)\frac{2\pi}{N}n}$ where $r \in \mathbb{Z}$

Hence, a linear combination of DT complex exponentials of the form

$$\sum_k a_k e^{jk\frac{2\pi}{N}n}$$

- is periodic with $N$
- and the summation needs to go over only $N$ successive values of $k$.

Let us now introduce the **DT Fourier series** representation :

$$x[n] = \sum_{k=k_0}^{k_0+N-1} a_k e^{jk\frac{2\pi}{N}n}.$$

- A representation of this form is called the **Fourier series** representation of periodic DT signal $x[n]$ with fundamental period $N$.

- Coefficients $a_k$ are called the Fourier series (FS) coefficients (or spectral coefficients).
- Fundamental, $N^{th}$ harmonic, and DC (constant) components of the FS representation are defined in the same way as in the CT FS representation.
- Note that the summation goes over $N$ consecutive values of $k$ starting at an arbitrary value $k_0$, which is commonly shown with the notation $\sum_{k=<N>}$.

**Existence of DT FS**

Unlike the existence of CT FS, the existence of DT FS is worry-free. The DT FS representation always exist for any finite periodic DT signal $x[n]$.

To see this, consider the set of $N$ linearly independent equations for coefficients $a_k$ obtainable from the DT FS representation :

$$\begin{aligned} x[0] &= \sum_{k=k_0}^{k_0+N-1} a_k \\ x[1] &= \sum_{k=k_0}^{k_0+N-1} a_k e^{jk\frac{2\pi}{N}} \\ &\vdots \qquad\qquad \vdots \\ x[N-1] &= \sum_{k=k_0}^{k_0+N-1} a_k e^{jk\frac{2\pi}{N}(N-1)} \end{aligned}$$

It can be shown that these $N$ equations are linearly independent (see Problem 3.32 in textbook) and thus FS coefficients $a_k$ exist uniquely.

**Determination of DT Fourier series representation**

For a given DT periodic sequence $x[n]$, one way to determine the DT FS coefficients $a_k$, is to solve the above set of linear equations. However, similar to the CT FS, it is also possible to obtain a closed form solution for the coefficients $a_k$ :

Hence, this pair of **equations** defines the **Fourier series** of a **periodic DT signal** :

$$\boxed{x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n}} \qquad \boxed{a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\frac{2\pi}{N}n}}$$

**Notes :**

- an important property of DT FS is that not only the sequence $x[n]$ is **periodic** with N, but also the DF FS coefficients $a_k$ are, i.e. $\underline{a_k = a_{k+N}}$ :

- the summations in both the analysis and synthesis equations in DT FS run over one period of $a_k$ and $x[n]$, respectively.

#### Examples of DT Fourier series

**Ex:** (Coefficient matching approach) Find FS representation of $x[n] = \cos(\frac{2\pi}{5}n)$.

**Ex:** (General approach) Find FS representation of DT periodic square wave periodic with $N$ defined over one period $-\frac{N}{2} - 1 \leq n \leq \frac{N}{2}$ as $x[n] = \begin{cases} 1, & -N_1 \leq n \leq N_1 \\ 0, & \text{otherwise.} \end{cases}$

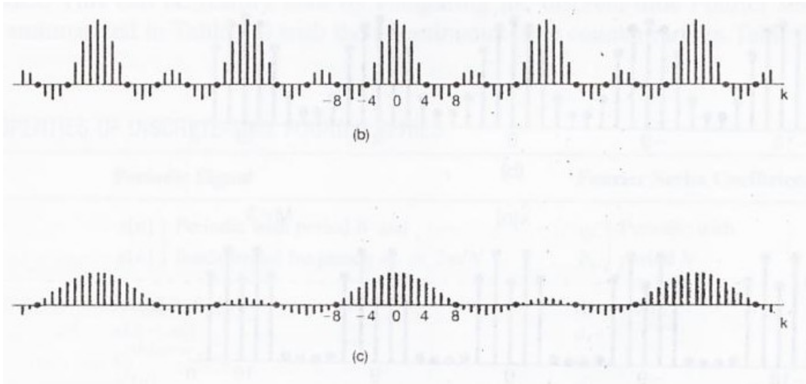



**Figure 3.17** Fourier series coefficients for the periodic square wave of Example 3.12; plots of $Na_k$ for $2N_1 + 1 = 5$ and (a) $N = 10$; (b) $N = 20$; and (c) $N = 40$.

## 5.2 DT Fourier Transform

### 5.2.1 Intuition and formal development of DT Fourier transform

DT Fourier transform representation is an extension of the DT FS representation to aperiodic DT signals. The derivation of DT FT is similar to that in CT. In other words, one starts with DT FS of a periodic sequence and then lets the period go to infinity.

Consider an aperiodic signal $x[n]$ and its periodic version $\tilde{x}[n]$, as shown in the Figure 5.1.

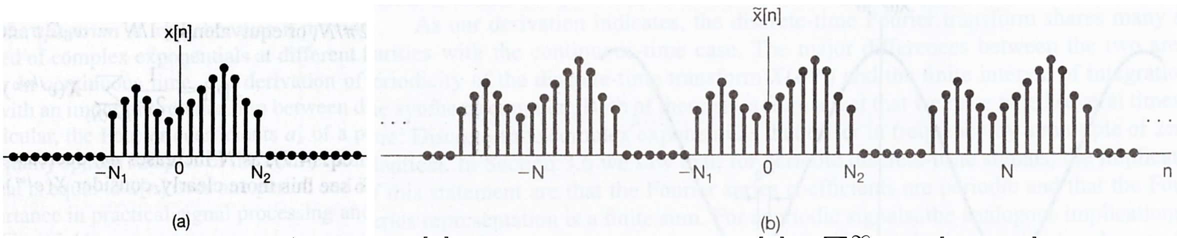



Figure 5.1: Aperiodic $x[n]$ and its periodic version $\tilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n-kN]$.

Let us start by remembering the FS analysis and synthesis equations for $\tilde{x}[n]$ :

Hence, this pair of **equations** defines the **DT Fourier transform** :

$$\boxed{x[n] = \frac{1}{2\pi}\int_{<2\pi>} X(\Omega)e^{j\Omega n}d\Omega} \qquad \boxed{X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}}$$

**Notes :**

- $X(\Omega)$ is called the DT Fourier transform (FT) of $x[n]$ (or spectrum of $x[n]$)
- These are also called **synthesis** and **analysis equations** of the DT FT.
- The DT FT $X(\Omega)$ is periodic with $2\pi$ :
- If DT FS and FT representations are compared :
  - Both represent signals as a **linear combination of complex exponentials**.
  - DT FS : complex exponentials occur at a **discrete set of harmonically related frequencies** $k\frac{2\pi}{N}$ and have amplitudes $a_k$
  - DT FT : complex exponentials occur at a **continuum of frequencies** over any continuous **interval of length** $2\pi$ and have amplitudes $\dfrac{X(\Omega)}{2\pi}$
  - From the development of DT FT, the DT FS coefficients $a_k$ of $\tilde{x}[n]$ are equally spaced samples of the DT FT of $x[n]$ : $a_k = \frac{1}{N}X(\Omega)|_{\Omega=k\frac{2\pi}{N}}$

### 5.2.2 Convergence of DT Fourier transform

Similar to CT FT, DT FT exists (i.e. analysis equation $\sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$ converges ) if **either**

- $x[n]$ has **finite energy**, i.e. $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$ **or**
- $x[n]$ is **absolutely summable**, i.e. $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$.

Note that the DT FT synthesis equation $(\dfrac{1}{2\pi}\int_{<2\pi>} X(\Omega)e^{j\Omega n}d\Omega)$ is integration over a finite period, and hence there is no convergence issue with it.

### 5.2.3 Examples of DT Fourier transform

**Ex:** $x[n] = a^n u[n], \quad 0 < a < 1$. Compute $X(\Omega)$ and plot its magnitude and phase.

**Ex:** $x[n] = a^{|n|}, \quad 0 < a < 1$. Compute $X(\Omega)$ and plot its magnitude and phase. (Exercise.)

**Ex:** Impulse response of an DT LTI system is $h[n] = \delta[n+1] + 2\delta[n] + \delta[n-1]$. Compute and plot $H(\Omega)$ which is called the frequency response of the LTI system.

**Ex:** Rectangular pulse $x[n] = \begin{cases} 1, |n| \leq N_1 \\ 0, \text{otherwise.} \end{cases}$ Compute $X(\Omega)$ and plot its magnitude and phase.

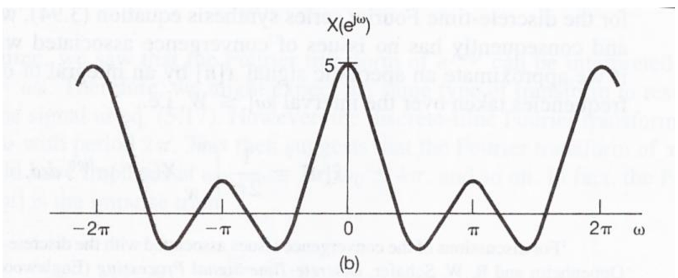



**Figure 5.6** (a) Rectangular pulse signal of Example 5.3 for $N_1 = 2$ and (b) its Fourier transform.

### 5.2.4 Response of LTI systems to complex exponentials (revisited)

Remember that $z^n$ is an **eigenfunction** of DT LTI systems and the eigenvalue was given by $H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$, where $z$ is a general complex number in polar form as $z = re^{j\Omega}$ :

Note that if $z$ has a magnitude of 1 (i.e. $z = e^{j\Omega}$), then the complex exponential $z^n = e^{j\Omega n}$ and its eigenvalue reduces to the DT FT of $h[n]$, i.e. we have $H(z)|_{z=e^{j\Omega}} = \mathscr{F}\{h[n]\} = H(\Omega)$.

**Ex:** Let $x_1[n] = \cos(\frac{\pi}{4}n)$ and $x_2[n] = \cos(\frac{3\pi}{4}n)$ be inputs to an LTI system with $\mathscr{F}\{h[n]\} = H(\Omega) = rect(\frac{\Omega}{\pi})$ , $|\Omega| < \frac{\pi}{2}$ and 0 otherwise in $|\Omega| < \pi$. Find outputs $y_1[n]$ and $y_2[n]$.

### 5.2.5 DT Fourier transform of periodic signals

As in the CT case, periodic DT signals can also be represented with the DT FT by allowing impulses in the transform. Hence, as in CT, the DT FT can serve as a common framework for representing both periodic and aperiodic DT signals.

Let us start with the result : the expression for the DT FT of a periodic DT signal $\tilde{x}[n]$ is very similar to the expression we obtained previously for the CT FT of a periodic CT signal $\tilde{x}(t)$.

Remember, in CT, we could obtain the CT FT of a periodic signal $\tilde{x}(t)$ with FS coefficients $a_k$ and fundamental frequency $\omega_0$ via an impulse train :

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longleftrightarrow X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

In DT, the DT FT of a periodic sequence $\tilde{x}[n]$ with periodic DT FS coefficients $a_k$ and fundamental frequency $\Omega_0 = \dfrac{2\pi}{N}$ can be obtained via a very similar impulse train :

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \longleftrightarrow X(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\Omega - k\Omega_0) \qquad \text{where} \qquad \Omega_0 = \frac{2\pi}{N}.$$

Let us derive this expression, by first considering the DT FT of $e^{j\Omega_0 n}$ :

- In CT, remember that the FT of $e^{j\omega_0 t}$ was : $\qquad e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega - \omega_0)$
- In DT, the FT must be periodic with $2\pi$, so let us try for the DT FT of $e^{j\Omega_0 n}$ a periodic impulse train

$$e^{j\Omega_0 n} \longleftrightarrow \sum_{l=\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0 - 2\pi l)$$

Now, consider the DT FS of a periodic DT signals $\tilde{x}[n]$ and use the above result :

In summary, to determine the **DT FT of a periodic signal**, (as we did in CT)

1. determine its periodic DT FS coefficients $a_k$
2. then write its DT FT as $X(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\Omega - k\Omega_0)$ where $\Omega_0 = \frac{2\pi}{N}$.

**Ex:** $\cos(\frac{2\pi}{5}n)$. Find its DT FT.

**Ex:** Find DT FT of DT periodic impulse train $x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$.

## 5.3 Properties of DT Fourier series and transform

To discuss the DT Fourier transform and series properties, we use the same convenient shorthand notation we used in CT FS and FT properties. In other words, to indicate the pairing of a signal and its transform, we use :

- signal $\longleftrightarrow$ FT ( e.g. $x[n] \longleftrightarrow X(\Omega)$ )

We sometimes also refer to a FT and inverse FT with the following notation :

- $\mathscr{F}\{\text{signal}\}$ and $\mathscr{F}^{-1}\{\text{FT}\}$ ( e.g. $X(\Omega) = \mathscr{F}\{x[n]\}$ and $x[n] = \mathscr{F}^{-1}\{X(\Omega)\}$)

For the following properties, let us assume that we have two signals $x[n]$ and $y[n]$ with corresponding DT FT $X(\Omega)$ and $Y(\Omega)$, i.e.

$$x[n] \longleftrightarrow X(\Omega) \qquad \text{and} \qquad y[n] \longleftrightarrow Y(\Omega).$$

Let us also assume we have two periodic signals $\tilde{x}[n]$ and $\tilde{y}[n]$, periodic with the same period $N$, with corresponding DT FS coefficients $a_k$ and $b_k$, i.e.

$$\tilde{x}[n] \longleftrightarrow a_k \qquad \text{and} \qquad \tilde{y}[n] \longleftrightarrow b_k.$$

### 5.3.1 Periodicity

$$\boxed{X(\Omega) = X(\Omega + 2\pi)}$$

(FS: $a_k = a_{k+N}$)

### 5.3.2 Linearity

$$\boxed{ax[n] + by[n] \longleftrightarrow aX(\Omega) + bY(\Omega)}$$

(FS: $A\tilde{x}[n] + B\tilde{y}[n] \longleftrightarrow Aa_k + Bb_k$ What if $\tilde{x}[n]$ and $\tilde{y}[n]$ do not have the same period ?)

### 5.3.3 Time Shifting and Frequency Shifting

$$\boxed{x[n-n_0] \longleftrightarrow e^{-j\Omega n_0}X(\Omega)} \qquad \text{and} \qquad \boxed{e^{j\Omega_0 n}x[n] \longleftrightarrow X(\Omega - \Omega_0)}$$

(FS: $\tilde{x}[n-n_0] \longleftrightarrow e^{-jk\frac{2\pi}{N}n_0} a_k$ and $e^{jk_0\frac{2\pi}{N}n}\tilde{x}[n] \longleftrightarrow a_{k-k_0}$ )

**Ex:** Consider the frequency response $H_{ILP}(\Omega)$ of the ideal DT low-pass filter with cut-off frequency $\Omega_c$, plotted below. Plot and discuss $H_{IHP}(\Omega) = H_{ILP}(\Omega - \pi)$. Also obtain $h_{IHP}[n]$ in terms of $h_{ILP}[n]$.

### 5.3.4 Conjugation and Conjugate Symmetry

$\boxed{x^*[n] \longleftrightarrow X^*(-\Omega)}$

Properties similar to the CT case can be obtained if $x[n]$ is real (i.e. $x[n] = x^*[n]$) :

- $X(\Omega) =$ or $X^*(\Omega) =$
- $Re\{X(\Omega)\} = Re\{X(-\Omega)\}$ and $Im\{X(\Omega)\} = -Im\{X(-\Omega)\}$
- $|X(\Omega)| = |X(-\Omega)|$ and $\angle X(\Omega) = -\angle X(-\Omega)$

Similar to the CT case, again,

- if $x[n]$ is both real and even (i.e. $x[n] = x^*[n] = x[-n]$), then $X(\Omega)$ is also both real and even
- if $x[n]$ is both real and odd (i.e. $x[n] = x^*[n] = -x[-n]$), then $X(\Omega)$ is purely imaginary and odd.

(FS : $\tilde{x}^*[n] \longleftrightarrow a^*_{-k}$ ) Similar properties exist in case $\tilde{x}[n]$ is real, real and even, etc.

For a summary of all conjugate symmetry properties of DT FT and FS, see the properties tables in the upcoming Section 5.3.12.

### 5.3.5 Differencing and Accumulation

$\boxed{x[n] - x[n-1] \longleftrightarrow (1 - e^{j\Omega})X(\Omega)}$ and $\boxed{\displaystyle\sum_{m=-\infty}^{n} x[m] \longleftrightarrow \frac{1}{1 - e^{j\Omega}} X(\Omega) + \pi X(0) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)}$

(FS: $\tilde{x}[n] - \tilde{x}[n-1] \longleftrightarrow (1 - e^{jk\frac{2\pi}{N}})a_k$ and $\sum_{m=-\infty}^{n} \tilde{x}[m] \longleftrightarrow \dfrac{1}{1 - e^{jk\frac{2\pi}{N}}} a_k$ )

Note that $\sum_{m=-\infty}^{n} \tilde{x}[m]$ is finite and period only if $a_0 = 0$.

**Ex:** Find $\mathscr{F}\{u[n]\}$.

### 5.3.6 Time Reversal

$\boxed{x[-n] \longleftrightarrow X(-\Omega)}$

(FS: $\tilde{x}[-n] \longleftrightarrow a_{-k}$)

### 5.3.7 Differentiation in Frequency

$\boxed{nx[n] \longleftrightarrow j\dfrac{d}{d\Omega} X(\Omega)}$

### 5.3.8 Time Expansion

Remember the CT FT result for time expansion (scaling) : $x(at) \longleftrightarrow \dfrac{1}{|a|} X(\dfrac{\omega}{a})$.

If we try to define similarly the signal $x[an]$, we run into difficulties if $a$ is not an integer.
In CT, <u>if $a > 1$</u>, then this speeds up the signal (i.e. compresses signal wrt origin.) Let's similarly, consider $x[an]$ for an integer $a$ such that $a > 1$. This not only speeds up the signal but also discards some samples of $x[n]$ :

Note that one can relate the $\mathscr{F}\{x[an]\}$ to $X(\Omega)$ but we will leave that to EE430.

In CT, <u>if $0 < a < 1$</u>, then this slows down the signal (i.e. expands signal wrt origin.) Let's similarly, consider $x_{(a)}[n]$ for an integer $a$ such that $a > 1$ :

$$x_{(a)}[n] = \begin{cases} x[\frac{n}{a}], & \text{if } n \text{ is a multiple of } a \\ 0, & \text{otherwise.} \end{cases}$$

This inserts $a-1$ zeros in between the samples of $x[n]$. Intuitively, one can think of $x_{(a)}[n]$ as a slowed-down version of $x[n]$.

Note that in this case, one can easily relate $\mathscr{F}\{x_{(a)}[n]\}$ to $X(\Omega)$ : $\boxed{x_{(a)}[n] \longleftrightarrow X(a\Omega)}$

The figure below shows examples of $x_{(a)}[n]$ and corresponding $X(a\Omega)$ for $a = 2$ and $a = 3$ :

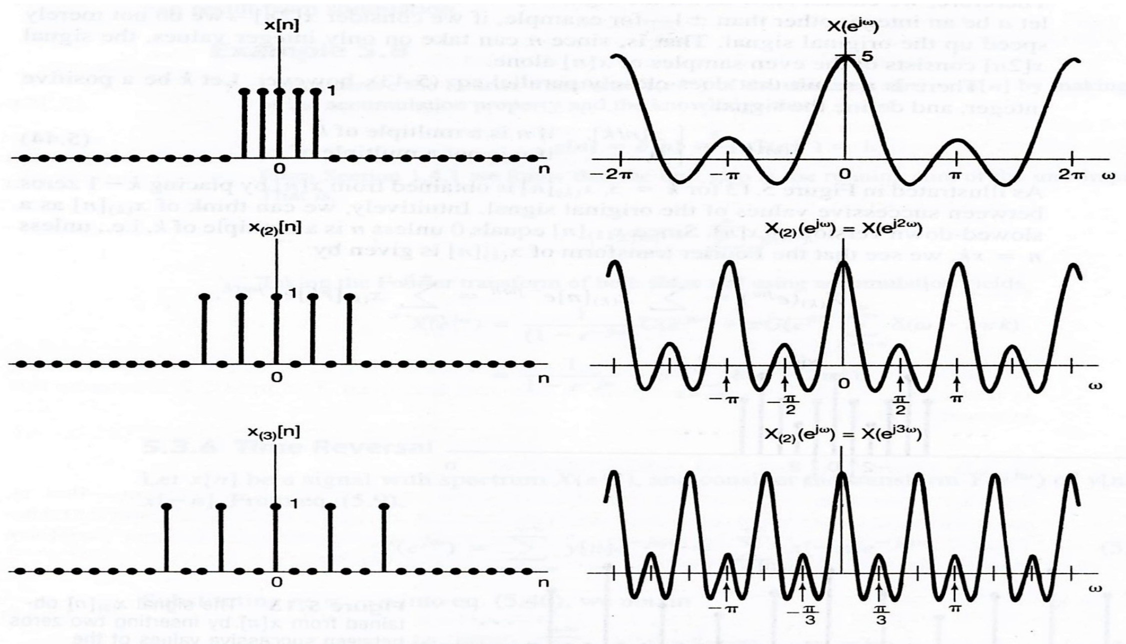



**Figure 5.14** Inverse relationship between the time and frequency domains: As $k$ increases, $x_{(k)}[n]$ spreads out while its transform is compressed.

Figure 5.2: (from textbook) Time expansion property of DT FT.

(FS : $\tilde{x}_{(a)}[n] \longleftrightarrow \frac{1}{a}a_k$ ) Note that $\tilde{x}_{(a)}[n]$ becomes periodic with $aN$, and so does $\frac{1}{a}a_k$.

**Ex:** Find $X(\Omega)$ in terms of $Y(\Omega)$ where $x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$.

### 5.3.9 Parseval's Relation

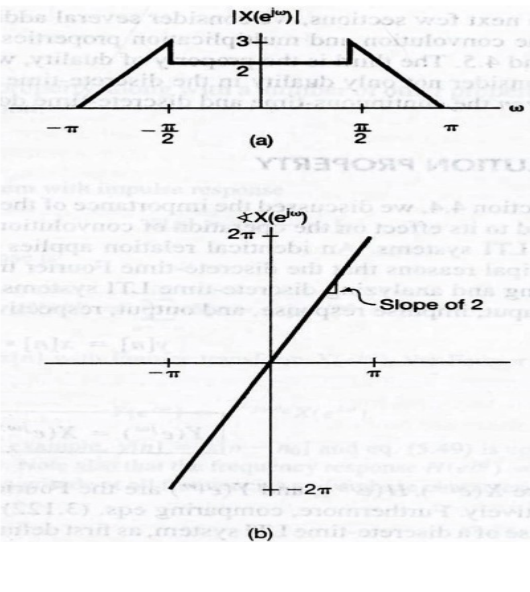
$$\boxed{\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{<2\pi>} |X(\Omega)|^2 d\Omega}$$

- The term on left side is the **total energy in the signal** $x[n]$.
- Parseval's relation indicates that the total signal energy can be computed either
  - from $x[n]$ by summing up $(|x[n]|^2)$ over all time samples
  - or from $X(\Omega)$ by integrating its energy per unit frequency $(\frac{|X(\Omega)|^2}{2\pi})$ over a full $2\pi$ interval of frequencies.
- $|X(\Omega)|^2$ is called the *energy-density spectrum* of signal $x[n]$.

Derivation is similar to the CT case.

(FS: $\frac{1}{N}\sum_{n=<N>} |\tilde{x}[n]|^2 = \sum_{k=<N>} |a_k|^2$ Left side of equation is the **average power of the periodic signal** $\tilde{x}[n]$, and equals to the sum of squared magnitudes of FS coefficients over one period.)

**Ex:** Consider the sequence $x[n]$ whose FT $X(\Omega)$ is shown for $-\pi \le \Omega \le \pi$ below. Determine whether or not $x[n]$ is periodic, real, even, and/or finite energy.



### 5.3.10 Convolution Property

$\boxed{x[n] * y[n] \longleftrightarrow X(\Omega) \cdot Y(\Omega)}$

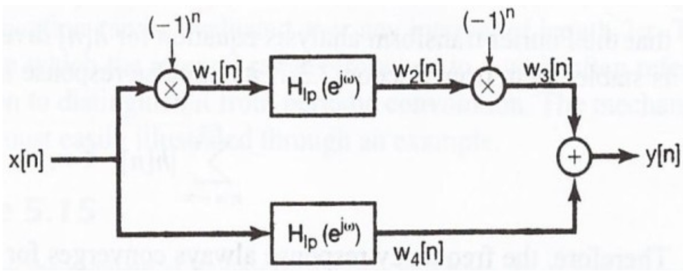
- The convolution property implies that we can analyze DT LTI systems also in frequency domain.

- $\mathscr{F}\{h[n]\} = H(\Omega)$ is called the **frequency response** of the DT LTI system.
  - $H(\Omega)$ may not be defined for all DT LTI systems. For $H(\Omega)$ to be defined, $h[n]$ must have FT (i.e. have finite energy or be absolutely summable.)
  - If an DT LTI system is stable (i.e. $\sum_{-\infty}^{\infty} |h[n]| < \infty$), then we typically say that it has a frequency response $H(\Omega)$. (Why?)
  - To analyze unstable LTI systems, we will learn the Z transform in upcoming chapters.

(FS: $\sum_{r=<N>} \tilde{x}[r]\tilde{y}[n-r] \longleftrightarrow N \cdot a_k \cdot b_k$ Convolution over one period corresponds to multiplication of FS coefficients.)

**Ex:** Consider an ideal low-pass filter $H_{ILP}(\Omega)$ with cut-off frequency $\Omega_c = \frac{\pi}{2}$. Find the output if the input is $e^{j\Omega_0 n}$, $\cos(\frac{\pi}{8}n)$, $\cos(\frac{3\pi}{4}n)$.

**Ex:** Consider the system shown in the figure. The LTI system with frequency response $H_{lp}(\Omega)$ is an ideal low-pass filter with cut-off frequency $\Omega_c = \frac{\pi}{4}$. Find $W_i(\Omega)$ for $i = 1, 2, 3, 4$ and also the overall frequency response of the system.



### 5.3.11 Multiplication property

$$\boxed{x[n] \cdot y[n] \longleftrightarrow \frac{1}{2\pi} \int_{<2\pi>} X(\theta) Y(\Omega - \theta) d\theta} \qquad \text{(convolution over one period, i.e. } 2\pi\text{)}$$

(FS: $\tilde{x}[n] \cdot \tilde{y}[n] \longleftrightarrow \sum_{l=<N>} a_l b_{k-l}$ convolution over one period )

**Ex:** Consider $x_1[n] = \dfrac{\sin(3\pi n/4)}{\pi n}$ , $x_2[n] = \dfrac{\sin(\pi n/2)}{\pi n}$ and $x[n] = x_1[n] \cdot x_2[n]$. Find $X(\Omega)$.

### 5.3.12 Table of properties of DT FT and FS

The following table from the textbook summarizes all DT FT and FS properties.

**TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM**

| Section | Property | Aperiodic Signal | Fourier Transform |
|---|---|---|---|
| | | $x[n]$<br>$y[n]$ | $\left.\begin{array}{l} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{array}\right\}$ periodic with period $2\pi$ |
| 5.3.2 | Linearity | $ax[n] + by[n]$ | $aX(e^{j\omega}) + bY(e^{j\omega})$ |
| 5.3.3 | Time Shifting | $x[n - n_0]$ | $e^{-j\omega n_0}X(e^{j\omega})$ |
| 5.3.3 | Frequency Shifting | $e^{j\omega_0 n}x[n]$ | $X(e^{j(\omega-\omega_0)})$ |
| 5.3.4 | Conjugation | $x^*[n]$ | $X^*(e^{-j\omega})$ |
| 5.3.6 | Time Reversal | $x[-n]$ | $X(e^{-j\omega})$ |
| 5.3.7 | Time Expansion | $x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$ | $X(e^{jk\omega})$ |
| 5.4 | Convolution | $x[n] * y[n]$ | $X(e^{j\omega})Y(e^{j\omega})$ |
| 5.5 | Multiplication | $x[n]y[n]$ | $\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$ |
| 5.3.5 | Differencing in Time | $x[n] - x[n-1]$ | $(1 - e^{-j\omega})X(e^{j\omega})$ |
| 5.3.5 | Accumulation | $\sum_{k=-\infty}^{n} x[k]$ | $\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$ |
| 5.3.8 | Differentiation in Frequency | $nx[n]$ | $j\frac{dX(e^{j\omega})}{d\omega}$ |
| 5.3.4 | Conjugate Symmetry for Real Signals | $x[n]$ real | $\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$ |
| 5.3.4 | Symmetry for Real, Even Signals | $x[n]$ real an even | $X(e^{j\omega})$ real and even |
| 5.3.4 | Symmetry for Real, Odd Signals | $x[n]$ real and odd | $X(e^{j\omega})$ purely imaginary and odd |
| 5.3.4 | Even-odd Decomposition of Real Signals | $x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$<br>$x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$ | $\mathcal{R}e\{X(e^{j\omega})\}$<br>$j\mathcal{I}m\{X(e^{j\omega})\}$ |
| 5.3.9 | Parseval's Relation for Aperiodic Signals | $\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$ | |

Figure 5.3: Properties of DT FT.

### 5.3.13 Table of basic signals and their DT FT and FS

The following table from the textbook summarizes the DT FT of some basic signals (and their DT FS if the signal is periodic) .

**TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES**

| Property | Periodic Signal | Fourier Series Coefficients |
|---|---|---|
| | $\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period $N$ and fundamental frequency $\omega_0 = 2\pi/N$ | $\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period $N$ |
| Linearity | $Ax[n] + By[n]$ | $Aa_k + Bb_k$ |
| Time Shifting | $x[n - n_0]$ | $a_k e^{-jk(2\pi/N)n_0}$ |
| Frequency Shifting | $e^{jM(2\pi/N)n} x[n]$ | $a_{k-M}$ |
| Conjugation | $x^*[n]$ | $a^*_{-k}$ |
| Time Reversal | $x[-n]$ | $a_{-k}$ |
| Time Scaling | $x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period $mN$) | $\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$ |
| Periodic Convolution | $\sum_{r=\langle N \rangle} x[r]y[n-r]$ | $Na_k b_k$ |
| Multiplication | $x[n]y[n]$ | $\sum_{l=\langle N \rangle} a_l b_{k-l}$ |
| First Difference | $x[n] - x[n-1]$ | $(1 - e^{-jk(2\pi/N)})a_k$ |
| Running Sum | $\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$ | $\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$ |
| Conjugate Symmetry for Real Signals | $x[n]$ real | $\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$ |
| Real and Even Signals | $x[n]$ real and even | $a_k$ real and even |
| Real and Odd Signals | $x[n]$ real and odd | $a_k$ purely imaginary and odd |
| Even-Odd Decomposition of Real Signals | $\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$ | $\Re e\{a_k\}$ <br> $j\Im m\{a_k\}$ |

**Parseval's Relation for Periodic Signals**

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

Figure 5.4: Properties of DT FS.

**TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS**

| Signal | Fourier Transform | Fourier Series Coefficients (if periodic) |
|---|---|---|
| $\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k$ |
| $e^{j\omega_0 n}$ | $2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$ | (a) $\omega_0 = \frac{2\pi m}{N}$<br>$a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$<br>(b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic |
| $\cos \omega_0 n$ | $\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$ | (a) $\omega_0 = \frac{2\pi m}{N}$<br>$a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$<br>(b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic |
| $\sin \omega_0 n$ | $\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$ | (a) $\omega_0 = \frac{2\pi r}{N}$<br>$a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \dots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$<br>(b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic |
| $x[n] = 1$ | $2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$ | $a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ |
| Periodic square wave<br>$x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$<br>and<br>$x[n+N] = x[n]$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]},\ k \ne 0, \pm N, \pm 2N, \dots$<br>$a_k = \frac{2N_1 + 1}{N},\ k = 0, \pm N, \pm 2N, \dots$ |
| $\sum_{k=-\infty}^{+\infty} \delta[n - kN]$ | $\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k = \frac{1}{N}$ for all $k$ |
| $a^n u[n], \quad \lvert a\rvert < 1$ | $\frac{1}{1 - ae^{-j\omega}}$ | — |
| $x[n] \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$ | $\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$ | — |
| $\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$<br>$0 < W < \pi$ | $X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$<br>$X(\omega)$ periodic with period $2\pi$ | — |
| $\delta[n]$ | $1$ | — |
| $u[n]$ | $\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$ | — |
| $\delta[n - n_0]$ | $e^{-j\omega n_0}$ | — |
| $(n+1)a^n u[n], \quad \lvert a\rvert < 1$ | $\frac{1}{(1 - ae^{-j\omega})^2}$ | — |
| $\frac{(n + r - 1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$ | $\frac{1}{(1 - ae^{-j\omega})^r}$ | — |

Figure 5.5: Basic DT FT pairs.

# Chapter 6

# Sampling

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This chapter discusses sampling, the process in which samples from a CT signal $x_c(t)$ are taken to generate a sequences of numbers ...,$x_c(0), x_c(T), x_c(2T)$, ..., called samples of the signal.

The samples can be seen as a DT signal ($x_d[n] = x_c(nT)$) and thus sampling can be seen as converting a CT signal to a DT signal. Much of the importance of sampling comes from this role as a bridge between CT and DT signals. In particular,

- nowadays, digital systems are preferred in many signal processing applications since they are more flexible, cheaper and easily re-programmable, and
- signal processing of CT signals is done with digital systems by first sampling the CT signal, then processing these samples as a DT signal, and then converting the processed DT signal back to a CT signal.

## 6.1 Representation of CT signals by its samples : the sampling theorem

Sampling is a process in which samples from a CT signal $x(t)$ are taken at regular intervals to generate a sequence of numbers ...,$x(0), x(T), x(2T), ...$, called samples of the signal.

In general, without any conditions or additional information, one cannot expect that a signal can be uniquely represented by its samples, as evident from the figure below.

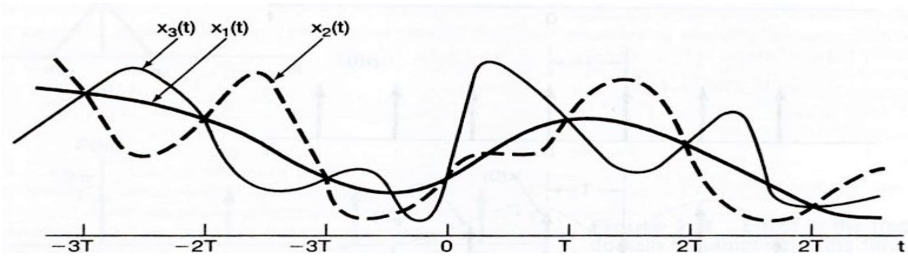



Figure 6.1: Three different CT signals with identical samples.

A signal $x(t)$ is band-limited if its spectrum $X(\omega)$ is zero outside a finite band of frequencies, i.e. $X(\omega) = 0,\ |\omega| > \omega_M$.

As we will see, if a CT signal $x(t)$ is

- band-limited and
- the samples are taken sufficiently close together,

then the samples uniquely represent, and can be used to recover, the original signal $x(t)$.

### 6.1.1 Impulse-train sampling

One convenient way to represent the sampling of a CT signal is to multiply the signal with a periodic impulse train. The samples of the signal will be the weights of the impulses.

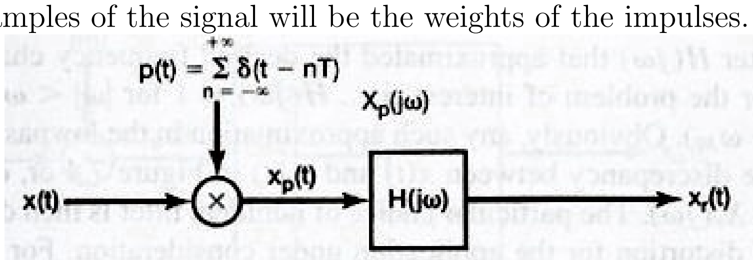



Figure 6.2: System for impulse train sampling and reconstruction of signal from samples.

Sampling in time domain :

Sampling in frequency domain :

Hence, when sampling frequency is sufficiently high, i.e. $\omega_s > 2\omega_M$, the original signal $x(t)$ can be perfectly recovered from $x_p(t)$ by an ideal low-pass filter with gain $T$ and cut-off frequency $\omega_c$ chosen from the range $\omega_M < \omega_c < \omega_s - \omega_M$.

**Sampling theorem :**

Let $x(t)$ be a band-limited signal with spectrum $X(\omega) = 0$ for $|\omega| > \omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm1, \pm2, ...$ if

$$\omega_s > 2\omega_M$$

where $\omega_s = \dfrac{2\pi}{T}$. (The frequency $2\omega_M$, which the sampling frequency $\omega_s$ must exceed, is called **Nyquist sampling frequency**.)

Given these samples, $x(t)$ can be recovered by

- generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values,
- and processing this impulse train with an idea low-pass filter with gain $T$ and cut-off frequency $\omega_c$ chosen from the range $\omega_M < \omega_c < \omega_s - \omega_M$.

**Note** that reconstructing the original CT signal $x(t)$ from its samples requires an ideal low-pass filter, which corresponds in time domain to convolving with a *sinc*() :

(Watch the demo on reconstruction by A.V. Oppenheim : `https://youtu.be/_WV4J1BOQro?t=569`)
In practical applications, since the *sinc*() signal is difficult to implement, a simpler low-pass filter may be used, which can give a reconstructed signal $x_r(t)$ not exactly equal to original $x(t)$ but the error may be tolerated in some applications.

(Demo on linear reconstruction by A.V. Oppenheim : `https://youtu.be/_WV4J1BOQro?t=1301`)

### 6.1.2 Sampling with a zero-order hold

Narrow, large amplitude pulses that approximate impulses are difficult to generate in practice. Therefore it is often more convenient to generate a sampled signal waveform known as zero-order hold.

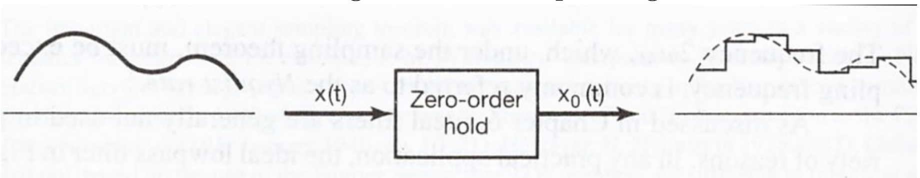



Figure 6.3: Sampling with a zero-order hold system.

Zero-order hold sampling system can be modeled with impulse train sampling followed by an LTI system with a rectangular impulse response $h_0(t)$:

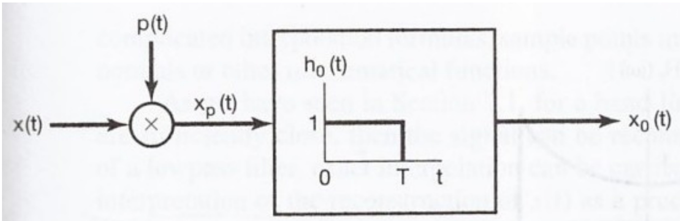



Figure 6.4: Model for zero-order hold sampling

## 6.2 Effect of undersampling

Let us discuss what to expect if sampling is performed below the Nyquist sampling frequency, i.e. $\omega_s < 2\omega_M$ :

This effect, where individual terms overlap, is called *aliasing*.

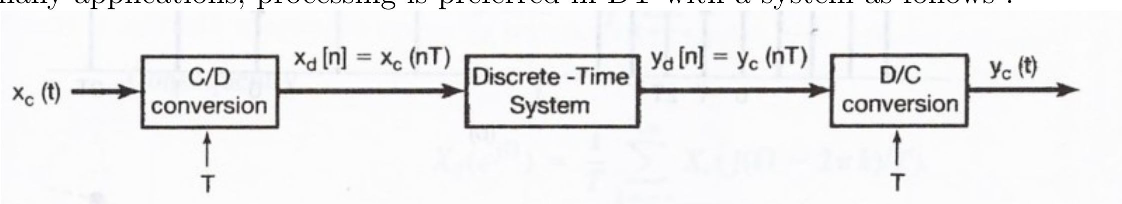
**Ex:** Consider sampling $x(t) = \cos(20\pi t)$ below the Nyquist sampling frequency and then attempting to reconstruct it back from the samples :

Let us watch a demo on aliasing from A.V. Oppenheim : https://youtu.be/P3eLer1edx8?t=817

## 6.3 DT processing of CT signals

We often have systems where input and output are CT signals :

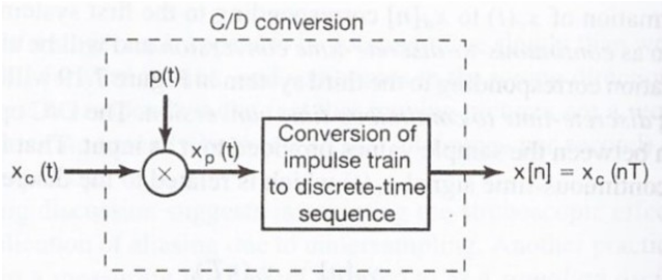
But in many applications, processing is preferred in DT with a system as follows :




These two systems can be made equivalent with proper choice of sampling frequency and filters.

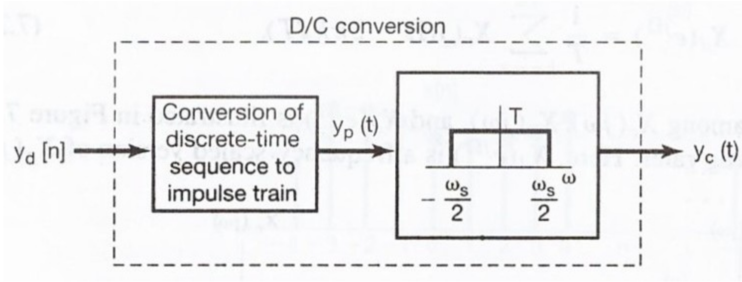
### 6.3.1 C/D conversion

A model for continuous-to-discrete (C/D) conversion is as follows:




### 6.3.2 D/C conversion

A model for discrete-to-continuous (D/C) conversion is as follows :




Overall, a system for filtering a CT signal with a DT filter is as follows :

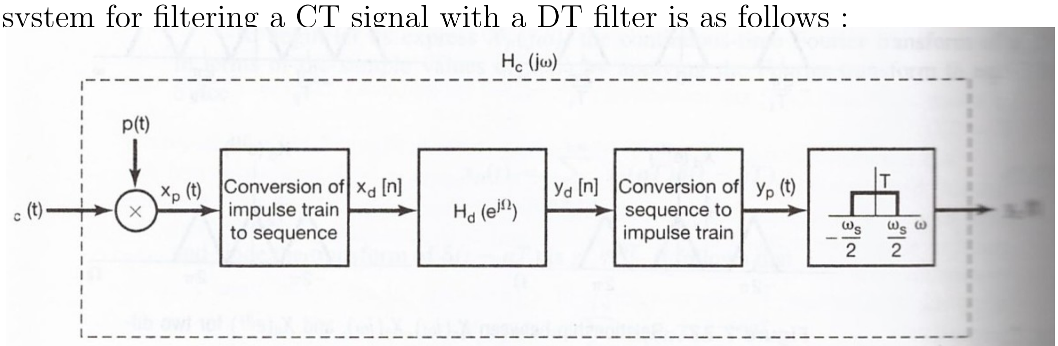



**Figure 7.24** Overall system for filtering a continuous-time signal using a discrete-time filter.

# Chapter 7

# The Z-transform

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In the preceding chapters, we have seen that Fourier series and transform representations are very useful in the study of many problems involving signals and LTI systems. This is mainly due to the facts that Fourier tools represent signals as a linear combination of complex exponentials of the form $e^{j\omega t}$ in CT and $e^{j\Omega n}$ in DT, which are eigenfunctions of LTI systems.

However, the eigenfunction property also applies to more general complex exponentials of the form $e^{st}$ with $s = \alpha + j\omega$ in CT and $z^n$ with $z = re^{j\Omega}$ in DT. This observation leads to generalization of the CT and DT Fourier transforms known as the Laplace transform in CT and z transform in DT.

As we will see, the Laplace and z transform have many of the beautiful properties of the Fourier transform but also provide additional tools and insights. In particular, for signals for which the

Fourier transforms do not exist, the Laplace or z transform may converge/exist, and thus they can be very useful in the stability analysis of systems.

This chapter discusses the z transform and the next chapter will discuss the Laplace transform.

## 7.1 The Z transform and its region of convergence (ROC)

Remember that DT complex exponentials of the general form $z^n = (re^{j\Omega})^n$ are *eigenfunctions of DT LTI* systems with *eigenvalues* given by $H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$ where $h[n]$ is the impulse response of the system:

The eigenvalue expression provides the definition of the z transform of a signal $x[n]$ where $z$ is a complex number :

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

We again will use shorthand notations for the z transform of a signal $x[n]$:

- $\mathscr{Z}\{x[n]\}$
- $x[n] \longleftrightarrow X(z)$

The z transform is a generalization of the Fourier transform and they are related as follows :

- $X(z)|_{z=re^{j\Omega}} = \mathscr{F}\{x[n]r^{-n}\}$

- $X(z)|_{z=e^{j\Omega}} = \mathscr{F}\{x[n]\}$ (i.e. z transform reduces to FT for values of $z$ on the unit circle of the z-plane)

The above relation indicates that the convergence/existence of z transform, requires the convergence/existence $\mathscr{F}\{x[n]r^{-n}\}$,

- which happens when $x[n]r^{-n}$ is absolutely summable, i.e. $\sum_{-\infty}^{\infty} |x[n]r^{-n}| < \infty$
- and thus z transform $X(z)$ converges/exists for some values of $r$ and does not for others.

Region of Convergence (ROC) : A range of $r$ (note that $|z| = r$) values for which z transform $X(z)$ converges.

**Ex:** $x[n] = \alpha^n u[n]$. Find $X(z)$ and its ROC.

Note that the above relation between z transform and Fourier transform are valid if $|z| = r$ is contained in the ROC of $X(z)$ :

In particular, if the ROC of $X(z)$ contains the unit circle $|z| = r = 1$, then the DT FT $X(\Omega)$ exists and is given by $X(z)|_{z=e^{j\Omega}}$ :

**Ex:** Previous example with $\alpha = \frac{1}{2}$ and $\alpha = \frac{3}{2}$.

**Ex:** $x[n] = -\alpha^n u[-n-1]$. Find $X(z)$ and its ROC.

**Notes :**

- It is helpful to remember the following frequently used signal and z transform pairs :

$$\begin{aligned} \alpha^n u[n] &\longleftrightarrow \frac{z}{z-\alpha}, \quad ROC : |z| > |\alpha| \\ -\alpha^n u[-n-1] &\longleftrightarrow \frac{z}{z-\alpha}, \quad ROC : |z| < |\alpha| \end{aligned}$$

- The specification of the z transform requires both
  - the algebraic expression for $X(z)$
  - and the associated ROC.

  (Without the ROC, $X(z)$ by itself does not uniquely specify a signal $x[n]$.)

**Ex:** $x[n] = (\frac{1}{2})^n u[n] + (\frac{1}{3})^n u[n]$. Find $X(z)$ and its ROC.

**Ex:** $x[n] = -(\frac{1}{2})^n u[-n-1] + (\frac{1}{3})^n u[n]$. Find $X(z)$ and its ROC.

## 7.2 Properties of ROC

**Property 1 :** The **ROC** of $X(z)$ depends only on $|z| = r$ and therefore consists of a **ring** in the z-plane centered at the origin.

Reason :

- The relation $X(z)|_{z=re^{j\Omega}} = \mathscr{F}\{x[n]r^{-n}\}$ indicates that the convergence/existence of z transform $X(z)$, requires the convergence/existence $\mathscr{F}\{x[n]r^{-n}\}$, which happens when $x[n]r^{-n}$ is absolutely summable, i.e. $\sum_{-\infty}^{\infty} |x[n]r^{-n}| = \sum_{-\infty}^{\infty} |x[n]| r^{-n} < \infty$.
- Thus, ROC of $X(z)$ depends only on $r = |z|$ and is independent of $\Omega$.

**Property 2 :** For rational $X(z)$, the **ROC** does not contain any poles.

Reason :

- Rational $X(z)$ means $X(z)$ is a ratio of polynomials of $z$.
- A pole of $X(z)$ is a root of the denominator and a zero of $X(z)$ is a root of the numerator.
- $X(z)$ is infinite at a pole and hence does not converge at a pole.

**Property 3 :** If $x[n]$ is of finite duration, then the ROC is the entire z-plane except possibly $z = 0$ and/or $z = \infty$.

Reason :

- A finite duration sequence has only a finite number of nonzero samples, e.g.

- For finite $z$, $X(z)$ is a finite sum of ...

- If $N_1 < 0$ and $N_2 > 0$, then ...
- If $0 < N_1 < N_2$, then ...
- If $N_1 < N_2 < 0$, then ...

**Property 4 :** If $x[n]$ is a right-sided sequence, then the ROC is the outside of a circle centered at the origin (excluding possibly $z = \infty$).

Reason :

- A sight-sided sequence is of the form ...

- Some $r$ is in ROC of $X(z)$ if $X(z) = \mathscr{F}\{x[n]r^{-n}\}$ converges which happens if $\sum_{-\infty}^{\infty} |x[n]|r^{-n} < \infty$.
- If some $r_0$ is in the ROC of $X(z)$, i.e. $\sum_{N_1}^{\infty} |x[n]r_0^{-n}| < \infty$, then for $N_1 > 0$ and some $r_1 > r_0$, we have $\sum_{N_1}^{\infty} |x[n]r_1^{-n}| < \infty$ since ...
  - Hence, $\mathscr{F}\{x[n]r_1^{-n}\}$ converges and thus $|z| = r_1$ is also in the ...
  - Thus the ROC must be the outside of a circle.
  - Note that if $N_1 < 0$, there is a finite sum of terms coming from negative $n$ (i.e. $\sum_{N_1}^{0} |x[n]r_1^{-n}|$), which will not cause a problem for convergence except possibly at for $z = \infty$.

**Property 5 :** If $x[n]$ is a left-sided sequence, then the ROC is the inside of a circle centered at the origin (excluding possibly $z = 0$).

Reason is similar to the previous property's.

**Property 6 :** If $x[n]$ is a two-sided sequence, then the ROC consists of a ring or is empty.

Reason :

- A two-sided sequence can be considered as the sum of ...
- The ROC of $X(z)$ of the sum of sequences is the intersection of the ROCs of the z transforms of each sequence.

**Ex:** $x[n] = \begin{cases} 2^n, n < 0 \\ (\frac{1}{3})^n, n = 0, 2, 4, .. \\ 0, n = 1, 3, 5, ... \end{cases}$. Find $X(z)$ and its ROC.

**Property 7 :** If $X(z)$ is rational, then its ROC is bounded by the poles or extends to infinity.

Reason :

- A signal $x[n]$ with rational $X(z)$ consist of a linear combination of exponentials $\alpha^n u[n]$ or $-\alpha^n u[-n-1]$ which have ROCs bounded by their poles.
- The ROC of $X(z)$ of the linear combination of exponentials thus is the intersection of ROCs bounded by poles.

**Property 8 :** If $X(z)$ is rational and $x[n]$ is right-sided, then its ROC is the outside of the outermost pole. (Furthermore, if $x[n]$ is also causal, then the ROC also includes $z = \infty$.)

Reason : ...

**Property 9 :** If $X(z)$ is rational and $x[n]$ is left-sided, then its ROC is the inside of the innermost pole. (Furthermore, if $x[n]$ is also anti-causal (i.e. $x[n] = 0$ for $n > 0$), then the ROC also includes $z = 0$.)

Reason : ...

## 7.3 Inversion of Z transforms

The inverse z transform expression contains integration around a circular contour on the z-plane and is typically difficult to compute and will not be applied in this course. However, there are a number of alternative procedures for obtaining a sequence from its z transform and associated ROC.

One useful procedure for rational $X(z)$ is to expand $X(z)$ into a **partial-fraction expansion**, and then to recognize the sequence associated with each term in the expansion.

**Ex:** $X(z) = \dfrac{3z}{(2-z)(2z-1)}$, ROC:$\frac{1}{2} < |z| < 2$. Find $x[n]$.

Another procedure is to use the **power series expansion of z transform** and determine $x[n]$ by inspection.

**Ex:** $X(z) = 4z^2 + 2 + 3z^{-1}$, ROC:$0 < |z| < \infty$. Find $x[n]$.

Power series expansion can be very useful for some non-rational $X(z)$.

**Ex:** $X(z) = \log(1 + z^{-1})$, ROC:$|z| > 1$. Find $x[n]$.

## 7.4 Properties of Z transform

To discuss the z transform properties, we use the same convenient shorthand notation we used for Fourier transform properties. In other words, to indicate the pairing of a signal and its z transform, we use :

- signal $\longleftrightarrow$ z transform ( e.g. $x[n] \longleftrightarrow X(z)$ )

We sometimes also refer to a z transform with the following notation :

- $\mathscr{Z}\{\text{signal}\}$ ( e.g. $X(z) = \mathscr{Z}\{x[n]\},\ ROC : |z| > a$)

For the following properties, let us assume that we have two signals $x[n]$ and $y[n]$ with corresponding z transforms $X(z)$ and $Y(z)$, and ROCs $R_x$ and $R_y$, respectively, i.e.

$$x[n] \longleftrightarrow X(z),\ ROC : R_x \qquad \text{and} \qquad y[n] \longleftrightarrow Y(z),\ ROC : R_y.$$

### 7.4.1 Linearity

$ax[n] + by[n] \longleftrightarrow aX(z) + bY(z), \quad ROC : R_x \cap R_y$

### 7.4.2 Time Shift

$x[n-n_0] \longleftrightarrow z^{-n_0}X(z),\ ROC: R_x$ except for possible inclusion or deletion of origin or infinity.

**Ex:** $x_1[n] = -\delta[n+1] + \delta[n]$ and $x_2[n] = x_1[n-1]$. Find the z transforms and associated ROCs.

### 7.4.3 Scaling in z domain (Frequency shifting)

$z_0^n x[n] \longleftrightarrow X(\frac{z}{z_0}),\ ROC: |z_0|R_x = \{z : \frac{z}{z_0} \in R_x\}$

Important special case for $z_0 = 1 \cdot e^{j\omega_0 n}$:
$e^{j\omega_0 n}x[n] \longleftrightarrow$

**Ex:** $R_x:\ \frac{1}{2} < |z| < 5$ and $|z_0| = 3$. Find $|z_0|R_x$.

### 7.4.4 Time Reversal

$x[-n] \longleftrightarrow X(\frac{1}{z}),\ ROC: \frac{1}{R_x} = \{z : \frac{1}{z} \in R_x\}$

**Ex:** $R_x:\ \frac{1}{2} < |z| < 5$. Find $\frac{1}{R_x}$.

### 7.4.5 Conjugation

$x^*[n] \longleftrightarrow X^*(z^*),\ ROC: R_x$

If $x[n]$ is real, (i.e. $x[n] = x^*[n]$)

- $X(z) = X^*(z^*)$
- thus, if $X(z)$ has a pole (zero) at $z = z_0$, then it must also have a pole (zero) at the complex conjugate point $z = z_0^*$.

**Ex:** Consider $X(z) = A(z)\dfrac{z-b}{z-a}$. Find $X^*(z^*)$ and its poles. Show that if $X(z) = X^*(z^*)$, then the poles and zeros must appear in complex conjugate pairs.

### 7.4.6 Convolution

$x[n] * y[n] \longleftrightarrow X(z) \cdot Y(z)$, $ROC:$ at least $R_x \cap R_y$ (pole zero cancellations may occur)

**Ex:** Consider $x_1[n]$ and $x_2[n]$ plotted below and find z transform of $x_1[n] * x_2[n]$.

### 7.4.7 Differentiation

$nx[n] \longleftrightarrow -z\dfrac{d}{dz}X(z)$, $ROC : R_x$with possible inclusion of$z = 0$.

**Ex:** $X(z) = \dfrac{az^{-1}}{(1-az^{-1})^2}$, $|z| > a$. Determine $x[n]$.

### 7.4.8 The initial value theorem

$x[0] = \lim_{z\to\infty} X(z)$ if $x[n]$ is causal, i.e. $x[n] = 0$ for $n < 0$.

### 7.4.9 Table of Z transform properties and some common z transform pairs

The following tables from the textbook summarize z transform properties and common pairs.

**TABLE 10.1** PROPERTIES OF THE $z$-TRANSFORM

| Section | Property | Signal | $z$-Transform | ROC |
|---|---|---|---|---|
| | | $x[n]$<br>$x_1[n]$<br>$x_2[n]$ | $X(z)$<br>$X_1(z)$<br>$X_2(z)$ | $R$<br>$R_1$<br>$R_2$ |
| 10.5.1 | Linearity | $ax_1[n] + bx_2[n]$ | $aX_1(z) + bX_2(z)$ | At least the intersection of $R_1$ and $R_2$ |
| 10.5.2 | Time shifting | $x[n-n_0]$ | $z^{-n_0}X(z)$ | $R$, except for the possible addition or deletion of the origin |
| 10.5.3 | Scaling in the $z$-domain | $e^{j\omega_0 n}x[n]$ | $X(e^{-j\omega_0}z)$ | $R$ |
| | | $z_0^n x[n]$ | $X\left(\frac{z}{z_0}\right)$ | $z_0 R$ |
| | | $a^n x[n]$ | $X(a^{-1}z)$ | Scaled version of $R$ (i.e., $\lvert a\rvert R$ = the set of points $\{\lvert a\rvert z\}$ for $z$ in $R$) |
| 10.5.4 | Time reversal | $x[-n]$ | $X(z^{-1})$ | Inverted $R$ (i.e., $R^{-1}$ = the set of points $z^{-1}$, where $z$ is in $R$) |
| 10.5.5 | Time expansion | $x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer $r$ | $X(z^k)$ | $R^{1/k}$ (i.e., the set of points $z^{1/k}$, where $z$ is in $R$) |
| 10.5.6 | Conjugation | $x^*[n]$ | $X^*(z^*)$ | $R$ |
| 10.5.7 | Convolution | $x_1[n] * x_2[n]$ | $X_1(z)X_2(z)$ | At least the intersection of $R_1$ and $R_2$ |
| 10.5.7 | First difference | $x[n] - x[n-1]$ | $(1-z^{-1})X(z)$ | At least the intersection of $R$ and $\lvert z\rvert > 0$ |
| 10.5.7 | Accumulation | $\sum_{k=-\infty}^{n} x[k]$ | $\frac{1}{1-z^{-1}}X(z)$ | At least the intersection of $R$ and $\lvert z\rvert > 1$ |
| 10.5.8 | Differentiation in the $z$-domain | $nx[n]$ | $-z\frac{dX(z)}{dz}$ | $R$ |
| 10.5.9 | | Initial Value Theorem<br>If $x[n] = 0$ for $n < 0$, then<br>$x[0] = \lim_{z\to\infty} X(z)$ | | |

**TABLE 10.2** SOME COMMON $z$-TRANSFORM PAIRS

| Signal | Transform | ROC |
|---|---|---|
| 1. $\delta[n]$ | $1$ | All $z$ |
| 2. $u[n]$ | $\frac{1}{1-z^{-1}}$ | $\lvert z\rvert > 1$ |
| 3. $-u[-n-1]$ | $\frac{1}{1-z^{-1}}$ | $\lvert z\rvert < 1$ |
| 4. $\delta[n-m]$ | $z^{-m}$ | All $z$, except 0 (if $m > 0$) or $\infty$ (if $m < 0$) |
| 5. $\alpha^n u[n]$ | $\frac{1}{1-\alpha z^{-1}}$ | $\lvert z\rvert > \lvert\alpha\rvert$ |
| 6. $-\alpha^n u[-n-1]$ | $\frac{1}{1-\alpha z^{-1}}$ | $\lvert z\rvert < \lvert\alpha\rvert$ |
| 7. $n\alpha^n u[n]$ | $\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$ | $\lvert z\rvert > \lvert\alpha\rvert$ |
| 8. $-n\alpha^n u[-n-1]$ | $\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$ | $\lvert z\rvert < \lvert\alpha\rvert$ |
| 9. $[\cos\omega_0 n]u[n]$ | $\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$ | $\lvert z\rvert > 1$ |
| 10. $[\sin\omega_0 n]u[n]$ | $\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$ | $\lvert z\rvert > 1$ |
| 11. $[r^n\cos\omega_0 n]u[n]$ | $\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$ | $\lvert z\rvert > r$ |
| 12. $[r^n\sin\omega_0 n]u[n]$ | $\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$ | $\lvert z\rvert > r$ |

## 7.5 LTI Systems and the Z transform

The z transform plays an important role in the analysis and representation of DT LTI systems :

- From the convolution property, we have : $Y(z) = H(z) \cdot X(z)$

- $\mathscr{Z}\{h[n]\} = H(z)$ is called the **system function** or **transfer function** of the LTI system.
- If ROC of $H(z)$ includes the unit circle, then for $z$ on the unit circle (i.e. $|z| = r = 1$), $H(z)$ reduces to frequency response $H(\Omega) = \mathscr{F}\{h[n]\}$ of the system,
  - i.e. $|z| = 1 \ \in \text{ROC} \longrightarrow H(z)|_{z=1\cdot e^{j\Omega}} = H(\Omega)$
- DT complex exponentials $z^n$ are eigenfunctions of DT LTI systems and the eigenvalues are given by the system function $H(z)$.

Many properties of and LTI system can be tied directly to characteristics of the poles, zeros and ROC of $H(z)$, as we discus below.

### 7.5.1 Causality

**Property 1 :** A DT LTI system is causal **if and only if** the ROC of $H(z)$ is the outside of a circle, including infinity.

Reason :

- Impulse response $h[n]$ of a causal system is right-sided, which implies that ROC of $H(z)$ must be ...
- $H(z) = \sum_{n=0}^{\infty} h[n]z^{-n} = h[0] + h[1]z^{-1} + h[2]z^{-2} + ...$ implies that ROC includes ...

**Property 2 :** A DT LTI system with rational $H(z)$ is causal **if and only if**

- the ROC of $H(z)$ is the outside of the outermost pole
- order$(a(z))\leq$ order$(b(z))$ where $H(z) = \dfrac{a(z)}{b(z)}$.

Reason :

- Property 8 of ROC properties
- If order$(a(z))>$ order$(b(z))$, consider for example $H(z) = \dfrac{2z^2+1}{z-1}$ =...

### 7.5.2 Stability

<u>**Property 1 :**</u> A DT LTI system is <u>stable</u> **if and only if** the ROC of $H(z)$ includes the unit circle, $|z| = 1$.

Reason :

- A DT LTI system is stable $\Longleftrightarrow$
- $\sum_{n=-\infty}^{\infty} |h[n]| < \infty \Longleftrightarrow \mathscr{F}\{h[n]\} = H(\Omega)$ converges ($1^{st}$ Dirichlet condition)
- $H(\Omega) = H(z)|_{z=1\cdot e^{j\Omega}}$

<u>**Property 2 :**</u> A <u>causal</u> DT LTI system with <u>rational $H(z)$</u> is <u>stable</u> **if and only if** all poles of $H(z)$ lie inside the unit circle (i.e. all poles have magnitudes smaller than 1.)

Reason :

- For a causal system with rational $H(z)$, ROC is ...
- For this ROC to include unit circle, ...

**Ex:** An LTI system satisfies the following difference equation :

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1].$$

Find the system function $H(z)$. Find all possible ROCs for this $H(z)$. For each possible ROC, find the corresponding $h[n]$.

**Ex:** Consider a stable and causal system with impulse response $h[n]$ and rational system function $H(z)$. It is known that $H(z)$ contains a pole at $z = \frac{1}{2}$ and a zero somewhere on the unit circle. The precise number and locations of other poles and zeros are unknown. Find if each of the following is *True, False or cannot be surely determined.*

- $\mathscr{F}\{(\frac{1}{2})^n h[n]\}$ converges
- $H(\Omega) = 0$ for some $\Omega$
- $h[n]$ has finite duration
- $h[n]$ is real
- $g[n] = n(h[n] * h[n])$ is the impulse response of a stable system.

# Chapter 8

# The Laplace transform

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This chapter discusses the Laplace transform, which plays a role for CT signals that is similar to the role of z transform for DT signals.

## 8.1 The Laplace transform and its region of convergence (ROC)

Remember that CT complex exponentials of the general form $e^{st} = e^{(\sigma+j\omega)t}$ are *eigenfunctions of CT LTI* systems with *eigenvalues* given by $H(s) = \int_{t=-\infty}^{\infty} h(t)e^{-st}dt$ where $h(t)$ is the impulse response of the system:

The eigenvalue expression provides the definition of the Laplace transform of a signal $x(t)$ where $s$ is a complex number :

$$\boxed{X(s) = \int_{t=-\infty}^{\infty} x(t)e^{-st}dt}$$

We again will use shorthand notations for the Laplace transform of a signal $x(t)$:

- $\mathscr{L}\{x(t)\}$
- $x(t) \longleftrightarrow X(s)$

The Laplace transform is a generalization of the Fourier transform and they are related as follows :

- $X(s)|_{s=\sigma+j\omega} = \mathscr{F}\{x(t)e^{-\sigma t}\}$

- $X(s)|_{s=j\omega} = \mathscr{F}\{x(t)\}$ (i.e. Laplace transform reduces to FT for values of $s$ on the imaginary axis of the s-plane)

The above relation indicates that the convergence/existence of Laplace transform, requires the convergence/existence $\mathscr{F}\{x(t)e^{-\sigma t}\}$,

- which happens when $x(t)e^{-\sigma t}$ is absolutely integrable, i.e. $\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| < \infty$
- and thus Laplace transform $X(s)$ converges/exists for some values of $\sigma$ and does not for others.

Region of Convergence (ROC) : A range of $\sigma$ (note that $\sigma = Re\{s\}$) values for which Laplace transform $X(s)$ converges.

**Ex:** $x(t) = e^{-\alpha t}u(t)$. Find $X(s)$ and its ROC.

Note that the above relation between Laplace transform and Fourier transform are valid if $Re\{s\} = \sigma$ is contained in the ROC of $X(s)$ :

In particular, if the ROC of $X(s)$ contains the $j\omega$-axis ($Re\{s\} = 0$), then the CT FT $X(\omega)$ exists and is given by $X(s)|_{s=j\omega}$ :

**Ex:** Previous example with $\alpha = 2$ and $\alpha = -2$.

**Ex:** $x(t) = -e^{-\alpha t}u(-t)$. Find $X(s)$ and its ROC.

**Notes :**

- It is helpful to remember the following frequently used signal and Laplace transform pairs :

$$e^{-\alpha t}u(t) \longleftrightarrow \frac{1}{s+\alpha}, \quad ROC : Re\{s\} > \alpha$$
$$-e^{-\alpha t}u(-t) \longleftrightarrow \frac{1}{s+\alpha}, \quad ROC : Re\{s\} < \alpha$$

- The specification of the Laplace transform requires both
  - the algebraic expression for $X(s)$
  - and the associated ROC.

  (Without the ROC, $X(s)$ by itself <u>does not uniquely</u> specify a signal $x(t)$.)

**Ex:** $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$. Find $X(s)$ and its ROC.

**Ex:** $x(t) = e^{-2t}u(t) + e^{-t}\cos(3t)u(t)$. Find $X(s)$ and its ROC.

## 8.2 Properties of ROC

**Property 1 :** The **ROC** of $X(s)$ depends only on $Re\{s\}$ and therefore consists of **strips** parallel to the $j\omega$-axis in the s-plane.

Reason :

- The relation $X(s)|_{s=\sigma+j\omega} = \mathscr{F}\{x(t)e^{-\sigma t}\}$ indicates that the convergence/existence of Laplace transform $X(s)$, requires the convergence/existence $\mathscr{F}\{x(t)e^{-\sigma t}\}$, which happens when $x(t)e^{-\sigma t}$ is absolutely integrable, i.e. $\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}|dt = \int_{-\infty}^{\infty} |x(t)|e^{-\sigma t}dt < \infty$.
- Thus, ROC of $X(s)$ depends only on $\sigma = Re\{s\}$ and is independent of $\omega$.

**Property 2 :** For rational $X(s)$, the **ROC** does not contain any poles.

Reason :

- Rational $X(s)$ means $X(s)$ is a ratio of polynomials of $s$.
- A pole of $X(s)$ is a root of the denominator and a zero of $X(s)$ is a root of the numerator.
- $X(s)$ is infinite at a pole and hence does not converge at a pole.

**Property 3 :** If $x(t)$ is of finite duration, then the ROC is either empty or the entire s-plane.

Reason :

- A finite duration signal $x(t)$ is nonzero only a finite interval, e.g.
- Let us assume that $X(s)$ converges for one $\sigma_0 = Re\{s\}$, then ...
- Then the ROC must also include any arbitrary vertical line at any $\sigma_1$, since
  - $\int_{T_1}^{T_2} |x(t)|e^{-\sigma_1 t}dt = ...$
  - and therefore ...

**Property 4 :** If $x(t)$ is a right-sided sequence, then the ROC is either empty or to the right of a vertical line.

Reason :

- A right-sided signal is such that $x(t) = 0$, $t < T_1$ e.g.

- Let us assume that $X(s)$ converges for one $\sigma_0 = Re\{s\}$, then ...
- Then the ROC must also include the vertical line at any $\sigma_1 > \sigma_0$, since
  - $\int_{T_1}^{\infty} |x(t)| e^{-\sigma_1 t} dt = ...$
  - and therefore ...

**Property 5 :** If $x(t)$ is a left-sided sequence, then the ROC is either empty or to the left of a vertical line.

Reason is similar to the previous property's.

**Property 6 :** If $x(t)$ is a two-sided signal, then the ROC is either empty or consists of a strip bounded from right and left.

Reason :

- A two-sided signal can be considered as the sum of ...
- The ROC of $X(s)$ of the sum of sequences is the intersection of the ROCs of the Laplace transforms of each sequence.

**Ex:** $x(t) = e^{-bt}$. Find $X(s)$ and the ROC if $b > 0$ and $b < 0$.

**Property 7 :** If $X(s)$ is rational, then its ROC is bounded by the poles or extends to infinity.

Reason :

- A signal $x(t)$ with rational $X(s)$ consist of a linear combination of exponentials $e^{-\alpha t}u(t)$ or $-e^{-\alpha t}u(-t)$ which have ROCs bounded by their poles.
- The ROC of $X(s)$ of the linear combination of exponentials thus is the intersection of ROCs bounded by poles.

**Property 8 :** If $X(s)$ is rational and $x(t)$ is right-sided, then its ROC is the right side of the right-most pole. If $X(s)$ is rational and $x(t)$ is left-sided, then its ROC is the left side of the left-most pole.

Reason : ...

## 8.3 Inversion of Laplace transforms

The inverse Laplace transform expression contains integration around a contour on the s-plane and is typically difficult to compute and will not be applied in this course.

However, one useful alternative procedure for obtaining a signal from its Laplace transform and associated ROC is to expand $X(s)$ into a **partial-fraction expansion**, and then to recognize the signal associated with each term in the expansion.

**Ex:** $X(s) = \dfrac{2}{s^2 - 4}$, ROC:$Re\{s\} > 2$. Find $x(t)$.

**Ex:** $X(s) = \dfrac{1}{(s+1)(s+2)}$. Find all possible ROCs for this $X(s)$. For each possible ROC, find the corresponding $x(t)$.

## 8.4 Properties of Laplace transform

To discuss the Laplace transform properties, we use the same convenient shorthand notation we used for Fourier transform properties. In other words, to indicate the pairing of a signal and its Laplace transform, we use :

- signal $\longleftrightarrow$ Laplace transform ( e.g. $x(t) \longleftrightarrow X(s)$ )

We sometimes also refer to a Laplace transform with the following notation :

- $\mathscr{L}\{\text{signal}\}$ ( e.g. $X(s) = \mathscr{L}\{x(t)\},\ ROC : Re\{s\} > a$)

For the following properties, let us assume that we have two signals $x(t)$ and $y(t)$ with corresponding Laplace transforms $X(s)$ and $Y(s)$, and ROCs $R_x$ and $R_x$, respectively, i.e.

$$x(t) \longleftrightarrow X(s),\ ROC : R_x \qquad \text{and} \qquad y(t) \longleftrightarrow Y(s),\ ROC : R_y.$$

### 8.4.1 Linearity

$ax(t) + by(t) \longleftrightarrow aX(s) + bY(s),\quad ROC : R_x \cap R_y$

### 8.4.2 Time Shift

$x(t - t_0) \longleftrightarrow e^{-st_0}X(s),\ ROC : R_x$

### 8.4.3 Frequency Shift

$e^{s_0 t}x(t) \longleftrightarrow X(s-s_0),\ ROC: Re\{s\}+R_x = \{s : s-s_0 \in R_x\}$

**Ex:** ROC of $X(s)$ is $Re\{s\} > 2$. Find $\mathscr{L}\{x(t-1)\}$ and the associated ROC.

### 8.4.4 Time Scaling

$x(at) \longleftrightarrow \dfrac{1}{|a|}X(\frac{s}{a}),\ ROC: \dfrac{R_x}{a} = \{s : \frac{s}{a} \in R_x\}$

**Ex:** ROC of $X(s)$ is $-3 < Re\{s\} < 2$. Find $\mathscr{L}\{x(2t)\}$ and the associated ROC.

### 8.4.5 Conjugation

$x^*(t) \longleftrightarrow X^*(s^*),\ ROC: R_x$

If $x(t)$ is real, (i.e. $x(t) = x^*(t)$)

- $X(s) = X^*(s^*)$
- thus, if $X(s)$ has a pole (zero) at $s = s_0$, then it must also have a pole (zero) at the complex conjugate point $s = s_0^*$.

### 8.4.6 Convolution

$x(t) * y(t) \longleftrightarrow X(s)\cdot Y(s),\ \ ROC:$ at least $R_x \cap R_y$ (pole zero cancellations may occur)

### 8.4.7 Differentiation in s domain

$-tx(t) \longleftrightarrow \dfrac{d}{ds}X(s),\ ROC: R_x$

**Ex:** $x(t) = te^{-at}u(t)$. Find $X(s)$ and ROC.

### 8.4.8 Differentiation in time domain

$\dfrac{d}{dt}x(t) \longleftrightarrow sX(s),\ ROC : R_x$with possible pole zero cancellation at$s = 0$.

### 8.4.9 Integration in time domain

$\int_{-\infty}^{t} x(\tau)d\tau \longleftrightarrow \dfrac{1}{s}X(s),\ ROC :$ At leatst $R_x \cap Re\{s\} > 0$.

### 8.4.10 Initial and Final Value Theorem

Under the conditions that $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher order singularities at the origin :

IVT : $x(0^+) = \lim_{s\to\infty} sX(s)$ FVT : $\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

IVT and FVT can be useful in checking correctness of Laplace transform calculations for a signal.

**Ex:** In a previous example we found that $X(s) = \dfrac{2s^2+5s+12}{(s^2+2s+10)(s+2)}$ for $x(t) = e^{-2t}u(t) + e^{-t}\cos(3t)u(t)$. We can check if the calculated $X(s)$ is correct by IVT :

### 8.4.11 Table of Laplace transform properties and common Laplace transform pairs

The following tables from the textbook summarize Laplace transform properties and common pairs.

**TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM**

| Section | Property | Signal | Laplace Transform | ROC |
|---|---|---|---|---|
| | | $x(t)$ | $X(s)$ | $R$ |
| | | $x_1(t)$ | $X_1(s)$ | $R_1$ |
| | | $x_2(t)$ | $X_2(s)$ | $R_2$ |
| 9.5.1 | Linearity | $ax_1(t) + bx_2(t)$ | $aX_1(s) + bX_2(s)$ | At least $R_1 \cap R_2$ |
| 9.5.2 | Time shifting | $x(t - t_0)$ | $e^{-st_0}X(s)$ | $R$ |
| 9.5.3 | Shifting in the $s$-Domain | $e^{s_0 t}x(t)$ | $X(s - s_0)$ | Shifted version of $R$ (i.e., $s$ is in the ROC if $s - s_0$ is in R) |
| 9.5.4 | Time scaling | $x(at)$ | $\frac{1}{\lvert a \rvert}X\left(\frac{s}{a}\right)$ | Scaled ROC (i.e., $s$ is in the ROC if $s/a$ is in $R$) |
| 9.5.5 | Conjugation | $x^*(t)$ | $X^*(s^*)$ | $R$ |
| 9.5.6 | Convolution | $x_1(t) * x_2(t)$ | $X_1(s)X_2(s)$ | At least $R_1 \cap R_2$ |
| 9.5.7 | Differentiation in the Time Domain | $\frac{d}{dt}x(t)$ | $sX(s)$ | At least $R$ |
| 9.5.8 | Differentiation in the $s$-Domain | $-tx(t)$ | $\frac{d}{ds}X(s)$ | $R$ |
| 9.5.9 | Integration in the Time Domain | $\int_{-\infty}^{t} x(\tau)d(\tau)$ | $\frac{1}{s}X(s)$ | At least $R \cap \{\Re e\{s\} > 0\}$ |

**Initial- and Final-Value Theorems**

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

**TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS**

| Transform pair | Signal | Transform | ROC |
|---|---|---|---|
| 1 | $\delta(t)$ | $1$ | All $s$ |
| 2 | $u(t)$ | $\frac{1}{s}$ | $\Re e\{s\} > 0$ |
| 3 | $-u(-t)$ | $\frac{1}{s}$ | $\Re e\{s\} < 0$ |
| 4 | $\frac{t^{n-1}}{(n-1)!}u(t)$ | $\frac{1}{s^n}$ | $\Re e\{s\} > 0$ |
| 5 | $-\frac{t^{n-1}}{(n-1)!}u(-t)$ | $\frac{1}{s^n}$ | $\Re e\{s\} < 0$ |
| 6 | $e^{-\alpha t}u(t)$ | $\frac{1}{s+\alpha}$ | $\Re e\{s\} > -\alpha$ |
| 7 | $-e^{-\alpha t}u(-t)$ | $\frac{1}{s+\alpha}$ | $\Re e\{s\} < -\alpha$ |
| 8 | $\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$ | $\frac{1}{(s+\alpha)^n}$ | $\Re e\{s\} > -\alpha$ |
| 9 | $-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$ | $\frac{1}{(s+\alpha)^n}$ | $\Re e\{s\} < -\alpha$ |
| 10 | $\delta(t - T)$ | $e^{-sT}$ | All $s$ |
| 11 | $[\cos \omega_0 t]u(t)$ | $\frac{s}{s^2 + \omega_0^2}$ | $\Re e\{s\} > 0$ |
| 12 | $[\sin \omega_0 t]u(t)$ | $\frac{\omega_0}{s^2 + \omega_0^2}$ | $\Re e\{s\} > 0$ |
| 13 | $[e^{-\alpha t}\cos \omega_0 t]u(t)$ | $\frac{s+\alpha}{(s+\alpha)^2 + \omega_0^2}$ | $\Re e\{s\} > -\alpha$ |
| 14 | $[e^{-\alpha t}\sin \omega_0 t]u(t)$ | $\frac{\omega_0}{(s+\alpha)^2 + \omega_0^2}$ | $\Re e\{s\} > -\alpha$ |
| 15 | $u_n(t) = \frac{d^n\delta(t)}{dt^n}$ | $s^n$ | All $s$ |
| 16 | $u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$ | $\frac{1}{s^n}$ | $\Re e\{s\} > 0$ |

## 8.5 LTI Systems and the Laplace transform

The Laplace transform plays an important role in the analysis and representation of CT LTI systems:

- From the convolution property, we have : $Y(s) = H(s) \cdot X(s)$

- $\mathscr{L}\{h(t)\} = H(s)$ is called the **system function** or **transfer function** of the LTI system.
- If ROC of $H(s)$ includes the $j\omega$-axis (i.e. $Re\{s\} = 0$), $H(s)$ reduces to frequency response $H(\omega) = \mathscr{F}\{h(t)\}$ of the system,
  - i.e. $Re\{s\} = 0 \ \in$ ROC $\longrightarrow H(s)|_{s=0+j\omega} = H(\omega)$
- CT complex exponentials $e^{st}$ are eigenfunctions of CT LTI systems and the eigenvalues are given by the system function $H(s)$.

Many properties of and LTI system can be tied directly to characteristics of the poles, zeros and ROC of $H(s)$, as we discus below.

### 8.5.1 Causality

**Property 1 :** The ROC associated with $H(s)$ of a causal CT LTI system is a right-half plane. (The converse is not necessarily true!)

Reason :

- Impulse response $h(t)$ of a causal system is right-sided, which implies that ROC of $H(s)$ must be ...

**Ex:** Consider causal LTI system with $h(t) = e^{-t}u(t)$. Find $H(s)$ and the associated ROC.
Next, consider $H(s) = \dfrac{e^s}{s+1}$, with ROC:$Re\{s\} > -1$. Find $h(t)$.

**Property 2 :** A CT LTI system with rational $H(s)$ is causal **if and only if** the ROC of $H(s)$ is the right side of the right most pole.

### 8.5.2 Stability

**Property 1 :** A CT LTI system is stable **if and only if** the ROC of $H(s)$ includes the entire $j\omega$-axis (i.e. $Re\{s\} = 0$ line).

Reason :

- A CT LTI system is stable $\Longleftrightarrow$
- $\int_{t=-\infty}^{\infty} |h(t)| < \infty \Longleftrightarrow \mathscr{F}\{h(t)\} = H(\omega)$ converges ($1^{st}$ Dirichlet condition)
- $H(\omega) = H(s)|_{s=0+j\omega}$

**Property 2 :** A causal CT LTI system with rational $H(s)$ is stable **if and only if** all poles of $H(s)$ lie on the left half of the s-plane (i.e. all poles have negative real parts $Re\{s\} < 0$.)

Reason :

- For a causal system with rational $H(s)$, ROC is ...
- For this ROC to include the $j\omega$-axis, ...

**Ex:** For LTI system with $H(s) = \dfrac{s-1}{(s+1)(s-2-j)(s-2+j)}$, determine all possible ROCs. For each possible ROC, discuss if the system causal and/or stable.

**Ex:** For an input $x(t) = e^{3t}u(t)$, the output of an LTI system is $y(t) = [e^{-t} - e^{-2t}]u(t)$. Determine system function $H(s)$.

- Can you also find the ROC of $H(s)$ and then $h(t)$ ?
- Is the system causal and/or stable ?

- Find the differential equation that can represent this LTI system.

**Ex:** Consider a stable and causal system with impulse response $h(t)$ and rational system function $H(s)$. It is known that $H(s)$ contains a pole at $s = -2$ and does not have a zero at the origin. The precise number and locations of other poles and zeros are unknown. Find if each of the following is *True, False or cannot be surely determined.*

- $\mathscr{F}\{e^{3t}h(t)\}$ converges
- $\int_{-\infty}^{\infty} h(t)dt = 0$
- $th(t)$ is the impulse response of a causal and stable system
- $\frac{d}{dt}h(t)$ contains at least one pole in its Laplace transform
- $h(t)$ has finite duration
- $H(s) = H(-s)$
- $\lim_{s\to\infty} H(s) = 2.$