MECHANICAL VIBRATIONS:

LECTURE NOTES FOR COURSE EML 4220

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Anil V. Rao earned his B.S. in mechanical engineering and A.B. in mathematics from Cornell University, his M.S.E. in aerospace engineering from the University of Michigan, and his M.A. and Ph.D. in mechanical and aerospace engineering from Princeton University. After earning his Ph.D., Dr. Rao joined the Flight Mechanics Department at The Aerospace Corporation in Los Angeles, where he was involved in mission support for U.S. Air Force launch vehicle programs and trajectory optimization software development. Subsequently, Dr. Rao joined The Charles Stark Draper Laboratory, Inc., in Cambridge, Massachusetts. As a Draper employee, Dr. Rao led numerous projects related to trajectory optimization, guidance, and navigation of both space flight and atmospheric flight vehicles. Concurrently, from 2001 to 2006 Dr. Rao was an Adjunct Professor of Aerospace and Mechanical Engineering at Boston University where he taught the core undergraduate engineering dynamics course. While at Boston University, Dr. Rao was voted the 2002 and 2006 Mechanical and Aerospace Engineering Faculty Member of the Year and was voted 2004 College of Engineering Professor of the Year for outstanding teaching.

Vakratunda Mahaakaaya Soorya Koti Samaprabha
Nirvighnam Kuru Mein Deva Sarva Kaaryashu Sarvadaa

Contents

Chapter 1

Response of Single Degree-of-Freedom Systems to Initial Conditions

In this chapter we begin the study of vibrations of mechanical systems. Generally speaking a vibration is a periodic or oscillatory motion of an object or a set of objects. Vibrating systems are ubiquitous in engineering and thus the study of vibrations is extremely important.

The most basic problem of interest is the study of the vibration of a one degree-of-freedom (i.e., a system whose motion can be described using a single scalar second-order ordinary differential equation). The generic model for a one degree-of-freedom system is a mass connected to a linear spring and a linear viscous damper (i.e., a mass-spring-damper system). Because of its mathematical form, the mass-spring-damper system will be used as the baseline for analysis of a one degree-of-freedom system. In particular, the differential equation of motion will be derived for the mass-spring-damper system. It will then be shown that the time response of this system is the sum of the *zero input response* and the *zero initial condition response*. In this chapter we will focus attention on the zero input response, i.e., the response of the system to a given set of initial conditions. Several examples of single degree-of-freedom systems will then be given. In each of these examples the differential equation will be derived and will be shown to have the same mathematical form as the generic mass-spring-damper system.

1.1 Mass-Spring-Damper System

The most basic system that is used as a model for vibrational analysis is a block of mass m connected to a linear spring (with spring constant K and unstretched length ℓ_0) and a viscous damper (with damping coefficient c). In addition, an external force $\mathbf{P}(t)$ is applied to the block and the displacement of the block is measured from the inertially fixed point O, where O is the point where the spring is unstretched. Finally, the spring and damper are both attached at the inertially fixed point Q. This system is shown in Fig. 1–1 Denoting unit vector in the direction from O to Q as $\mathbf{E}_x$ and the inertial reference frame of the ground by $\mathcal{F}$, the inertial acceleration of the block is given as

$$ {}^{\mathcal{F}}\mathbf{a} = \ddot{x}\mathbf{E}_x \tag{1–1} $$

Next, the forces exerted by the spring and damper are given, respectively, as

$$ \mathbf{F}_s = -K(\ell - \ell_0)\mathbf{u}_s \tag{1–2} $$

$$ \mathbf{F}_f = -c\mathbf{v}_{\text{rel}} \tag{1–3} $$

First, because the spring is attached at point Q, we have

$$ \ell = \|\mathbf{r} - \mathbf{r}_Q\| \tag{1–4} $$

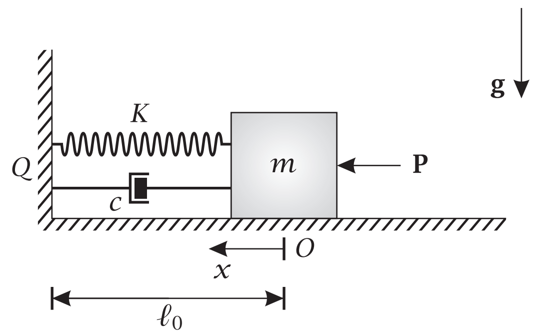

Figure 1–1 Block of mass m sliding without friction along a horizontal surface connected to a linear spring and a linear viscous damper.

where $\mathbf{r}$ and $\mathbf{r}_Q$ are the positions of the block and the attachment points of the spring, respectively. Using a coordinate system with its origin at point O at $\mathbf{E}_x$ as the first principal direction, we have

$$\mathbf{r} = x\mathbf{E}_x \quad (1\text{–}5)$$

$$\mathbf{r}_Q = \ell_0\mathbf{E}_x \quad (1\text{–}6)$$

Therefore,

$$\ell = \|x\mathbf{E}_x - \ell_0\mathbf{E}_x\| = \|(x-\ell_0)\mathbf{E}_x\| = |x-\ell_0| \quad (1\text{–}7)$$

Then, because $x < \ell_0$ we have

$$|x-\ell_0| = \ell_0 - x \quad (1\text{–}8)$$

Finally, the unit vector in the direction from the attachment point of the spring to the position of the block is

$$\mathbf{u}_s = \frac{\mathbf{r}-\mathbf{r}_Q}{\|\mathbf{r}-\mathbf{r}_Q\|} = \frac{(x-\ell_0)\mathbf{E}_x}{\ell_0 - x} = -\mathbf{E}_x \quad (1\text{–}9)$$

The force in the linear spring is then given as

$$\mathbf{F}_s = -K(\ell_0 - x - \ell_0)(-\mathbf{E}_x) = -Kx\mathbf{E}_x \quad (1\text{–}10)$$

Next, because the ground is already assumed to be inertial, the relative velocity between the block and the ground is simply the velocity of the block, i.e.,

$$\mathbf{v}_{\text{rel}} = {}^{\mathcal{F}}\mathbf{v} = \dot{x}\mathbf{E}_x \quad (1\text{–}11)$$

Therefore, the force exerted by the viscous damper is obtained as

$$\mathbf{F}_f = -c\dot{x}\mathbf{E}_x \quad (1\text{–}12)$$

The resultant external force acting on the particle is then obtained as

$$\mathbf{F} = \mathbf{P} + \mathbf{F}_s + \mathbf{F}_f = P\mathbf{E}_x - Kx\mathbf{E}_x - c\dot{x}\mathbf{E}_x = (P - Kx - c\dot{x})\mathbf{E}_x \quad (1\text{–}13)$$

Applying Newton's second law to the particle, we obtain

$$(P - Kx - c\dot{x})\mathbf{E}_x = m\ddot{x}\mathbf{E}_x \quad (1\text{–}14)$$

Dropping $\mathbf{E}_x$ from Eq. (1–14) and rearranging, we obtain the differential equation of motion as

$$m\ddot{x} + c\dot{x} + Kx = P \quad (1\text{–}15)$$

Now historically it has been the case that the differential equation has been written in a form that is normalized by the mass, i.e., we divide Eq. (1–15) by m to obtain

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{K}{m}x = \frac{P}{m} = p(t) \tag{1–16}$$

where $p(t) = P(t)/m$. Furthermore, it is common practice to define the quantities K/m and c/m as follows:

$$\begin{aligned} \omega_n^2 &= \frac{K}{m} \\ 2\zeta\omega_n &= \frac{c}{m} \end{aligned}$$

The quantities ω_n and ζ are called the *natural frequency* and *damping ratio* of the system, respectively. In terms of the natural frequency and damping ratio, the differential equation of motion for the mass-spring-damper system can be written in the so called *standard form* as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = p(t) \tag{1–17}$$

It is seen that Eq. (1–17) is a second-order linear constant coefficient ordinary differential equation. Often, the term “constant coefficient” is replaced with the term *time-invariant*, i.e., we say that Eq. (1–17) is a called a second-order *linear time-invariant* (LTI) ordinary differential equation. The terminology “time invariant” stems from the fact that, for a given input $p(t)$ and a given set of initial conditions $(x(t_0), \dot{x}(t_0) = (x_0, \dot{x}_0)$ at the initial time $t = t_0$ is the same as the solution to the input $p(t+\tau)$ for the initial conditions $(x(t_0+\tau), \dot{x}(t_0+\tau) = (x_0, \dot{x}_0)$ at the (shifted) initial time $t = t_0 + \tau$. Because of this fact associated with an LTI system, without loss of generality we can assume that the initial time is zero, i.e., $t_0 = 0$. Thus, when studying the zero input response of an LTI system we can restrict our attention to initial conditions $(x(0), \dot{x}(0) = (x_0, \dot{x}_0)$.

1.2 General Solution of a Second-Order LTI Differential Equation

Eq. (1–17) can be written as

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2 x = p(t) \tag{1–18}$$

which can be further written as

$$\left(\frac{d^2}{dt^2} + 2\zeta\omega_n\frac{d}{dt} + \omega_n^2\right)x = p(t) \tag{1–19}$$

Now let

$$L = \frac{d^2}{dt^2} + 2\zeta\omega_n\frac{d}{dt} + \omega_n^2 \tag{1–20}$$

Then we can view the system of Eq. (1–17) as a system of the form

$$Lx = f \tag{1–21}$$

It is seen that the operator L defined in Eq. (1–20) is *linear* because

$$L(\alpha x_1 + \beta x_2) = \alpha L(x_1) + \beta L(x_2) \tag{1–22}$$

for all constants α and β. Then it is seen that Eq. (1–21) is a linear system whose general solution is of then form Eq. (1–17) is given as

$$x(t) = x_h(t) + x_p(t) \tag{1–23}$$

here $x_h(t)$ is the *homogeneous solution* (i.e., the solution for a particular set of initial conditions $(x(t_0), \dot{x}(t_0)) = (x_0, \dot{x}_0)$ with a *zero* input function $p(t) \equiv 0$) while $x_p(t)$ is the *particular solution* (i.e., the solution for *zero* initial conditions $(x(t_0), \dot{x}(t_0) = (0, 0)$ and an *arbitrary* input function $p(t) \neq 0$). The homogeneous solution and particular solutions are also called the *zero input response* and *zero initial condition response*, respectively. The general solution $x(t)$ to a second-order LTI system is then given as the *sum* of the zero input response and the zero initial condition response. Because the zero input response satisfies Eq. (1–17) when $p(t) \equiv 0$, we have

$$\ddot{x}_h + 2\zeta\omega_n\dot{x}_h + \omega_n^2 x_h = 0 \tag{1–24}$$

Contrariwise, because the zero initial condition response satisfies Eq. (1–17) when $p(t) \neq 0$ and the initial conditions are zero, we have

$$\ddot{x}_p + 2\zeta\omega_n\dot{x}_p + \omega_n^2 x_p = p(t) \tag{1–25}$$

From the preceding discussion, it is seen that studying the general response of a second-order LTI system amounts to studying *independently* the zero input response and the zero initial condition response. Consequently, the study of single degree-of-freedom vibrations amounts to quantifying the zero input response and the zero initial condition response. In this remainder of this chapter we study in detail the zero input response of a second-order LTI system that arises in the study of mechanical vibrations.

1.3 General Solution to Second-Order Homogeneous LTI System

We now focus on the zero input response of the second-order LTI system of Eq. (1–17), i.e., we focus on the system

$$\ddot{x}_h + 2\zeta\omega_n\dot{x}_h + \omega_n^2 x_h = 0 \tag{1–26}$$

Suppose that we guess the solution to Eq. (1–26) as

$$x_h(t) = e^{\lambda t} \tag{1–27}$$

where λ is constant that has yet to be determined. Differentiating the assumed solution of Eq. (1–27) twice, we have

$$\dot{x}_h(t) = \lambda e^{\lambda t} \tag{1–28}$$

$$\ddot{x}_h(t) = \lambda^2 e^{\lambda t} \tag{1–29}$$

Substituting the results of Eqs. (1–28) and (1–29) into (1–26), we obtain

$$\lambda^2 e^{\lambda t} + 2\zeta\omega_n\lambda e^{\lambda t} + \omega_n^2 e^{\lambda t} = 0 \tag{1–30}$$

Then, because $e^{\lambda t}$ is not zero as a function of time, it can be dropped from Eq. (1–30) to give

$$\lambda^2 + 2\zeta\omega_n + \omega_n^2 = 0 \tag{1–31}$$

Equation (1–31) is called the *characteristic equation* whose roots give the behavior of the zero input response of Eq. (1–17). Using the quadratic formula, the roots of Eq. (1–31) are given as

$$\lambda_{1,2} = -\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{1–32}$$

It can be seen that the types of roots admitted by Eq. (1–31) depend upon the value of ζ. In particular, the types of roots are governed by the quantity $\zeta^2 - 1$. We have three cases to consider: (1) $0 \leq \zeta < 1$, (2) $\zeta = 1$, and (3) $\zeta > 1$. We now consider each of these cases in turn.

Case 1: $0 \le \zeta < 1$ **(Underdamping)**

When $0 \le \zeta < 1$ the zero input response is said to be *underdamped*. For an underdamped system the quantity $\zeta^2 - 1 < 0$ which implies that $\sqrt{\zeta^2 - 1} = i\sqrt{1-\zeta^2}$. The roots of the characteristic equation for an underdamped system are then given as

$$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2} \tag{1–33}$$

It is seen from Eq. (1–33) that the roots of the characteristic equation for an underdamped system are *complex*. Furthermore, the general zero input response for an underdamped system is given as

$$x_h(t) = e^{-\zeta\omega_n t}\left[c_1\cos\left(\omega_n\sqrt{1-\zeta^2}t\right) + c_2\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right] \tag{1–34}$$

Eq. (1–34) can be written as

$$x_h(t) = e^{-\zeta\omega_n t}\left(c_1\cos\omega_d t + c_2\sin\omega_d t\right) \tag{1–35}$$

where the quantity $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is called the *damped natural frequency* of the system. The constants c_1 and c_2 can be solved for by using the initial conditions $(x(0), \dot{x}(0)) = (x_0, \dot{x}_0)$ as follows. First, substituting the initial condition $x(0) = x_0$ into Eq. (1–35), we obtain c_1 as

$$x_h(0) = x_0 = c_1 \tag{1–36}$$

Next, differentiating $x_h(t)$ in Eq. (1–35), we obtain

$$\begin{aligned}\dot{x}_h(t) &= -\zeta\omega_n e^{-\zeta\omega_n t}\left(c_1\cos\omega_d t + c_2\sin\omega_d t\right)\\ &\quad + e^{-\zeta\omega_n t}\left(-c_1\omega_d\sin\omega_d t + c_2\omega_d\cos\omega_d t\right)\end{aligned} \tag{1–37}$$

Applying the initial condition $\dot{x}(0) = \dot{x}_0$, we obtain

$$\dot{x}_h(0) = \dot{x}_0 = -\zeta\omega_n c_1 + \omega_d c_2 \tag{1–38}$$

Substituting the result for c_1 from Eq. (1–36) into Eq. (1–38), we obtain

$$\dot{x}_0 = -x_0\zeta\omega_n + \omega_d c_2 \tag{1–39}$$

Solving for c_2 we have

$$c_2 = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d} \tag{1–40}$$

The zero input response for an underdamped system is then given as

$$x_h(t) = e^{-\zeta\omega_n t}\left(x_0\cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right) \tag{1–41}$$

A schematic of the underdamped zero input response for various values of $0 \le \zeta <$ is shown in Fig. 1–2.

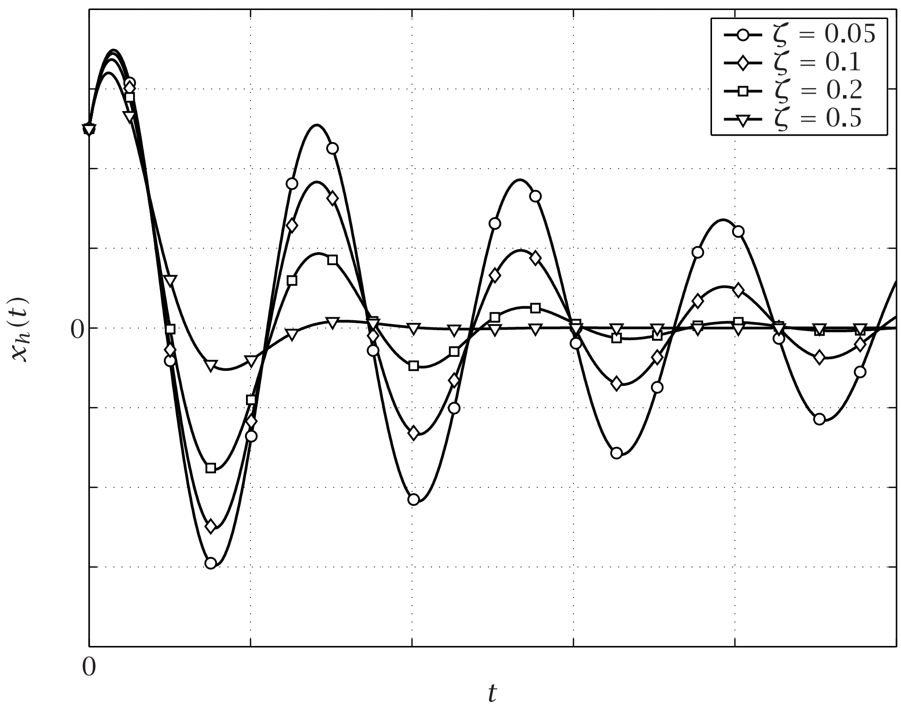

Figure 1–2 Schematic of the zero input response of an underdamped second-order linear time-invariant system.

Case 2: $\zeta = 1$ (Critical Damping)

When $\zeta = 1$ the zero input response is said to be *critically damped*. For critically damped system the quantity $\zeta^2 - 1 = 0$ which implies that $\sqrt{\zeta^2 - 1} = 0$. The roots of the characteristic equation for an underdamped system are then given as

$$\lambda_{1,2} = -\zeta\omega_n = -\omega_n \tag{1–42}$$

It is seen from Eq. (1–42) that the roots of the characteristic equation for a critically damped system are *real* and *repeated* (i.e., the two roots are the same). Furthermore, the general zero input response for a critically damped system is given as

$$x_h(t) = e^{-\omega_n t}(c_1 + c_2 t) \tag{1–43}$$

The constants c_1 and c_2 can be solved for by using the initial conditions $(x(0), \dot{x}(0)) = (x_0, \dot{x}_0)$ as follows. First, applying the initial condition $x(0) = x_0$ into Eq. (1–43), we have

$$x_h(0) = x_0 = c_1 \tag{1–44}$$

Next, differentiating Eq. (1–43), we obtain

$$\dot{x}_h(t) = -\omega_n e^{-\omega_n t}(c_1 + c_2 t) + c_2 e^{-\omega_n t} \tag{1–45}$$

Applying the initial condition $\dot{x}(0) = \dot{x}_0$, we obtain

$$\dot{x}_h(0) = \dot{x}_0 = -\omega_n c_1 + c_2 \tag{1–46}$$

Substituting the result for c_1 from Eq. (1–44), we have

$$\dot{x}_h(0) = \dot{x}_0 = -\omega_n x_0 + c_2 \tag{1–47}$$

Solving Eq. (1–47) for c_2 gives

$$c_2 = \dot{x}_0 + \omega_n x_0 \tag{1–48}$$

The zero input response for an critically damped system is then given as

$$x_h(t) = e^{-\omega_n t}\left[x_0 + (\dot{x}_0 + \omega_n x_0)t\right] \tag{1–49}$$

A schematic of a critically damped zero input response is shown in Fig. 1–3.

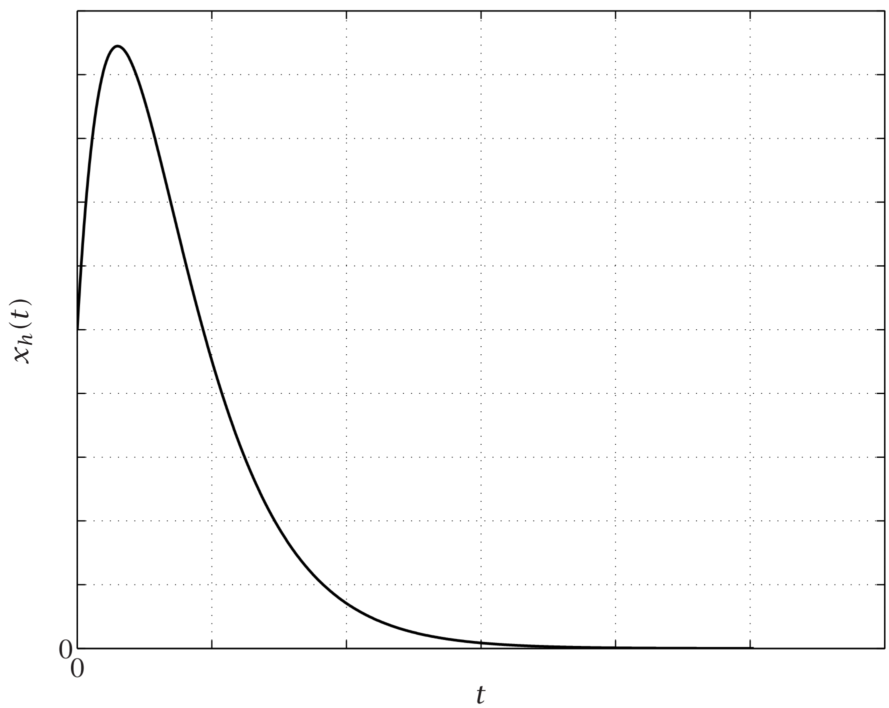

Figure 1–3 Schematic of the zero input response of a critically damped second-order linear time-invariant system.

Case 3: $\zeta > 1$ (Overdamping)

When $\zeta > 1$ the zero input response is said to be *overdamped*. For an overdamped system the quantity $\zeta^2 - 1 > 0$ which implies that $\sqrt{\zeta^2 - 1} > 0$. The roots of the characteristic equation for an underdamped system are then given as

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{1–50}$$

It is seen from Eq. (1–50) that the roots of an overdamped system are *real* and *distinct*. Furthermore, the general zero input response for an overdamped system is given as

$$x_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \tag{1–51}$$

The constants c_1 and c_2 can be solved for by using the initial conditions $(x(0), \dot{x}(0)) = (x_0, \dot{x}_0)$ as follows. First, applying the initial condition $x(0) = x_0$, we obtain

$$x_h(0) = x_0 = c_1 + c_2 \tag{1–52}$$

Next, differentiating Eq. (1–51) gives

$$\dot{x}_h(t) = -c_1\lambda_1 e^{\lambda_1 t} + c_2\lambda_2 e^{\lambda_2 t} \tag{1–53}$$

Then, applying the initial condition $\dot{x}(0) = \dot{x}_0$, we obtain

$$\dot{x}_h(0) = \dot{x}_0 = -c_1\lambda_1 + c_2\lambda_2 \tag{1–54}$$

Equations (1–52) and (1–54) can then be solved simultaneously for c_1 and c_2 to give

$$c_1 = \frac{x_0\lambda_2 - \dot{x}_0}{\lambda_1 + \lambda_2} \tag{1–55}$$

$$c_2 = \frac{x_0\lambda_1 + \dot{x}_0}{\lambda_1 + \lambda_2} \tag{1–56}$$

The general zero input response for an overdamped system is then given as

$$x_h(t) = \frac{x_0\lambda_2 - \dot{x}_0}{\lambda_1 + \lambda_2} e^{\lambda_1 t} + \frac{x_0\lambda_1 + \dot{x}_0}{\lambda_1 + \lambda_2} e^{\lambda_2 t} \tag{1–57}$$

A schematic of an overdamped zero input response for various values of $\zeta > 1$ is shown in Fig. 1–4.

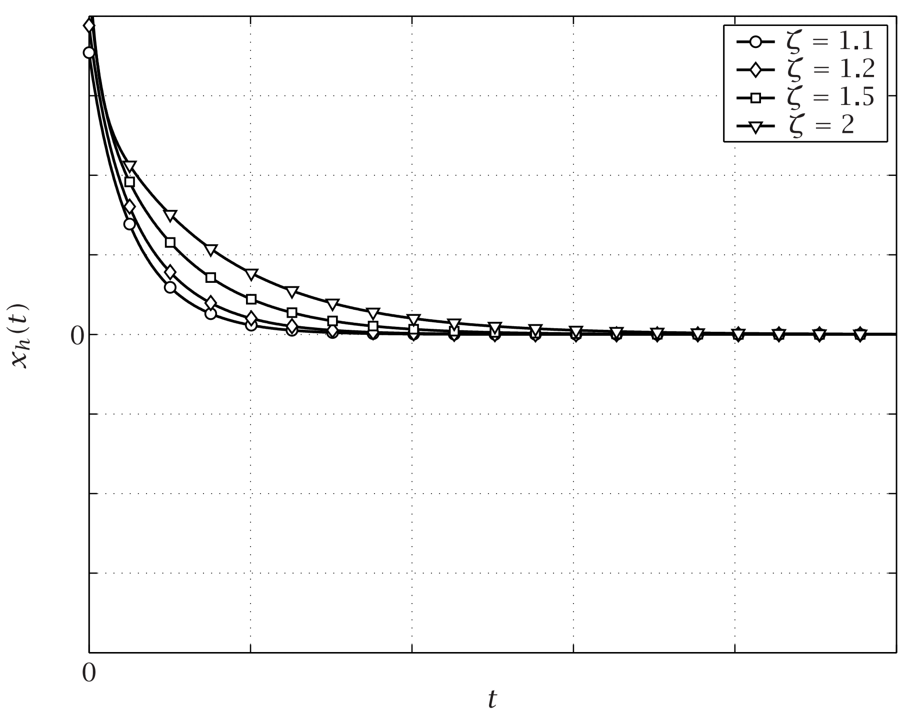

Figure 1–4 Schematic of the zero input response of an overdamped second-order linear time-invariant system.

Chapter 2

Forced Response of Single Degree-of-Freedom Systems

2.1 Response of Single Degree-of-Freedom Systems to Nonperiodic Inputs

2.2 Physics of Impulsive Motion

Recall from dynamics that the principle of impulse and momentum for a particle states that

$$\hat{\mathbf{F}} = {}^{\mathcal{N}}\mathbf{G}' - {}^{\mathcal{N}}\mathbf{G} \tag{2–1}$$

where ${}^{\mathcal{N}}\mathbf{G}$ is the linear momentum of the particle as viewed by an observer in an inertial reference frame $\mathcal{N}$. Suppose now that we consider the following system. A block of mass m is connected to a linear spring with spring constant K and unstretched length ℓ_0 and a viscous linear damper with damping coefficient c as shown in Fig. 2–1. The block is *initially at rest* (i.e., its initial velocity is zero) at its static equilibrium position (i.e., the spring is initially unstressed) when a horizontal impulse $\hat{\mathbf{P}}$ is applied. We are interested here in determining the velocity of the block immediately after the application of the impulse $\hat{\mathbf{P}}$.

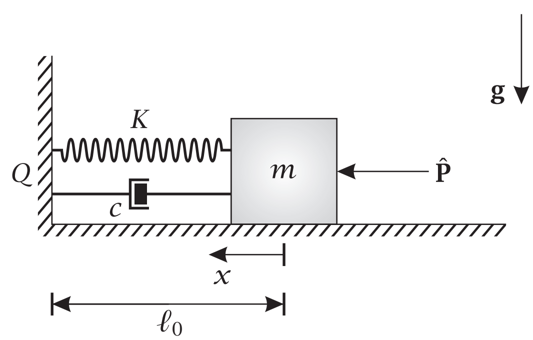

Figure 2–1 Block of mass m connected to linear spring and linear damper struck by horizontal impulse $\hat{\mathbf{P}}$.

The solution of the above problem is found as follows. First, let $\mathcal{F}$ be the ground. Then, choose the following coordinate system fixed in $\mathcal{F}$:

	Origin at block when $x = 0$	
$\mathbf{E}_x$	$=$	To the left
$\mathbf{E}_z$	$=$	Into page
$\mathbf{E}_y$	$=$	$\mathbf{E}_z \times \mathbf{E}_x$

Then, the position of the block is given in terms of the displacement x as

$$\mathbf{r} = x\mathbf{E}_x \tag{2–2}$$

Because $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ is a fixed basis, the velocity of the block in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\mathbf{v} = \frac{^{\mathcal{F}}d\mathbf{r}}{dt} = \dot{x}\mathbf{E}_x = v\mathbf{E}_x \tag{2–3}$$

Now because we are going to apply the principle of linear impulse and momentum to this problem, we do not need the acceleration of the block. Instead, we know that neither the spring nor the damper can apply an instantaneous impulse. Therefore, the only impulse applied to the system at $t = 0$ is that due to $\hat{\mathbf{P}}$. Consequently, the external impulse acting on the system at $t = 0$ is

$$\hat{\mathbf{F}} = \hat{\mathbf{P}} = \hat{P}\mathbf{E}_x \tag{2–4}$$

Furthermore, the linear momentum of the block the instant before the impulse is applied is zero (i.e., the block is initially at rest) while the linear momentum of the block the instant after the impulse is applied is given as

$$^{\mathcal{F}}\mathbf{G}' = m\,^{\mathcal{F}}\mathbf{v}' = mv'\mathbf{E}_x \tag{2–5}$$

Setting $\hat{\mathbf{F}}$ equal to $^{\mathcal{F}}\mathbf{G}'$, we obtain

$$\hat{P} = mv' \equiv mv(t = 0^+) \tag{2–6}$$

Solving for $v(t = 0^+)$, we obtain

$$v(t = 0^+) = \frac{\hat{P}}{m} \tag{2–7}$$

The result of this analysis shows that the response of a resting second-order linear system to an impulsive force $\hat{\mathbf{F}}$ is equivalent to giving the system the initial velocity shown in Eq. (2–7).

2.3 Impulse Response of Second-Order Linear System

Suppose now that we consider the general motion of the system in Fig. 2–1, i.e., we consider motion to a general force $\mathbf{F}(t)$. Then, recalling the result from earlier, the differential equation of motion is given as

$$m\ddot{x} + c\dot{x} + Kx = F(t) + K\ell_0 \tag{2–8}$$

It is noted that the equilibrium point of the system in Eq. (2–8) is $x_{eq} = \ell_0$, we can define the variable $y = x - x_{eq}$ and rewrite Eq. (2–8) in terms of y to give

$$m\ddot{y} + c\dot{y} + Ky = F(t) \tag{2–9}$$

Now suppose that $F(t)$ is the following function:

$$F(t) = \hat{F}\delta(t) \tag{2–10}$$

where $\delta(t)$ is defined as follows:

$$\delta(t - a) = \begin{cases} \infty & , \quad t = a \\ 0 & , \quad t \neq \tau \end{cases} \tag{2–11}$$

The function $\delta(t)$ is called the *Dirac delta function* or the *unit impulse function*. It is known that the Dirac delta function satisfies the following properties:

$$\int_{-\infty}^{\infty} \delta(t-a)dt = 1 \tag{2–12}$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a) \tag{2–13}$$

where $f(t)$ is an arbitrary function. For simplicity, consider the case where $\hat{F} = 1$, i.e., the case of unit impulse being applied to the system. Also, let $g(t)$ be the response to the input $\delta(t)$, i.e., consider the system

$$m\ddot{g} + c\dot{g} + Kg = \delta(t) \tag{2–14}$$

Let T be a value of t such that $T > 0$. Then, integrating Eq. (2–14) from zero to T, we have

$$\int_0^T [m\ddot{g} + c\dot{g} + Kg]\, dt = \int_0^T \delta(t)dt \tag{2–15}$$

Now we have the following

$$\int_0^T m\ddot{g}dt = m\dot{g}(t)|_0^T \tag{2–16}$$

$$\int_0^T c\dot{g}dt = mg(t)|_0^T \tag{2–17}$$

Taking the limit as $T \to 0$ from above, we obtain

$$\lim_{T\to 0^+} m\dot{g}(t)\,|_0^T = \lim_{T\to 0^+} m\left[\dot{g}(T)| - \dot{g}(0)\right] = m\dot{g}(0^+) \tag{2–18}$$

$$\lim_{T\to 0^+} mg(t)\,|_0^T = \lim_{T\to 0^+} m\left[g(T) - g(0)\right] = 0 \tag{2–19}$$

Furthermore, because the position of the mass cannot change during the application of an instantaneous impulse, we see that

$$\lim_{T\to 0^+} \int_0^T Kg(t)dt = \lim_{T\to 0^+} Kg(0)t\,|_0^T = \lim_{T\to 0^+} Kg(0) = 0 \tag{2–20}$$

Using the results of Eqs. (2–18), (2–19) and (2–20) in Eq. (2–15), we obtain

$$m\dot{g}(0^+) = 1 \tag{2–21}$$

Solving Eq. (2–21) for $\dot{g}(0^+)$, we obtain

$$\dot{g}(0^+) = \frac{1}{m} \tag{2–22}$$

It is seen that, for the case where $\hat{P} \equiv 1$, the results of Eq. (2–7) and Eq. (2–22) are *identical.* More specifically, as we saw above, the effect of a unit impulsive force on a resting particle of mass m is to provide an initial velocity of magnitude $1/m$ while the *response* of a second-order linear system to a unit impulse function (i.e., the Dirac delta function) is to provide an initial velocity of magnitude $1/m$. Consequently, the physics of an impulsive force on a resting particle is identical to the mathematics of the impulse response of the system to a unit impulse.

Now that we know that the response of a second-order resting system is to change the velocity (while leaving position unchanged), we can use this fact to obtain the *impulse response* $g(t)$. In particular, assuming an underdamped system, we know that the general form of the free response is given as

$$g(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) \tag{2–23}$$

where $\omega_n = \sqrt{k/m}$ is the natural frequency, ζ is the damping ratio, and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency. Differentiating this last equation, we have

$$\dot{g}(t) = -\zeta\omega_n e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + e^{-\zeta\omega_n t}(-A\omega_d\sin\omega_d t + B\omega_d\cos\omega_d t) \quad (2\text{–}24)$$

Noting that $g(0) = 0$ and that $\dot{g}(0^+) = 1/m$, we have

$$A = 0 \quad (2\text{–}25)$$

$$B = \frac{1}{m\omega_d} \quad (2\text{–}26)$$

Therefore, the response of the system to a unit impulse at $t = 0$ is given as

$$g(t) = \begin{cases} \frac{1}{m\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t & , \quad t > 0 \\ 0 & , \quad t \leq 0 \end{cases} \quad (2\text{–}27)$$

It is seen that, for an underdamped system, the impulse response is a decaying sinusoid with a zero phase (i.e., the applied impulse did not result in a nonzero phase shift). A schematic of the impulse response is shown in Fig. 2–2.

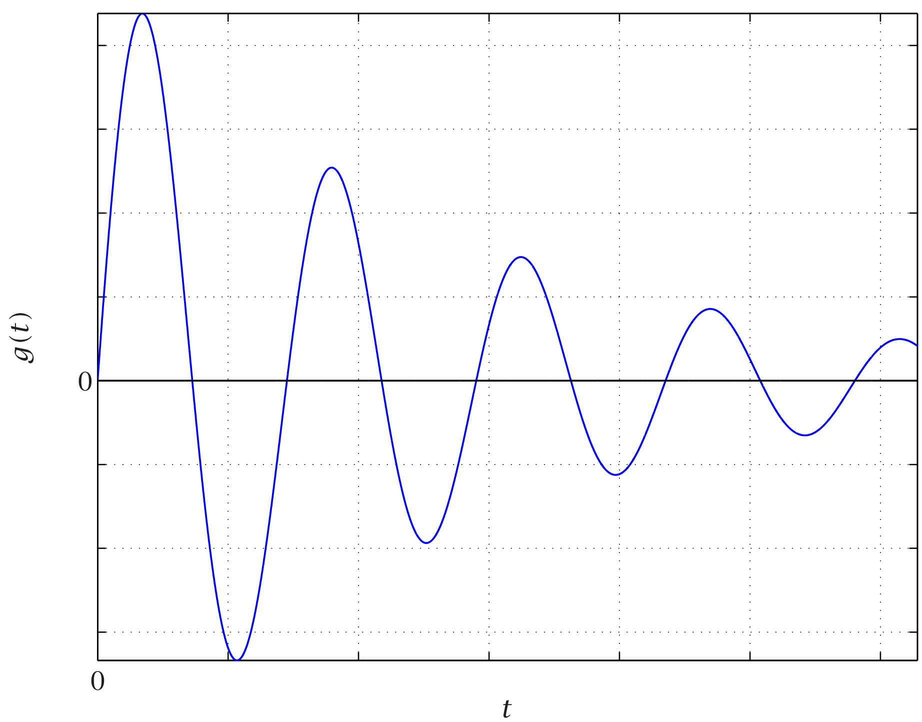

Figure 2–2 Schematic of Impulse Response of Underdamped Second-Order Linear System.

2.4 Step Response of Second-Order Linear System

After the unit impulse function, the next fundamental function of importance in the analysis of vibratory systems is the *unit step function*. The unit step function, denoted $u(t)$, is defined as

$$u(t-a) = \begin{cases} 0 & , \quad t \le a \\ 1 & , \quad t > a \end{cases} \tag{2–28}$$

Recalling the unit impulse function $\delta(t)$ from Eq. (2–11), it is seen that $u(t)$ is related to $\delta(t)$ as follows:

$$u(t-a) = \int_{-\infty}^{t} \delta(\tau - a)d\tau \tag{2–29}$$

where τ is a dummy variable of integration. Now suppose we want to determine the response, $s(t)$, of the system of Eq. (2–9) to a unit step input at $t = 0$. The function $s(t)$ is called the *step response* and, from Eq. (2–9), satisfies

$$m\ddot{s} + c\dot{s} + Ks = u(t) \tag{2–30}$$

It is noted that Eq. (2–30) can be written as

$$m\frac{d^2 s}{dt^2} + c\frac{ds}{dt} + Ks = u(t) \tag{2–31}$$

We can obtain $s(t)$ as follows. Consider again the relationship that holds between the unit impulse and the impulse response, i.e.,

$$m\ddot{g} + c\dot{g} + Kg = \delta(t) \tag{2–32}$$

Then, from Eq. (2–29), we have

$$\frac{du(t-a)}{dt} = \delta(t-a) \tag{2–33}$$

Therefore, for a unit step function at $t = 0$, we have

$$m\ddot{g} + c\dot{g} + Kg = \frac{du}{dt} \tag{2–34}$$

Integrating both sides of Eq. (2–34) gives

$$\int_{-\infty}^{t} \left[\frac{d^2 g}{d\tau^2} + c\frac{dg}{d\tau} + Kg\right] d\tau = \int_{-\infty}^{t} \frac{du(a)}{da} da = u(t) \tag{2–35}$$

Now from the fundamental theorem of calculus we have

$$\int_{-\infty}^{t} \frac{d^2 g}{d\tau^2} = \frac{d^2}{dt^2}\int_{-\infty}^{t} g(\tau)d\tau \tag{2–36}$$

$$\int_{-\infty}^{t} \frac{dg}{d\tau} = \frac{d}{dt}\int_{-\infty}^{t} g d\tau \tag{2–37}$$

Therefore, Eq. (2–35) can be rewritten as

$$\left[\frac{d^2}{d\tau^2} + c\frac{d}{d\tau} + K\right]\int_{-\infty}^{t} g(\tau)d\tau = u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \tag{2–38}$$

Now if we compare Eq. (2–38) to Eq. (2–31), it is seen that

$$s(t) = \int_{-\infty}^{t} g(\tau)d\tau \tag{2–39}$$

In other words, the response of the system of Eq. (2–9) to a unit step function is the *integral* of the response of the system to a unit impulse[1]. We can then use the result of Eq. (2–39) and the

[1]More generally, it is the case that the response of any linear time-invariant system to the integral of a function $f(t)$ is equal to the integral of the response to the original function $f(t)$.

impulse response given in Eq. (2–27) as follows. First, for $t \le 0$ we have $s(t) = 0$. For $t > 0$, we have

$$s(t) = \int_0^t \frac{1}{m\omega_d} e^{-\zeta\omega_n\tau} \sin\omega_d\tau d\tau = \frac{1}{m\omega_d}\int_0^t e^{-\zeta\omega_n\tau}\sin\omega_d\tau d\tau \tag{2–40}$$

Now from DeMoivre's theorem we have

$$\sin\omega_d\tau = \frac{e^{i\omega_d\tau} - e^{-i\omega_d\tau}}{2i} \tag{2–41}$$

Therefore,

$$\begin{aligned} s(t) &= \frac{1}{2im\omega_d}\int_0^t e^{-\zeta\omega_n\tau}\left[e^{i\omega_d\tau} - e^{-i\omega_d\tau}\right] \\ &= \frac{1}{2im\omega_d}\int_0^t \left[e^{-(\zeta\omega_n - i\omega_d)\tau} - e^{-(\zeta\omega_n + i\omega_d)\tau}\right] \\ &= \frac{1}{2im\omega_d}\left[-\frac{e^{-(\zeta\omega_n - i\omega_d)\tau}}{\zeta\omega_n - i\omega_d} + \frac{e^{-(\zeta\omega_n + i\omega_d)\tau}}{\zeta\omega_n + i\omega_d}\right]_0^t \\ &= -\frac{e^{-\zeta\omega_n\tau}}{2im\omega_d}\left[\frac{e^{i\omega_d\tau}}{\zeta\omega_n - i\omega_d} - \frac{e^{-i\omega_d\tau}}{\zeta\omega_n + i\omega_d}\right]_0^t \\ &= -\frac{e^{-\zeta\omega_n\tau}}{2im\omega_d}\left[\frac{(\zeta\omega_n + i\omega_d)e^{i\omega_d\tau} - (\zeta\omega_n - i\omega_d)e^{-i\omega_d\tau}}{\zeta^2\omega_n^2 + \omega_d^2}\right]_0^t \end{aligned} \tag{2–42}$$

Now, noting that $\omega_d = \omega_n\sqrt{1-\zeta^2}$, we have

$$\begin{aligned} s(t) &= -\frac{1}{2im\omega_d\omega_n^2}\left[(\zeta\omega_n + i\omega_d)e^{-(\zeta\omega_n - i\omega_d)\tau} - (\zeta\omega_n - i\omega_d)e^{-(\zeta\omega_n + i\omega_d)\tau}\right]_0^t \\ &= \frac{1}{2im\omega_d\omega_n^2}\left[(\zeta\omega_n + i\omega_d)\left(1 - e^{-(\zeta\omega_n - i\omega_d)t}\right) - (\zeta\omega_n - i\omega_d)\left(1 - e^{-(\zeta\omega_n + i\omega_d)t}\right)\right] \\ &= \frac{1}{2im\omega_d\omega_n^2}\left[2i\omega_d - e^{-\zeta\omega_n t}\left\{\zeta\omega_n\left(e^{i\omega_d t} - e^{-i\omega_d t}\right) + i\omega_d\left(e^{i\omega_d t} + e^{-i\omega_d t}\right)\right\}\right] \\ &= \frac{1}{m\omega_d\omega_n^2}\left[\omega_d - e^{-\zeta\omega_n t}\left(\zeta\omega_n\frac{e^{i\omega_d t} - e^{-i\omega_d t}}{2i} + \omega_d\frac{e^{i\omega_d t} + e^{-i\omega_d t}}{2}\right)\right] \end{aligned} \tag{2–43}$$

Now we have

$$\frac{e^{i\omega_d t} + e^{-i\omega_d t}}{2} = \cos\omega_d t \tag{2–44}$$

Using Eq. (2–44) together with Eq. (2–41), we have

$$\begin{aligned} s(t) &= \frac{1}{m\omega_d\omega_n^2}\left[\omega_d - e^{-\zeta\omega_n t}\left(\zeta\omega_n\sin\omega_d t + \omega_d\cos\omega_d t\right)\right] \\ &= \frac{1}{m\omega_n^2}\left[1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right)\right] \end{aligned} \tag{2–45}$$

It is noted that the expression in Eq. (2–45) is valid when $t > 0$. Therefore, the response of the system of Eq. (2–9) to a unit step function is given as

$$s(t) = \begin{cases} 0 & , \quad t \le 0 \\ \frac{1}{m\omega_n^2}\left[1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right)\right] & , \quad t > 0 \end{cases} \tag{2–46}$$

2.5 Response of Single Degree-of-Freedom Systems to Periodic Inputs

Recall the standard form of the differential equation that describes the motion of a damped single degree-of-freedom system subject from Eq. (1–17) as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = p(t) \tag{2–47}$$

Suppose now that $p(t)$ has the general form

$$p(t) = \omega_n^2 f(t) \tag{2–48}$$

Then Eq. (2–47) can be written as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 f(t) \tag{2–49}$$

Suppose further that $f(t)$ is a periodic function of the form $f(t) = Ae^{i\omega t}$. We then have

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 Ae^{i\omega t} \tag{2–50}$$

The function $f(t) = Ae^{i\omega t}$ will be called the *normalized input function*.

2.5.1 General Solution to Second-Order Linear Differential Equation

It is known that the *general* solution to Eq. (2–50) is the sum of the *homogeneous* and *particular* solutions, i.e.,

$$x(t) = x_h(t) + x_p(t) \tag{2–51}$$

where $x_h(t)$ satisfies the equation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \tag{2–52}$$

and $x_p(t)$ is the particular solution that satisfies Eq. (2–50). In this analysis we are interested in determining the particular solution of Eq. (2–50).

2.5.2 Particular Solution to Complex Periodic Input

Suppose now that we want to determine the particular solution to Eq. (2–50). Given that the input $F(t) = \omega_n^2 f(t) = \omega_n^2 Ae^{i\omega t}$ is an exponential with exponent $i\omega t$, the particular solution will itself have the form

$$x_p(t) = X(\omega)e^{i\omega t} \tag{2–53}$$

where we note that the coefficient X is a function of the input frequency ω. Differentiating $x_p(t)$ in Eq. (2–53), we obtain

$$\dot{x}_p(t) = i\omega Xe^{i\omega t} \tag{2–54}$$

$$\ddot{x}_p(t) = -\omega^2 Xe^{i\omega t} \tag{2–55}$$

Substituting $x_p(t)$, $\dot{x}_p(t)$, and $\ddot{x}_p(t)$ from Eqs. (2–53)–(2–55), respectively, into Eq. (2–50), we have

$$-\omega^2 Xe^{i\omega t} + i2\zeta\omega_n\omega Xe^{i\omega t} + \omega_n^2 Xe^{i\omega t} = \omega_n^2 Ae^{i\omega t} \tag{2–56}$$

Rearranging Eq. (2–56) gives

$$Xe^{i\omega t}\left[(\omega_n^2 - \omega^2) + i2\zeta\omega_n\omega\right] = \omega_n^2 Ae^{i\omega t} \tag{2–57}$$

Observing that $e^{i\omega t}$ is not zero as a function of time, it can be dropped from Eq. (2–57) to give

$$\left[(\omega_n^2 - \omega^2) + i2\zeta\omega_n\omega\right] = \omega_n^2 A \tag{2–58}$$

Rearranging Eq. (2–58), we obtain

$$\frac{X(\omega)}{A} = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + i2\zeta\omega_n\omega} \tag{2–59}$$

Suppose now that we let

$$G(i\omega) = \frac{X(\omega)}{A} = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + i2\zeta\omega_n\omega} \tag{2–60}$$

Finally, we can divide the numerator and denominator of Eq. (2–60) by ω_n^2 to obtain

$$G(i\omega) = \frac{X(\omega)}{A} = \frac{1}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2 + i2\zeta\dfrac{\omega}{\omega_n}} \tag{2–61}$$

The quantity $G(i\omega)$ is called the *transfer function* of the system to the input $Ae^{i\omega t}$. It is seen that the transfer function is the ratio of the amplitude of the output to the amplitude of the input. It is seen that the transfer function of the system of Eq. (2–50) is a function of the frequency, ω, of the input $F(t) = \omega_n^2 Ae^{i\omega t}$

Now since the transfer function $G(i\omega)$ is complex, it can be written as

$$G(i\omega) = \alpha + i\beta \tag{2–62}$$

where

$$\alpha = \operatorname{Re}[G(i\omega)] \tag{2–63}$$
$$\beta = \operatorname{Im}[G(i\omega)] \tag{2–64}$$

where Re [·] and Im [·] are the real and imaginary parts of G. From complex analysis, we know that any complex number can be written as

$$z = \alpha + i\beta = |z|e^{-i\phi} \tag{2–65}$$

where

$$|z| = \sqrt{z\bar{z}} = \sqrt{\alpha^2 + \beta^2} \tag{2–66}$$
$$\phi = \tan^{-1}\left(\frac{-\beta}{\alpha}\right) \tag{2–67}$$

and $\bar{z} = \alpha - i\beta$ is the complex conjugate of z. It is noted in Eq. (2–66) that $\bar{z}$ is the complex conjugate of z (i.e., $\bar{z} = \alpha - i\beta$) and the negative sign in Eq. (2–67) is associated with the numerator. Using the result of Eq. (2–65), we can write $G(i\omega)$ as

$$G(i\omega) = |G(i\omega)|e^{-i\phi(\omega)} \tag{2–68}$$

where

$$|G(i\omega)| = \sqrt{G(i\omega)\bar{G}(i\omega)} \tag{2–69}$$
$$\phi(\omega) = \frac{-\operatorname{Im}[G(i\omega)]}{\operatorname{Re}[G(i\omega)]} \tag{2–70}$$

Returning to the particular solution $x_p(t)$, we note that

$$x_p(t) = Xe^{i\omega t} = AG(i\omega)e^{i\omega t} = A|G(i\omega)|e^{i(\omega t - \phi)} \tag{2–71}$$

2.5.3 Response of Second-Order System to Sine and Cosine Inputs

In section 2.5.2 we obtained the response of the second-order system of Eq. (2–47) to a complex periodic input of the form $p(t) = \omega_n^2 A e^{i\omega t}$. However, actual physical systems are *real*, not complex. Consequently, it would never actually be the case that the input to a physical system would be complex.

A question that arises from the fact that only a real function would be an input to a physical system is, what is the particular solution of the system Eq. (2–47) to a real periodic input? This question is answered as follows. We know that the two fundamental periodic functions are $\cos\omega t$ and $\sin\omega t$. Using the normalization $A\omega_n^2$, the real question being asked is, what are the particular solutions of Eq. (2–47) to the inputs $A\omega_n^2\cos\omega t$ and $A\omega_n^2\sin\omega t$? We can obtain these two particular solutions as follows. First, from Eq. (2–71) we know from De'Moivre's theorem that

$$e^{i(\omega t-\phi)} = \cos(\omega t-\phi) + i\sin(\omega t-\phi) \tag{2–72}$$

Therefore, the particular solution $x_p(t)$ in Eq. (2–71) can be written as $A\omega_n^2 e^{i\omega t}$ can be written as

$$x_p(t) = Xe^{i\omega t} = AG(i\omega)e^{i\omega t} = A|G(i\omega)|\cos(\omega t-\phi) + iA|G(i\omega)|\sin(\omega t-\phi) \tag{2–73}$$

Expanding Eq. (2–73), we obtain

$$x_p(t) = Xe^{i\omega t} = A|G(i\omega)|\cos(\omega t-\phi) + iA|G(i\omega)|\sin(\omega t-\phi) \tag{2–74}$$

Now, by the principle of superposition we know that the particular solution of Eq. (2–47) to the *sum* of two inputs $p_1(t) + p_2(t)$ is the *sum* of the responses, i.e., if $x_1(t)$ is the particular solution to the input $p_1(t)$ and $x_2(t)$ is the particular solution to the input $p_2(t)$, then $x_1(t) + x_2(t)$ is the particular solution to the input $p_1(t) + p_2(t)$. Now suppose we rewrite the general complex input $A\omega_n^2 e^{i\omega t}$ as

$$A\omega_n^2 e^{i\omega t} = A\omega_n^2\cos\omega t + iA\omega_n^2\sin\omega t = f_r(t) + if_i(t) \tag{2–75}$$

where

$$f_r(t) = A\omega_n^2\cos\omega t \tag{2–76}$$

$$f_i(t) = A\omega_n^2\sin\omega t \tag{2–77}$$

Now observe that $f_r(t)$ and $f_i(t)$ are the real and imaginary parts of $A\omega_n^2 e^{i\omega t}$, respectively. Furthermore, observe from Eq. (2–73) that $A|G(i\omega)|\cos(\omega t-\phi)$ and $A|G(i\omega)|\sin(\omega t-\phi)$ are the real and complex parts, respectively, of the response $x_p(t)$ to $A\omega_n^2 e^{i\omega t}$. Then, by the principle of superposition we know that the response of Eq. (2–47) to $f_r(t)$ must be the *real part* of $x_p(t)$ in Eq. (2–73), i.e.,

$$x_r(t) = \operatorname{Re}\left\{A|G(i\omega)|e^{i(\omega t-\phi)}\right\} = A|G(i\omega)|\cos(\omega t-\phi) \tag{2–78}$$

Similarly, the response of Eq. (2–47) to $f_i(t)$ is the *imaginary* part of $x_p(t)$ in Eq. (2–73), i.e.,

$$x_i(t) = \operatorname{Im}\left\{A|G(i\omega)|e^{i(\omega t-\phi)}\right\} = A|G(i\omega)|\sin(\omega t-\phi) \tag{2–79}$$

2.5.4 Frequency Response to Periodic Input

We now turn to a more detailed analysis of the response of the system of Eq. (2–50) to a periodic input. In particular, we are interested in the *amplitude* and *phase* of the output as a function of input frequency. Generally speaking, the amplitude is determined as the ratio of the output amplitude to the input amplitude. Recall that the transfer function $G(i\omega)$ was defined as $G(i\omega) = X(\omega)/A$ where $X(\omega)$ is the output amplitude (i.e., the amplitude of the particular solution) and A is the input amplitude of the normalized input function $f(t) = Ae^{i\omega t}$. The frequency response to a periodic input is defined as the combination of the magnitude and phase of the ratio of the output to the input. Recall the magnitude and phase of $G(i\omega)$ from Eqs. (2–69) and (2–70). Furthermore, recall from Eq. (2–61) that

$$G(i\omega) = \frac{X(\omega)}{A} = \frac{1}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2 + i2\zeta\dfrac{\omega}{\omega_n}} \tag{2–80}$$

Then the magnitude of $G(i\omega)$ is given as

$$|G(i\omega)| = \left[G(i\omega)\bar{G}(i\omega)\right]^{1/2} = \left\{\left[\frac{1}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2 + i2\zeta\dfrac{\omega}{\omega_n}}\right]\left[\frac{1}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2 - i2\zeta\dfrac{\omega}{\omega_n}}\right]\right\}^{1/2} \tag{2–81}$$

where

$$\bar{G}(i\omega) = \frac{1}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2 - i2\zeta\dfrac{\omega}{\omega_n}} \tag{2–82}$$

Eq. (2–81) can be simplified to give

$$|G(i\omega)| = \frac{1}{\left\{\left[1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\dfrac{\omega}{\omega_n}\right]^2\right\}^{1/2}} \tag{2–83}$$

Next, the phase of $G(i\omega)$ can be obtained as follows. First, we know that

$$G(i\omega) = G(i\omega)\frac{\bar{G}(i\omega)}{\bar{G}(i\omega)} = \frac{|G(i\omega)|^2}{\bar{G}(i\omega)} \tag{2–84}$$

Substituting $|G(i\omega)|$ and $\bar{G}(i\omega)$ from Eqs. (2–61) and (2–82), we obtain

$$G(i\omega) = \frac{1 - \left(\dfrac{\omega}{\omega_n}\right)^2 - i2\zeta\dfrac{\omega}{\omega_n}}{\left[1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\dfrac{\omega}{\omega_n}\right]^2} \tag{2–85}$$

Extracting the real and imaginary parts of $G(i\omega)$ from Eq. (2–85), we have

$$\mathrm{Re}\left[G(i\omega)\right] = \frac{1 - \left(\dfrac{\omega}{\omega_n}\right)^2}{\left[1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\dfrac{\omega}{\omega_n}\right]^2} \tag{2–86}$$

$$\mathrm{Im}\left[G(i\omega)\right] = \frac{-2\zeta\dfrac{\omega}{\omega_n}}{\left[1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\dfrac{\omega}{\omega_n}\right]^2} \tag{2–87}$$

The phase is then obtained as

$$\phi(\omega) = \tan^{-1}\frac{-\mathrm{Im}\left[G(i\omega)\right]}{\mathrm{Re}\left[G(i\omega)\right]} = \tan^{-1}\left[\frac{2\zeta\dfrac{\omega}{\omega_n}}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2}\right] \tag{2–88}$$

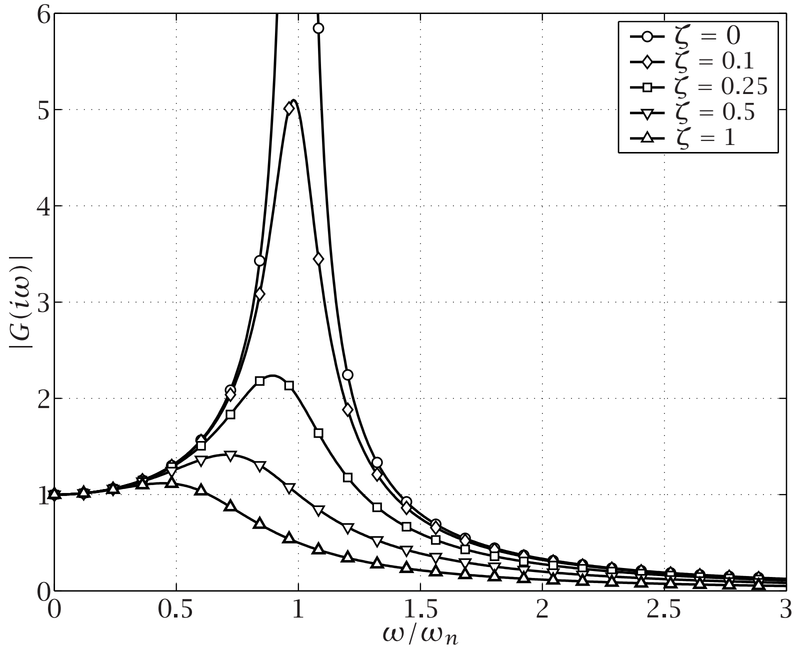

Figure 2–3 Magnitude of Frequency Response of a Single Degree-of-Freedom Linear System to an Input $f(t) = Ae^{i\omega t}$.

The magnitude and phase of $G(i\omega)$ are shown in Figs. 2–3 and 2–4, respectively, for various values of the damping ratio ζ. It is seen from Fig. 2–3 that the amplitude of the response approaches ∞ as $\zeta \to 0$, i.e.,

$$\lim_{\zeta \to 0} |G(i\omega)| = \infty \tag{2–89}$$

In general, it can be shown that the maximum value of $|G(i\omega)|$ is given as

$$|G(i\omega)|_{\max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{2–90}$$

Furthermore, it is seen that as ζ approaches zero, the value at which $|G(i\omega)|$ is maximum approaches unity, i.e.,

$$\lim_{\zeta \to 0} \arg\max |G(i\omega)| = 1 \tag{2–91}$$

Turning attention to the phase of $G(i\omega)$ (i.e., $\phi(\omega)$), it is seen that all of the curves pass through the point $\omega/\omega_n = 1$ and $\phi = \pi/2$. Furthermore, it is seen that ϕ approaches zero and ∞ as ω/ω_n approaches zero and ω/ω_n approaches ∞, respectively, i.e.,

$$\lim_{\omega/\omega_n \to \infty} \phi(\omega) = 0 \tag{2–92}$$

$$\lim_{\omega/\omega_n \to \infty} \phi(\omega) = \pi \tag{2–93}$$

It is noted that, for the special case of $\zeta = 0$, the phase has a discontinuity at $\omega/\omega_n = 1$ (this is not shown in Fig. 2–4). Finally, for the case where $\zeta = 0$ and $\omega/\omega_n = 1$ the system is at resonance with a phase angle of $\pi/2$.

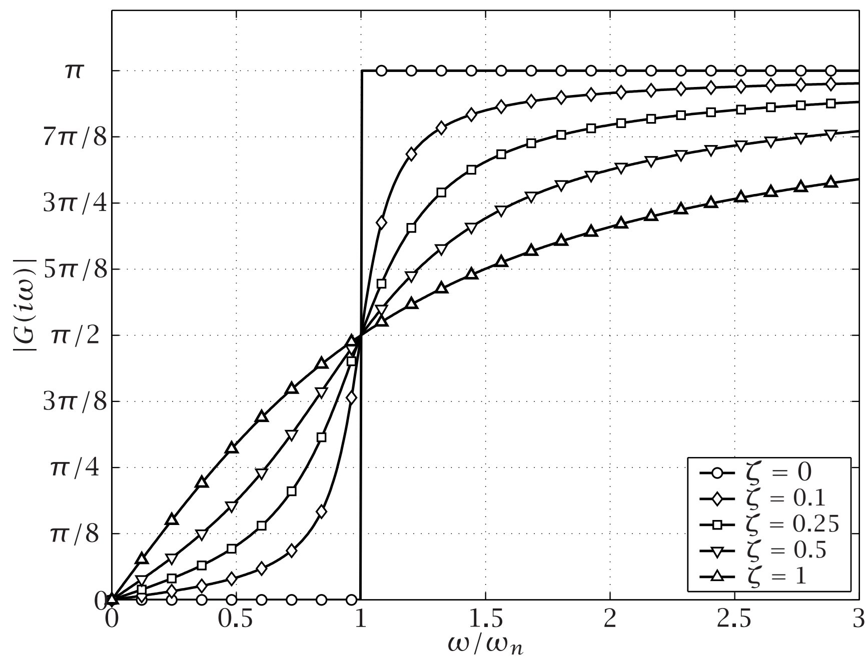

Figure 2–4 Phase of Frequency Response of a Single Degree-of-Freedom Linear System to an Input $f(t) = Ae^{i\omega t}$.

2.5.5 Transfer Functions of Second-Order System to Sine and Cosine Inputs

In section 2.5.4 the transfer function of the second-order differential equation given in Eq. (2–47) to the input $A\omega_n^2 e^{i\omega t}$ was derived. In this section we determine the transfer functions of Eq. (2–47) to the inputs $A\omega_n^2 \cos \omega t$ and $A\omega_n^2 \sin \omega t$. First, the response of Eq. (2–47) to the input $A\omega_n^2 \cos \omega t$ is given from Eq. (2–78) as

$$x_r(t) = A|G(i\omega)| \cos(\omega t - \phi) \tag{2–94}$$

Now we know that $x_r(t)$ can be written as

$$x_r(t) = X_r \cos(\omega t - \beta) \tag{2–95}$$

where X_r and β are the amplitude and phase, respectively, of $x_r(t)$. Comparing Eq. (2–94) and (2–95) it is seen that

$$X_r = A|G(i\omega)| \tag{2–96}$$
$$\beta = \phi \tag{2–97}$$

Therefore, the magnitude and phase of $x_r(t)$ is the same as the magnitude and phase $x_p(t)$ where $x_p(t)$ is given from Eq. (2–71). Now because a complex number is defined completely from its magnitude and phase, we have

$$G_r(i\omega) = G(i\omega) \tag{2–98}$$

In other words, the transfer function of that the $A\omega_n^2 \cos \omega t$ is *identical* to the transfer function of Eq. (2–98) to the input $A\omega_n^2 e^{i\omega t}$. Next, the response of Eq. (2–47) to the input $A\omega_n^2 \sin \omega t$ is given from Eq. (2–79) as

$$x_i(t) = A|G(i\omega)| \sin(\omega t - \phi) \tag{2–99}$$

Now we know that $x_i(t)$ can be written as

$$x_i(t) = X_i \sin(\omega t - \gamma) \tag{2–100}$$

where X_i and γ are the amplitude and phase, respectively, of $x_i(t)$. Comparing Eq. (2–99) and (2–100) it is seen that

$$X_i = A|G(i\omega)| \tag{2–101}$$
$$\gamma = \phi \tag{2–102}$$

Therefore, the magnitude and phase of $x_i(t)$ is the same as the magnitude and phase $x_p(t)$ where $x_p(t)$ is given from Eq. (2–71). Again, because a complex number is defined completely from its magnitude and phase, we have

$$G_i(i\omega) = G(i\omega) \tag{2–103}$$

In other words, the transfer function of that the $A\omega_n^2 \sin \omega t$ is *identical* to the transfer function of Eq. (2–103) to the input $A\omega_n^2 e^{i\omega t}$.

2.5.6 Comments on Complex Periodic Input vs. Real Periodic Input

The results of section 2.5.5 demonstrate an important fact. The transfer function (i.e., the magnitude and phase of the output $x(t)$ over the input $p(t)$ where $p(t)$ is a periodic function of time) to the input $A\omega_n^2 e^{i\omega t}$ is the *same* as the transfer function to the inputs $A\omega_n^2 \cos \omega t$ and $A\omega_n^2 \sin \omega t$. The reason the transfer function is the same regardless of whether complex or real periodic inputs are used is because the responses to $A\omega_n^2 \cos \omega t$ and $A\omega_n^2 \sin \omega t$ have the same magnitude and phase as does the response to $A\omega_n^2 e^{i\omega t}$. This was the reason that we studied the response to the complex periodic input in the first place. Therefore, it is not necessary to analyze the response to the sine and cosine functions separately; they can be combined into a single analysis using a complex periodic input.

Example 2–1

A collar of mass m slides without friction along a circular arc portion of a rigid massless structure as shown in Fig. 2–5. The structure consists of two arms, one oriented horizontally and the other oriented at a constant angle α from the downward direction. The entire structure, centered at point Q, translates with *known* horizontal displacement $q(t)$ along a rod, where q is measured from a track-fixed point O. A collar of mass m slides along the arc of the structure circular part of the structure. The position of the collar relative to the structure is measured by the angle θ, where θ is measured from the downward direction. Attached to the collar is a curvilinear spring with spring constant K and unstretched length $\ell_0 = R\alpha$. Also, a viscous friction force with viscous friction coefficient c is exerted by the circular arc on the collar. The spring and friction forces are given, respectively, as

$$\begin{aligned}\mathbf{F}_s &= -K(\ell - \ell_0)\mathbf{e}_t \\ \mathbf{F}_f &= -c\mathbf{v}_{\text{rel}}\end{aligned}$$

where $\mathbf{e}_t$ is the tangent vector to the track at the location of the collar and $\mathbf{v}_{\text{rel}}$ is the velocity of the collar relative to the track. Assuming *no gravity*, determine (a) the differential equation of motion; (b) the static equilibrium value θ_{eq} for the system; (c) the differential equation of motion relative to the static equilibrium point found in (b); (d) the standard form of the differential equation obtained in part (c); (e) the transfer function for Θ/Q where Θ is the amplitude of the output(i.e., the amplitude of θ) $q(t) = (QK/\omega^2)e^{i\omega t}$; (f) the time response of the system to the sinusoidal input q given in part (e).

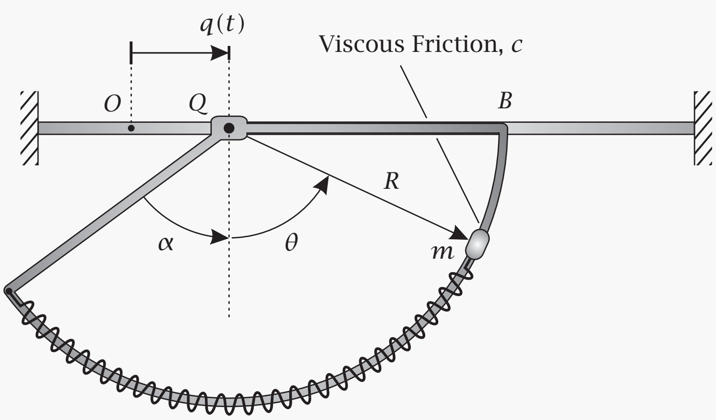

Figure 2–5 Collar of mass m moving on circular part of a structure, where the structure slides with horizontal displacement $q(t)$.

Solution to Example 2–1

(a) Differential Equation of Motion

Kinematics

Let $\mathcal{F}$ be fixed to the track. Then choose the following coordinate system fixed in $\mathcal{F}$:

$$\begin{array}{ccc} & \text{Origin at } Q \text{ when} q = 0 & \\ \mathbf{E}_x & = & \text{to the right} \\ \mathbf{E}_z & = & \text{out of page} \\ \mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x \end{array}$$

Next, let $\mathcal{A}$ be fixed to the structure. Then choose the following coordinate system fixed in $\mathcal{A}$:

$$\begin{array}{ccc} & \text{Origin at } Q & \\ \mathbf{e}_x & = & \text{to the right} \\ \mathbf{e}_z & = & \text{out of page} \\ \mathbf{e}_y & = & \mathbf{e}_z \times \mathbf{e}_x \end{array}$$

Finally, let $\mathcal{B}$ be fixed to the direction Qm. Then choose the following coordinate system fixed in $\mathcal{B}$:

$$\begin{array}{ccc} & \text{Origin at } Q & \\ \mathbf{e}_r & = & \text{along } Qm \\ \mathbf{e}_z & = & \text{out of page} \\ \mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r \end{array}$$

Then the position of point Q is given as

$$\mathbf{r}_Q = q\mathbf{E}_x = q\mathbf{e}_x \tag{2–104}$$

Furthermore, the position of the collar relative to point Q is given as

$$\mathbf{r}_{m/Q} = R\mathbf{e}_r \tag{2–105}$$

Then the position of the collar is obtained as

$$\mathbf{r} = \mathbf{r}_m = \mathbf{r}_Q + \mathbf{r}_{m/Q} = q\mathbf{e}_x + R\mathbf{e}_r \tag{2–106}$$

Next, the angular velocity of reference frame $\mathcal{B}$ in reference frame $\mathcal{F}$ is

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\theta}\mathbf{e}_z \tag{2–107}$$

Now the velocity and acceleration of point Q in reference frame $\mathcal{F}$ are

$$^{\mathcal{F}}\mathbf{v}_Q = \dot{q}\mathbf{e}_x \tag{2–108}$$
$$^{\mathcal{F}}\mathbf{a}_Q = \ddot{q}\mathbf{e}_x \tag{2–109}$$

The velocity of the collar relative to point Q in reference frame $\mathcal{F}$ is obtained from the transport theorem as

$$^{\mathcal{F}}\mathbf{v}_{m/Q} = \frac{^{\mathcal{F}}d}{dt}\left(\mathbf{r}_{m/Q}\right) = \frac{^{\mathcal{B}}d}{dt}\left(\mathbf{r}_{m/Q}\right) + {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}}} \times \mathbf{r}_{m/Q} \tag{2–110}$$

where

$$\frac{^{\mathcal{B}}d}{dt}\left(\mathbf{r}_{m/Q}\right) = \mathbf{0} \tag{2–111}$$
$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{m/Q} = \dot{\theta}\mathbf{e}_z \times R\mathbf{e}_r = R\dot{\theta}\mathbf{e}_\theta \tag{2–112}$$

Consequently,

$$^{\mathcal{F}}\mathbf{v}_{m/Q} = R\dot{\theta}\mathbf{e}_\theta \tag{2–113}$$

The acceleration of the collar relative to point Q in reference frame $\mathcal{F}$ is obtained from the transport theorem as

$$^{\mathcal{F}}\mathbf{a}_{m/Q} = \frac{^{\mathcal{F}}d}{dt}\left({^{\mathcal{F}}\mathbf{a}_{m/Q}}\right) = \frac{^{\mathcal{B}}d}{dt}\left({^{\mathcal{F}}\mathbf{v}_{m/Q}}\right) + {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}}} \times {^{\mathcal{F}}\mathbf{v}_{m/Q}} \tag{2–114}$$

where

$$\frac{^{\mathcal{B}}d}{dt}\left({^{\mathcal{F}}\mathbf{v}_{m/Q}}\right) = R\ddot{\theta}\mathbf{e}_\theta \tag{2–115}$$
$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {^{\mathcal{F}}\mathbf{v}_{m/Q}} = \dot{\theta}\mathbf{e}_z \times R\dot{\theta}\mathbf{e}_\theta = -R\dot{\theta}^2\mathbf{e}_r \tag{2–116}$$

Consequently,

$$^{\mathcal{F}}\mathbf{a}_{m/Q} = -R\dot{\theta}^2\mathbf{e}_r + R\ddot{\theta}\mathbf{e}_\theta \tag{2–117}$$

Finally, the acceleration of the collar in reference frame $\mathcal{F}$ is

$$^{\mathcal{F}}\mathbf{a} = {}^{\mathcal{F}}\mathbf{a}_Q + {}^{\mathcal{F}}\mathbf{a}_{m/Q} = \ddot{q}\mathbf{e}_x - R\dot{\theta}^2\mathbf{e}_r + R\ddot{\theta}\mathbf{e}_\theta \tag{2–118}$$

Kinetics

The free body diagram of the collar is shown in Fig. 2–6.

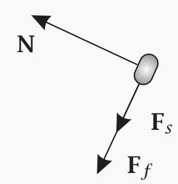

Figure 2–6 Free body diagram for Example 2–1.

Now the forces acting on the particle are

$$\begin{aligned} \mathbf{N} &= \text{Reaction force of track on collar} \\ \mathbf{F}_s &= \text{Force of curvilinear spring} \\ \mathbf{F}_f &= \text{Force of viscous friction} \end{aligned}$$

Resolving these forces, we have

$$\mathbf{N} = N\mathbf{e}_r \tag{2–119}$$

$$\mathbf{F}_s = -K(\ell - \ell_0)\mathbf{e}_t \tag{2–120}$$

$$\mathbf{F}_f = -c\mathbf{v}_{\text{rel}} \tag{2–121}$$

Now

$$\mathbf{e}_t = \mathbf{e}_\theta \tag{2–122}$$

$$\ell = R(\alpha + \theta) \tag{2–123}$$

$$\ell_0 = R\alpha \tag{2–124}$$

$$\mathbf{v}_{\text{rel}} = {}^{\mathcal{F}}\mathbf{v} - {}^{\mathcal{F}}\mathbf{v}_Q = {}^{\mathcal{F}}\mathbf{v}_{m/Q} = R\dot{\theta}\mathbf{e}_\theta \tag{2–125}$$

Then the spring and friction forces are given as

$$\mathbf{F}_s = -K(R(\alpha + \theta) - R\alpha)\mathbf{e}_\theta = -KR\theta\mathbf{e}_\theta \tag{2–126}$$

$$\mathbf{F}_f = -cR\dot{\theta}\mathbf{e}_\theta \tag{2–127}$$

The resultant force acting on the particle is then given as

$$\mathbf{F} = \mathbf{N} + \mathbf{F}_s + \mathbf{F}_f = N\mathbf{e}_r - KR\theta\mathbf{e}_\theta - cR\dot{\theta}\mathbf{e}_\theta \tag{2–128}$$

Applying Newton's second law, we obtain

$$N\mathbf{e}_r - KR\theta\mathbf{e}_\theta - cR\dot{\theta}\mathbf{e}_\theta = m(\ddot{q}\mathbf{e}_x - R\dot{\theta}^2\mathbf{e}_r + R\ddot{\theta}\mathbf{e}_\theta) = m\ddot{q}\mathbf{e}_x - mR\dot{\theta}^2\mathbf{e}_r + mR\ddot{\theta}\mathbf{e}_\theta \tag{2–129}$$

Now it is convenient to substitute $\mathbf{e}_x$ in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ as

$$\mathbf{e}_x = \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta \tag{2–130}$$

Therefore,

$$\begin{aligned} N\mathbf{e}_r - (KR\theta + cR\dot{\theta})\mathbf{e}_\theta &= m\ddot{q}\mathbf{e}_x - mR\dot{\theta}^2\mathbf{e}_r + mR\ddot{\theta}\mathbf{e}_\theta \\ &= m\ddot{q}(\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta) - mR\dot{\theta}^2\mathbf{e}_r + mR\ddot{\theta}\mathbf{e}_\theta \\ &= m\ddot{q}\sin\theta\,\mathbf{e}_r + m\ddot{q}\cos\theta\,\mathbf{e}_\theta - mR\dot{\theta}^2\mathbf{e}_r + mR\ddot{\theta}\mathbf{e}_\theta \\ &= (m\ddot{q}\sin\theta - mR\dot{\theta}^2)\mathbf{e}_r + (m\ddot{q}\cos\theta + +mR\ddot{\theta})\mathbf{e}_\theta \end{aligned} \qquad (2\text{–}131)$$

Setting the $\mathbf{e}_r$ and $\mathbf{e}_\theta$ components equal, we obtain

$$N = m\ddot{q}\sin\theta - mR\dot{\theta}^2 \qquad (2\text{–}132)$$

$$-(KR\theta + cR\dot{\theta}) = m\ddot{q}\cos\theta + +mR\ddot{\theta} \qquad (2\text{–}133)$$

It is seen that the second of these equation is the differential equation of motion. Rearranging, we obtain

$$mR\ddot{\theta} + cR\dot{\theta} + KR\theta = -m\ddot{q}\cos\theta \qquad (2\text{–}134)$$

(b) Static Equilibrium Point

Let θ_{eq} be the static equilibrium value of θ. Setting $\dot{\theta}_{eq}$, $\ddot{\theta}_{eq}$, and $q(t)$ equal to zero, we see that the static equilibrium point is given as

$$KR\theta_{eq} = 0 \qquad (2\text{–}135)$$

Equation (2–135) implies that

$$\theta_{eq} = 0 \qquad (2\text{–}136)$$

(c) Differential Equation Linearized Relative to θ_{eq}

It is seen that it is not necessary to change $\dot{\theta}$ and $\ddot{\theta}$ because the static equilibrium point is $\theta_{eq} = 0$. Now the linearized value of $\cos\theta$ is

$$\cos\theta \approx 1 \qquad (2\text{–}137)$$

for values of θ near zero. Therefore, the linearized differential equation for values of θ near θ_{eq} is

$$mR\ddot{\theta} + cR\dot{\theta} + KR\theta = -m\ddot{q} \qquad (2\text{–}138)$$

(d) Standard Form of Differential Equation

Dividing the linearized differential equation by mR, we obtain

$$\ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{K}{m}\theta = -\frac{\ddot{q}}{R} \qquad (2\text{–}139)$$

(e) Transfer Function for Input $q(t) = QKe^{i\omega t}/\omega^2$

Differentiating $q(t)$, we obtain

$$\dot{q}(t) = iQKe^{i\omega t}/\omega \qquad (2\text{–}140)$$

$$\ddot{q}(t) = -QKe^{i\omega t} \qquad (2\text{–}141)$$

Then the differential equation is

$$\ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{K}{m}\theta = \frac{QK}{R}e^{i\omega t} = \frac{Qm\omega_n^2}{R}e^{i\omega t} = \frac{Qm}{R}\omega_n^2 e^{i\omega t} \qquad (2\text{–}142)$$

where, because $\omega_n^2 = K/m$, we have $K = m\omega_n^2$. Now let

$$A = \frac{Qm}{R} \tag{2–143}$$

Furthermore, let

$$\theta(t) = \Theta e^{i\omega t} \tag{2–144}$$

which implies

$$\dot{\theta} = i\omega\Theta e^{i\omega t} \tag{2–145}$$

$$\ddot{\theta} = -\omega^2\Theta e^{i\omega t} \tag{2–146}$$

Therefore,

$$\Theta e^{i\omega t}\left[-\omega^2 + i2\zeta\omega_n\omega + \omega_n^2\right] = A\omega_n^2 e^{i\omega t} \tag{2–147}$$

Therefore,

$$\frac{\Theta}{A} = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + i2\zeta\omega_n\omega} \tag{2–148}$$

which implies that

$$\frac{\Theta}{Q} = \frac{\frac{A}{Q}\omega_n^2}{\omega_n^2 - \omega^2 + i2\zeta\omega_n\omega} = \frac{m}{R}\frac{\omega_n^2}{\omega_n^2 - \omega^2 + i2\zeta\omega_n\omega} = \frac{m}{R}\frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + i2\zeta\frac{\omega}{\omega_n}} \tag{2–149}$$

Consequently,

$$\frac{\Theta}{Q} = \frac{m}{R}G(i\omega) \tag{2–150}$$

where

$$G(i\omega) = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + i2\zeta\frac{\omega}{\omega_n}} \tag{2–151}$$

(f) Time Response to Input Given in Part (e)

The time response for the standard system

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = A\omega_n^2 e^{i\omega t} \tag{2–152}$$

is given as

$$x(t) = A|G(i\omega)|e^{i(\omega t - \phi)} \tag{2–153}$$

where $|G(i\omega)|$ and ϕ are the magnitude and phase of $G(i\omega)$. Now our input amplitude is

$$A = \frac{Qm}{R} \tag{2–154}$$

Therefore, the time response for this problem is

$$\theta(t) = \frac{Qm}{R}|G(i\omega)|e^{i(\omega t - \phi)} \tag{2–155}$$

■

Example 2–2

A collar of mass m slides along an inertially fixed circular track of radius R as shown in Fig. 2–7. Attached to the collar is a curvilinear spring with spring constant K and unstretched length $\ell_0 = R\theta_0$. The position of the collar on the track is measured by the angle θ, where θ is measured from the inertially fixed downward direction. Furthermore, the contact between the track and the collar creates a viscous friction force with friction coefficient c. The forces exerted by the curvilinear spring and the viscous damper are given, respectively, as

$$\begin{aligned} \mathbf{F}_s &= -K(\ell - \ell_0)\mathbf{e}_t \\ \mathbf{F}_f &= -c\mathbf{v}_{\text{rel}} \end{aligned}$$

where $\mathbf{e}_t$ is the tangent vector to the track at the location of the collar and $\mathbf{v}_{\text{rel}}$ is the velocity of the collar relative to the track. Finally, attached to the other end of the spring is a massless collar that moves with *specified* displacement described the angle $\phi(t)$, where, like θ, ϕ is also measured from inertially fixed downward direction. *Assuming no gravity*, determine (a) the differential equation of motion; (b) the value θ_{eq} for which the system is in static equilibrium; (c) the differential equation of motion relative to the static equilibrium found in part (b); (d) the standard form of the differential equation obtained in part (d); (e) the natural frequency, damping ratio, and damped natural frequency of the system (assuming that the system is underdamped); (f) the transfer function $\bar{\alpha}/P$ where $\bar{\alpha}$ is the amplitude of the output of $\alpha(t)$ and $\phi(t) = PK\sin\omega t$; (g) the time response of the system to the sinusoidal input ϕ given in part (f).

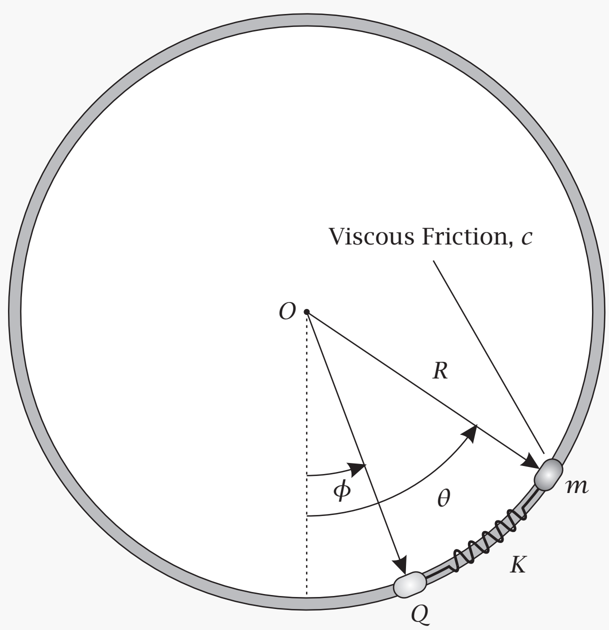

Figure 2–7 Collar sliding on fixed circular track attached to a linear spring with moving attachment point and viscous friction.

Solution to Example 2–2

(a) Differential Equation of Motion

Kinematics

Let $\mathcal{F}$ be fixed to the circular track. Then choose the following coordinate system fixed in $\mathcal{F}$:

$$
\begin{array}{ccc}
 & \text{Origin at } O & \\
\mathbf{E}_x & = & \text{along } Om \text{ when } \theta = 0 \\
\mathbf{E}_z & = & \text{out of page} \\
\mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x
\end{array}
$$

Next, let $\mathcal{A}$ be fixed to the direction Om. Then choose the following coordinate system fixed in $\mathcal{A}$:

$$
\begin{array}{ccc}
 & \text{Origin at } O & \\
\mathbf{e}_r & = & \text{along } Om \\
\mathbf{e}_z & = & \text{out of page} \\
\mathbf{e}_z & = & \mathbf{e}_z \times \mathbf{e}_r
\end{array}
$$

Then the position of the collar is given as

$$\mathbf{r} = R\mathbf{e}_r \tag{2–156}$$

which implies that

$$^{\mathcal{F}}\mathbf{v} = \frac{^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{^{\mathcal{A}}d\mathbf{r}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} \tag{2–157}$$

where $^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \dot{\theta}\mathbf{e}_z$. Now we have

$$\frac{^{\mathcal{A}}d\mathbf{r}}{dt} = \mathbf{0} \tag{2–158}$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} = \dot{\theta}\mathbf{e}_z \times R\mathbf{e}_r = R\dot{\theta}\mathbf{e}_\theta \tag{2–159}$$

which implies that

$$^{\mathcal{F}}\mathbf{v} = R\dot{\theta}\mathbf{e}_\theta \tag{2–160}$$

The acceleration of the collar as viewed by an observer fixed to the track is then given as

$$^{\mathcal{F}}\mathbf{a} = \frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) = \frac{^{\mathcal{A}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} \tag{2–161}$$

Now we have

$$\frac{^{\mathcal{A}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) = R\ddot{\theta}\mathbf{e}_\theta \tag{2–162}$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} = \dot{\theta}\mathbf{e}_z \times R\dot{\theta}\mathbf{e}_\theta = -R\dot{\theta}^2\mathbf{e}_r \tag{2–163}$$

which implies that

$$^{\mathcal{F}}\mathbf{a} = -R\dot{\theta}^2\mathbf{e}_r + R\ddot{\theta}\mathbf{e}_\theta \tag{2–164}$$

Kinetics

From the free body diagram, the following forces act on the collar:

$$
\begin{array}{ccl}
\mathbf{N} & = & \text{Reaction force of track} \\
\mathbf{F}_s & = & \text{Force of curvilinear spring} \\
\mathbf{F}_f & = & \text{Force of viscous friction}
\end{array}
$$

Now we have

$$\mathbf{N} = N\mathbf{e}_r \quad (2\text{–}165)$$

$$\mathbf{F}_s = -K(\ell - \ell_0)\mathbf{e}_t \quad (2\text{–}166)$$

$$\mathbf{F}_f = -c\mathbf{v}_{\text{rel}} \quad (2\text{–}167)$$

Using the fact that $\mathbf{e}_t = \mathbf{e}_\theta$ and that the surface is absolutely fixed, we obtain $\mathbf{v}_{\text{rel}} = {}^{\mathcal{F}}\mathbf{v}$. Consequently,

$$\mathbf{N} = N\mathbf{e}_r \quad (2\text{–}168)$$

$$\mathbf{F}_s = -K(\ell - \ell_0)\mathbf{e}_\theta \quad (2\text{–}169)$$

$$\mathbf{F}_f = -cR\dot{\theta}\mathbf{e}_\theta \quad (2\text{–}170)$$

Finally, we know that

$$\ell = R(\theta - \phi) \quad (2\text{–}171)$$

and that $\ell_0 = R\theta_0$ which implies

$$\mathbf{F}_s = -K(R(\theta - \phi) - R\theta_0)\mathbf{e}_\theta = -KR(\theta - \phi - \theta_0)\mathbf{e}_\theta \quad (2\text{–}172)$$

The resultant force acting on the collar is then obtained as

$$\mathbf{F} = N\mathbf{e}_r - KR(\theta - \phi - \theta_0)\mathbf{e}_\theta - cR\dot{\theta}\mathbf{e}_\theta \quad (2\text{–}173)$$

Applying Newton's second law to the collar, we obtain

$$N\mathbf{e}_r - KR(\theta - \phi - \theta_0)\mathbf{e}_\theta - cR\dot{\theta}\mathbf{e}_\theta = m\left[-R\dot{\theta}^2\mathbf{e}_r + R\ddot{\theta}\mathbf{e}_\theta\right] \quad (2\text{–}174)$$

which yields the following two scalar equations:

$$-mR\dot{\theta}^2 = N \quad (2\text{–}175)$$

$$mR\ddot{\theta} = -KR(\theta - \phi - \theta_0) - cR\dot{\theta} \quad (2\text{–}176)$$

It is seen that the second of these last two equations has no unknown reaction forces and, thus, is the differential equation. Rearranging this equation, we obtain

$$mR\ddot{\theta} + cR\dot{\theta} + KR\theta = KR\theta_0 + KR\phi \quad (2\text{–}177)$$

Dropping the common factor of R gives

$$m\ddot{\theta} + c\dot{\theta} + K\theta = K\theta_0 + K\phi \quad (2\text{–}178)$$

(b) Static Equilibrium Point

Let θ_{eq} be the static equilibrium point. Then we have $\dot{\theta}_{eq} = \ddot{\theta}_{eq} = 0$. Also, setting $\phi = 0$, we obtain

$$K\theta_{eq} = K\theta_0 \quad (2\text{–}179)$$

which implies

$$\theta_{eq} = \theta_0 \quad (2\text{–}180)$$

(c) Differential Equation Relative to Equilibrium Point

Let $\alpha = \theta - \theta_0$. Then $\dot{\alpha} = \dot{\theta}$ and $\ddot{\alpha} = \ddot{\theta}$ which implies that

$$m\ddot{\alpha} + c\dot{\alpha} + K(\alpha + \theta_0) = K\theta_0 + K\phi \quad (2\text{–}181)$$

Simplifying this last equation gives

$$m\ddot{\alpha} + c\dot{\alpha} + K\alpha = K\phi \quad (2\text{–}182)$$

(d) Standard Form of Differential Equation

Dividing the last differential equation by m gives

$$\ddot{\alpha} + \frac{c}{m}\dot{\alpha} + \frac{K}{m}\alpha = \frac{K}{m}\phi \tag{2–183}$$

(e) Natural Frequency, Damping Ratio, and Damped Natural Frequency

The natural frequency is given as

$$\omega_n = \sqrt{K/m} \tag{2–184}$$

The damping ratio is found by solving

$$2\zeta\omega_n = \frac{c}{m} \tag{2–185}$$

which implies that the damping ratio is given as

$$\zeta = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{mK}} \tag{2–186}$$

The damped natural frequency is given as

$$\omega_d = \sqrt{1-\zeta^2}\omega_n \tag{2–187}$$

where ζ and ω_n are as computed above.

(f) Transfer Function for Periodic Input $\phi(t) = PK\sin\omega t$

We know that the transfer function for an input of the form $\sin\omega t$ is the same as the transfer function for an input $e^{i\omega t}$. Therefore, for this part of the problem let $\phi(t) = PKe^{i\omega t}$. Also, let

$$\alpha(t) = \bar{\alpha}e^{i\omega t} \tag{2–188}$$

Then

$$\dot{\alpha}(t) = i\omega bar\alpha e^{i\omega t} \tag{2–189}$$

$$\ddot{\alpha}(t) = -\omega^2 bar\alpha e^{i\omega t} \tag{2–190}$$

Substituting into the differential equation, we obtain

$$\bar{\alpha}e^{i\omega t}\left[-\omega^2 + i2\zeta\omega_n\omega + \omega_n^2\right] = \frac{K}{m}PKe^{i\omega t} = PK\omega_n^2 e^{i\omega t} \tag{2–191}$$

Now let

$$A = PK \tag{2–192}$$

Then,

$$\bar{\alpha}e^{i\omega t}\left[\omega_n^2 - \omega^2 + i2\zeta\omega_n\omega\right] = A\omega_n^2 e^{i\omega t} \tag{2–193}$$

Rearranging gives

$$\frac{\bar{\alpha}}{A} = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + i2\zeta\omega_n\omega} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + i2\zeta\frac{\omega}{\omega_n}} = G(i\omega) \tag{2–194}$$

Therefore,

$$\frac{\bar{\alpha}}{P} = \frac{\bar{\alpha}}{A}\frac{A}{P} = KG(i\omega) \tag{2–195}$$

(g) Time Response to $\phi(t) = AK\sin\omega t$

We know that the time response to the standard system

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = A\omega_n^2 e^{i\omega t} \tag{2–196}$$

is given as

$$x(t) = A|G(i\omega)|e^{i(\omega t-\phi)} \tag{2–197}$$

Now in our case we have

$$A = PK \tag{2–198}$$

which implies that

$$\alpha(t) = PK|G(i\omega)|e^{i(\omega t-\phi)} \tag{2–199}$$

Therefore, the response of the system to the input $AK\sin\omega t$ is the *imaginary* part of $\alpha(t)$, i.e.,

$$\alpha_r(t) = \text{Im}\,[\alpha(t)] = PK|G(i\omega)|\sin(\omega t - \phi) \tag{2–200}$$

■

Example 2–3

A massless cart moves horizontally along the ground with a known displacement $q(t)$, where q is measured from a point O fixed to the ground as shown in Fig. 2–8. A block of mass m slides along the surface of the cart. Attached to the block are a linear spring with spring constant K and unstretched length ℓ_0 and a viscous damper with damping coefficient c. The spring and damper are attached at point Q, where Q is located on the vertical support of the cart. Knowing that x describes the displacement of the block relative to the cart and that gravity acts downward, determine (a)the differential equation of motion for the system; (b) (b) the static equilibrium value x_{eq} for the differential equation given in part (a); (c) the differential equation of motion relative to the static equilibrium found in part (b); (d) the standard form of the differential equation obtained in part (c); (e) the natural frequency, damping ratio, and damped natural frequency of the system in terms of the parameters K and c (assuming that the system is underdamped); (f) the transfer function associated with the ratio of the amplitude Y/Q where Y is the amplitude of the output $y(t)$ and $q(t) = QKe^{ia\omega t}$; (g) the time response, denoted $z(t)$, of the system to the periodic input $q(t) = QK(\cos a\omega t)$.

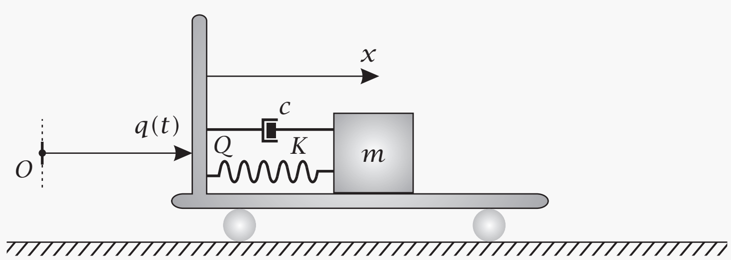

Figure 2–8 Block sliding on horizontally moving cart with linear spring and viscous damper.

Solution to Example 2–3

(a) Differential Equation of Motion

Kinematics

Let $\mathcal{F}$ be fixed to the ground. Then choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$
\begin{array}{ccc}
 & \text{Origin at } O & \\
\mathbf{E}_x & = & \text{to the right} \\
\mathbf{E}_z & = & \text{into page} \\
\mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x
\end{array}
$$

Next, $\mathcal{A}$ be fixed to the block. Then choose the following coordinate system fixed in reference frame $\mathcal{A}$:

$$
\begin{array}{ccc}
 & \text{Origin at } Q & \\
\mathbf{e}_x & = & \text{along } Qm \\
\mathbf{e}_z & = & \mathbf{E}_z \\
\mathbf{e}_y & = & \mathbf{e}_z \times \mathbf{e}_x
\end{array}
$$

Now, because the block is in pure translation, the position of the support Q is given as

$$\mathbf{r}_Q = q\mathbf{E}_x \tag{2–201}$$

Next, the position of the block relative to the upper support is given as

$$\mathbf{r}_{P/Q} = x\mathbf{e}_x \tag{2–202}$$

Therefore, the position of the block relative to the ground is obtained as

$$\mathbf{r} = \mathbf{r}_P = \mathbf{r}_Q + \mathbf{r}_{P/Q} = q\mathbf{E}_x + x\mathbf{e}_x = (q + x)\mathbf{e}_x \tag{2–203}$$

where we note that $\mathbf{E}_x = \mathbf{e}_x$. Then the velocity and acceleration of the block in reference frame $\mathcal{F}$ are given as block are given, respectively, as

$$^{\mathcal{F}}\mathbf{v} = (\dot{q} + \dot{x})\mathbf{e}_x \tag{2–204}$$

$$^{\mathcal{F}}\mathbf{a} = (\ddot{q} + \ddot{x})\mathbf{e}_x \tag{2–205}$$

Kinetics

The free body diagram of the block is shown in Fig. 2–9.

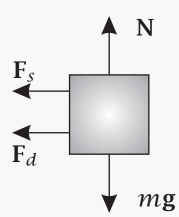

Figure 2–9 Free body diagram for block sliding on horizontally moving cart with linear spring and viscous damper.

where

$$
\begin{array}{ccl}
\mathbf{F}_s & = & \text{Force exerted by spring} \\
\mathbf{F}_d & = & \text{Force exerted by damper} \\
m\mathbf{g} & = & \text{Force of gravity} \\
\mathbf{N} & = & \text{Reaction force of cart on block}
\end{array}
$$

Now we have

$$\mathbf{F}_s = -K(\ell - \ell_0)\mathbf{u}_s \tag{2–206}$$
$$\mathbf{F}_d = -c\mathbf{v}_{\text{rel}} \tag{2–207}$$
$$m\mathbf{g} = -mg\mathbf{e}_y \tag{2–208}$$
$$N = N\mathbf{e}_y \tag{2–209}$$

Observing that the spring is attached at point Q, we have

$$\mathbf{r} - \mathbf{r}_Q = q\mathbf{E}_x + x\mathbf{e}_x - q\mathbf{E}_x = x\mathbf{e}_x \tag{2–210}$$
$$\ell = \|\mathbf{r} - \mathbf{r}_Q\| = x \tag{2–211}$$
$$\mathbf{u}_s = \frac{\mathbf{r} - \mathbf{r}_Q}{\|\mathbf{r} - \mathbf{r}_Q\|} = \frac{x\mathbf{e}_x}{x} = \mathbf{e}_x \tag{2–212}$$
(2–213)

Therefore,

$$\mathbf{F}_s = -K(x - \ell_0)\mathbf{e}_x \tag{2–214}$$

Next, the relative velocity $\mathbf{v}_{\text{rel}}$ is computed as

$$\mathbf{v}_{\text{rel}} = {}^{\mathcal{F}}\mathbf{v} - {}^{\mathcal{F}}\mathbf{v}_Q = (\dot{q} + \dot{x})\mathbf{e}_x - \dot{q}\mathbf{e}_x = \dot{x}\mathbf{e}_x \tag{2–215}$$

where it is noted that ${}^{\mathcal{F}}\mathbf{v}_Q = \dot{q}\mathbf{E}_x = \dot{q}\mathbf{e}_x$. Therefore, the force of the damper is given as

$$\mathbf{F}_d = -c\mathbf{v}_{\text{rel}} = -c\dot{x}\mathbf{e}_x \tag{2–216}$$

The resultant force acting on the particle is then given as

$$\mathbf{F} = \mathbf{F}_s + +\mathbf{F}_d + m\mathbf{g} + \mathbf{N} = -K(x - \ell_0)\mathbf{e}_x - c\dot{x}\mathbf{e}_x - mg\mathbf{e}_y + N\mathbf{e}_y \tag{2–217}$$

Then, applying Newton's second law (i.e., $\mathbf{F} = m^{\mathcal{F}}\mathbf{a}$), we have

$$-K(x - \ell_0)\mathbf{e}_x - c\dot{x}\mathbf{e}_x - mg\mathbf{e}_y + N\mathbf{e}_y = m(\ddot{q} + \ddot{x})\mathbf{e}_x \tag{2–218}$$

Separating this last equation into $\mathbf{e}_x$ and $\mathbf{e}_y$ components gives

$$-\left[K(x - \ell_0) + c\dot{x}\right]\mathbf{e}_x + \left[N - mg\right]\mathbf{e}_y = m(\ddot{q} + \ddot{x})\mathbf{e}_x \tag{2–219}$$

Equating $\mathbf{e}_x$ and $\mathbf{e}_y$ components gives

$$-\left[K(x - \ell_0) + c\dot{x}\right] = m(\ddot{q} + \ddot{x}) \tag{2–220}$$
$$N - mg = 0 \tag{2–221}$$

It is seen that Eq. (2–220) has no unknown reaction forces and, thus, is the differential equation of motion. Rearranging Eq. (2–220), we obtain

$$m\ddot{x} + c\dot{x} + Kx = K\ell_0 - m\ddot{q} \tag{2–222}$$

(b) Static Equilibrium Point

Let x_{eq} be the static equilibrium point. Then

$$\dot{x}_{eq} = 0 \tag{2–223}$$
$$\ddot{x}_{eq} = 0 \tag{2–224}$$

Furthermore, in order to find the static equilibrium point, we need to set $q(t) = 0$. Substituting the equilibrium conditions into Eq. (2–222) gives

$$Kx_{eq} = K\ell_0 \tag{2–225}$$

Solving for x_{eq}, we obtain

$$x_{eq} = \ell_0 \tag{2–226}$$

(c) Differential Equation Relative to Static Equilibrium Point

Suppose we define

$$y = x - x_{eq} \Longrightarrow x = y + x_{eq} = y + \ell_0 \tag{2–227}$$

We then have

$$\dot{y} = \dot{x} \tag{2–228}$$
$$\ddot{y} = \ddot{x} \tag{2–229}$$

Then the differential equation can be written in terms of y as

$$m\ddot{y} + c\dot{y} + K(y + \ell_0) = K\ell_0 - m\ddot{q} \tag{2–230}$$

Simplifying this last equation gives

$$m\ddot{y} + c\dot{y} + Ky = -m\ddot{q} \tag{2–231}$$

(d) Standard Form of Differential Equation

Dividing Eq. (2–231) by m, we obtain the standard form of the differential equation as

$$\ddot{y} + \frac{c}{m}\dot{y} + \frac{K}{m}y = -\ddot{q} \tag{2–232}$$

(e) Natural Frequency, Damping Ratio, and Damped Natural Frequency

The natural frequency is given as

$$\omega_n = \sqrt{\frac{K}{m}} \tag{2–233}$$

The damping ratio is found by solving

$$2\zeta\omega_n = \frac{c}{m} \tag{2–234}$$

for ζ. We have

$$2\zeta\omega_n = 2\zeta\sqrt{\frac{K}{m}} = \frac{c}{m} \tag{2–235}$$

Solving for ζ gives

$$\zeta = \frac{c}{2m}\sqrt{\frac{m}{K}} = \frac{c}{2\sqrt{mK}} \tag{2–236}$$

Finally, the damped natural frequency is given as

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} \tag{2–237}$$

(f) Transfer Function for Periodic Input $q(t) = QKe^{ia\omega t}$

Differentiating $q(t) = QKe^{ia\omega t}$ twice gives

$$\dot{q}(t) = ia\omega QKe^{ia\omega t} \tag{2–238}$$
$$\ddot{q}(t) = -a^2\omega^2 QKe^{ia\omega t} \tag{2–239}$$

Substituting $\ddot{q}(t)$ into Eq. (2–232) and using the generic expressions for ω_n and ζ, we obtain

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = a^2\omega^2 QKe^{ia\omega t} \tag{2–240}$$

Now let the output $y(t)$ be given as

$$y(t) = Ye^{ia\omega t} \tag{2–241}$$

Then

$$\dot{y} = ia\omega Ye^{i\omega t} \tag{2–242}$$
$$\ddot{y} = -a^2\omega^2 Ye^{i\omega t} \tag{2–243}$$

Substituting y, $\dot{y}$, and $\ddot{y}$ into Eq. (2–240) gives

$$\left[-a^2\omega^2 + i2a\zeta\omega_n\omega + \omega_n^2\right]Ye^{ia\omega t} = a^2\omega^2 QKe^{ia\omega t} \tag{2–244}$$

Observing that $K = m\omega_n^2$, we have

$$\left[\omega_n^2 - a^2\omega^2 + ia2\zeta\omega_n\omega\right]Ye^{ia\omega t} = a^2\omega^2 QKe^{ia\omega t} = \left(Qma^2\omega^2\right)\omega_n^2 e^{i\omega t} \tag{2–245}$$

Now let

$$A = Qma^2\omega^2 \tag{2–246}$$

Then, dropping the common factor of $e^{i\omega t}$ and dividing through by $-a^2\omega^2 + i2a\zeta\omega_n\omega + \omega_n^2$ gives

$$Y = \frac{A\omega_n^2}{\omega_n^2 - a^2\omega^2 + i2a\zeta\omega_n\omega} \tag{2–247}$$

Then, dividing numerator and denominator by ω_n^2 gives

$$Y = \frac{A}{1 - \left(\frac{a\omega}{\omega_n}\right)^2 + i2\zeta\frac{a\omega}{\omega_n}} \tag{2–248}$$

Dividing both sides by Q, we obtain the transfer function Y/A as

$$\frac{Y}{Q} = \frac{A/Q}{1 - \left(\frac{a\omega}{\omega_n}\right)^2 + i2\zeta\frac{a\omega}{\omega_n}} \tag{2–249}$$

Now let

$$\Omega = a\omega \tag{2–250}$$

Then, in terms of Ω we can let

$$G(i\Omega) = \frac{1}{1 - \left(\frac{\Omega}{\omega_n}\right)^2 + i2\zeta\frac{\Omega}{\omega_n}} \tag{2–251}$$

Then

$$\frac{Y}{Q} = \frac{A}{Q}G(i\Omega) = m\Omega^2 G(i\Omega) \tag{2–252}$$

(g) Time Response $z(t)$ to $q(t) = QK\cos a\omega t$

We know that for a system in the standard form

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = A\omega_n^2 e^{i\omega t} \tag{2–253}$$

The time response is

$$x(t) = A|G(i\omega)|e^{i(\omega t - \phi)} \tag{2–254}$$

In our case we have the amplitude A from Eq. (2–246) as

$$A = Qma^2\omega^2 = Qm\Omega^2 \tag{2–255}$$

where we recall that $\Omega = a\omega$. Therefore, the time response to the input $QKe^{ia\omega t}$ is

$$y(t) = Qm\Omega^2|G(i\Omega)|e^{i(\Omega t-\phi)} \tag{2–256}$$

Then the response to the input $QK\cos a\omega t$ is the *real* part of $y(t)$, i.e.,

$$\begin{aligned} z(t) &= \text{Re}\left[Qm\Omega^2|G(i\Omega)|e^{i(\Omega t-\phi)}\right] \\ &= Qm\Omega^2|G(i\Omega)|\cos(\Omega t-\phi) \\ &= Qma^2\omega^2|G(ia\omega)|\cos(a\omega t-\phi) \end{aligned} \tag{2–257}$$

■

Example 2–4

A collar of mass m_1 slides along an inertially fixed track. The displacement of the collar is measured relative to the fixed point O by the variable x. The collar is attached to a linear spring with spring constant K, a linear damper with damping coefficient c, and a rigid massless arm of length L. Attached to the other end of the arm is a particle of mass m_2. Knowing that the arm rotates with a constant angular rate Ω, (a) derive the differential equation of motion for the system in terms of the displacement x; (b) determine the equilibrium point of the system; (c) write the differential equation in part (a) relative to the equilibrium point found in part (b); and (d) determine the time response of the collar.

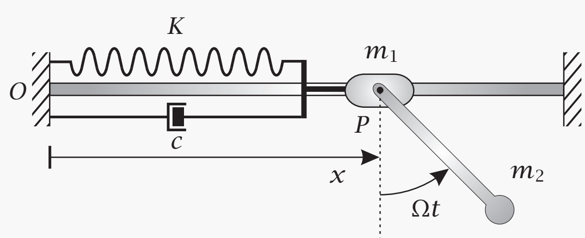

Figure 2–10 Collar on Spring and Damper with Imbalanced Mass.

Solution to Example 2–4

(a) Differential Equation of Motion

Kinematics

First, let $\mathcal{F}$ be the track. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

	Origin at point O	
$\mathbf{E}_x$	=	Along OP
$\mathbf{E}_z$	=	Out of page
$\mathbf{E}_y$	=	$\mathbf{E}_z \times \mathbf{E}_x$

Next, let $\mathcal{R}$ be a reference frame fixed to the arm. Then, choose the following coordinate system fixed in reference frame $\mathcal{R}$:

$$\begin{array}{ccc} & \text{Origin at } O & \\ \mathbf{e}_r & = & \text{Along } m_1 m_2 \\ \mathbf{e}_z & = & \text{Out of page} \\ \mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r \end{array}$$

Then the angular velocity of arm as viewed by an observer fixed to the track is given as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} = \Omega \mathbf{e}_z \tag{2–258}$$

Next, the position of the collar is given as

$$\mathbf{r}_1 = x\mathbf{E}_x \tag{2–259}$$

The velocity and acceleration of the collar in reference frame $\mathcal{F}$ are given, respectively, as

$$^{\mathcal{F}}\mathbf{v}_1 = \dot{x}\mathbf{E}_x \tag{2–260}$$

$$^{\mathcal{F}}\mathbf{a}_1 = \ddot{x}\mathbf{E}_x \tag{2–261}$$

Now in order to solve this problem, we also need the acceleration of the particle attached to the arm. The position of the particle is given as

$$\mathbf{r}_2 = \mathbf{r}_1 + \mathbf{r}_{2/1} \tag{2–262}$$

where $\mathbf{r}_{2/1}$ is the position of the particle relative to the collar. Now we know that $\mathbf{r}_{2/1}$ is given as

$$\mathbf{r}_{2/1} = L\mathbf{e}_r \tag{2–263}$$

Then, the velocity and acceleration of the particle are given, respectively, as

$$^{\mathcal{F}}\mathbf{v}_2 = {^{\mathcal{F}}\mathbf{v}_1} + {^{\mathcal{F}}\mathbf{v}_{2/1}} \tag{2–264}$$

$$^{\mathcal{F}}\mathbf{v}_2 = {^{\mathcal{F}}\mathbf{a}_1} + {^{\mathcal{F}}\mathbf{a}_{2/1}} \tag{2–265}$$

Now we already have $^{\mathcal{F}}\mathbf{v}_1$ and $^{\mathcal{F}}\mathbf{a}_1$ from Eqs. (2–260) and (2–261), respectively. Computing the velocity of the particle relative to the collar, we have

$$^{\mathcal{F}}\mathbf{v}_{2/1} = \frac{^{\mathcal{F}}d}{dt}\left(\mathbf{r}_{2/1}\right) = \frac{^{\mathcal{R}}d}{dt}\left(\mathbf{r}_{2/1}\right) + {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}} \times \mathbf{r}_{2/1} \tag{2–266}$$

Now we have

$$\frac{^{\mathcal{R}}d}{dt}\left(\mathbf{r}_{2/1}\right) = \mathbf{0} \tag{2–267}$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{2/1} = \Omega\mathbf{e}_z \times L\mathbf{e}_r = L\Omega\mathbf{e}_\theta \tag{2–268}$$

Adding the expressions in Eqs. (2–267) and (2–268), we obtain

$$^{\mathcal{F}}\mathbf{v}_{2/1} = L\Omega\mathbf{e}_\theta \tag{2–269}$$

Next, the acceleration of the particle relative to the collar in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\mathbf{a}_{2/1} = \frac{^{\mathcal{F}}d}{dt}\left({^{\mathcal{F}}\mathbf{v}_{2/1}}\right) = \frac{^{\mathcal{R}}d}{dt}\left({^{\mathcal{F}}\mathbf{v}_{2/1}}\right) + {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}} \times {^{\mathcal{F}}\mathbf{v}_{2/1}} \tag{2–270}$$

Noting that L and Ω are constant, we have

$$\frac{^{\mathcal{R}}d}{dt}\left({^{\mathcal{F}}\mathbf{v}_{2/1}}\right) = \mathbf{0} \tag{2–271}$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times {^{\mathcal{F}}\mathbf{v}_{2/1}} = \Omega\mathbf{e}_z \times L\Omega\mathbf{e}_\theta = -L\Omega^2\mathbf{e}_r \tag{2–272}$$

Adding Eqs. (2–271) and (2–272), we obtain the acceleration of the particle relative to the collar in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{a}_{2/1} = -L\Omega^2\mathbf{e}_r \tag{2–273}$$

Adding Eqs. (2–261) and (2–273), the acceleration of the particle is given as

$$^{\mathcal{F}}\mathbf{a}_2 = \ddot{x}\mathbf{E}_x - L\Omega^2\mathbf{e}_r \tag{2–274}$$

Then the acceleration of the center of mass of the collar-particle system is obtained as

$$^{\mathcal{F}}\bar{\mathbf{a}} = \frac{m_1{}^{\mathcal{F}}\mathbf{a}_1 + m_2{}^{\mathcal{F}}\mathbf{a}_2}{m_1 + m_2} = \ddot{x}\mathbf{E}_x - \frac{m_2}{m_1 + m_2}L\Omega^2\mathbf{e}_r \tag{2–275}$$

Kinetics

The free body diagram of the system consisting of the collar *and* the particle is shown in Fig. 2–11.

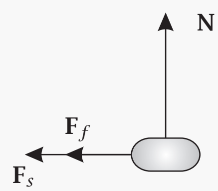

Figure 2–11 Free body diagram for collar attached to spring, damper, and rotating arm with particle.

It is noted explicitly that the reaction force exerted by the arm on the collar is *not* included in the free body diagram of the collar-particle system because this reaction force is *internal* to the system. Consequently, the forces acting on the collar particle system are

$\mathbf{F}_s$	=	Spring force
$\mathbf{F}_f$	=	Force of viscous friction
$\mathbf{N}$	=	Reaction force of track on system

Now we know that the reaction force $\mathbf{N}$ must act in the direction orthogonal to the track. Furthermore, because the motion is planar, the force $\mathbf{N}$ must lie in the plane of motion. Consequently, $\mathbf{N}$ must lie in the direction of $\mathbf{E}_y$ and can be expressed as

$$\mathbf{N} = N\mathbf{E}_y \tag{2–276}$$

Next, because the friction force is viscous, we have

$$\mathbf{F}_f = -c\mathbf{v}_{\text{rel}} \tag{2–277}$$

where$\mathbf{v}_{\text{rel}}$ is the velocity of the collar relative to the track (because the track is the surface on which the particle slides and the attachment point of the spring and damper is fixed to the track). Therefore,

$$\mathbf{v}_{\text{rel}} = {}^{\mathcal{F}}\mathbf{v}_1 = \dot{x}\mathbf{E}_x \tag{2–278}$$

which implies that the force of viscous friction is given as

$$\mathbf{F}_f = -c\dot{x}\mathbf{E}_x \tag{2–279}$$

Next, the spring force is given as

$$\mathbf{F}_s = -K(\ell - \ell_0)\mathbf{u}_s \tag{2–280}$$

Now from the geometry we have $\ell = \|bfr_1 - \mathbf{r}_O\| = \|x\mathbf{E}_x\| = x$. Next,

$$\mathbf{u}_s = \frac{\mathbf{r}_1 - \mathbf{r}_O}{\|\mathbf{r}_1 - \mathbf{r}_O\|} = \frac{x\mathbf{E}_x}{x} = \mathbf{E}_x \tag{2–281}$$

Therefore, the spring force is given as

$$\mathbf{F}_s = -K(x - \ell_0)\mathbf{E}_x \tag{2–282}$$

Adding Eqs. (2–276), (2–279), and (2–282), the force acting on the collar-particle system is given as

$$\mathbf{F} = \mathbf{N} + \mathbf{F}_f + \mathbf{F}_s = N\mathbf{E}_y - c\dot{x}\mathbf{E}_x - K(x - \ell_0)\mathbf{E}_x = -\left[c\dot{x} + K(x - \ell_0)\right]\mathbf{E}_x + N\mathbf{E}_y \tag{2–283}$$

Setting $\mathbf{F}$ in Eq. (2–283) equal to $(m_1 + m_2)^{\mathcal{F}}\bar{\mathbf{a}}$ using the expression for ${}^{\mathcal{F}}\bar{\mathbf{a}}$ from Eq. (2–275), we obtain

$$-\left[c\dot{x} + K(x - \ell_0)\right]\mathbf{E}_x + N\mathbf{E}_y = (m_1 + m_2)\ddot{x}\mathbf{E}_x - m_2 L\Omega^2\mathbf{e}_r \tag{2–284}$$

Now we note that $\mathbf{e}_r$ is given in terms of $\mathbf{E}_x$ and $\mathbf{E}_y$ as

$$\mathbf{e}_r = \sin\Omega t\mathbf{E}_x - \cos\Omega t\mathbf{E}_y \tag{2–285}$$

Substituting the expression for $\mathbf{e}_r$ into Eq. (2–284), we have

$$-\left[c\dot{x} + K(x - \ell_0)\right]\mathbf{E}_x + N\mathbf{E}_y = (m_1 + m_2)\ddot{x}\mathbf{E}_x - m_2 L\Omega^2(\sin\Omega t\mathbf{E}_x - \cos\Omega t\mathbf{E}_y) \tag{2–286}$$

Eq. (2–286) simplifies to

$$-\left[c\dot{x} + K(x - \ell_0)\right]\mathbf{E}_x + N\mathbf{E}_y = \left[(m_1 + m_2)\ddot{x} - m_2 L\Omega^2\sin\Omega t\right]\mathbf{E}_x + m_2 L\Omega^2\cos\Omega t\mathbf{E}_y \tag{2–287}$$

Equating $\mathbf{E}_x$ and $\mathbf{E}_y$ components in Eq. (2–287), we obtain the following two scalar equations:

$$(m_1 + m_2)\ddot{x} - m_2 L\Omega^2\sin\Omega t = -\left[c\dot{x} + K(x - \ell_0)\right]\mathbf{E}_x \tag{2–288}$$

$$m_2 L\Omega^2\cos\Omega t = N \tag{2–289}$$

Observing that Eq. (2–288) has no unknown reaction forces and all other quantities (with the exception of x) are known, the differential equation of motion is given as

$$(m_1 + m_2)\ddot{x} - m_2 L\Omega^2\sin\Omega t = -\left[c\dot{x} + K(x - \ell_0)\right] \tag{2–290}$$

Rearranging Eq. (2–290), we obtain

$$\ddot{x} + \frac{c}{m_1 + m_2}\dot{x} + \frac{K}{m_1 + m_2}x = \frac{m_2 L\Omega^2}{m_1 + m_2}\sin\Omega t + \frac{K}{m_1 + m_2}\ell_0 \tag{2–291}$$

Suppose now that we define $M = m_1 + m_2$. Then the differential equation of Eq. (2–291) can be written as

$$\ddot{x} + \frac{c}{M}\dot{x} + \frac{K}{M}x = \frac{m_2 L\Omega^2}{M}\sin\Omega t + \frac{K}{M}\ell_0 \tag{2–292}$$

(b) Static Equilibrium Point of System

Setting $\dot{x}$ and $\ddot{x}$ to zero and shutting off the input (in this case the rotation of the arm), the condition for static equilibrium of the collar is given as

$$\frac{K}{M}x_{eq} = \frac{K}{M}\ell_0 \tag{2–293}$$

which implies that

$$x_{eq} = \ell_0 \tag{2–294}$$

(c) Differential Equation Relative to Equilibrium Point

Setting $z = x - x_{eq} = x - \ell_0$, the differential equation becomes

$$\ddot{z} + \frac{c}{M}\dot{z} + \frac{K}{M}z = \frac{m_2 L\Omega^2}{M}\sin\Omega t \tag{2–295}$$

(d) Time Response of Collar

It is seen that the input applied to the system is

$$F(t) = \frac{m_2 L\Omega^2}{M}\sin\Omega t \tag{2–296}$$

Recall that the standard form of the input is given as

$$f(t) = \omega_n^2 A e^{i\omega t} \tag{2–297}$$

Now for this problem we have

$$\omega_n^2 = \frac{K}{M} \tag{2–298}$$

Therefore,

$$\omega_n^2 A = \frac{K}{M}A = \frac{m_2 L\Omega^2}{M} \tag{2–299}$$

Solving for A, we obtain

$$A = \frac{m_2 L\Omega^2}{K} = \frac{m_2}{M}\left(\frac{\Omega}{\omega_n}\right)^2 L \tag{2–300}$$

Next, recall the standard second-order linear time-invariant system

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f(t) = \omega_n^2 A e^{i\omega t} \tag{2–301}$$

where

$$f(t) = \omega_n^2 A e^{i\omega t} \tag{2–302}$$

is the normalized input function. The transfer function $G(i\omega) = X/A$ for the system of Eq. (2–301) is given as

$$G(i\omega) = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + i2\zeta\frac{\omega}{\omega_n}} \tag{2–303}$$

where the magnitude and phase of $G(i\omega)$ are given, respectively, as

$$|G(i\omega)| = \left\{\left[\frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + i2\zeta\frac{\omega}{\omega_n}}\right]\left[\frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 - i2\zeta\frac{\omega}{\omega_n}}\right]\right\}^{1/2} \tag{2–304}$$

$$\phi(\omega) = \tan^{-1}\frac{-\mathrm{Im}\,[G(i\omega)]}{\mathrm{Re}\,[G(i\omega)]} = \tan^{-1}\left[\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right] \tag{2–305}$$

Also, recall the time response for the system of Eq. (2–301) is given as

$$x(t) = A|G(i\omega)|e^{i(\omega t - \phi)} \tag{2–306}$$

Suppose now that we let $F(t)$ be defined as

$$F(t) = \frac{m_2 L\Omega^2}{M} e^{i\Omega t} = \omega_n^2 \left[\frac{m_2}{M} \left(\frac{\Omega}{\omega_n} \right)^2 L \right] e^{i\Omega t} = \left[\frac{m_2}{M} \left(\frac{\Omega}{\omega_n} \right)^2 L \right] f(t) \tag{2–307}$$

Therefore, the response of the system

$$\ddot{q} + \frac{c}{M}\dot{q} + \frac{K}{M}q = f(t) \tag{2–308}$$

to the input $f(t)$ from Eq. (2–307) is given as

$$q(t) = \left[\frac{m_2}{M} \left(\frac{\Omega}{\omega_n} \right)^2 L \right] |G(i\Omega)| e^{i(\Omega t - \phi)} \tag{2–309}$$

Finally, we are interested in the response to the input

$$\frac{m_2 L\Omega^2}{M} \sin \Omega t = \left[\frac{m_2}{M} \left(\frac{\Omega}{\omega_n} \right)^2 \right] \sin \Omega t \tag{2–310}$$

Observing that

$$e^{i(\Omega t - \phi)} = \cos\left(i(\Omega t - \phi)\right) + i \sin\left(\Omega t - \phi\right) \tag{2–311}$$

The time response of the system in Eq. (2–292) is the imaginary part of $q(t)$, i.e.,

$$z(t) = \mathrm{Im}[q(t)] = \left[\frac{m_2}{M} \left(\frac{\Omega}{\omega_n} \right)^2 L \right] |G(i\Omega)| \sin\left(\Omega t - \phi\right) \tag{2–312}$$

■

2.6 Base Motion Isolation

An important problem in vibratory systems is *base motion isolation*. The problem of base motion isolation is as follows. Consider an object in vibratory (i.e., connected to a linear spring and a viscous damper) such that the spring and damper are connected at the other end to a system that is itself vibrating. The objective is to isolate the motion of the mass from this other system. A good example of a base motion isolation system is the suspension of an automobile where it is desired to isolate the vibration of the automobile from undulations in the road. In this section we derive the frequency response of a base motion isolation system.

The basic model for a base isolation system is shown in Fig. 2–12. The primary object is a collar of mass m. The collar slides along an inertially fixed horizontal track. The displacement of the collar is given by $x(t)$ and is measured relative to a point O, where O is fixed to the track. Attached to the collar is a linear spring with spring constant K and unstretched length ℓ_0 and a viscous damper with damping coefficient c. Attached to the other end of the spring and damper is a *base* that slides with known displacement $q(t)$ (again, measured from the inertially fixed point O) where $q(t)$ is assumed to be a periodic function of the form

$$q(t) = Ae^{i\omega t}$$

The objective of this study is to determine the frequency response of the system to the motion of the base and to understand how this frequency response can be used to isolate the motion of the base from the collar.

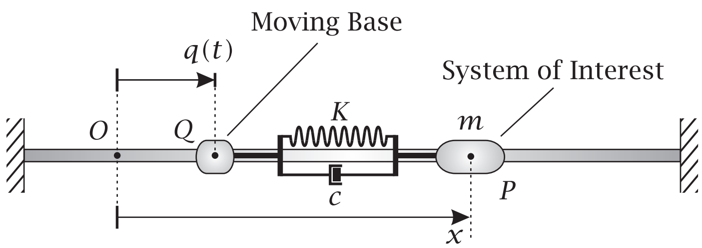

Figure 2–12

We begin by deriving the differential equation of motion for the system. Choosing the horizontal shaft and an inertial reference frame (denoted $\mathcal{F}$), we can define the following coordinate system fixed in reference frame $\mathcal{F}$:

$$\begin{array}{ccc} & \text{Origin at } O & \\ \mathbf{E}_x & = & \text{To the right} \\ \mathbf{E}_z & = & \text{Out of page} \\ \mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x \end{array}$$

The positions of the base and collar are then given as

$$\mathbf{r}_Q = q\mathbf{E}_x \tag{2–313}$$

$$\mathbf{r} = x\mathbf{E}_x \tag{2–314}$$

The corresponding velocities and accelerations in reference frame $\mathcal{F}$ are then given, respectively, as

$${}^{\mathcal{F}}\mathbf{v}_Q = \dot{q}\mathbf{E}_x \tag{2–315}$$

$${}^{\mathcal{F}}\mathbf{v} = \dot{x}\mathbf{E}_x \tag{2–316}$$

$$
\begin{aligned}
{}^{\mathcal{F}}\mathbf{a}_Q &= \ddot{q}\mathbf{E}_x && (2\text{–}317)\\
{}^{\mathcal{F}}\mathbf{a} &= \ddot{x}\mathbf{E}_x && (2\text{–}318)
\end{aligned}
$$

Next, the free body diagram of the collar is shown in Fig. 2–13 (where we assume that all motion takes place in the horizontal plane and thus there is no gravity). It is seen that the forces acting

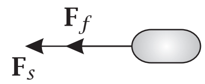

Figure 2–13 Free Body Diagram of Base Motion Isolation System.

on the collar are due to the spring and gravity. The spring force is given as

$$\mathbf{F}_s = -K(\ell - \ell_0)\mathbf{u}_s \tag{2–319}$$

In this case the length of the spring and the direction along which the spring acts are obtained, respectively, as

$$
\begin{aligned}
\ell &= \|\mathbf{r} - \mathbf{r}_Q\| = \|x\mathbf{E}_x - q\mathbf{E}_x\| = |x - q| = x - q && (2\text{–}320)\\
\mathbf{u}_s &= \frac{\mathbf{r} - \mathbf{r}_Q}{\|\mathbf{r} - \mathbf{r}_Q\|} = \frac{(x - q)\mathbf{E}_x}{x - q} = \mathbf{E}_x && (2\text{–}321)
\end{aligned}
$$

Therefore,

$$\mathbf{F}_s = -K(x - q - \ell_0)\mathbf{E}_x \tag{2–322}$$

Next, the force exerted by the viscous damper is given as

$$\mathbf{F}_f = -c\mathbf{v}_{\text{rel}} \tag{2–323}$$

In this case $\mathbf{v}_{\text{rel}}$ is obtained as

$$\mathbf{v}_{\text{rel}} = {}^{\mathcal{F}}\mathbf{v} - {}^{\mathcal{F}}\mathbf{v}_Q = \dot{x}\mathbf{E}_x - \dot{q}\mathbf{E}_x = (\dot{x} - \dot{q})\mathbf{E}_x \tag{2–324}$$

Therefore,

$$\mathbf{F}_f = -c(\dot{x} - \dot{q})\mathbf{E}_x \tag{2–325}$$

The resultant force acting on the particle is then given as

$$\mathbf{F} = \mathbf{F}_s + \mathbf{F}_f = -K(x - q - \ell_0)\mathbf{E}_x - c(\dot{x} - \dot{q})\mathbf{E}_x \tag{2–326}$$

Setting $\mathbf{F}$ equal to $m\,{}^{\mathcal{F}}\mathbf{a}$, we obtain

$$-K(x - q - \ell_0)\mathbf{E}_x - c(\dot{x} - \dot{q})\mathbf{E}_x = m\ddot{x}\mathbf{E}_x \tag{2–327}$$

which leads to the scalar equation

$$-K(x - q - \ell_0) - c(\dot{x} - \dot{q}) = m\ddot{x} \tag{2–328}$$

Rearranging Eq. (2–328) gives

$$m\ddot{x} + c\dot{x} + Kx = c\dot{q} + K(q + \ell_0) \tag{2–329}$$

Finally, defining $y = x - \ell_0$, we can rewrite Eq. (2–329) in terms of y to give

$$m\ddot{y} + c\dot{y} + Ky = c\dot{q} + Kq \tag{2–330}$$

It is seen that the motion of the base affects the motion of the collar through both the spring and the damper. Rewriting Eq. (2–330) in standard form, we have

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 2\zeta\omega_n\dot{q} + \omega_n^2 q \tag{2–331}$$

Now assume that the input to the system is given as

$$q(t) = Qe^{i\omega t} \tag{2–332}$$

Furthermore, assume that the output has the form

$$y(t) = Ye^{i\omega t} \tag{2–333}$$

Substituting $q(t)$ and $y(t)$ into Eq. (2–331) gives

$$-\omega^2 Ye^{i\omega t} + i2\zeta\omega_n\omega Ye^{i\omega t} + \omega_n^2 Ye^{i\omega t} = i2\zeta\omega_n\omega Qe^{i\omega t} + \omega_n^2 Qe^{i\omega t} \tag{2–334}$$

Noting that $e^{i\omega t}$ is not zero, Eq. (2–334) simplifies to

$$\left[-\omega^2 + i2\zeta\omega_n\omega + \omega_n^2\right] Y = \left[i2\zeta\omega_n\omega + \omega_n^2\right] Q \tag{2–335}$$

Rearranging Eq. (2–335) gives

$$\frac{Y}{Q} = \frac{1 + i2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + i2\zeta\frac{\omega}{\omega_n}} \tag{2–336}$$

Now, using the expression for $G(i\omega)$ from Eq. (2–61), Y/Q can be written as

$$\frac{Y}{Q} = \left(1 + i2\zeta\frac{\omega}{\omega_n}\right) G(i\omega) \tag{2–337}$$

Now since $y(t)$ is complex, we know that

$$y(t) = |Y(i\omega)|e^{-i\phi(\omega)} \tag{2–338}$$

Where we can obtain the magnitude and phase of $y(t)$ as follows. First, we have

$$Y(i\omega) = Y(i\omega)\frac{\bar{G}(i\omega)}{\bar{G}(i\omega)} = \left[1 + i2\zeta\frac{\omega}{\omega_n}\right] G(i\omega)\frac{\bar{G}(i\omega)}{\bar{G}(i\omega)} \tag{2–339}$$

Eq. (2–339) can be rewritten as

$$Y(i\omega) = \left[1 + i2\zeta\frac{\omega}{\omega_n}\right]\frac{|G(i\omega)|^2}{\bar{G}(i\omega)}Q \tag{2–340}$$

Then, using the expression for $G(i\omega)$ from Eq. (2–61), we have

$$Y(i\omega) = \left[1 + i2\zeta\frac{\omega}{\omega_n}\right]\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 - i2\zeta\frac{\omega}{\omega_n}\right]|G(i\omega)|^2 Q \tag{2–341}$$

Expanding Eq. (2–341) gives

$$Y(i\omega) = \left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2 - i2\zeta\left(\frac{\omega}{\omega_n}\right)^3\right]|G(i\omega)|^2 Q \tag{2–342}$$

Using Eq. (2–342), the magnitude and phase of $y(t)$ are given as

$$|Y(i\omega)| = \left[1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right]^{1/2}|G(i\omega)|Q \tag{2–343}$$

$$\phi(\omega) = \tan^{-1}\left[\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)^3}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}\right] \tag{2–344}$$

where it is noted that $\phi(\omega)$ is obtained as

$$\phi(\omega) = \tan^{-1}\left[\frac{-\text{Im}\,(H(i\omega))}{\text{Re}\,(H(i\omega))}\right] \tag{2–345}$$

where

$$H(i\omega) = 1 - \left(\frac{\omega}{\omega_n}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2 - i2\zeta\left(\frac{\omega}{\omega_n}\right)^3 \tag{2–346}$$

and

$$\text{Re}\,(H(i\omega)) = 1 - \left(\frac{\omega}{\omega_n}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2 \tag{2–347}$$

$$\text{Im}\,(H(i\omega)) = -2\zeta\left(\frac{\omega}{\omega_n}\right)^3 \tag{2–348}$$

Therefore,

$$\frac{|Y(i\omega)|}{Q} = \left[1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right]^{1/2}|G(i\omega)| \tag{2–349}$$

Substituting $|G(i\omega)|$ from Eq. (2–69), we obtain

$$T(i\omega) = \frac{|Y(i\omega)|}{Q} = \left\{\frac{1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}\right\}^{1/2} \tag{2–350}$$

The quantity $T(i\omega)$ is called the *transmittibility* and gives a measure of the amount of the input (i.e., the motion of the base) that is transmitted to the output. We note several features of the transmittibility function. First, it is seen that

$$T(i\omega) = \begin{cases} > 1 & , \quad \omega/\omega_n < 1 \\ = 1 & , \quad \omega/\omega_n = 1 \\ < 1 & , \quad \omega/\omega_n > 1 \end{cases} \tag{2–351}$$

In other words, the motion transmitted by the base to the system is *amplified* for low frequencies and it *attenuated* for high frequencies. Therefore, when designing a base motion isolation system the parameters ζ and ω_n must be chosen correctly in order to attenuate the input signal. Finally, it is seen that the phase $\phi(\omega)$ is zero for low frequencies and all values of ζ, and approach π for large values of ω/ω_n (although it approach π very slowly when ζ is close to unity). Therefore, the response is in phase with the input when ω/ω_n is small and is out of phase with the input when ω/ω_n is large.

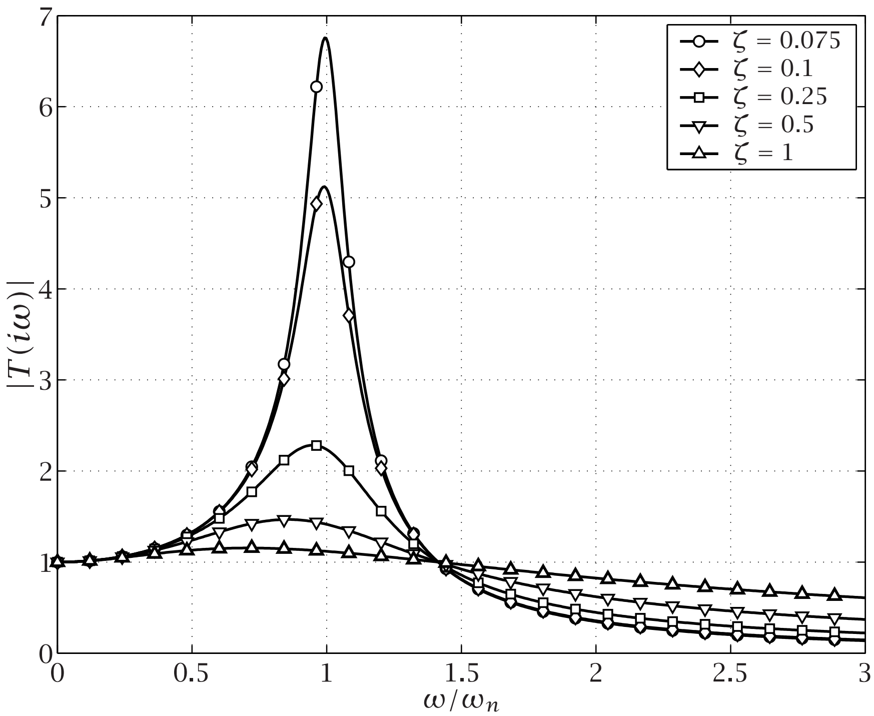

Figure 2–14 Magnitude of Transmittibility Function for a System Under the Influence of a Base Motion Input $q(t) = Qe^{i\omega t}$.

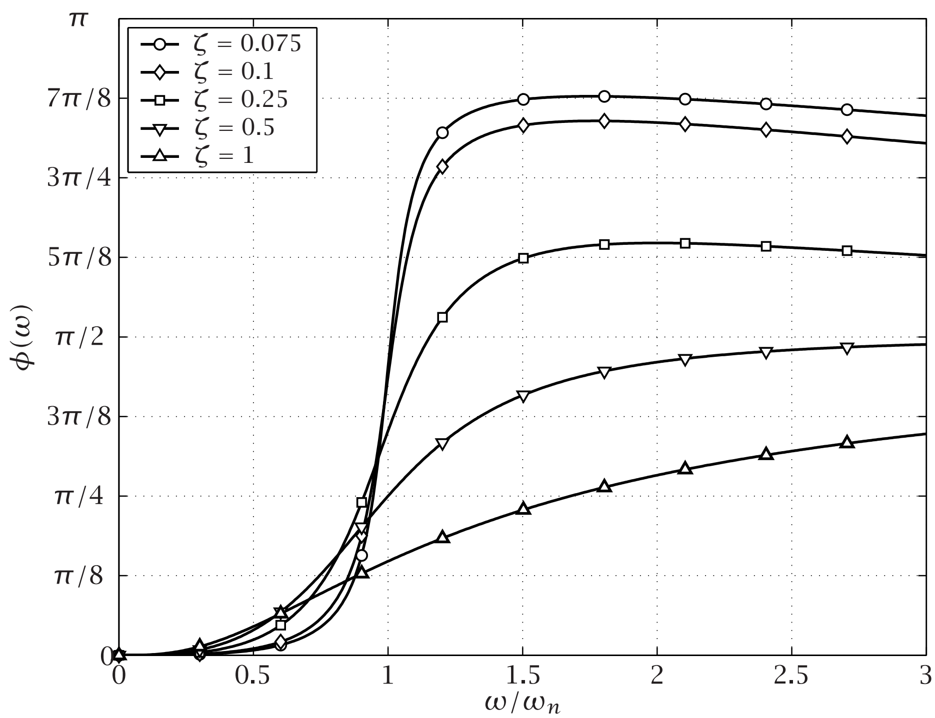

Figure 2–15 Phase of Transmittibility Function for a System Under the Influence of a Base Motion Input $q(t) = Qe^{i\omega t}$.

2.7 Fourier Series Representation of an Arbitrary Periodic Function

Consider now an *arbitrary* periodic function $f(t)$ with period T, i.e., $f(t)$ satisfies the property

$$f(t+nT)=f(t),\ \forall_{n=\mathbb{Z}} \tag{2–352}$$

where $\mathbb{Z}$ is the set of integers. Examples of arbitrary periodic functions include a square-wave (see Fig. 2–16) and a sawtooth (see Fig. 2–17). It is known that any arbitrary periodic function

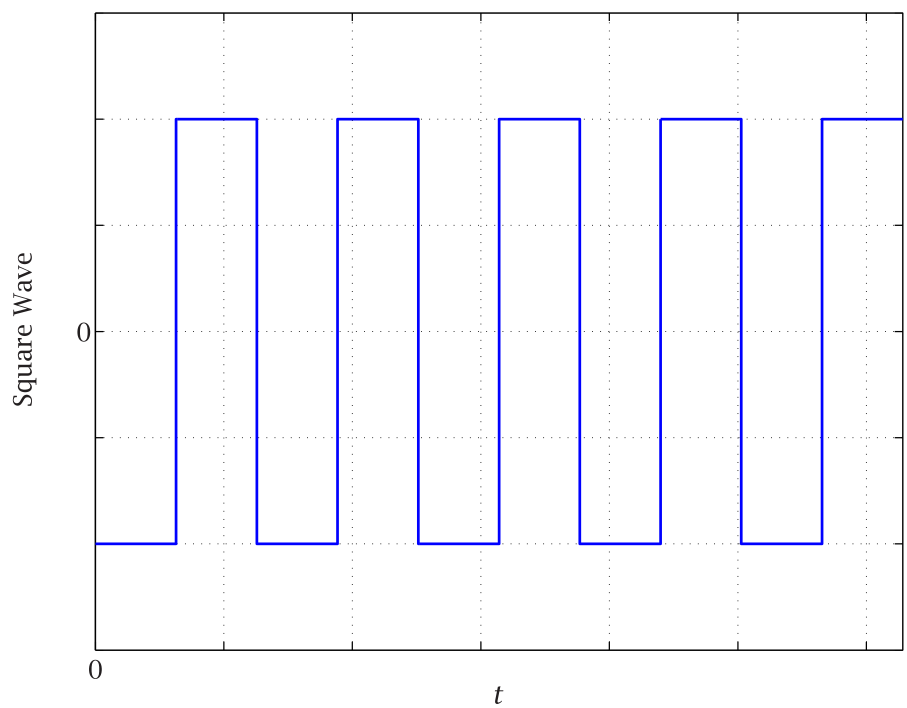

Figure 2–16 Square-Wave Function.

can be expressed as an infinite series of sines and cosines. This infinite series is called a *Fourier series*. Suppose now that we consider a function $f(t)$ that is periodic with period T on the interval from zero to T. Then, in terms of a Fourier series expansion, the periodic function $f(t)$ can be written as

$$f(t)=\sum_{k=-\infty}^{\infty}c_k e^{ik\Omega t} \tag{2–353}$$

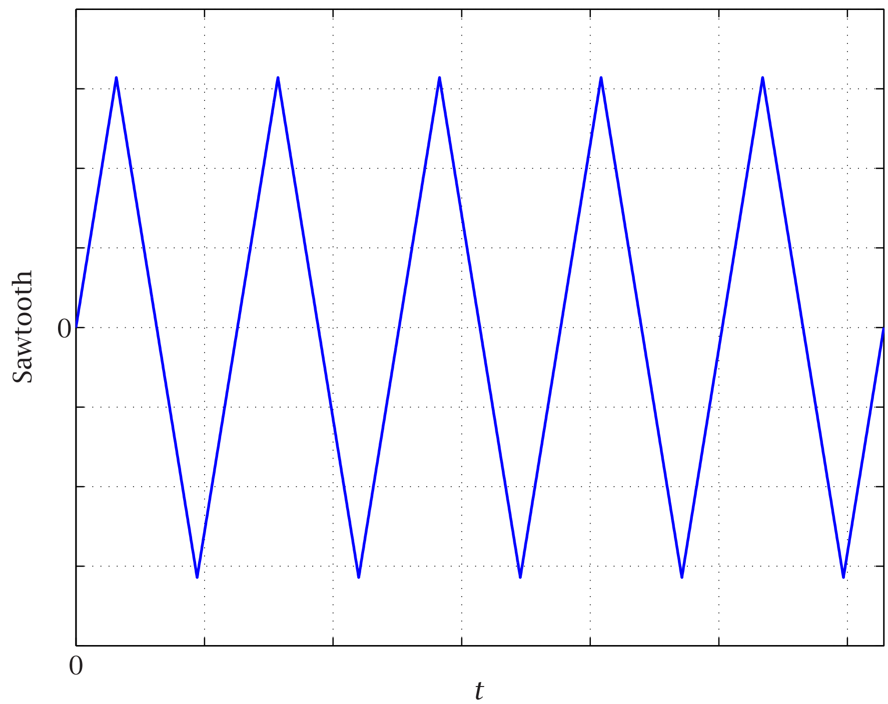

Figure 2–17 Sawtooth Function.

where $\Omega = 2\pi/T$ is the *fundamental frequency*. It is known that the functions $e^{ik\Omega t}$, $(k = 0, \pm 1, \pm 2, \ldots)$ are *orthogonal* over the time interval $t \in [0, T]$, i.e.,

$$\begin{aligned}
\int_0^T e^{ik\Omega t} e^{il\Omega t} dt &= \int_0^T e^{i(k+l)\Omega t} dt \\
&= \frac{1}{i(k+l)\Omega} \left[e^{i(k+l)\Omega t} \right]_0^T \\
&= \frac{1}{i(k+l)\Omega} \left[e^{i(k+l)\Omega T} - 1 \right] \\
&= \frac{1}{i(k+l)\Omega} \left[e^{2\pi i(k+l)T} - 1 \right] \\
&= \frac{1}{i(k+l)\Omega} \left[e^{2\pi i(k+l)T} - 1 \right] \\
&= \frac{1}{i(k+l)\Omega} [1 - 1] = 0
\end{aligned} \tag{2–354}$$

The coefficients c_k, $(k = 0, \pm 1, \pm 2, \ldots)$ are obtained as follows. Suppose we multiply both sides of Eq. (2–353) by $e^{-il\Omega t}$ (where $l \in \mathbb{I}$) and integrate over the period of the function (i.e., from zero to T). We then obtain

$$\int_0^T f(t) e^{-il\Omega t} dt = \int_0^T \left[\sum_{k=-\infty}^{\infty} c_k e^{ik\Omega t} \right] e^{-il\Omega t} dt = \sum_{k=-\infty}^{\infty} c_k \int_0^T e^{i(k-l)\Omega t} dt \tag{2–355}$$

where

$$\int_0^T e^{i(k-l)\Omega t} dt = \frac{1}{i(k-l)\Omega} \left[e^{i(k-l)\Omega t} \right]_0^T \tag{2–356}$$

Noting that $\Omega = 2\pi/T$, we have

$$\int_0^T e^{i(k-l)\Omega t}dt = \frac{1}{i(k-l)\Omega}\left[e^{i(k-l)2\pi} - 1\right] \tag{2–357}$$

Suppose now that we let $m = k - l$ (we note that, because k and l are integers, m is also an integer). Then when $m \neq 0$ we have

$$\frac{1}{im\Omega}\left[e^{2im\pi} - 1\right] = 0, \qquad (m \neq 0) \tag{2–358}$$

Furthermore, for the case that $m = 0$, we need to take the limit as $m \to 0$ as

$$\lim_{m\to 0}\frac{1}{im\Omega}\left[e^{2im\pi} - 1\right] \tag{2–359}$$

Because both the numerator and denominator approach zero as $m \to 0$, we can use L'Hopital's rule to obtain

$$\lim_{m\to 0}\frac{e^{2im\pi} - 1}{im\Omega} = \lim_{m\to 0}\frac{2i\pi e^{2im\pi}}{i\Omega} = \frac{2\pi}{\Omega}\lim_{m\to 0}e^{2im\pi} = \frac{2\pi}{2\pi/T} = T \tag{2–360}$$

Noting that the condition $m = 0$ is equivalent to the condition that $k = l$, we have

$$\int_0^T f(t)e^{-ik\Omega t}dt = Tc_k \tag{2–361}$$

which implies

$$c_k = \frac{1}{T}\int_0^T f(t)e^{-ik\Omega t}dt, \qquad (k = 0, \pm 1, \pm 2, \ldots) \tag{2–362}$$

The expression for c_k from Eq. (2–362) can then be used in Eq. (2–353) to obtain the Fourier series expansion of the periodic function $f(t)$.

It is noted that a Fourier series can be written in *real form* as follows. First, we note that

$$c_{-k} = \frac{1}{T}\int_0^T f(t)e^{ik\Omega t}dt \tag{2–363}$$

Then, we have

$$c_{-k} + c_k = \frac{1}{T}\int_0^T f(t)\left[e^{-ik\Omega t} + e^{ik\Omega t}\right]dt \tag{2–364}$$

Now we have

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{2–365}$$

from which we obtain

$$c_{-k} + c_k = \frac{2}{T}\int_0^T f(t)\cos(k\Omega t) \tag{2–366}$$

Similarly,

$$c_{-k} - c_k = \frac{1}{T}\int_0^T f(t)\left[e^{ik\Omega t} - e^{-ik\Omega t}\right]dt \tag{2–367}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{2–368}$$

from which we obtain

$$c_{-k} - c_k = \frac{2i}{T}\int_0^T f(t)\sin(k\Omega t)dt \tag{2–369}$$

Then we can define

$$a_k = c_{-k} + c_k \tag{2–370}$$

$$ib_k = c_{-k} - c_k \tag{2–371}$$

Solving for c_k and c_{-k} in terms of a_k and b_k, we obtain

$$c_{-k} = \frac{a_k + ib_k}{2} \tag{2–372}$$

$$c_k = \frac{a_k - ib_k}{2} \tag{2–373}$$

We can then write

$$\begin{aligned} f(t) &= \sum_{k=-\infty}^{\infty} c_k e^{ik\Omega t} = \sum_{k=-\infty}^{-1} c_k e^{ik\Omega t} + c_0 + \sum_{k=1}^{\infty} c_k e^{ik\Omega t} \\ &= c_0 + \sum_{k=1}^{\infty} c_{-k} e^{-ik\Omega t} + \sum_{k=1}^{\infty} c_k e^{ik\Omega t} \end{aligned} \tag{2–374}$$

Substituting the expressions for c_k and c_{-k} into this last equation, we obtain

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \frac{a_k + ib_k}{2} e^{-ik\Omega t} + \sum_{k=1}^{\infty} \frac{a_k - ib_k}{2} e^{ik\Omega t} \tag{2–375}$$

Rearranging, we have

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \frac{e^{ik\Omega t} + e^{-ik\Omega t}}{2} - i \sum_{k=1}^{\infty} b_k \frac{e^{ik\Omega t} - e^{-ik\Omega t}}{2} \\ &= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \frac{e^{ik\Omega t} + e^{-ik\Omega t}}{2} + \sum_{k=1}^{\infty} b_k \frac{e^{ik\Omega t} - e^{-ik\Omega t}}{2i} \end{aligned} \tag{2–376}$$

Then, using Eqs. (2–365) and (2–368), we obtain

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\Omega t) + \sum_{k=1}^{\infty} b_k \sin(k\Omega t) \tag{2–377}$$

Eq. (2–377) is a *real* form of a Fourier series for an arbitrary periodic function $f(t)$.

Example 2–5

Consider the following function:

$$f(t+kT) = \begin{cases} 1 & , \quad kT \leq t + kT < kT + T/2 \\ -1 & , \quad kT + T/2 \leq t + kTleq(k+1)T \end{cases} , \qquad (k = 0, \pm 1, \pm 2, \ldots) \tag{2–378}$$

Determine both the complex and real form of the Fourier series expansion of $f(t)$

Solution to Example 2–5

From Eq. (2–362), the coefficients of a Fourier expansion of a periodic function are given in complex form as

$$c_k = \frac{1}{T} \int_0^T f(t) e^{-ik\Omega t} dt, \qquad (k = 0, \pm 1, \pm 2, \ldots) \tag{2–379}$$

Now because the square-wave takes on values of 1 and -1 on the intervals $t \in [0, T/2)$ and $t \in [T/2, T)$, respectively, we need to compute the integral in two parts, i.e.,

$$c_k = \frac{1}{T} \left[\int_0^{T/2} e^{-ik\Omega t} dt + \int_{T/2}^{T} -e^{-ik\Omega t} dt \right] = \frac{1}{T} \left[\int_0^{T/2} e^{-ik\Omega t} dt - \int_{T/2}^{T} e^{-ik\Omega t} dt \right] \tag{2–380}$$

Computing the first integral in Eq. (2–380), we have

$$\int_0^{T/2} e^{-ik\Omega t}dt = -\frac{1}{ik\Omega}\left[e^{-ik\Omega t}\right]_0^{T/2} = -\frac{1}{ik\Omega}\left[e^{-ik\Omega T/2} - 1\right] \quad (2\text{–}381)$$

Noting that $\Omega = 2\pi/T$, we have

$$\int_0^{T/2} e^{-ik\Omega t}dt = -\frac{1}{ik\Omega}\left[e^{-ik\pi} - 1\right] \quad (2\text{–}382)$$

Now we note that

$$e^{-ik\pi} = \begin{cases} -1 & , \quad k = 1,3,5,\ldots \\ 1 & , \quad k = 2,4,6,\ldots \end{cases} \quad (2\text{–}383)$$

Therefore,

$$\int_0^{T/2} e^{-ik\Omega t}dt = \begin{cases} \frac{2}{ik\Omega} & , \quad k = 1,3,5,\ldots \\ 0 & , \quad k = 2,4,6,\ldots \end{cases} \quad (2\text{–}384)$$

Computing the second integral, we have

$$\int_{T/2}^{T} e^{-ik\Omega t}dt = -\frac{1}{ik\Omega}\left[e^{-ik\Omega t}\right]_{T/2}^{T} = -\frac{1}{ik\Omega}\left[e^{-ik\Omega T} - e^{-ik\Omega T/2}\right] \quad (2\text{–}385)$$

Again, using the fact that $\Omega = 2\pi/T$, we obtain

$$\int_{T/2}^{T} -e^{-ik\Omega t}dt = -\frac{1}{ik\Omega}\left[e^{-2ik\pi} - e^{-ik\pi}\right] \quad (2\text{–}386)$$

Now we know that $e^{-2ik\pi} = 1$. Furthermore, we can apply the result of Eq. (2–383) to obtain

$$\int_{T/2}^{T} e^{-ik\Omega t}dt = \begin{cases} -\frac{2}{ik\Omega} & , \quad k = 1,3,5,\ldots \\ 0 & , \quad k = 2,4,6,\ldots \end{cases} \quad (2\text{–}387)$$

Substituting the result of Eqs. (2–384) and (2–387) into Eq. (2–380), we obtain

$$c_k = \begin{cases} \frac{4}{ik\Omega T} & , \quad l = 1,3,5,\ldots \\ 0 & , \quad k = 2,4,6,\ldots \end{cases} \quad (2\text{–}388)$$

Now we consider the special case of $k = 0$ (which was not easily done earlier because k appears in the denominator of the anti-derivative of $e^{-ik\Omega t}$. In the case where $k = 0$, we have

$$c_0 = \int_0^T f(t)dt = \int_0^{T/2} dt + \int_{T/2}^{T} -dt = 0 \quad (2\text{–}389)$$

It is noted that the value of c_0 could have been deduced from the fact that the function is odd. Now in order to obtain only the odd values of l in the Fourier series, we can make the substitution

$$k = 2m - 1 \quad (2\text{–}390)$$

where $m = 0, \pm 1, \pm 2, \ldots$. Then, using the fact that $\Omega = 2\pi/T$, the Fourier series representation of the square-wave function of Eq. (2–378) is given as

$$f(t) = \frac{2}{i\pi}\sum_{m=-\infty}^{\infty} e^{i(2m-1)\Omega t} \quad (2\text{–}391)$$

Next, using Eqs. (2–370) and (2–371), we can write the Fourier series of Eq. (2–391) in real form as follows. First, we have

$$a_k = c_{-k} + c_k = \frac{4}{-ik\Omega T} + \frac{4}{ik\Omega T} = \frac{4ik\Omega T - 4ik\Omega T}{-k^2\Omega^2 T^2} = 0 \quad (2\text{–}392)$$

Next,

$$ib_k = c_{-k} - c_k = \frac{4}{-ik\Omega T} - \frac{4}{ik\Omega T} = -\frac{8}{ik\Omega T} \tag{2–393}$$

which implies that

$$b_k = \frac{1}{i}\left[-\frac{8}{ik\Omega T}\right] = \frac{4}{k\pi} \tag{2–394}$$

Then, using the real form of the Fourier series as given in Eq. (2–377), we obtain

$$f(t) = \sum_{k=1}^{\infty} \frac{4}{k\pi}\sin(k\Omega t) \tag{2–395}$$

■

2.8 Response of a Single Degree-of-Freedom System to an Arbitrary Periodic Input

Using the results of Section 2.7, we can now obtain the response of the second-order differential equation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f(t) \tag{2–396}$$

to a general periodic input $f(t)$. First, recall the response of the system of Eq. (2–71) [i.e., the particular solution] to the complex periodic input $A\omega_n^2 e^{i\omega t}$ as

$$x(t) = AG(i\omega)e^{i\omega t} = A|G(i\omega)|e^{i(\omega t-\phi)} \tag{2–397}$$

where $|G(i\omega)|$ and $\phi(\omega)$ were the magnitude and phase of the transfer function $G(i\omega)$, where $G(i\omega)$ was given from Eq. (2–61) as

$$G(i\omega) = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2 + i2\zeta\frac{\omega}{\omega_n}} \tag{2–398}$$

Correspondingly, $|G(i\omega)$ and $\phi(\omega)$ were given from Eqs. (2–83) and (2–88), respectively, as

$$|G(i\omega)| = \frac{1}{\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{1/2}} \tag{2–399}$$

$$\phi(\omega) = \tan^{-1}\left[\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] \tag{2–400}$$

Suppose now that we let $f(t)$ be a periodic function

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\Omega t} \tag{2–401}$$

Then, because the system of Eq. (2–396) is linear, the principal of superposition applies, i.e., the particular solution to the input of Eq. (2–401) is the *sum* of the terms in the infinite series. First, let us determine the response of the system of Eq. (2–396) to the input

$$f_k(t) = c_k e^{ik\Omega t} \tag{2–402}$$

In order to obtain the response to $f_k(t)$, it is convenient to write Eq. (2–402) as

$$f_k(t) = c_k e^{ik\Omega t} = \omega_n^2 A_k e^{ik\Omega t} \tag{2–403}$$

Then, the response of the system of Eq. (2–396) to the input of Eq. (2–403) is given as

$$x_k(t) = A_k |G(i\omega_k)| e^{i(k\Omega t - \phi(k\Omega))} \tag{2–404}$$

where $|G(ik\Omega)|$ and $\phi(k\Omega)$ are obtained from Eqs. (2–399) and (2–400), respectively, as

$$|G(ik\Omega)| = \frac{1}{\left\{\left[1-\left(\dfrac{k\Omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\dfrac{k\Omega}{\omega_n}\right]^2\right\}^{1/2}} \tag{2–405}$$

$$\phi(k\Omega) = \tan^{-1}\left[\frac{2\zeta\dfrac{k\Omega}{\omega_n}}{1-\left(\dfrac{k\Omega}{\omega_n}\right)^2}\right] \tag{2–406}$$

Now for simplicity we can write

$$\omega_k = k\Omega = \frac{2\pi k}{T} \tag{2–407}$$

$$\phi_k = \phi(k\Omega) \tag{2–408}$$

Then, applying the principal of superposition and using the Fourier series representation of an arbitrary periodic input $f(t)$ with period T [where $f(t)$ is given by Eq. (2–353)], the response is given as

$$x(t) = \sum_{k=0}^{\infty} x_k(t) = \sum_{k=0}^{\infty} A_k |G(i\omega_k)| e^{i(\omega_k t - \phi_k)} \tag{2–409}$$

In other words, the response of the system of Eq. (2–396) to the periodic input $f(t)$ is the *sum* of the responses of Eq. (2–396) to the individual periodic inputs $c_k e^{ik\Omega t} \equiv \omega_n^2 A_k e^{jk\Omega t}$ that are the terms in the Fourier series expansion of $f(t)$.

Chapter 3

Response of Multiple Degree-of-Freedom Systems to Initial Conditions

We now turn our attention to vibrating systems with more than one degree-of-freedom. As opposed to single degree-of-freedoms systems, whose dynamics are described by a single differential equation, systems with n degrees of freedom systems are described by a *system* of n differential equations. Moreover, this system of differential equations is, in general, coupled (meaning that the dynamics of each object in the system depend on one another). In this chapter we will begin the study of vibrations of multiple degree-of-freedom systems by studying systems without any time-varying external forcing. The study of unforced two degree-of-freedom systems will itself be divided into two parts: (1) systems without damping and (2) systems with damping.

3.1 Unforced Undamped Multiple Degree-of-Freedom Systems

The most basic class of systems in the study of multiple degree-of-freedom vibratory systems is the class of *undamped and unforced* systems. In particular, in this section we develop a generic mathematical model for linear time-invariant (LTI) undamped and unforced multiple degree-of-freedom systems and develop the mathematics associated with characterizing the response of these systems.

3.1.1 Model Problem: Blocks with Attached to Linear Springs

Consider the system shown in Fig. 3–1 of two blocks of mass m_1 and m_2 is connected in tandem to three linear springs with spring constants K_1, K_2, and K_3 and corresponding unstretched lengths ℓ_{10}, ℓ_{20}, and ℓ_{30}.
The blocks slide without friction along a horizontal surface of length ℓ and the displacements of each collar, denoted x_1 and x_2, respectively, are measured relative to the inertially fixed point O, where O is located on a vertical wall located at the left end of the surface. The objective of this part of this analysis is to derive a system of two differential equations for the blocks in terms of x_1 and x_2.

First, taking the ground as an inertial reference frame (denoted $\mathcal{F}$), we note that the accelerations of the blocks in reference frame $\mathcal{F}$ are given, respectively, as

$$^{\mathcal{F}}\mathbf{a}_1 = \ddot{x}_1\mathbf{E}_x \quad (3\text{–}1)$$
$$^{\mathcal{F}}\mathbf{a}_2 = \ddot{x}_2\mathbf{E}_x \quad (3\text{–}2)$$

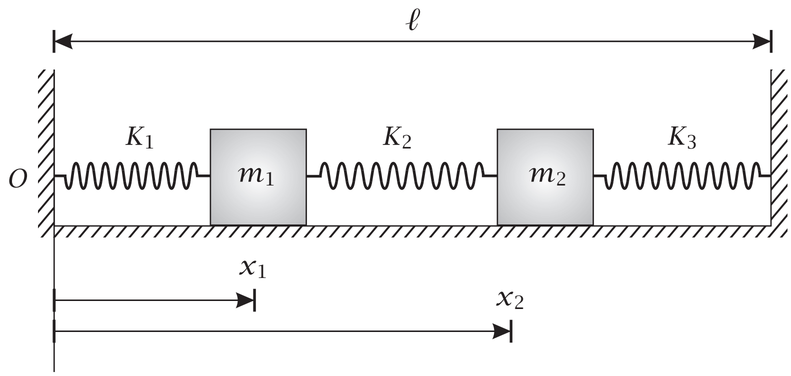

Figure 3–1 Two blocks of mass m_1 and m_2 connected in tandem to three springs with spring constants K_1, K_2, and K_3 with corresponding unstretched lengths ℓ_{10}, ℓ_{20}, and ℓ_{30}.

where $\mathbf{E}_x$ is the unit vector in the rightward direction. Next, the forces acting on each collar are given, respectively, as

$$\mathbf{F}_1 = \mathbf{F}_{s1} + \mathbf{F}_{s2} \tag{3–3}$$
$$\mathbf{F}_2 = -\mathbf{F}_{s2} + \mathbf{F}_{s3} \tag{3–4}$$

where we note that, because spring 2 lies between the two blocks, the force exerted by spring 2 on m_1 is equal and opposite the force exerted by spring 2 on m_2 (i.e., because $\mathbf{F}_{s2}$ acts on m_1, $-\mathbf{F}_{s2}$ acts on m_2). Now the forces exerted by each of the three springs are given, respectively, as

$$\mathbf{F}_{s1} = -K_1(\ell_1 - \ell_{10})\mathbf{u}_{s1} \tag{3–5}$$
$$\mathbf{F}_{s2} = -K_2(\ell_2 - \ell_{20})\mathbf{u}_{s2} \tag{3–6}$$
$$\mathbf{F}_{s3} = -K_3(\ell_3 - \ell_{30})\mathbf{u}_{s3} \tag{3–7}$$

First, the lengths of each of the springs are given, respectively, as

$$\ell_1 = x_1 \tag{3–8}$$
$$\ell_2 = x_2 - x_1 \tag{3–9}$$
$$\ell_3 = \ell - x_2 \tag{3–10}$$

where ℓ is the length of the track. Next, the unit vectors in the directions from the attachment points of each spring to the corresponding blocks are given, respectively, as

$$\mathbf{u}_{s1} = \mathbf{E}_x \tag{3–11}$$
$$\mathbf{u}_{s2} = -\mathbf{E}_x \tag{3–12}$$
$$\mathbf{u}_{s3} = -\mathbf{E}_x \tag{3–13}$$

We note that $\mathbf{u}_{s2} = \mathbf{u}_{s3} = -\mathbf{E}_x$ because the attachment points of springs 2 and 3 lie *ahead* of the positions of the first and second block, respectively. Then the spring forces are given as

$$\mathbf{F}_{s1} = -K_1(x_1 - \ell_{10})\mathbf{E}_x \tag{3–14}$$
$$\mathbf{F}_{s2} = K_2(x_2 - x_1 - \ell_{20})\mathbf{E}_x \tag{3–15}$$
$$\mathbf{F}_{s3} = K_3(\ell - x_2 - \ell_{30})\mathbf{E}_x \tag{3–16}$$

Then, Newton's 2^{nd} law for the first block is given as

$$\mathbf{F}_1 = \mathbf{F}_{s1} + \mathbf{F}_{s2} = m_1 {}^{\mathcal{F}}\mathbf{a}_1 \tag{3–17}$$

which implies that

$$-K_1(x_1 - \ell_{10})\mathbf{E}_x + K_2(x_2 - x_1 - \ell_{20})\mathbf{E}_x = m_1\ddot{x}_1\mathbf{E}_x \tag{3–18}$$

Dropping $\mathbf{E}_x$ from this last equation gives

$$-K_1(x_1 - \ell_{10}) + K_2(x_2 - x_1 - \ell_{20}) = m_1\ddot{x}_1 \tag{3–19}$$

Rearranging, we obtain

$$m_1\ddot{x}_1 + K_1x_1 - K_2(x_2 - x_1) = K_1\ell_{10} - K_2\ell_{20} \tag{3–20}$$

Equation 3–20) can be rewritten as

$$m_1\ddot{x}_1 + (K_1 + K_2)x_1 - K_2x_2 = K_1\ell_{10} - K_2\ell_{20} \tag{3–21}$$

Newton's 2^{nd} law for the second collar is given as

$$\mathbf{F}_2 = -\mathbf{F}_{s2} + \mathbf{F}_{s3} = m_2 {}^{\mathcal{F}}\mathbf{a}_2 \tag{3–22}$$

which implies that

$$-K_2(x_2 - x_1 - \ell_{20})\mathbf{E}_x + K_3(\ell - x_2 - \ell_{30})\mathbf{E}_x = m_2\ddot{x}_2\mathbf{E}_x \tag{3–23}$$

Dropping $\mathbf{E}_x$ from this last equation gives

$$-K_2(x_2 - x_1 - \ell_{20}) + K_3(\ell - x_2 - \ell_{30}) = m_2\ddot{x}_2 \tag{3–24}$$

Rearranging, we obtain

$$m_2\ddot{x}_2 + K_2(x_2 - x_1) + K_3x_2 = K_2\ell_{20} + K_3(\ell - \ell_{30}) \tag{3–25}$$

Equation 3–25) can be rewritten as

$$m_2\ddot{x}_2 - K_2x_1 + (K_2 + K_3)x_2 = K_2\ell_{20} + K_3(\ell - \ell_{30}) \tag{3–26}$$

The system of two differential equations describing the motion of the two collars is then given as

$$m_1\ddot{x}_1 + (K_1 + K_2)x_1 - K_2x_2 = K_1\ell_{10} - K_2\ell_{20} \tag{3–27}$$
$$m_2\ddot{x}_2 - K_2x_1 + (K_2 + K_3)x_2 = K_2\ell_{20} + K_3(\ell - \ell_{30}) \tag{3–28}$$

The system of differential equations given in Eqs. (3–27) and (3–28) can be written in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 + K_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} K_1\ell_{10} - K_2\ell_{20} \\ -K_2\ell_{20} + K_3(\ell - \ell_{30}) \end{bmatrix} \tag{3–29}$$

$$M\ddot{\mathbf{X}} + K\mathbf{X} = \mathbf{b} \tag{3–30}$$

Now it is seen that the condition for static equilibrium of the system in Eq. (3–30) is given as

$$K\mathbf{X}_{eq} = \mathbf{b} \tag{3–31}$$

Now let

$$\mathbf{Y} = \mathbf{X} - \mathbf{X}_{eq} \tag{3–32}$$

We then have

$$\dot{\mathbf{Y}} = \dot{\mathbf{X}} \tag{3–33}$$

$$\ddot{\mathbf{Y}} = \ddot{\mathbf{X}} \tag{3–34}$$

Substituting the expression for **b** into Eq. (3–30), we obtain

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{K}\mathbf{X} = K\mathbf{X}_{eq} \tag{3–35}$$

This last equation can be rewritten as

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{K}(\mathbf{X} - \mathbf{X}_{eq}) = \mathbf{0} \tag{3–36}$$

Noting that $\mathbf{Y} = \mathbf{X} - \mathbf{X}_{eq}$ and $\ddot{\mathbf{X}} = \ddot{\mathbf{Y}}$, Eq. (3–36) can be rewritten as

$$\mathbf{M}\ddot{\mathbf{Y}} + \mathbf{K}\mathbf{Y} = \mathbf{0} \tag{3–37}$$

It is seen that Eq. (3–37) has a similar mathematical form to the single degree-of-freedom system, the difference being that in this case we have a matrix and column-vector differential equation (or, alternatively, a system of differential equations) as opposed to a scalar differential equation. Consequently, the solution to Eq. (3–37) will itself be a column vector. For the general case of n degrees of freedom the quantities **M**, **K**, and **Y** are given as follows:

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix} \tag{3–38}$$

$$\mathbf{M} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix} \tag{3–39}$$

(3–40)

3.1.2 General Solution to Undamped Multiple Degree-of-Freedom System

In a manner analogous to the single degree-of-freedom system, suppose we let

$$\mathbf{Y} = q\mathbf{u} \tag{3–41}$$

where **u** is a constant vector. Differentiating $\mathbf{Y}(t)$ in Eq. (3–41), we obtain

$$\dot{\mathbf{Y}}(t) = \dot{q}\mathbf{u} \tag{3–42}$$

$$\ddot{\mathbf{Y}}(t) = \ddot{q}\mathbf{u} \tag{3–43}$$

Substituting the expressions from Eqs. (3–41) and (3–43) into Eq. (3–37) gives

$$\mathbf{M}\ddot{q}\mathbf{u} + \mathbf{K}q\mathbf{u} = \mathbf{0} \tag{3–44}$$

Noting that q is a scalar, Eq. (3–44) can be rewritten as

$$\mathbf{M}\mathbf{u}\ddot{q} + \mathbf{K}\mathbf{u}q = \mathbf{0} \tag{3–45}$$

Multiplying both sides of Eq. (3–44) by $\mathbf{u}^T$, we obtain

$$\mathbf{u}^T\mathbf{M}\mathbf{u}\ddot{q} + \mathbf{u}^T\mathbf{K}\mathbf{u}q = \mathbf{0} \tag{3–46}$$

Now because $\mathbf{u}$ is a column vector, we see that the quantities $\mathbf{u}^T\mathbf{M}\mathbf{u}$ and $\mathbf{u}^T\mathbf{M}\mathbf{u}$ are *scalars*. Suppose now that we let

$$\lambda = \frac{\mathbf{u}^T\mathbf{K}\mathbf{u}}{\mathbf{u}^T\mathbf{M}\mathbf{u}} \Longrightarrow \mathbf{u}^T\mathbf{K}\mathbf{u} = \lambda(\mathbf{u}^T\mathbf{M}\mathbf{u}) \tag{3–47}$$

Now it can be shown that the matrices $\mathbf{M}$ and $\mathbf{K}$ are symmetric and positive definite (see Appendix A for the definition of a positive definite matrix). Consequently, we have

$$\mathbf{u}^T\mathbf{K}\mathbf{u} \quad > \quad 0 \ \forall \mathbf{u} \neq 0 \tag{3–48}$$

$$\mathbf{u}^T\mathbf{M}\mathbf{u} \quad > \quad 0 \ \forall \mathbf{u} \neq 0 \tag{3–49}$$

Consequently, $\lambda > 0$. We then obtain

$$\mathbf{u}^T\mathbf{M}\mathbf{u}\ddot{q} + \lambda\mathbf{u}^T\mathbf{M}\mathbf{u}q = \mathbf{0} \tag{3–50}$$

Factoring out $\mathbf{u}^T\mathbf{M}\mathbf{u}$ in Eq. (3–50) gives

$$\mathbf{u}^T\mathbf{M}\mathbf{u}(\ddot{q} + \lambda q) = \mathbf{0} \tag{3–51}$$

Equation (3–51) implies that

$$\ddot{q} + \lambda q = 0 \Longrightarrow \ddot{q} = -\lambda q \tag{3–52}$$

Substituting the result of Eq. (3–52) into Eq. (3–37), we obtain

$$-\lambda\mathbf{M}\mathbf{u} + \mathbf{K}\mathbf{u} = \mathbf{0} \tag{3–53}$$

Rearranging Eq. (3–53) gives

$$\mathbf{K}\mathbf{u} = \lambda\mathbf{M}\mathbf{u} \tag{3–54}$$

Equation (3–54) is a *weighted eigenvalue problem* (see Appendix A) in the matrices $\mathbf{K}$ and $\mathbf{M}$. whose eigenvalues are obtained from the condition

$$\det(\lambda\mathbf{M} - \mathbf{K}) = 0 \tag{3–55}$$

Furthermore, because the eigenvalues must be positive, the general solution of Eq. (3–52) is given as

$$q(t) = \sum_{k=1}^{n} C_{1k}\cos\omega_k t + C_{2k}\sin\omega_k t = C_k\cos(\omega_k t - \phi_k) \tag{3–56}$$

where $\omega_k^2 = \lambda_k$ and the constants C_{1k} and C_{2k} (equivalently, C_k and ϕ_k) are determined from the initial conditions. Observing that there will be *two* eigenvalues and eigenvectors in Eq. (3–54), we obtain

$$\mathbf{Y}(t) = \sum_{k=1}^{n} q_k\mathbf{u}_k\mathbf{U}\mathbf{q} \tag{3–57}$$

where

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} \tag{3–58}$$

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \tag{3–59}$$

Suppose now that we return to Eq. (3–41). Then for each eigenvector of Eq. (3–54) we have

$$\mathbf{K}\mathbf{u}_k = \lambda_k\mathbf{M}\mathbf{u}_k, \qquad (k = 1, \ldots, n) \tag{3–60}$$

which implies that

$$\mathbf{KU} = \mathbf{MU}\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix} \tag{3–61}$$

Now let

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix} \tag{3–62}$$

We then obtain

$$\mathbf{KU} = \mathbf{MU}\Lambda \tag{3–63}$$

Multiplying both sides by $\mathbf{U}^T$ gives

$$\mathbf{U}^T\mathbf{KU} = \mathbf{U}^T\mathbf{MU}\boldsymbol{\Lambda} \tag{3–64}$$

Returning to the original differential equation and Eq. (3–57), we see that

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{KX} = \mathbf{MU}\ddot{\mathbf{q}} + \mathbf{KUq} = \mathbf{0} \tag{3–65}$$

Multiplying both sides of Eq. (3–65) by $\mathbf{U}^T$, we obtain

$$\mathbf{U}^T\mathbf{MU}\ddot{\mathbf{q}} + \mathbf{U}^T\mathbf{KUq} = \mathbf{0} \tag{3–66}$$

Next, using the result of Eq. (3–64) gives

$$\mathbf{U}^T\mathbf{MU}\ddot{\mathbf{q}} + \mathbf{U}^T\mathbf{MU}\boldsymbol{\Lambda}\mathbf{q} = \mathbf{0} \tag{3–67}$$

Factoring out the quantity $\mathbf{U}^T\mathbf{MU}$, we have

$$\mathbf{U}^T\mathbf{MU}(\ddot{\mathbf{q}} + \boldsymbol{\Lambda}\mathbf{q}) = \mathbf{0} \tag{3–68}$$

Observing that $\mathbf{U}^T\mathbf{MU} \neq \mathbf{0}$, Eq. (3–68) implies that

$$\ddot{\mathbf{q}} + \boldsymbol{\Lambda}\mathbf{q} = \mathbf{0} \tag{3–69}$$

Finally, because $\boldsymbol{\Lambda}$ is diagonal, we can write Eq. (3–69) as a set of scalar equations of the form

$$\ddot{q}_k + \lambda_k q_k = 0, \qquad (k = 1, \ldots, n) \tag{3–70}$$

Now in order to solve Eq. (3–70), we need initial conditions. In general we will be given initial conditions on $\mathbf{Y}$ of the form

$$\mathbf{Y}(0) \quad = \quad \mathbf{Y}_0 = \begin{bmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{n0} \end{bmatrix} \tag{3–71}$$

$$\dot{\mathbf{Y}}(0) \quad = \quad \dot{\mathbf{Y}}_0 = \begin{bmatrix} \dot{y}_{10} \\ \dot{y}_{20} \\ \vdots \\ \dot{y}_{n0} \end{bmatrix} \tag{3–72}$$

Then, from Eq. (3–57) and the fact that $\mathbf{U}$ is nonsingular,

$$\mathbf{q}(0) = \mathbf{U}^{-1}\mathbf{Y}(0) = \begin{bmatrix} q_{10} \\ q_{20} \\ \vdots \\ q_{n0} \end{bmatrix} \tag{3–73}$$

$$\dot{\mathbf{q}}(0) = \mathbf{U}^{-1}\mathbf{Y}(0) = \begin{bmatrix} \dot{q}_{10} \\ \dot{q}_{20} \\ \vdots \\ \dot{q}_{n0} \end{bmatrix} \tag{3–74}$$

3.1.3 Solution Procedure for Multiple Degree-of-Freedom Undamped System

Using the results of section 3.1.2, we now provide a procedure for determining the solution of the two degree-of-freedom undamped system

$$\mathbf{M}\ddot{\mathbf{Y}} + \mathbf{K}\mathbf{Y} = \mathbf{0} \tag{3–75}$$

subject to the initial conditions

$$\mathbf{Y}(0) \quad = \quad \mathbf{Y}_0 = \begin{bmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{n0} \end{bmatrix} \tag{3–76}$$

$$\dot{\mathbf{Y}}(0) \quad = \quad \dot{\mathbf{Y}}_0 = \begin{bmatrix} \dot{y}_{10} \\ \dot{y}_{20} \\ \vdots \\ \dot{y}_{n0} \end{bmatrix} \tag{3–77}$$

Step 1: Determine the Eigenvalues

The characteristic equation for the differential equation of Eq. (3–75) is given from the following determinant:

$$\det(\lambda\mathbf{M} - \mathbf{K}) = 0 \tag{3–78}$$

The determinant of Eq. (3–78) leads to a polynomial of degree n which has the general form

$$p(\lambda) = \sum_{k=1}^{n} a_k \lambda^k \tag{3–79}$$

The eigenvalues are then the roots of the characteristic polynomial of Eq. (3–79), It is important to note that the eigenvalues of an undamped multiple degree-of-freedom problem should be real and positive because otherwise the solution would not make physical sense. Finally, the natural frequencies of the two degree of freedom system are then obtained by taking the square-roots of the eigenvalues, i.e.,

$$\begin{array}{rcl} \omega_1^2 & = & \lambda_1 \\ \omega_2^2 & = & \lambda_2 \\ \vdots & \vdots & \vdots \\ \omega_n^2 & = & \lambda_n \end{array} \tag{3–80}$$

Step 2: Determine the Eigenvectors

For each eigenvalue obtained in Step 1, we have from the weighted eigenvalue problem that

$$\mathbf{K}\mathbf{w}_k = \lambda_k \mathbf{M}\mathbf{w}_k, \qquad (k = 1, \ldots, n) \tag{3-81}$$

Consequently,

$$(\lambda_i \mathbf{M} - \mathbf{K})\mathbf{w}_i = \mathbf{0}, \qquad (k = 1, \ldots, n) \tag{3-82}$$

where $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are the eigenvectors.

Step 3: Normalization of Eigenvectors

In general, the eigenvectors obtained in Step 2 are *not normalized.* While it is not necessary to normalize the eigenvectors, it is usually convenient to obtain a set of normalized eigenvectors. The most common normalizations are either *mass normalization* or *stiffness normalization.* If mass normalization is chosen, the each normalized eigenvector will have be given as

$$\begin{aligned} \mathbf{u}_1 &= \frac{\mathbf{w}_1}{\sqrt{\mathbf{w}_1^T \mathbf{M}\mathbf{w}_1}} \\ \mathbf{u}_2 &= \frac{\mathbf{w}_2}{\sqrt{\mathbf{w}_2^T \mathbf{M}\mathbf{w}_2}} \\ \vdots \;\; &\;\; \vdots \quad \vdots \\ \mathbf{u}_n &= \frac{\mathbf{w}_n}{\sqrt{\mathbf{w}_n^T \mathbf{M}\mathbf{w}_n}} \end{aligned} \tag{3-83}$$

If stiffness normalization is chosen, then each normalized eigenvector will be given as

$$\begin{aligned} \mathbf{u}_1 &= \frac{\mathbf{w}_1}{\sqrt{\mathbf{w}_1^T \mathbf{K}\mathbf{w}_1}} \\ \mathbf{u}_2 &= \frac{\mathbf{w}_2}{\sqrt{\mathbf{w}_2^T \mathbf{K}\mathbf{w}_2}} \\ \vdots \;\; &\;\; \vdots \quad \vdots \\ \mathbf{u}_n &= \frac{\mathbf{w}_n}{\sqrt{\mathbf{w}_n^T \mathbf{K}\mathbf{w}_n}} \end{aligned} \tag{3-84}$$

Step 4: Assemble the Eigenvector Matrix

Using the eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ obtained in Step 2 can then be assembled to give the eigenvector matrix

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} \tag{3-85}$$

Step 5: Determine the Initial Conditions in Modal Coordinates

The initial conditions in modal coordinates are given as

$$\mathbf{q}(0) = \mathbf{U}^{-1}\mathbf{Y}(0) = \begin{bmatrix} q_{10} \\ q_{20} \\ \vdots \\ q_{n0} \end{bmatrix} \tag{3-86}$$

$$\dot{\mathbf{q}}(0) = \mathbf{U}^{-1}\dot{\mathbf{Y}}(0) = \begin{bmatrix} \dot{q}_{10} \\ \dot{q}_{20} \\ \vdots \\ \dot{q}_{n0} \end{bmatrix} \tag{3-87}$$

Step 6: Determine the Solutions in Modal Coordinates

The differential equations in modal coordinates are given as

$$\ddot{q}_k + \omega_k^2 q_k = 0, \qquad (k = 1, \ldots, n) \tag{3–88}$$

subject to the initial conditions given in Eqs. (3–86) and (3–87), i.e., the initial conditions for each $k = 1, \ldots, n$ are given as

$$\begin{array}{rcl} q_k(0) & = & q_{k0} \\ \dot{q}_k(0) & = & \dot{q}_{k0} \end{array}, \qquad (k = 1, \ldots, n) \tag{3–89}$$

The general solution to this differential equation is given as

$$q_k(t) = c_{1k}\cos(\omega_k t) + c_{2k}\sin(\omega_k t), \qquad (k = 1, \ldots, n) \tag{3–90}$$

where the constants c_{1k} and c_{2k} are given as

$$\begin{array}{rcl} c_{1k} & = & q_{k0} \\ c_{2k} & = & \dot{q}_{k0}/\omega_k \end{array}, \qquad (k = 1, \ldots, n) \tag{3–91}$$

Step 7: Transform the Modal Coordinate Solution to the Original Coordinates

In vector form, the result of Step 5 is

$$\mathbf{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix} \tag{3–92}$$

Then, recalling from Eq. (3–57) that $\mathbf{X} = \mathbf{U}\mathbf{q}$, we have

$$\mathbf{X}(t) = \mathbf{U}\mathbf{q}(t) \tag{3–93}$$

where $\mathbf{q}(t)$ is the column vector assembled from the solution given in Step 5.

Example 3–1

Consider the undamped two degree-of-freedom system with the following mass and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3–94}$$

$$\mathbf{K} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{3–95}$$

Determine the solution to the undamped differential equation

$$\mathbf{M}\ddot{\mathbf{Y}} + \mathbf{K}\mathbf{Y} = \mathbf{0}$$

with the initial conditions

$$\mathbf{Y}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \dot{\mathbf{Y}}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Solution to Example 3–1

We will obtain the solution to this problem using the six-step procedure described in section 3.1.3. Following Step 1, we compute the eigenvalues of the weighted eigenvalue problem as

$$\det(\lambda \mathbf{M} - \mathbf{K}) = \det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right) = 0 \tag{3–96}$$

Equation (3–96) can be rewritten as

$$\det\left(\begin{bmatrix} \lambda - 2 & 1 \\ 1 & \lambda - 2 \end{bmatrix}\right) = 0 \tag{3–97}$$

Computing the determinant in Eq. (3–97), we have

$$\det[\lambda \mathbf{M} - \mathbf{K}] = (\lambda - 2)^2 - 1 = 0 \Longleftarrow (\lambda - 2)^2 = 1 \tag{3–98}$$

Solving for λ in Eq. (3–98), we obtain the eigenvalues as

$$\begin{array}{rcl} \lambda_1 & = & 1 \\ \lambda_2 & = & 3 \end{array} \tag{3–99}$$

Equation (3–99) implies that the natural frequencies are given as

$$\begin{array}{rcl} \omega_1 & = & \sqrt{\lambda_1} = 1 \\ \omega_2 & = & \sqrt{\lambda_2} = \sqrt{3} \end{array} \tag{3–100}$$

Following Steps 2 and 3, the eigenvectors of the weighted eigenvalue problem are obtained from the condition

$$[\lambda_i \mathbf{M} - \mathbf{K}]\,\mathbf{u}_i = \mathbf{0}, \qquad (i = 1, 2) \tag{3–101}$$

For the eigenvalue $\lambda_1 = 1$, we have

$$\begin{bmatrix} \lambda_1 - 2 & 1 \\ 1 & \lambda_1 - 2 \end{bmatrix}\mathbf{u}_1 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\mathbf{u}_1 = \mathbf{0} \tag{3–102}$$

Suppose that we denote the first *unnormalized* eigenvector by $\mathbf{w}_1$. Then

$$\mathbf{w}_1 = \begin{bmatrix} w_{11} \\ w_{12} \end{bmatrix} \tag{3–103}$$

We then have

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} w_{11} \\ w_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3–104}$$

It is seen that a set of values of w_{11} and w_{22} that satisfy Eq. (3–104) are

$$\begin{array}{rcl} w_{11} & = & 1 \\ w_{21} & = & 1 \end{array} \tag{3–105}$$

Therefore, the first eigenvector *before normalization* is given as

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{3–106}$$

Then, choosing a mass normalization of $\mathbf{w}_1$, we have

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\sqrt{\mathbf{w}_1^T \mathbf{M} \mathbf{w}_1}} \tag{3–107}$$

Now we see that

$$\mathbf{w}_1^T\mathbf{M}\mathbf{w}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \qquad (3\text{–}108)$$

which implies that

$$\sqrt{\mathbf{w}_1^T\mathbf{M}\mathbf{w}_1} = \sqrt{2} \qquad (3\text{–}109)$$

Therefore, the first normalized eigenvector is given as

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\sqrt{\mathbf{w}_1^T\mathbf{M}\mathbf{w}_1}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad (3\text{–}110)$$

The second eigenvector is obtained in a manner similar to that used to obtain the first eigenvector. In particular, for the eigenvalue $\lambda_2 = 3$, we have

$$\begin{bmatrix} \lambda_2 - 2 & 1 \\ 1 & \lambda_2 - 2 \end{bmatrix}\mathbf{u}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{u}_1 = \mathbf{0} \qquad (3\text{–}111)$$

Suppose that we denote the *unnormalized* second eigenvector as $\mathbf{w}_2$. Then

$$\mathbf{w}_2 = \begin{bmatrix} w_{21} \\ w_{22} \end{bmatrix} \qquad (3\text{–}112)$$

We then have

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} w_{21} \\ w_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (3\text{–}113)$$

It is seen that a set of values of w_{21} and w_{22} that satisfy Eq. (3–113) are

$$\begin{aligned} w_{21} &= 1 \\ w_{22} &= -1 \end{aligned} \qquad (3\text{–}114)$$

Therefore, the second eigenvector *before normalization* is given as

$$\mathbf{w}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad (3\text{–}115)$$

Then, choosing a mass normalization of $\mathbf{w}_1$, we have

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\sqrt{\mathbf{w}_2^T\mathbf{M}\mathbf{w}_2}} \qquad (3\text{–}116)$$

Now we see that

$$\mathbf{w}_2^T\mathbf{M}\mathbf{w}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \qquad (3\text{–}117)$$

which implies that

$$\sqrt{\mathbf{w}_2^T\mathbf{M}\mathbf{w}_2} = \sqrt{2} \qquad (3\text{–}118)$$

Therefore, the first normalized eigenvector is given as

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\sqrt{\mathbf{w}_2^T\mathbf{M}\mathbf{w}_2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad (3\text{–}119)$$

It is observed that $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors and are orthogonal with respect to both $\mathbf{M}$ and $\mathbf{K}$, i.e.,

$$\mathbf{u}_1^T\mathbf{M}\mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = 0 \qquad (3\text{–}120)$$

$$\mathbf{u}_1^T\mathbf{K}\mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = 0 \qquad (3\text{–}121)$$

Following Step 4, the eigenvector matrix $\mathbf{U}$ is given as

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad (3\text{–}122)$$

From Eq. (3–122) we see that the eigenvector matrix for this example is orthogonal, i.e.,

$$\mathbf{U}^{-1} = \mathbf{U}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad (3\text{–}123)$$

Following Step 5, the initial conditions in modal coordinates are given as

$$\begin{aligned} \mathbf{q}(0) &= \mathbf{U}^{-1}\mathbf{Y}(0) = \mathbf{U}^T\mathbf{Y}(0) = \begin{bmatrix} q_{10} \\ q_{20} \end{bmatrix} && (3\text{–}124) \\ \dot{\mathbf{q}}(0) &= \mathbf{U}^{-1}\dot{\mathbf{Y}}(0) = \mathbf{U}^T\dot{\mathbf{Y}}(0) = \begin{bmatrix} \dot{q}_{10} \\ \dot{q}_{20} \end{bmatrix} && (3\text{–}125) \\ & && (3\text{–}126) \end{aligned}$$

Using the initial conditions given in the problem statement, we have

$$\begin{aligned} \mathbf{q}(0) &= \mathbf{U}^T\mathbf{Y}(0) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix} = \begin{bmatrix} q_{10} \\ q_{20} \end{bmatrix} && (3\text{–}127) \\ \dot{\mathbf{q}}(0) &= \mathbf{U}^T\dot{\mathbf{Y}}(0) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \dot{q}_{10} \\ \dot{q}_{20} \end{bmatrix} && (3\text{–}128) \end{aligned}$$

Following Step 5, we can now solve the differential equations in modal coordinates, i.e., solve

$$\ddot{q}_k + \omega_k^2 q_k = 0, \qquad (k = 1, \dots, n) \qquad (3\text{–}129)$$

subject to the initial conditions

$$\begin{aligned} (q_1(0), q_2(0) &= (q_{10}, q_{20}) = \left(0, -\sqrt{2}\right) && (3\text{–}130) \\ (\dot{q}_1(0), \dot{q}_2(0) &= (\dot{q}_{10}, \dot{q}_{20}) = \left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) && (3\text{–}131) \end{aligned}$$

where the initial conditions are reiterated from Eqs. (3–127 and (3–128). Solving the differential equation corresponding to $k = 1$, we have

$$q_1(t) = c_{11}\cos\omega_1 t + c_{21}\sin\omega_1 t \qquad (3\text{–}132)$$

where, from Eq. (3–91), we have

$$\begin{aligned} c_{11} &= q_{10} = 0 \\ c_{21} &= \dot{q}_{10}/\omega_1 = (1/\sqrt{2})/1 = 1/\sqrt{2} \end{aligned} \qquad (3\text{–}133)$$

Consequently,

$$q_1(t) = \frac{1}{\sqrt{2}}\sin t \qquad (3\text{–}134)$$

Next, solving the differential equation corresponding to $i = 2$, we have

$$q_1(t) = c_{12}\cos\omega_1 t + c_{22}\sin\omega_1 t \qquad (3\text{–}135)$$

where, from Eq. (3–91), we have

$$\begin{aligned} c_{12} &= q_{20} = -\sqrt{2} \\ c_{22} &= \dot{q}_{20}/\omega_2 = (3/\sqrt{2})/\sqrt{3} = \sqrt{\tfrac{3}{2}} \end{aligned} \qquad (3\text{–}136)$$

Consequently,

$$q_2(t) = -\sqrt{2}\cos\sqrt{3}t + \sqrt{\frac{3}{2}}\sin\sqrt{3}t \tag{3-137}$$

The vector solution in modal coordinates is then given from Eqs. (3–134) and (3–137) as

$$\mathbf{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}\sin t \\ -\sqrt{2}\cos\sqrt{3}t + \sqrt{\frac{3}{2}}\sin\sqrt{3}t \end{bmatrix} \tag{3-138}$$

Following Step 6, we can now transform the solution in modal coordinates to the original coordinates (i.e., the variable $\mathbf{Y}$) using Eq. (3–57), i.e., we can obtain $\mathbf{Y}(t)$ as

$$\mathbf{Y}(t) = \mathbf{U}\mathbf{q}(t) \tag{3-139}$$

In particular, we can substitute $\mathbf{U}$ and $\mathbf{q}(t)$ from Eqs. (3–122) and (3–138) into Eq. (3–139) to obtain

$$\begin{aligned} \mathbf{Y}(t) &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}}\sin t \\ -\sqrt{2}\cos\sqrt{3}t + \sqrt{\frac{3}{2}}\sin\sqrt{3}t \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}\sin t - \cos\sqrt{3}t + \frac{\sqrt{3}}{2}\cos\sqrt{3}t \\ \frac{1}{2}\sin t + \sin\sqrt{3}t - \frac{\sqrt{3}}{2}\cos\sqrt{3}t \end{bmatrix} \end{aligned} \tag{3-140}$$

It is important to understand the difference in behavior between the solution in modal coordinates (i.e., $\mathbf{q}$) as compared with the solution in the coordinates of interest (i.e., $\mathbf{Y}$). First, it is seen that each modal coordinate (i.e., the solutions $q_1(t)$ and $q_2(t)$), contains only a single frequency. Specifically, the modal coordinate $q_1(t)$ contains only the frequency $\omega_1 = 1$ while the modal coordinate $q_2(t)$ contains only the frequency $\omega_2 = sqrt3$. This "purity" in the modal coordinate solutions is shown in Fig. 3–2

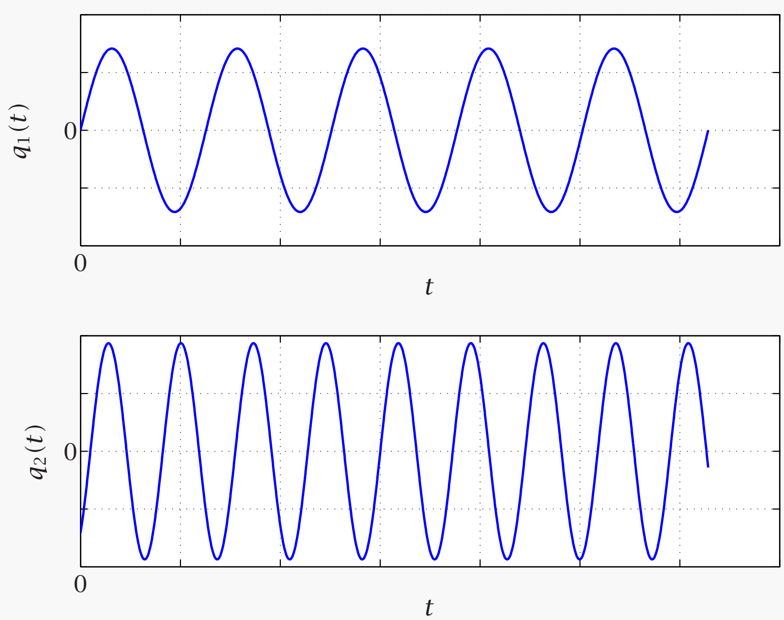

Figure 3–2 Modal coordinate solutions $q_1(t)$ and $q_2(t)$ for Example 3–1.

Contrariwise, each component of the solution $\mathbf{Y}(t)$ (which is the solution we care about) contains *both* frequencies $\omega_1 = 1$ and $\omega_2 = \sqrt{3}$. This "impurity" in the non-modal coordinate solutions is shown in Fig. 3–3

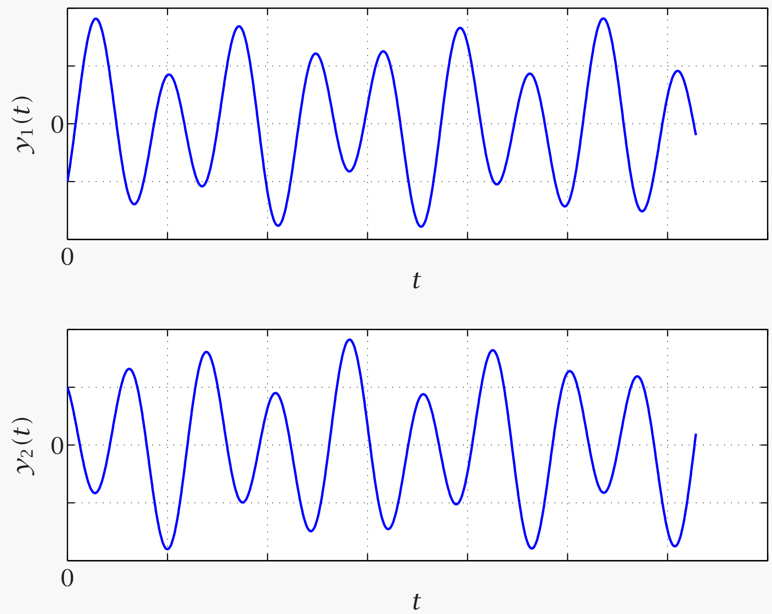

Figure 3–3 Non-modal coordinate solutions $y_1(t)$ and $y_2(t)$ for Example 3–1.

The pure behavior of the modal coordinate solution as opposed to the impure behavior of the non-modal coordinate solution is characteristic of undamped multiple degree-of-freedom systems. Essentially, in modal coordinates the solution is being viewed as a system of uncoupled harmonic oscillators whereas in non-modal coordinates the solution is being viewed as a set of *coupled* oscillators. Since the original problem is coupled (through the stiffness matrix **K**) we expect that the non-modal coordinate solution will exhibit mixed (i.e., impure) behavior. On the other hand, because the eigenvector matrix decouples the mass and stiffness matrices, we expect that each component of the modal coordinate solution will exhibit non-mixed (i.e., pure) behavior.

■

Example 3–2

Consider the undamped two degree-of-freedom system with the following mass and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \tag{3–141}$$

$$\mathbf{K} = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \tag{3–142}$$

Determine the solution to the undamped differential equation

$$\mathbf{M}\ddot{\mathbf{Y}} + \mathbf{K}\mathbf{Y} = \mathbf{0}$$

with the initial conditions

$$\mathbf{Y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \dot{\mathbf{Y}}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solution to Example 3–2

We will obtain the solution to this problem using the seven-step procedure described in section 3.1.3. Following Step 1, we compute the eigenvalues of the weighted eigenvalue problem as

$$\det(\lambda\mathbf{M} - \mathbf{K}) = \det\left(\lambda\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\right) = 0 \tag{3–143}$$

Equation (3–143) can be rewritten as

$$\det\left(\begin{bmatrix} \lambda - 2 & 1 \\ 1 & 2\lambda - 3 \end{bmatrix}\right) = 0 \tag{3–144}$$

Computing the determinant in Eq. (3–144), we have

$$\det[\lambda\mathbf{M} - \mathbf{K}] = (\lambda - 2)(2\lambda - 3) - 1 = 0 \tag{3–145}$$

Equation (3–145) implies that

$$\det[\lambda\mathbf{M} - \mathbf{K}] = 2\lambda^2 - 7\lambda + 5 = 0 \tag{3–146}$$

Solving for λ in Eq. (3–146) by applying the quadratic formula, we obtain

$$\lambda_{1,2} = \frac{7 \pm \sqrt{7^2 - 4(2)(5)}}{2(2)} = \frac{7 \pm \sqrt{49 - 40}}{4} = \frac{7 \pm 3}{4} = 1, 5/2 \tag{3–147}$$

Therefore, the eigenvalues of the symmetric weighted eigenvalue problem are

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 5/2 \end{aligned} \tag{3–148}$$

Equation (3–148) implies that the natural frequencies are given as

$$\begin{aligned} \omega_1 &= \sqrt{\lambda_1} = 1 \\ \omega_2 &= \sqrt{\lambda_2} = \sqrt{\tfrac{5}{2}} \end{aligned} \tag{3–149}$$

Following Steps 2 and 3, the eigenvectors of the weighted eigenvalue problem are obtained from the condition

$$[\lambda_i\mathbf{M} - \mathbf{K}]\,\mathbf{w}_i = \mathbf{0}, \qquad (i = 1, 2) \tag{3–150}$$

For the eigenvalue $\lambda_1 = 1$, we have

$$\begin{bmatrix} \lambda_1 - 2 & 1 \\ 1 & 2\lambda_1 - 3 \end{bmatrix}\mathbf{w}_1 = \tag{3–151}$$

where $\mathbf{w}_1$ denotes the first *unnormalized* eigenvector. Now we can write $\mathbf{w}_1$ as

$$\mathbf{w}_1 = \begin{bmatrix} w_{11} \\ w_{12} \end{bmatrix} \tag{3–152}$$

We then have

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} w_{11} \\ w_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3–153}$$

It is seen that a set of values of w_{11} and w_{22} that satisfy Eq. (3–153) are

$$\begin{aligned} w_{11} &= 1 \\ w_{21} &= 1 \end{aligned} \tag{3–154}$$

Therefore, the first eigenvector *before normalization* is given as

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{3–155}$$

Then, choosing a mass normalization of $\mathbf{w}_1$, we have

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\sqrt{\mathbf{w}_1^T\mathbf{M}\mathbf{w}_1}} \tag{3–156}$$

Now we see that

$$\mathbf{w}_1^T\mathbf{M}\mathbf{w}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \tag{3–157}$$

which implies that

$$\sqrt{\mathbf{w}_1^T\mathbf{M}\mathbf{w}_1} = \sqrt{3} \tag{3–158}$$

Therefore, the first normalized eigenvector is given as

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\sqrt{\mathbf{w}_1^T\mathbf{M}\mathbf{w}_1}} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \tag{3–159}$$

The second eigenvector is obtained in a manner similar to that used to obtain the first eigenvector. In particular, for the eigenvalue $\lambda_2 = 5/2$, we have

$$\begin{bmatrix} \lambda_2 - 2 & 1 \\ 1 & 2\lambda_2 - 3 \end{bmatrix}\mathbf{u}_1 = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 2 \end{bmatrix}\mathbf{w}_2 = \mathbf{0} \tag{3–160}$$

where $\mathbf{w}_2$ denotes the *unnormalized* second eigenvector. Now we have

$$\mathbf{w}_2 = \begin{bmatrix} w_{21} \\ w_{22} \end{bmatrix} \tag{3–161}$$

We then have

$$\begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} w_{21} \\ w_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3–162}$$

It is seen that a set of values of w_{21} and w_{22} that satisfy Eq. (3–162) are

$$\begin{aligned} w_{21} &= 1 \\ w_{22} &= -\tfrac{1}{2} \end{aligned} \tag{3–163}$$

Therefore, the second eigenvector *before normalization* is given as

$$\mathbf{w}_2 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \tag{3–164}$$

Then, choosing a mass normalization of $\mathbf{w}_1$, we have

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\sqrt{\mathbf{w}_2^T\mathbf{M}\mathbf{w}_2}} \tag{3–165}$$

Now we see that

$$\mathbf{w}_2^T\mathbf{M}\mathbf{w}_2 = \begin{bmatrix} 1 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} = \frac{3}{2} \tag{3-166}$$

which implies that

$$\sqrt{\mathbf{w}_2^T\mathbf{M}\mathbf{w}_2} = \sqrt{\frac{3}{2}} \tag{3-167}$$

Therefore, the first normalized eigenvector is given as

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\sqrt{\mathbf{w}_2^T\mathbf{M}\mathbf{w}_2}} = \begin{bmatrix} \sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} \tag{3-168}$$

It is observed that $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors and are orthogonal with respect to both $\mathbf{M}$ and $\mathbf{K}$, i.e.,

$$\mathbf{u}_1^T\mathbf{M}\mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} = 0 \tag{3-169}$$

$$\mathbf{u}_1^T\mathbf{K}\mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} \sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} = 0 \tag{3-170}$$

Following Step 4, the eigenvector matrix $\mathbf{U}$ is given as

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \tag{3-171}$$

It is seen for this example that the eigenvector matrix is *not* orthogonal because the eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are not orthogonal, i.e.,

$$\mathbf{u}_1^T\mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} \sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} = \frac{\sqrt{2}}{3} - \frac{1}{\sqrt{18}} = \frac{\sqrt{2}}{3} - \frac{1}{3\sqrt{2}} \neq 0 \tag{3-172}$$

Now, because we will need it shortly, we compute the inverse of $\mathbf{U}$ as

$$\mathbf{U}^{-1} = -\sqrt{2}\begin{bmatrix} -\frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 2 \\ \sqrt{2} & -\sqrt{2} \end{bmatrix} \tag{3-173}$$

Following Step 5, the initial conditions in modal coordinates are given as

$$\mathbf{q}(0) = \mathbf{U}^{-1}\mathbf{Y}(0) = \begin{bmatrix} q_{10} \\ q_{20} \end{bmatrix} \tag{3-174}$$

$$\dot{\mathbf{q}}(0) = \mathbf{U}^{-1}\dot{\mathbf{Y}}(0) = \begin{bmatrix} \dot{q}_{10} \\ \dot{q}_{20} \end{bmatrix} \tag{3-175}$$

(3-176)

Using the initial conditions given in the problem statement, we have

$$\mathbf{q}(0) = \mathbf{U}^{-1}\mathbf{Y}(0) = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 2 \\ \sqrt{2} & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix} = \begin{bmatrix} q_{10} \\ q_{20} \end{bmatrix} \tag{3-177}$$

$$\dot{\mathbf{q}}(0) = \mathbf{U}^{-1}\dot{\mathbf{Y}}(0) = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 2 \\ \sqrt{2} & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} \end{bmatrix} = \begin{bmatrix} \dot{q}_{10} \\ \dot{q}_{20} \end{bmatrix} \tag{3-178}$$

Following Step 5, we can now solve the differential equations in modal coordinates, i.e., solve

$$\ddot{q} + \omega_i^2 q = 0, \qquad (i = 1, 2) \tag{3–179}$$

subject to the initial conditions

$$(q_1(0), \dot{q}_1(0)) = (q_{10}, \dot{q}_{10}) = \left(\sqrt{3}, \tfrac{4}{\sqrt{3}}\right) \tag{3–180}$$

$$(q_2(0), \dot{q}_2(0)) = (q_{20}, \dot{q}_{20}) = \left(0, \sqrt{\tfrac{2}{3}}\right) \tag{3–181}$$

where the initial conditions are reiterated from Eqs. (3–177 and (3–178). Solving the differential equation corresponding to $i = 1$, we have

$$q_1(t) = c_1^{(1)} \cos \omega_1 t + c_2^{(1)} \sin \omega_1 t \tag{3–182}$$

where, from Eq. (3–91), we have

$$\begin{aligned} c_1^{(1)} &= q_{10} = \sqrt{3} \\ c_2^{(1)} &= \dot{q}_{10}/\omega_1 = (4/\sqrt{3})/1 = \tfrac{4}{\sqrt{3}} \end{aligned} \tag{3–183}$$

Consequently,

$$q_1(t) = \sqrt{3} \cos t + \frac{4}{\sqrt{3}} \sin t = \sqrt{3} \cos t + \frac{4\sqrt{3}}{3} \sin t \tag{3–184}$$

Next, solving the differential equation corresponding to $i = 2$, we have

$$q_2(t) = c_1^{(2)} \cos \omega_2 t + c_2^{(2)} \sin \omega_2 t \tag{3–185}$$

where, from Eq. (3–91), we have

$$\begin{aligned} c_1^{(2)} &= q_{20} = 0 \\ c_2^{(2)} &= \dot{q}_{20}/\omega_2 = (\sqrt{2/3})/\sqrt{5/2} = \tfrac{2}{\sqrt{15}} \end{aligned} \tag{3–186}$$

Consequently,

$$q_2(t) = \frac{2}{\sqrt{15}} \sin \sqrt{\frac{5}{2}} t = \frac{2\sqrt{15}}{15} \sin \sqrt{\frac{5}{2}} t \tag{3–187}$$

The vector solution in modal coordinates is then given from Eqs. (3–184) and (3–187) as

$$\mathbf{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} \sqrt{3} \cos t + \frac{4\sqrt{3}}{3} \cos t \\ \frac{2\sqrt{15}}{15} \sin \sqrt{\frac{5}{2}} t \end{bmatrix} \tag{3–188}$$

Following Step 6, we can now transform the solution in modal coordinates to the original coordinates (i.e., the variable $\mathbf{Y}$) using Eq. (3–57), i.e., we can obtain $\mathbf{Y}(t)$ as

$$\mathbf{Y}(t) = \mathbf{U}\mathbf{q}(t) \tag{3–189}$$

In particular, we can substitute $\mathbf{U}$ and $\mathbf{q}(t)$ from Eqs. (3–171) and (3–188) into Eq. (3–189) to obtain

$$\begin{aligned} \mathbf{Y}(t) &= \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{3} \cos t + \frac{4\sqrt{3}}{3} \sin t \\ \frac{2\sqrt{15}}{15} \sin \sqrt{\frac{5}{2}} t \end{bmatrix} \\ &= \begin{bmatrix} \cos t + \frac{4}{3} \sin t + \frac{2\sqrt{10}}{15} \sin \sqrt{\frac{5}{2}} t \\ \cos t + \frac{4}{3} \sin t - \frac{2\sqrt{10}}{30} \sin \sqrt{\frac{5}{2}} t \end{bmatrix} \end{aligned} \tag{3–190}$$

As with Example 3–1, again we see the key difference between the solution in modal coordinates (i.e., $\mathbf{q}$) and the solution in the original coordinates (i.e., $\mathbf{Y}$). First, it is seen from Fig. 3–4 that each modal coordinate contains only a single frequency.

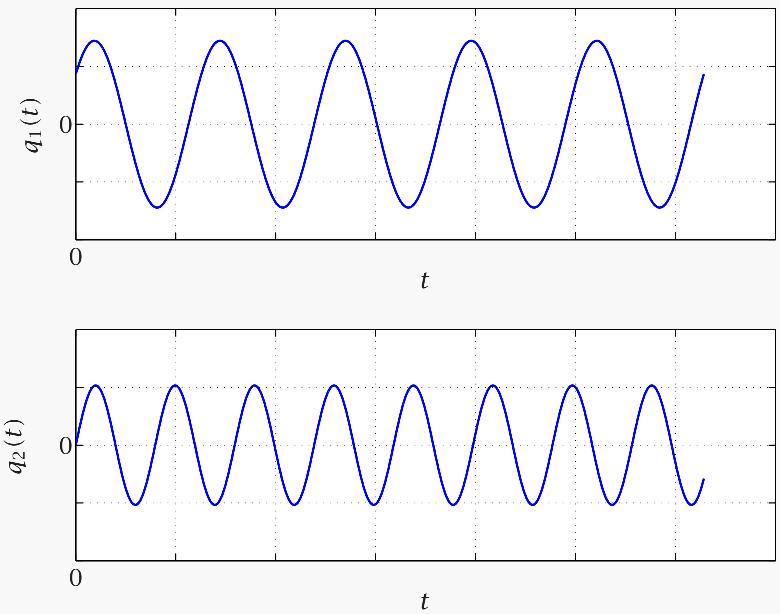

Figure 3–4 Modal coordinate solutions $q_1(t)$ and $q_2(t)$ for Example 3–2.

Contrariwise, examining Fig. 3–3 each component of the solution $\mathbf{Y}(t)$ contains *both* frequencies $\omega_1 = 1$ and $\omega_2 = \sqrt{5/2}$.

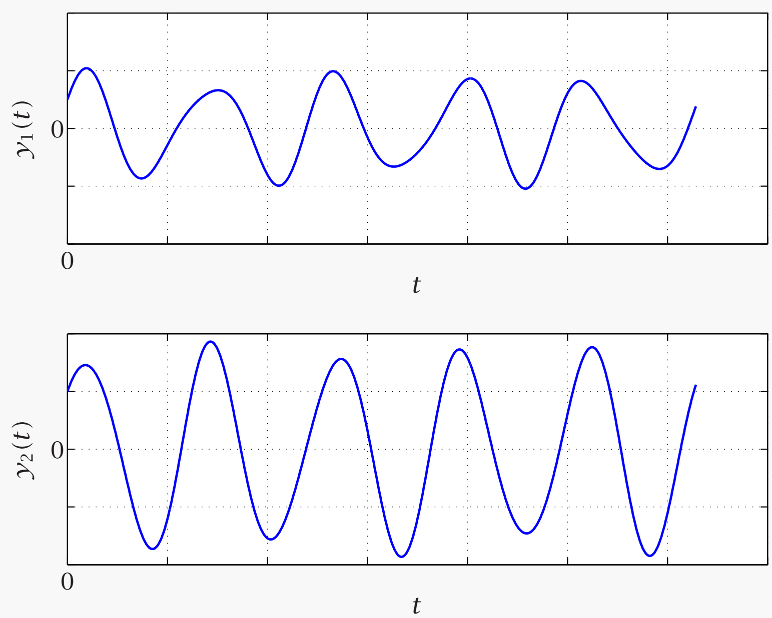

Figure 3–5 Non-modal coordinate solutions $y_1(t)$ and $y_2(t)$ for Example 3–2.

■

3.2 Unforced Damped Multiple Degree-of-Freedom Systems

In section 3.1 we studied the response of undamped and unforced multiple degree-of-freedom LTI systems. In the process of studying this class of systems, a key result was obtained that the response was a linear combination of terms that involved the product of periodic functions with the modal vectors where the frequencies and modal vectors were the *eigenvalues* and *eigenvectors*, respectively, of the weighted eigenvalue problem $\mathbf{Ku} = \lambda\mathbf{Mu}$. Thus, the response of an undamped multiple degree-of-freedom LTI system is characterized completely by mass and stiffness matrices $\mathbf{M}$ and $\mathbf{K}$.

We now turn our attention to unforced but *damped* multiple degree-of-freedom systems. The key difference between damped and undamped systems is that the eigenvalues and eigenvectors of the weighted eigenvalue problem of the undamped system do *not* decouple the system into modal coordinates. Instead, the presence of damping makes it such that no general decoupling can be obtained. However, a particular class of damping exists called *modal damping* for which the differential equations can be transformed to a decoupled form. In this section we develop the general model for a damped multiple degree-of-freedom system, show why the equations cannot be decoupled in the case of general damping, and develop the results for the case of modal damping.

3.2.1 Model Problem: Two Blocks with Linear Springs and Dampers

Consider the system shown in Fig. 3–1 of two blocks of mass m_1 and m_2 connected in tandem to three linear springs with spring constants K_1, K_2, and K_3 and corresponding unstretched lengths ℓ_{10}, ℓ_{20}, and ℓ_{30}, and three viscous dampers with damping coefficients c_1, c_2, and c_3, respectively.

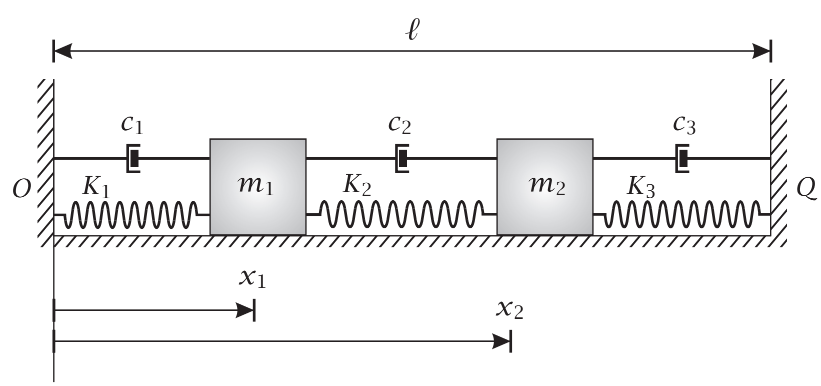

Figure 3–6 Two blocks of mass m_1 and m_2 connected in tandem to three springs with spring constants K_1, K_2, and K_3 and corresponding unstretched lengths ℓ_{10}, ℓ_{20}, and ℓ_{30}, and three viscous dampers with damping coefficients c_1, c_2, and c_3.

The blocks slide along a horizontal surface of length ℓ and the displacements of each collar, denoted x_1 and x_2, respectively, are measured relative to the inertially fixed point O, where O is located on a vertical wall located at the left end of the surface. Finally, assume that the surface is frictionless. The objective of this part of this analysis is to derive a system of two differential equations for the blocks in terms of x_1 and x_2.

First, taking the ground as an absolutely fixed inertial reference frame (denoted $\mathcal{F}$), the velocities and accelerations of the blocks in reference frame $\mathcal{F}$ are given, respectively, as

$$
\begin{aligned}
{}^{\mathcal{F}}\mathbf{v}_1 &= \dot{x}_1\mathbf{E}_x & (3\text{–}191)\\
{}^{\mathcal{F}}\mathbf{v}_2 &= \dot{x}_2\mathbf{E}_x & (3\text{–}192)\\
{}^{\mathcal{F}}\mathbf{a}_1 &= \ddot{x}_1\mathbf{E}_x & (3\text{–}193)\\
{}^{\mathcal{F}}\mathbf{a}_2 &= \ddot{x}_2\mathbf{E}_x & (3\text{–}194)
\end{aligned}
$$

where $\mathbf{E}_x$ is the unit vector in the rightward direction. Next, the forces acting on each collar are given, respectively, as

$$
\begin{aligned}
\mathbf{F}_1 &= \mathbf{F}_{s1} + \mathbf{F}_{s2} + \mathbf{F}_{f1} + \mathbf{F}_{f2} & (3\text{–}195)\\
\mathbf{F}_2 &= -\mathbf{F}_{s2} + \mathbf{F}_{s3} - \mathbf{F}_{f2} + \mathbf{F}_{f3} & (3\text{–}196)
\end{aligned}
$$

where $\mathbf{F}_{s1}$, $\mathbf{F}_{s2}$, and $\mathbf{F}_{s3}$ are the forces exerted by each of the three linear springs and $\mathbf{F}_{f1}$, $\mathbf{F}_{f2}$, and $\mathbf{F}_{f3}$ are the forces exerted by each of the three viscous dampers. Now we note that, because spring 2 lies between the two blocks, the force exerted by spring 2 on m_1 is equal and opposite the force exerted by spring 2 on m_2 (i.e., because $\mathbf{F}_{s2}$ acts on m_1, $-\mathbf{F}_{s2}$ acts on m_2). Similarly, because the second damper lies between the two blocks, the force exerted by the second viscous friction on m_1 is equal and opposite the force exerted by the second damper on m_2. The forces exerted by each of the three springs are given, respectively, as

$$
\begin{aligned}
\mathbf{F}_{s1} &= -K_1(\ell_1 - \ell_{10})\mathbf{u}_{s1} & (3\text{–}197)\\
\mathbf{F}_{s2} &= -K_2(\ell_2 - \ell_{20})\mathbf{u}_{s2} & (3\text{–}198)\\
\mathbf{F}_{s3} &= -K_3(\ell_3 - \ell_{30})\mathbf{u}_{s3} & (3\text{–}199)
\end{aligned}
$$

The lengths of each of the springs are given, respectively, as

$$
\begin{aligned}
\ell_1 &= x_1 & (3\text{–}200)\\
\ell_2 &= x_2 - x_1 & (3\text{–}201)\\
\ell_3 &= \ell - x_2 & (3\text{–}202)
\end{aligned}
$$

where ℓ is the length of the track. Next, the unit vectors in the directions from the attachment points of each spring to the corresponding blocks are given, respectively, as

$$
\begin{aligned}
\mathbf{u}_{s1} &= \mathbf{E}_x & (3\text{–}203)\\
\mathbf{u}_{s2} &= -\mathbf{E}_x & (3\text{–}204)\\
\mathbf{u}_{s3} &= -\mathbf{E}_x & (3\text{–}205)
\end{aligned}
$$

We note that $\mathbf{u}_{s2} = \mathbf{u}_{s3} = -\mathbf{E}_x$ because the attachment points of springs 2 and 3 lie *ahead* of the positions of the first and second block, respectively. Then the spring forces are given as

$$
\begin{aligned}
\mathbf{F}_{s1} &= -K_1(x_1 - \ell_{10})\mathbf{E}_x & (3\text{–}206)\\
\mathbf{F}_{s2} &= K_2(x_2 - x_1 - \ell_{20})\mathbf{E}_x & (3\text{–}207)\\
\mathbf{F}_{s3} &= K_3(\ell - x_2 - \ell_{30})\mathbf{E}_x & (3\text{–}208)
\end{aligned}
$$

Next, the force exerted by each of the dampers is given as

$$
\begin{aligned}
\mathbf{F}_{f1} &= -c_1\mathbf{v}_{\text{rel},1} & (3\text{–}209)\\
\mathbf{F}_{f2} &= -c_2\mathbf{v}_{\text{rel},2} & (3\text{–}210)\\
\mathbf{F}_{f3} &= -c_3\mathbf{v}_{\text{rel},3} & (3\text{–}211)
\end{aligned}
$$

Now we have

$$
\begin{aligned}
\mathbf{v}_{\text{rel},1} &= {}^{\mathcal{F}}\mathbf{v}_1 - {}^{\mathcal{F}}\mathbf{v}_O & (3\text{–}212)\\
\mathbf{v}_{\text{rel},2} &= {}^{\mathcal{F}}\mathbf{v}_2 - {}^{\mathcal{F}}\mathbf{v}_1 & (3\text{–}213)\\
\mathbf{v}_{\text{rel},3} &= {}^{\mathcal{F}}\mathbf{v}_2 - {}^{\mathcal{F}}\mathbf{v}_Q & (3\text{–}214)
\end{aligned}
$$

where we have taken into account the velocity of each block relative to the attachment point of the respective damper. Now because points O and Q are absolutely fixed, we have ${}^{\mathcal{F}}\mathbf{v}_O = {}^{\mathcal{F}}\mathbf{v}_Q = \mathbf{0}$. Furthermore, using the expressions for ${}^{\mathcal{F}}\mathbf{v}_1$, ${}^{\mathcal{F}}\mathbf{v}_2$, and ${}^{\mathcal{F}}\mathbf{v}_3$ from Eqs. (3–191) and (3–192), we obtain

$$\begin{aligned} \mathbf{v}_{\text{rel},1} &= \dot{x}_1\mathbf{E}_x && (3\text{–}215)\\ \mathbf{v}_{\text{rel},2} &= \dot{x}_2\mathbf{E}_x - \dot{x}_1\mathbf{E}_x = (\dot{x}_2 - \dot{x}_1)\mathbf{E}_x && (3\text{–}216)\\ \mathbf{v}_{\text{rel},3} &= \dot{x}_2\mathbf{E}_x && (3\text{–}217) \end{aligned}$$

We then obtain the force exert by each damper as

$$\begin{aligned} \mathbf{F}_{f1} &= -c_1\dot{x}_1\mathbf{E}_x && (3\text{–}218)\\ \mathbf{F}_{f2} &= -c_2(\dot{x}_2 - \dot{x}_1)\mathbf{E}_x && (3\text{–}219)\\ \mathbf{F}_{f3} &= -c_3\dot{x}_2\mathbf{E}_x && (3\text{–}220) \end{aligned}$$

Newton's 2^{nd} law for the first block is then given as

$$\mathbf{F}_1 = \mathbf{F}_{s1} + \mathbf{F}_{s2} + \mathbf{F}_{f1} + \mathbf{F}_{f2} = m_1{}^{\mathcal{F}}\mathbf{a}_1 \quad (3\text{–}221)$$

which implies that

$$-K_1(x_1 - \ell_{10})\mathbf{E}_x + K_2(x_2 - x_1 - \ell_{20})\mathbf{E}_x - c_1\dot{x}_1\mathbf{E}_x - c_2(\dot{x}_2 - \dot{x}_1)\mathbf{E}_x = m_1\ddot{x}_1\mathbf{E}_x \quad (3\text{–}222)$$

Dropping $\mathbf{E}_x$ from this last equation gives

$$-K_1(x_1 - \ell_{10}) + K_2(x_2 - x_1 - \ell_{20}) - c_1\dot{x}_1 - c_2(\dot{x}_2 - \dot{x}_1) = m_1\ddot{x}_1 \quad (3\text{–}223)$$

Rearranging, we obtain

$$m_1\ddot{x}_1 + (c_1 - c_2)\dot{x}_1 + c_2\dot{x}_2 + K_1x_1 - K_2(x_2 - x_1) = K_1\ell_{10} - K_2\ell_{20} \quad (3\text{–}224)$$

Equation (3–224) can be rewritten as

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + c_2\dot{x}_2 + (K_1 + K_2)x_1 - K_2x_2 = K_1\ell_{10} - K_2\ell_{20} \quad (3\text{–}225)$$

Newton's 2^{nd} law for the second collar is given as

$$\mathbf{F}_2 = -\mathbf{F}_{s2} + \mathbf{F}_{s3} - \mathbf{F}_{f2} + \mathbf{F}_{f3} = m_2{}^{\mathcal{F}}\mathbf{a}_2 \quad (3\text{–}226)$$

which implies that

$$-K_2(x_2 - x_1 - \ell_{20})\mathbf{E}_x + K_3(\ell - x_2 - \ell_{30})\mathbf{E}_x + c_2(\dot{x}_2 - \dot{x}_1)\mathbf{E}_x - c_3\dot{x}_2\mathbf{E}_x = m_2\ddot{x}_2\mathbf{E}_x \quad (3\text{–}227)$$

Dropping $\mathbf{E}_x$ from this last equation gives

$$-K_2(x_2 - x_1 - \ell_{20}) + K_3(\ell - x_2 - \ell_{30}) + c_2(\dot{x}_2 - \dot{x}_1) - c_3\dot{x}_2 = m_2\ddot{x}_2 \quad (3\text{–}228)$$

Rearranging, we obtain

$$m_2\ddot{x}_2 + c_2(\dot{x}_1 - \dot{x}_2) + c_3\dot{x}_2 + K_2(x_2 - x_1) + K_3x_2 = K_2\ell_{20} + K_3(\ell - \ell_{30}) \quad (3\text{–}229)$$

Equation (3–229) can be rewritten as

$$m_2\ddot{x}_2 + c_2\dot{x}_1 + (c_3 - c_2)\dot{x}_2 - K_2x_1 + (K_2 + K_3)x_2 = K_2\ell_{20} + K_3(\ell - \ell_{30}) \quad (3\text{–}230)$$

The system of two differential equations describing the motion of the two collars is then given as

$$\begin{aligned} m_1\ddot{x}_1 + (c_1 - c_2)\dot{x}_1 + c_2\dot{x}_2 + (K_1 + K_2)x_1 - K_2x_2 &= K_1\ell_{10} - K_2\ell_{20} && (3\text{–}231)\\ m_2\ddot{x}_2 + c_2\dot{x}_1 + (c_3 - c_2)\dot{x}_2 - K_2x_1 + (K_2 + K_3)x_2 &= K_2\ell_{20} + K_3(\ell - \ell_{30}) && (3\text{–}232) \end{aligned}$$

Equations (3–231) and (3–232) can be written in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 - c_2 & c_2 \\ c_2 & c_3 - c_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 + K_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} K_1 \ell_{10} - K_2 \ell_{20} \\ -K_2 \ell_{20} + K_3(\ell - \ell_{30}) \end{bmatrix} \tag{3–233}$$

which has the general matrix-vector form

$$M\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + K\mathbf{X} = \mathbf{b} \tag{3–234}$$

Now it is seen that the condition for static equilibrium of the system in Eq. (3–234) is given as

$$K\mathbf{X}_{eq} = \mathbf{b} \tag{3–235}$$

Now let

$$\mathbf{Y} = \mathbf{X} - \mathbf{X}_{eq} \tag{3–236}$$

We then have

$$\dot{\mathbf{Y}} = \dot{\mathbf{X}} \tag{3–237}$$

$$\ddot{\mathbf{Y}} = \ddot{\mathbf{X}} \tag{3–238}$$

Substituting the expression for $\mathbf{b}$ into Eq. (3–234), we obtain

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = K\mathbf{X}_{eq} \tag{3–239}$$

This last equation can be rewritten as

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}(\mathbf{X} - \mathbf{X}_{eq}) = \mathbf{0} \tag{3–240}$$

Noting that $\mathbf{Y} = \mathbf{X} - \mathbf{X}_{eq}$ and $\ddot{\mathbf{X}} = \ddot{\mathbf{Y}}$, Eq. (3–240) can be rewritten as

$$\mathbf{M}\ddot{\mathbf{Y}} + \mathbf{C}\dot{\mathbf{Y}} + \mathbf{K}\mathbf{Y} = \mathbf{0} \tag{3–241}$$

It is seen that Eq. (3–241) has a similar mathematical form to the single degree-of-freedom system, the difference being that in this case we have a matrix and column-vector differential equation (or, alternatively, a system of differential equations) as opposed to a scalar differential equation. Consequently, the solution to Eq. (3–241) will itself be a column vector.

3.2.2 Analysis of Unforced Damped Multiple Degree-of-Freedom Systems

Consider now a free damped multiple degree-of-freedom system relative to a static equilibrium point given as

$$\mathbf{M}\ddot{\mathbf{Y}} + \mathbf{C}\dot{\mathbf{Y}} + \mathbf{K}\mathbf{Y} = \mathbf{0} \tag{3–242}$$

It is seen that the difference betwen Eq. (3–241) and (3–37) on page 58 is that Eq. (3–241) has the additional term $\mathbf{C}\dot{\mathbf{Y}}$. This addition term is due to the *damping* that may be part of some two degree-of-freedom vibratory systems. Now, we recall from section 3.1.2 that two degree-of-freedom systems without damping [i.e., systems that satisfy Eq. (3–37)] can be decoupled by determining the eigenvectors of the weighted eigenvalues problem $\mathbf{Ku} = \lambda\mathbf{Mu}$. In particular, using the fact that the eigenvectors are orthogonal to both the mass and stiffness matrices, we can write

$$\mathbf{U}^T\mathbf{K}\mathbf{U} = \mathbf{U}^T\mathbf{M}\mathbf{U}\mathbf{\Lambda} \tag{3–243}$$

Then, using a mass-normalized eigenvector matrix $\mathbf{U}$, we have

$$\mathbf{U}^T\mathbf{M}\mathbf{U} = \mathbf{I} \tag{3–244}$$

which implies that

$$\mathbf{U}^T\mathbf{K}\mathbf{U} = \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{3–245}$$

Then, in terms of modal coordinates we had

$$\ddot{\mathbf{q}} + \mathbf{\Lambda}\mathbf{q} = \mathbf{0} \tag{3–246}$$

Alternatively, in scalar form we had the system of two decoupled differential equations

$$\ddot{q}_k + \lambda_k q_k = 0, \qquad (k = 1, \ldots, n) \tag{3–247}$$

Now, in the case where the system is damped, it is seen in general that the eigenvector matrix will *not* diagonalize the damping matrix, i.e., in general it is the case that

$$\mathbf{U}^T\mathbf{C}\mathbf{U} \neq \tilde{\mathbf{C}} \tag{3–248}$$

where $\tilde{\mathbf{C}}$ would be a $n \times n$ diagonal matrix, i.e., *if* $\mathbf{C}$ were diagonalizable by the eigenvector matrix then we would have

$$\tilde{\mathbf{C}} = \mathbf{I} \tag{3–249}$$

Then, *if* $\mathbf{C}$ was diagonalizable by $\mathbf{U}$, we would be able to write

$$\ddot{\mathbf{q}} + \tilde{\mathbf{C}}\dot{\mathbf{q}} + \mathbf{\Lambda}\mathbf{q} = \mathbf{0} \tag{3–250}$$

As a result, in the case where $\mathbf{U}$ diagonalizes $\mathbf{C}$, we would have n scalar equations of the form

$$\ddot{q}_k + \tilde{c}_k\dot{\mathbf{q}} + \lambda_k q_k = 0, \qquad (i = 1, \ldots, n) \tag{3–251}$$

However, because $\mathbf{C}$ is in general *not* diagonalizable by $\mathbf{U}$, the second term in Eq. (3–250) will not lead to the form of Eq. (3–251). In other words, when using the transformation

$$\mathbf{Y} = \mathbf{U}\mathbf{q} \tag{3–252}$$

where $\mathbf{U}$ is the eigenvector matrix, Eq. (3–241) becomes

$$\ddot{\mathbf{q}} + \mathbf{\Gamma}\dot{\mathbf{q}} + \mathbf{\Lambda}\mathbf{q} = \mathbf{0} \tag{3–253}$$

where $\mathbf{\Gamma}$ is given as

$$\mathbf{\Gamma} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nn} \end{bmatrix} \tag{3–254}$$

Because the matrix $\mathbf{\Gamma}$ is not diagonal, the eigenvector $\mathbf{U}$ matrix cannot be used to obtain a modal form for a multiple degree-of-freedom damped linear system.

3.2.3 Modal Damping (Proportional Damping)

Suppose now that we consider the special case where the damping matrix $\mathbf{C}$ is given as

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K} \tag{3–255}$$

Then Eq. (3–241) can be written as

$$\mathbf{M}\ddot{\mathbf{Y}} + (\alpha\mathbf{M} + \beta\mathbf{K})\dot{\mathbf{Y}} + \mathbf{K}\mathbf{Y} = \mathbf{0} \tag{3–256}$$

Suppose now that we transform Eq. (3–256) using the eigenvector matrix $\mathbf{U}$ of the weighted eigenvalue problem $\mathbf{K}\mathbf{u} = \lambda\mathbf{M}\mathbf{u}$, i.e., we let

$$\mathbf{Y} = \mathbf{U}\mathbf{q} \tag{3–257}$$

Then, we have

$$\mathbf{MU\ddot{q}} + (\alpha\mathbf{M} + \beta\mathbf{K})\mathbf{U\dot{q}} + \mathbf{KUq} = \mathbf{0} \quad (3\text{–}258)$$

Multiplying on the left-hand side by $\mathbf{U}^T$ gives

$$\mathbf{U}^T\mathbf{MU\ddot{q}} + \mathbf{U}^T(\alpha\mathbf{M} + \beta\mathbf{K})\mathbf{U\dot{q}} + \mathbf{U}^T\mathbf{KUq} = \mathbf{0} \quad (3\text{–}259)$$

Equation (3–259) can be rewritten as

$$\mathbf{U}^T\mathbf{MU\ddot{q}} + (\alpha\mathbf{U}^T\mathbf{MU} + \beta\mathbf{U}^T\mathbf{KU})\mathbf{\dot{q}} + \mathbf{U}^T\mathbf{KUq} = \mathbf{0} \quad (3\text{–}260)$$

Suppose now that we choose a mass-normalized eigenvector matrix, i.e., $\mathbf{U}$ has the property that

$$\mathbf{U}^T\mathbf{MU} = \mathbf{I} \quad (3\text{–}261)$$
$$\mathbf{U}^T\mathbf{KU} = \Lambda \quad (3\text{–}262)$$

Then Eq. (3–260) can be written as

$$\mathbf{\ddot{q}} + (\alpha\mathbf{I} + \beta\mathbf{\Lambda})\mathbf{\dot{q}} + \mathbf{\Lambda q} = \mathbf{0} \quad (3\text{–}263)$$

Because $\mathbf{\Lambda}$ is diagonal, it is seen that the matrix $\mathbf{I} + \beta\mathbf{\Lambda}$ is also diagonal. In scalar form we have the following two differential equations:

$$\ddot{q}_k + (\alpha + \beta\lambda_k)\dot{q} + \lambda_k q = 0, \qquad (i = 1, \ldots, n) \quad (3\text{–}264)$$

Suppose now that we consider the case where $\alpha + \beta\lambda_i > 0$ for $i = 1, 2$. Furthermore, using the earlier notation, we can write Eq. (3–264) as

$$\ddot{q}_k + \gamma_k\dot{q}_k + \lambda_k q_k = 0, \qquad (i = 1, \ldots, n) \quad (3\text{–}265)$$

where

$$\gamma_k = \alpha + \beta\lambda_k, \qquad (i = 1, \ldots, n) \quad (3\text{–}266)$$

It is seen that Eq. (3–265) is a system of two *decoupled* second-order LTI differential equations, each of which can be solved by the techniques for single degree-of-freedom systems. The form of damping given in Eq. (3–255) is called *modal* damping because it is diagonalizable by the eigenvector matrix of the weighted eigenvalue problem. Suppose now that we define

$$\begin{array}{rcl} \lambda_k & = & \omega_k^2 \\ \gamma_k & = & 2\zeta_k\omega_k \end{array}, \qquad (k = 1, \ldots, n) \quad (3\text{–}267)$$

The quantities ω_k, $(i = 1, \ldots, n)$ and ζ_k, $(i = 1, \ldots, n)$ are the *modal natural frequencies* and *modal damping ratios*, respectively. In terms of the modal natural frequencies and modal damping ratios, we can write

$$\ddot{q}_k + 2\zeta_k\omega_k\dot{q} + \omega_k^2 q = 0, \qquad (k = 1, \ldots, n) \quad (3\text{–}268)$$

It can be seen that the each differential equation in Eq. (3–268) is in the *standard form* and thus can be solved via the techniques for a single degree-of-freedom LTI.

Example 3–3

Consider the free damped two degree-of-freedom system

$$\mathbf{M}\ddot{\mathbf{Y}} + \mathbf{C}\dot{\mathbf{Y}} + \mathbf{K}\mathbf{Y} = \mathbf{0}$$

where the mass, damping, and stiffness matrices, are given, respectively, as follows

$$\begin{aligned} \mathbf{M} &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\ \mathbf{C} &= \begin{bmatrix} -3 & 2 \\ 2 & -4 \end{bmatrix} \\ \mathbf{K} &= \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \end{aligned}$$

Determine a system of two uncoupled differential equations of the form

$$\ddot{\mathbf{q}} + \boldsymbol{\Gamma}\dot{\mathbf{q}} + \boldsymbol{\Lambda}\mathbf{q} = \mathbf{0}$$

where the matrices $\boldsymbol{\Gamma}$ and $\boldsymbol{\Lambda}$ are diagonal.

Solution to Example 3–3

Using the mass, damping, and stiffness matrices given in the problem statement, it is seen that

$$\mathbf{C} = \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + 2\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = -\mathbf{M} + 2\mathbf{K} \tag{3–269}$$

Consequently, $\mathbf{C}$ is a modal damping matrix for this problem. We can then use the eigenvector matrix associated with the weighted eigenvalue problem to decouple the original differential equations. Now recall that the matrices $\mathbf{M}$ and $\mathbf{K}$ are the same as those used in Example 3–2. In particular, recall that the mass-normalized eigenvector matrix in Example 3–2 is given from Eq. (3–171) on page 71 as

$$\mathbf{U} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \tag{3–270}$$

Now we know from Example 3–2 that $\mathbf{U}$ decouples the undamped system, i.e.,

$$\mathbf{U}^T\mathbf{M}\mathbf{U} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3–271}$$

$$\mathbf{U}^T\mathbf{K}\mathbf{U} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{5}{2} \end{bmatrix} \tag{3–272}$$

Furthermore, for this example it is seen that

$$\mathbf{U}^T\mathbf{C}\mathbf{U} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \tag{3–273}$$

which implies that the eigenvector matrix also decouples the damping matrix $\mathbf{C}$. Now let us return to the original system. Applying the transformation $\mathbf{Y} = \mathbf{U}\mathbf{q}$, we obtain

$$\mathbf{M}\mathbf{U}\ddot{\mathbf{q}} + \mathbf{C}\mathbf{U}\dot{\mathbf{q}} + \mathbf{K}\mathbf{U}\mathbf{q} = \mathbf{0} \tag{3–274}$$

Then, pre-multiplying by $\mathbf{U}^T$ gives

$$\mathbf{U}^T\mathbf{M}\mathbf{U}\ddot{\mathbf{q}} + \mathbf{U}^T\mathbf{C}\mathbf{U}\dot{\mathbf{q}} + \mathbf{U}^T\mathbf{K}\mathbf{U}\mathbf{q} = \mathbf{0} \tag{3–275}$$

Then, substituting the expressions for $\mathbf{U}^T\mathbf{M}\mathbf{U}$, $\mathbf{U}^T\mathbf{K}\mathbf{U}$, and $\mathbf{U}^T\mathbf{C}\mathbf{U}$ from Eqs. (3–271), (3–271), and (3–271), respectively, we obtain

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\mathbf{q}} + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\dot{\mathbf{q}} + \begin{bmatrix} 1 & 0 \\ 0 & \frac{5}{2} \end{bmatrix}\mathbf{q} = \mathbf{0} \tag{3–276}$$

Then, setting $(q_1, q_2) = \mathbf{q}$, Eq. (3–276) is equivalent to the following two scalar differential equations:

$$\ddot{q}_1 + \dot{q}_1 + q_1 = 0 \tag{3–277}$$

$$\ddot{q}_2 + 4\dot{q}_2 + \tfrac{5}{2}q_2 = 0 \tag{3–278}$$

It can be seen that Eqs. (3–277) and (3–278) are decoupled, consistent with the fact that we have a modally damped system. Finally, recall from Eq. (3–269) that the damping matrix for this example is given as $\mathbf{C} = -\mathbf{M} + 2\mathbf{K}$ which implies that $\alpha = -1$ and $\beta = 2$. Therefore, from Eq. (3–266) we have

$$\gamma_1 = \alpha + \beta\lambda_1 \tag{3–279}$$

$$\gamma_1 = \alpha + \beta\lambda_2 \tag{3–280}$$

Now the eigenvalues of the undamped problem are the same as those given in Eq. (3–148) on page 69 of Example 3–2, i.e., the eigenvalues are

$$\lambda_1 = 1 \tag{3–281}$$

$$\lambda_2 = \tfrac{5}{2} \tag{3–282}$$

Therefore,

$$\gamma_1 = -1 + 2\lambda_1 = -1 + 2(1) = 1 \tag{3–283}$$

$$\gamma_2 = -1 + 2\lambda_2 = -1 + 2(5/2) = 4 \tag{3–284}$$

Examining Eqs. (3–277) and (3–278) it is seen that the coefficients multiplying the $\dot{q}_1$ and $\dot{q}_2$ terms are 1 and 4, respectively. Consequently, because we have a modally damped system in this example, determining the values γ_1 and γ_2 is equivalent to computing the transformation $\mathbf{U}^T\mathbf{C}\mathbf{U}$.

■

Example 3–4

Consider the system of two nonlinear differential equations

$$\begin{aligned}(m_1 + m_2)\ddot{x} + k_1 x + m_2 l\ddot{\theta}\cos\theta - m_2 l\dot{\theta}^2\sin\theta &= 0 \\ m_2\ddot{x}\cos\theta + m_2 l\ddot{\theta} + m_2 g\sin\theta &= 0\end{aligned}$$

Determine (a) the static equilibrium point for the system, (b) the differential equations of motion for values of x and θ near the static equilibrium point found in part (a), and (c) the differential equations in modal coordinates for the case $m_1 = m_2 = 1$, $l = 1$, and $g = 1$.

Solution to Example 3–4

(a) Static Equilibrium Point

Let (x_{eq}, θ_{eq}) be the static equilibrium point. Then we have

$$\begin{aligned} \dot{x}_{eq} = 0 \quad &, \quad \ddot{x}_{eq} = 0 \\ \dot{\theta}_{eq} = 0 \quad &, \quad \ddot{\theta}_{eq} = 0 \end{aligned}$$

Substituting these results into the system of differential equations, we obtain

$$\begin{aligned} k_1 x &= 0 \Longrightarrow x = 0 \\ m_2 g \sin\theta &= 0 \Longrightarrow \sin\theta = 0 \Longrightarrow \theta_{eq} = 0 \end{aligned} \tag{3–285}$$

Therefore, the equilibrium point for this system is $(x_{eq}, \theta_{eq}) = (0, 0)$.

(b) Linearization of Differential Equations Near Equilibrium Point

Then, we can linearize the differential equations of motion for values of x and θ near the equilibrium values as follows. First, let

$$\begin{aligned} \delta x &= x - x_{eq} \\ \delta\theta &= \theta - \theta_{eq} \end{aligned} \tag{3–286}$$

Then,

$$\begin{aligned} \delta\dot{x} &= \dot{x} \\ \delta\ddot{x} &= \ddot{x} \\ \delta\dot{\theta} &= \dot{\theta} \\ \delta\ddot{\theta} &= \ddot{\theta} \end{aligned} \tag{3–287}$$

Furthermore, we have

$$\begin{aligned} \cos\theta &\approx \cos\theta_{eq} - \sin\theta_{eq}\delta\theta = \cos(0) = 1 \\ \sin\theta &\approx \sin\theta_{eq} + \cos\theta_{eq}\delta\theta = \sin(0) + \cos(0)\delta\theta = \delta\theta \\ \dot{\theta}^2 &\approx 2\dot{\theta}_{eq}\delta\dot{\theta} = 0 \end{aligned} \tag{3–288}$$

Therefore, the differential equations near the equilibrium point are given as

$$\begin{aligned} (m_1 + m_2)\delta\ddot{x} + k_1\delta x + m_2 l\delta\ddot{\theta} &= 0 \\ m_2\delta\ddot{x} + m_2 l\delta\ddot{\theta} + m_2 g\delta\theta &= 0 \end{aligned}$$

Dividing the first differential equation by l (in order to make the system symmetric), we obtain

$$\begin{aligned} \frac{m_1+m_2}{l}\delta\ddot{x} + \frac{k_1}{l}\delta x + m_2\delta\ddot{\theta} &= 0 \\ m_2\delta\ddot{x} + m_2 l\delta\ddot{\theta} + m_2 g\delta\theta &= 0 \end{aligned}$$

These last two equations can be written in vector-matrix form as

$$\begin{bmatrix} \frac{m_1+m_2}{l} & m_2 \\ m_2 & m_2 l \end{bmatrix} \begin{bmatrix} \delta\ddot{x} \\ \delta\ddot{\theta} \end{bmatrix} + \begin{bmatrix} \frac{k_1}{l} & 0 \\ 0 & m_2 g \end{bmatrix} \begin{bmatrix} \delta x \\ \delta\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3–289}$$

(c) Differential Equations in Modal Coordinates

Let

$$\mathbf{X} = \begin{bmatrix} \delta x \\ \delta\theta \end{bmatrix} \tag{3–290}$$

$$\mathbf{M} = \begin{bmatrix} \frac{m_1+m_2}{l} & m_2 \\ m_2 & m_2 l \end{bmatrix} \tag{3–291}$$

$$\mathbf{K} = \begin{bmatrix} \frac{k_1}{l} & 0 \\ 0 & m_2 g \end{bmatrix} \tag{3–292}$$

Then, substituting the given values of $m_1 = m_2 = 1$, $l = 1$, $k_1 = 1$, and $g = 1$, we obtain

$$\mathbf{M} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \tag{3–293}$$

$$\mathbf{K} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \tag{3–294}$$

We then can obtain the modal natural frequencies by solving the weighted eigenvalue problem $\mathbf{Ku} = \lambda\mathbf{Mu}$. In particular, the eigenvalues are found as

$$\det(\lambda\mathbf{M} - \mathbf{K}) = 0 \tag{3–295}$$

which for this problem implies that

$$\det\left(\lambda\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} 2\lambda - 2 & \lambda \\ \lambda & \lambda - 1 \end{bmatrix} = 0 \tag{3–296}$$

We then obtain

$$(2\lambda - 2)(\lambda - 1) - \lambda^2 = \lambda^2 - 5\lambda + 2 = 0 \tag{3–297}$$

The eigenvalues are then given as

$$\lambda_{1,2} = \frac{5 \pm \sqrt{25 - 4(1)(2)}}{2} = \frac{5 \pm \sqrt{17}}{2} \tag{3–298}$$

It is noted that $5 > \sqrt{17}$ which implies that both λ_1 and λ_2 are positive (which they should be because this is an undamped oscillatory system). Then we know that the equations in modal coordinates are given as

$$\ddot{q}_1 + \lambda_1 q_1 = 0 \tag{3–299}$$

$$\ddot{q}_2 + \lambda_2 q_2 = 0 \tag{3–300}$$

which for this problem implies that

$$\ddot{q}_1 + \frac{5 + \sqrt{17}}{2} q_1 = 0 \tag{3–301}$$

$$\ddot{q}_2 + \frac{5 - \sqrt{17}}{2} q_2 = 0 \tag{3–302}$$

(3–303)

■

3.3 Non-Symmetric Mass and Stiffness Matrices

Until now all of the theory that we have been discussed has relied on the assumption that the mass and stiffness matrices are symmetric. In particular, the weighted eigenvalue problem $\mathbf{K}\mathbf{u} = \lambda\mathbf{M}\mathbf{u}$ produces positive real eigenvalues and orthogonal eigenvectors only in the case where $\mathbf{M}$ is symmetric and positive definite while $\mathbf{K}$ is symmetric and positive semi-definite. Unfortunately, the ability to obtain symmetric mass and stiffness matrices depends upon the approach used to derive the differential equations. In more advanced courses in vibrations the differential equations are derived using advanced methods involving Lagrangian mechanics. However, in our study we have focused on Newtonian formulations. When using Newtonian mechanics, it is often possible to obtain a system of differential equations which, when linearized about the static equilibrium point, results in non-symmetric mass and stiffness matrices. In order to demonstrate this point, we will now explore an example for which this lack of symmetry exists. Now, while no systematic procedure exists from which the system can be made symmetric, we will proceed to discuss briefly through this example how a symmetric form can be obtained.

Example 3–5

A collar of mass m_1 is constrained to slide along a frictionless horizontal track as shown in Fig. 3–7. Attached to the collar is a linear spring with spring constant K and unstretched length ℓ_0. The collar is attached to one end of a rigid massless arm of length l while a particle of mass m_2 is attached to the other end of the arm. Knowing that x describes the displacement of the collar relative to the track, that the angle θ is measured from the downward direction, that the spring is unstretched when $x = 0$, and that gravity acts downward, determine (a) a system of two differential equations that describes the motion of the collar-particle system; (b) a system of differential equations linearized about the static equilibrium point; (c) an alternate system of differential equations via algebraic manipulation of the system obtained in part (a); and (d) a system of differential equations from part (c) linearized about the static equilibrium point.

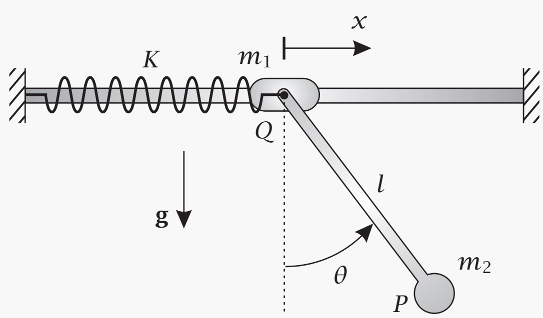

Figure 3–7 Particle on rigid massless arm attached to sliding collar on spring.

Solution to Example 3–5

Kinematics

First, let $\mathcal{F}$ be a reference frame fixed to the track. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

	Origin at Q when $x = 0$	
$\mathbf{E}_x$	=	To the right
$\mathbf{E}_z$	=	Out of page
$\mathbf{E}_y$	=	$\mathbf{E}_z \times \mathbf{E}_x$

Next, let $\mathcal{A}$ be a reference frame fixed to the arm. Then, choose the following coordinate system fixed in reference frame $\mathcal{A}$:

$$\begin{array}{ccc} & \text{Origin at } Q & \\ \mathbf{e}_r & = & \text{Along } QP \\ \mathbf{e}_z & = & \text{Out of page} \\ \mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r \end{array}$$

The geometry of the bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is shown in Fig. 3–8.

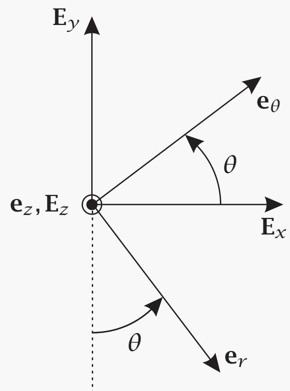

Figure 3–8 Geometry of bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ for Example 3–5.

Using Fig. 3–8, we have

$$\mathbf{e}_r = \sin\theta\,\mathbf{E}_x - \cos\theta\,\mathbf{E}_y \tag{3–304}$$

$$\mathbf{e}_\theta = \cos\theta\,\mathbf{E}_x + \sin\theta\,\mathbf{E}_y \tag{3–305}$$

Now, consistent with the discussion at the beginning of this problem, we establish the kinematics relevant to the system consisting of the collar and the system consisting of the collar and the particle.

Kinematics of Collar

The position of the collar is given as

$$\mathbf{r}_1 = x\mathbf{E}_x \tag{3–306}$$

Computing the rate of change of $\mathbf{r}_1$ in reference frame $\mathcal{F}$, we obtain the velocity of the collar in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{v}_1 = \frac{^{\mathcal{F}}d}{dt}(\mathbf{r}_1) = \dot{x}\mathbf{E}_x \tag{3–307}$$

Finally, computing the rate of change of $^{\mathcal{F}}\mathbf{v}_1$ in reference frame $\mathcal{F}$, we obtain the acceleration of the collar in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{a}_1 = \frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}_1\right) = \ddot{x}\mathbf{E}_x \tag{3–308}$$

Kinematics of Particle

The kinematics of the collar-particle system are governed by the motion of the center of mass of the system. Consequently, in order to determine the kinematics of the center of mass of the collar-particle system, it is first necessary to determine the position, velocity, and The position of the particle is given as

$$\mathbf{r}_2 = \mathbf{r}_1 + \mathbf{r}_{2/1} \tag{3–309}$$

Now we have

$$\mathbf{r}_{2/1} = l\mathbf{e}_r \tag{3–310}$$

Then, adding Eqs. (3–310) and (3–309), we obtain the position of the particle as

$$\mathbf{r}_2 = x\mathbf{E}_x + l\mathbf{e}_r \tag{3–311}$$

Next, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \dot{\theta}\mathbf{e}_z \tag{3–312}$$

Consequently, the velocity of the particle in reference frame $\mathcal{F}$ is obtained as

$${}^{\mathcal{F}}\mathbf{v}_2 = \frac{{}^{\mathcal{F}}d}{dt}(\mathbf{r}_2) = \frac{{}^{\mathcal{F}}d}{dt}(\mathbf{r}_1) + \frac{{}^{\mathcal{F}}d}{dt}(\mathbf{r}_{2/1}) = {}^{\mathcal{F}}\mathbf{v}_1 + {}^{\mathcal{F}}\mathbf{v}_{2/1} \tag{3–313}$$

We already have ${}^{\mathcal{F}}\mathbf{v}_1$ from Eq. (3–307). Applying the rate of change transport theorem to $\mathbf{r}_{2/1}$ between reference frames $\mathcal{A}$ and $\mathcal{F}$ gives

$${}^{\mathcal{F}}\mathbf{v}_{2/1} = \frac{{}^{\mathcal{F}}d}{dt}(\mathbf{r}_{2/1}) = \frac{{}^{\mathcal{A}}d}{dt}(\mathbf{r}_{2/1}) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_{2/1} \tag{3–314}$$

Now we have

$$\frac{{}^{\mathcal{A}}d}{dt}(\mathbf{r}_{2/1}) = \mathbf{0} \tag{3–315}$$

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_{2/1} = \dot{\theta}\mathbf{e}_z \times l\mathbf{e}_r = l\dot{\theta}\mathbf{e}_\theta \tag{3–316}$$

Adding Eqs. (3–315) and (3–316), we obtain the velocity of the particle relative to the collar in reference frame $\mathcal{F}$ as

$${}^{\mathcal{F}}\mathbf{v}_{2/1} = l\dot{\theta}\mathbf{e}_\theta \tag{3–317}$$

Then, substituting the results of Eqs. (3–307) and (3–317) into Eq. (3–313), we obtain the velocity of the particle in reference frame $\mathcal{F}$ as

$${}^{\mathcal{F}}\mathbf{v}_2 = \dot{x}\mathbf{E}_x + l\dot{\theta}\mathbf{e}_\theta \tag{3–318}$$

Computing the rate of change of ${}^{\mathcal{F}}\mathbf{v}_2$ in reference frame $\mathcal{F}$ using the general expression for ${}^{\mathcal{F}}\mathbf{v}_2$ as given in Eq. (3–313), the acceleration of the particle in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\mathbf{a}_2 = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_2\right) = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_1\right) + \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{2/1}\right) = {}^{\mathcal{F}}\mathbf{a}_1 + {}^{\mathcal{F}}\mathbf{a}_{2/1} \tag{3–319}$$

Now we already have ${}^{\mathcal{F}}\mathbf{a}_1$ from Eq. (3–308). Applying the rate of change transport theorem between reference frames $\mathcal{A}$ and $\mathcal{F}$, we obtain ${}^{\mathcal{F}}\mathbf{a}_{2/1}$ as

$${}^{\mathcal{F}}\mathbf{a}_{2/1} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{2/1}\right) = \frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{2/1}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v}_{2/1} \tag{3–320}$$

Now we have

$$^{\mathcal{A}}\frac{d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{2/1}\right) = l\ddot{\theta}\mathbf{e}_\theta \tag{3–321}$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v}_{2/1} = \dot{\theta}\mathbf{e}_z \times l\dot{\theta}\mathbf{e}_\theta = -l\dot{\theta}^2\mathbf{e}_r \tag{3–322}$$

Adding Eqs. (3–321) and (3–322), we obtain

$$^{\mathcal{F}}\mathbf{a}_{2/1} = -l\dot{\theta}^2\mathbf{e}_r + l\ddot{\theta}\mathbf{e}_\theta \tag{3–323}$$

Finally, adding Eqs. (3–323) and (3–308), we obtain the acceleration of the particle in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{a}_2 = \ddot{x}\mathbf{E}_x - l\dot{\theta}^2\mathbf{e}_r + l\ddot{\theta}\mathbf{e}_\theta \tag{3–324}$$

Kinematics of Center of Mass of Collar-Particle System

The position of the center of mass of the collar-particle system is given as

$$\bar{\mathbf{r}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \tag{3–325}$$

Substituting the expressions for $\mathbf{r}_1$ and $\mathbf{r}_2$ from Eqs. (3–306) and (3–309), respectively, into (3–325), we obtain $\bar{\mathbf{r}}$ as

$$\bar{\mathbf{r}} = \frac{m_1 x\mathbf{E}_x + m_2(x\mathbf{E}_x + l\mathbf{e}_r)}{m_1 + m_2} = x\mathbf{E}_x + \frac{m_2}{m_1 + m_2}l\mathbf{e}_r \tag{3–326}$$

Next, the velocity of the center of mass of the collar-particle system in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\bar{\mathbf{v}} = \frac{m_1{}^{\mathcal{F}}\mathbf{v}_1 + m_2{}^{\mathcal{F}}\mathbf{v}_2}{m_1 + m_2} \tag{3–327}$$

Substituting the expression for ${}^{\mathcal{F}}\mathbf{v}_1$ from Eq. (3–307) and the expression for ${}^{\mathcal{F}}\mathbf{v}_2$ from Eq. (3–318) into (3–327), we obtain ${}^{\mathcal{F}}\bar{\mathbf{v}}$ as

$$^{\mathcal{F}}\bar{\mathbf{v}} = \frac{m_1\dot{x}\mathbf{E}_x + m_2(\dot{x}\mathbf{E}_x + l\dot{\theta}\mathbf{e}_\theta)}{m_1 + m_2} = \dot{x}\mathbf{E}_x + \frac{m_2}{m_1 + m_2}l\dot{\theta}\mathbf{e}_\theta \tag{3–328}$$

Finally, the acceleration of the center of mass of the collar-particle system in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\bar{\mathbf{a}} = \frac{m_1{}^{\mathcal{F}}\mathbf{a}_1 + m_2{}^{\mathcal{F}}\mathbf{a}_2}{m_1 + m_2} \tag{3–329}$$

Substituting the expressions for ${}^{\mathcal{F}}\mathbf{a}_1$ and ${}^{\mathcal{F}}\mathbf{a}_2$ from Eqs. (3–308) and (3–324), respectively, into (3–329), we obtain

$$^{\mathcal{F}}\bar{\mathbf{a}} = \frac{m_1\ddot{x}\mathbf{E}_x + m_2(\ddot{x}\mathbf{E}_x - l\dot{\theta}^2\mathbf{e}_r + l\ddot{\theta}\mathbf{e}_\theta)}{m_1 + m_2} = \ddot{x}\mathbf{E}_x + \frac{m_2}{m_1 + m_2}(-l\dot{\theta}^2\mathbf{e}_r + l\ddot{\theta}\mathbf{e}_\theta) \tag{3–330}$$

Kinetics

In order to solve this problem, it is convenient to analyze the following two systems: (1) the collar and (2) the collar and the particle. The kinetic relationships for each of these two systems is now established.

Kinetics of Collar

The free body diagram of the collar is shown in Fig. 3–9.

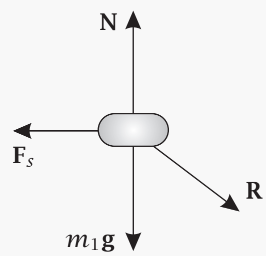

Figure 3–9 Free body diagram of collar for Example 3–5.

Using Fig. 3–9, it is seen that the following forces act on the collar:

$$\begin{array}{rcl}\mathbf{N} & = & \text{Reaction force of track on collar} \\ \mathbf{R} & = & \text{Tension force in arm due to particle} \\ \mathbf{F}_s & = & \text{Force of linear spring} \\ m_1\mathbf{g} & = & \text{Force of gravity}\end{array}$$

The forces acting on the collar are given in terms of the bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ as

$$\mathbf{N} = N\mathbf{E}_y \tag{3–331}$$

$$\mathbf{R} = R\mathbf{e}_r \tag{3–332}$$

$$\mathbf{F}_s = -K(\ell - \ell_0)\mathbf{u}_s = -K(x + \ell_0 - \ell_0)\mathbf{E}_x = -Kx\mathbf{E}_x \tag{3–333}$$

$$m_1\mathbf{g} = -m_1 g\mathbf{E}_y \tag{3–334}$$

It is noted that, because the spring is unstretched when $x = 0$, the length of the spring is $\ell = x + \ell_{10}$. Furthermore, it is noted in Eq. (3–332) that, from the strong form of Newton's 3^{rd} law, the force $\mathbf{R}$ must lie along the line of action connecting the collar and the particle. The resultant force acting on the collar is then given as

$$\mathbf{F}_1 = \mathbf{N} + \mathbf{R} + \mathbf{F}_s + m_1\mathbf{g} = N\mathbf{E}_y + R\mathbf{e}_r - Kx\mathbf{E}_x - m_1 g\mathbf{E}_y \tag{3–335}$$

Then, substituting the expression for $\mathbf{e}_r$ from Eq. (3–304) into (3–335), we have

$$\begin{aligned}\mathbf{F}_1 &= N\mathbf{E}_y + R(\sin\theta\mathbf{E}_x - \cos\theta\mathbf{E}_y) - Kx\mathbf{E}_x - m_1 g\mathbf{E}_y \\ &= (R\sin\theta - Kx)\mathbf{E}_x + (N - R\cos\theta - m_1 g)\mathbf{E}_y\end{aligned} \tag{3–336}$$

Applying Newton's 2^{nd} law to the collar by setting $\mathbf{F}_1$ in Eq. (3–336) equal to $m_1{}^{\mathcal{F}}\mathbf{a}_1$ using the expression for ${}^{\mathcal{F}}\mathbf{a}_1$ from Eq. (3–308), we obtain

$$(R\sin\theta - Kx)\mathbf{E}_x + (N - R\cos\theta - m_1 g)\mathbf{E}_y = m_1\ddot{x}\mathbf{E}_x \tag{3–337}$$

Equation (3–337) yields the following two scalar equations:

$$R\sin\theta - Kx = m_1\ddot{x} \tag{3–338}$$

$$N - R\cos\theta - m_1 g = 0 \tag{3–339}$$

Kinetics of Collar-Particle System

The free body diagram of the collar-particle system is shown in Fig. 3–10.

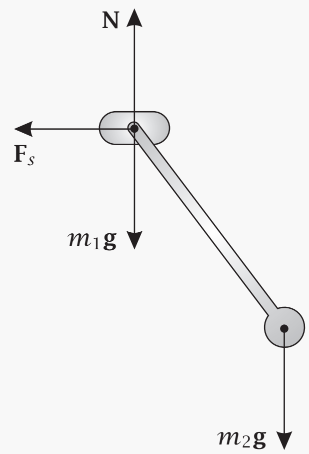

Figure 3–10 Free body diagram of collar-particle system for Example 3–5.

Using Fig. 3–10, it is seen that the following forces act on the collar-particle system:

$$
\begin{array}{rcl}
\mathbf{N} & = & \text{Reaction force of track on collar} \\
\mathbf{F}_s & = & \text{Force in linear spring} \\
(m_1+m_2)\mathbf{g} & = & \text{Force of gravity}
\end{array}
$$

We already have $\mathbf{N}$ and $\mathbf{F}_s$ from Eqs. (3–331) and (3–333), respectively. Furthermore, the force of gravity acting on the collar-particle system is given as

$$(m_1+m_2)\mathbf{g} = -(m_1+m_2)g\mathbf{E}_y \tag{3–340}$$

Consequently, the resultant force acting on the collar is given as

$$
\begin{aligned}
\mathbf{F} &= \mathbf{N} + \mathbf{F}_s + (m_1+m_2)\mathbf{g} \\
&= N\mathbf{E}_y - Kx\mathbf{E}_x - (m_1+m_2)g\mathbf{E}_y \\
&= -Kx\mathbf{E}_x + \left[N-(m_1+m_2)g\right]\mathbf{E}_y
\end{aligned} \tag{3–341}
$$

Applying Newton's 2^{nd} law to the collar-particle system by setting $\mathbf{F}$ in Eq. (3–341) equal to $(m_1+m_2)\,^{\mathcal{F}}\bar{\mathbf{a}}$ using the expression for $^{\mathcal{F}}\bar{\mathbf{a}}$ from Eq. (3–330), we obtain

$$-Kx\mathbf{E}_x + \left[N-(m_1+m_2)g\right]\mathbf{E}_y = (m_1+m_2)\left[\ddot{x}\mathbf{E}_x + \frac{m_2}{m_1+m_2}(-l\dot{\theta}^2\mathbf{e}_r + l\ddot{\theta}\mathbf{e}_\theta)\right] \tag{3–342}$$

Equation (3–342) can be rewritten as

$$-Kx\mathbf{E}_x + \left[N-(m_1+m_2)g\right]\mathbf{E}_y = (m_1+m_2)\ddot{x}\mathbf{E}_x - m_2l\dot{\theta}^2\mathbf{e}_r + m_2l\ddot{\theta}\mathbf{e}_\theta \tag{3–343}$$

Then, substituting the expressions for $\mathbf{e}_r$ and $\mathbf{e}_\theta$ from Eqs. (3–304) and (3–305), respectively, into Eq. (3–343), we obtain

$$
\begin{aligned}
-Kx\mathbf{E}_x + \left[N-(m_1+m_2)g\right]\mathbf{E}_y &= (m_1+m_2)\ddot{x}\mathbf{E}_x - m_2l\dot{\theta}^2(\sin\theta\,\mathbf{E}_x - \cos\theta\,\mathbf{E}_y) \\
&\quad + m_2l\ddot{\theta}(\cos\theta\,\mathbf{E}_x + \sin\theta\,\mathbf{E}_y)
\end{aligned} \tag{3–344}
$$

Rearranging Eq. (3–344), we obtain

$$
\begin{aligned}
-Kx\mathbf{E}_x + \left[N - (m_1+m_2)g\right]\mathbf{E}_y = &\left[(m_1+m_2)\ddot{x} + m_2 l\ddot{\theta}\cos\theta - m_2 l\dot{\theta}^2\sin\theta\right]\mathbf{E}_x \\
&+ \left[m_2 l\ddot{\theta}\sin\theta + m_2 l\dot{\theta}^2\cos\theta\right]\mathbf{E}_y
\end{aligned}
\tag{3–345}
$$

Equation (3–345) yields the following two scalar equations:

$$-Kx = (m_1+m_2)\ddot{x} + m_2 l\ddot{\theta}\cos\theta - m_2 l\dot{\theta}^2\sin\theta \tag{3–346}$$

$$N - (m_1+m_2)g = m_2 l\ddot{\theta}\sin\theta + m_2 l\dot{\theta}^2\cos\theta \tag{3–347}$$

(a) System of Two Differential Equations

Using the results of Eqs. (3–338), (3–339), (3–346), and (3–347), a system of two differential equations is now determined. Because Eq. (3–346) has no unknown forces, it is the first differential equation. The second differential equation is obtained as follows. Multiplying Eqs. (3–338) and (3–339) by $\cos\theta$ and $\sin\theta$, respectively, we have

$$(R\sin\theta - Kx)\cos\theta = m_1\ddot{x}\cos\theta \tag{3–348}$$

$$N\sin\theta - R\cos\theta\sin\theta - m_1 g\sin\theta = 0 \tag{3–349}$$

Adding Eqs. (3–348) and (3–349), we obtain

$$-Kx\cos\theta + N\sin\theta - m_1 g\sin\theta = m_1\ddot{x}\cos\theta \tag{3–350}$$

Next, multiplying Eq. (3–347) by $\sin\theta$ gives

$$N\sin\theta - (m_1+m_2)g\sin\theta = m_2 l\ddot{\theta}\sin^2\theta + m_2 l\dot{\theta}^2\cos\theta\sin\theta \tag{3–351}$$

Then, subtracting Eq. (3–351) from (3–350), we obtain

$$m_2 g\sin\theta - Kx\cos\theta = m_1\ddot{x}\cos\theta - m_2 l\ddot{\theta}\sin^2\theta - m_2 l\dot{\theta}^2\cos\theta\sin\theta \tag{3–352}$$

Rearranging Eq. (3–352), we obtain the second differential equation of motion as

$$m_1\ddot{x}\cos\theta - m_2 l\ddot{\theta}\sin^2\theta - m_2 l\dot{\theta}^2\cos\theta\sin\theta - m_2 g\sin\theta + Kx\cos\theta = 0 \tag{3–353}$$

A system of two differential equations that describes the motion of the collar-particle system is then given as

$$(m_1+m_2)\ddot{x} + m_2 l\ddot{\theta}\cos\theta - m_2 l\dot{\theta}^2\sin\theta + Kx = 0 \tag{3–354}$$

$$m_1\ddot{x}\cos\theta - m_2 l\ddot{\theta}\sin^2\theta - m_2 l\dot{\theta}^2\cos\theta\sin\theta - m_2 g\sin\theta + Kx\cos\theta = 0 \tag{3–355}$$

(b) Linearization of Differential Equations About Static Equilibrium Point

Let (x_{eq}, θ_{eq}) be the static equilibrium point. Then, substituting x_{eq} and θ_{eq}, along with the relationships $\dot{x}_{eq} = \ddot{x}_{eq} = 0$ and $\dot{\theta}_{eq} = \ddot{\theta}_{eq} = 0$, into Eq. (3–354) and (3–355), we obtain

$$Kx_{eq} = 0 \tag{3–356}$$

$$m_2 g\sin\theta_{eq} + Kx_{eq}\cos\theta_{eq} = 0 \tag{3–357}$$

It is seen from Eqs. (3–356) and (3–357) that the static equilibrium point is $(x_{eq}, \theta_{eq}) = (0, 0)$. Next, let

$$
\begin{aligned}
\delta x &= x - x_{eq} \equiv x \\
\delta\theta &= \theta - \theta_{eq} \equiv \theta
\end{aligned}
\tag{3–358}
$$

Then, assuming that δx and $\delta\theta$ are small, we know that all terms involving products of δx and $\delta\theta$ (and products involving derivatives of δx and $\delta\theta$) are negligible. Furthermor, we know that $\cos\theta = \cos\delta\theta \approx 1$ andsin $\theta = \sin\delta\theta \approx \delta\theta$. The differential equations linearized relative to the static equilibrium point are then given as

$$(m_1+m_2)\delta\ddot{x} + m_2 l\delta\ddot{\theta} + K\delta x = 0 \qquad (3\text{–}359)$$

$$m_1\delta\ddot{x} - m_2 g\delta\theta + K\delta x = 0 \qquad (3\text{–}360)$$

Equations (3–359) and (3–360) can be written in matrix form as

$$\begin{bmatrix} m_1+m_2 & m_2 l \\ m_1 & 0 \end{bmatrix}\begin{bmatrix} \delta\ddot{x} \\ \delta\ddot{\theta} \end{bmatrix} + \begin{bmatrix} K & 0 \\ K & -m_2 g \end{bmatrix}\begin{bmatrix} \delta x \\ \delta\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (3\text{–}361)$$

It can be seen that the mass and stiffness matrices in Eq. (3–361) are *not* symmetric. Consequently, this form of the linearized differential equations is not suitable for eigenvalue-eigenvector analysis using the symmetric weighted eigenvalue problem. In order to use the aforementioned techniques to decouple the differential equations, it is necessary to obtain a symmetric form for the linearized dynamics. We now show how to obtain a symmetric form of the linearized differential equations.

(c) Alternate System of Differential Equations

Multiplying Eq. (3–354) by $\cos\theta$, we have

$$(m_1+m_2)\ddot{x}\cos\theta + m_2 l\ddot{\theta}\cos^2\theta - m_2 l\dot{\theta}^2\sin\theta\cos\theta + Kx\cos\theta = 0 \qquad (3\text{–}362)$$

$$m_1\ddot{x}\cos\theta - m_2 l\ddot{\theta}\sin^2\theta - m_2 l\dot{\theta}^2\cos\theta\sin\theta - m_2 g\sin\theta + Kx\cos\theta = 0 \qquad (3\text{–}363)$$

Then, subtracting Eq. (3–363) from (3–362), we obtain

$$m_2\ddot{x}\cos\theta + m_2 l\ddot{\theta}(\cos^2\theta + \sin^2\theta) + m_2 g\sin\theta = 0 \qquad (3\text{–}364)$$

Using the fact that $\cos^2\theta + \sin^2\theta \equiv 1$, Eq. (3–364) simplifies to

$$m_2\ddot{x}\cos\theta + m_2 l\ddot{\theta} + m_2 g\sin\theta = 0 \qquad (3\text{–}365)$$

Now, because the two differential equations in Eqs. (3–354) and (3–355) are independent and we have obtained Eqs. (3–354) and (3–365) via a nonsingular transformation, the two differential equations in Eqs. (3–354) and (3–365) are also independent. Consequently, an alternate system of two differential equations describing the motion of the collar-particle system is given as

$$(m_1+m_2)\ddot{x} + m_2 l\ddot{\theta}\cos\theta - m_2 l\dot{\theta}^2\sin\theta + Kx = 0 \qquad (3\text{–}366)$$

$$m_2\ddot{x}\cos\theta + m_2 l\ddot{\theta} + m_2 g\sin\theta = 0 \qquad (3\text{–}367)$$

(d) Linearization of Alternate System of Differential Equations

First, it is important to note that the static equilibrium point for the system obtain in Eqs. (3–366) and (3–367) is the same as that which was obtained previously, i.e., the static equilibrium point is $(x_{eq},\theta_{eq}) = (0,0)$. Then, the alternate system of differential equations obtained in Eqs. (3–366) and (3–367) can be linearized in a manner similar to that which was performed for the original system of differential equations. In particular, neglecting all higher order terms involving products of δx and $\delta\theta$ and products of their derivatives, we obtain

$$(m_1+m_2)\delta\ddot{x} + m_2 l\delta\ddot{\theta} + K\delta x = 0 \qquad (3\text{–}368)$$

$$m_2\delta\ddot{x} + m_2 l\delta\ddot{\theta} + m_2 g\delta\theta = 0 \qquad (3\text{–}369)$$

where we have again used the approximations $\cos\theta = \cos\delta\theta \approx 1$ and $\sin\theta = \sin\delta\theta \approx \delta\theta$. Next, dividing Eq. (3–368 by l yields the system

$$\begin{aligned} \frac{m_1+m_2}{l}\delta\ddot{x} + m_2\delta\ddot{\theta} + \frac{K}{l}\delta x &= 0 && \text{(3–370)} \\ m_2\delta\ddot{x} + m_2 l\delta\ddot{\theta} + m_2 g\delta\theta &= 0 && \text{(3–371)} \end{aligned}$$

Equations (3–370) and (3–371) can be written in matrix form as

$$\begin{bmatrix} (m_1+m_2)/l & m_2 \\ m_2 & m_2 l \end{bmatrix}\begin{bmatrix} \delta\ddot{x} \\ \delta\ddot{\theta} \end{bmatrix} + \begin{bmatrix} K/l & 0 \\ 0 & m_2 g \end{bmatrix}\begin{bmatrix} \delta x \\ \delta\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3–372}$$

Unlike the matrix differential equation that was obtain in Eq. (3–361), it is seen that the mass and stiffness matrices in Eq. (3–372) are both symmetric. Furthermore (and equally important) **M** is positive definite while **K** is positive semi-definite, thereby making it possible to analyze the system in Eq. (3–372) using the eigenvalue-eigenvector techniques described previously.

■

Chapter 4

Forced Response of Multiple Degree-of-Freedom Systems

In Chapter 3 we studied the response of a multiple degree-of-freedom system to initial conditions. We now turn our attention to the response of multiple degree-of-freedom systems to external time-varying forcing functions. In particular, in this chapter we start with a general system of linear time-invariant second-order differential equations subject to a general forcing function. We then divide the analysis into two parts. In the first part of this chapter we study the response of a multiple degree-of-freedom system to *nonperiodic* inputs whereas in the second part of this chapter we study the response of multiple degree-of-freedom systems to *periodic* inputs.

4.1 Generic Model for Forced Multiple Degree-of-Freedom System

The general mathematical model for a forced multiple degree-of-freedom system subject to a time-varying forcing function is given as

$$\mathbf{M}\ddot{\mathbf{Y}} + \mathbf{C}\dot{\mathbf{Y}} + \mathbf{K}\mathbf{Y} = \mathbf{f}(t) \tag{4–1}$$

where $\mathbf{f}(t)$ is an external vector forcing function of time. Similar to a single degree-of-freedom system, the function $\mathbf{f}(t)$ is *not* a function of $\mathbf{Y}$ and its derivatives, but is an *explicit* function of time.

4.2 Response of Modally Damped Systems to Nonperiodic Inputs

Now consider the case of a modally damped system and an input function $\mathbf{f}(t)$ that is *nonperiodic*. Because $\mathbf{C}$ is a modal damping matrix, we know that

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K} \tag{4–2}$$

Then, from Eq. (3–263) we know that the eigenvector matrix $\mathbf{U}$ of the weighted eigenvalue problem $\mathbf{K}\mathbf{u} = \lambda\mathbf{M}\mathbf{u}$ can be used to decouple the homogeneous differential equation $\mathbf{M}\ddot{\mathbf{Y}} + \mathbf{C}\dot{\mathbf{Y}} + \mathbf{K}\mathbf{Y} = \mathbf{0}$

$$\ddot{\mathbf{q}} + (\alpha\mathbf{I} + \beta\boldsymbol{\Lambda})\dot{\mathbf{q}} + \boldsymbol{\Lambda}\mathbf{q} = \mathbf{0} \tag{4–3}$$

where $\mathbf{Y} = \mathbf{U}\mathbf{q}$. However, in this case we have a nonhomogeneous differential equation. Then, from Eq. (4–1) we have

$$\mathbf{M}\mathbf{U}\ddot{\mathbf{q}} + \mathbf{C}\mathbf{U}\dot{\mathbf{q}} + \mathbf{K}\mathbf{U}\mathbf{q} = \mathbf{f}(t) \tag{4–4}$$

Multiplying on the left-hand side by $\mathbf{U}^T$ gives

$$\mathbf{U}^T\mathbf{M}\mathbf{U}\ddot{\mathbf{q}} + \mathbf{U}^T\mathbf{C}\mathbf{U}\dot{\mathbf{q}} + \mathbf{U}^T\mathbf{K}\mathbf{U}\mathbf{q} = \mathbf{U}^T\mathbf{f}(t) \tag{4–5}$$

Now assuming that $\mathbf{U}$ is *mass-normalized*, $\mathbf{U}^T\mathbf{M}\mathbf{U} = \mathbf{I}$ we obtain

$$\ddot{\mathbf{q}} + (\alpha\mathbf{I} + \beta\mathbf{\Lambda})\dot{\mathbf{q}} + \mathbf{\Lambda}\mathbf{q} = \mathbf{U}^T\mathbf{f}(t) \tag{4–6}$$

where we know for a mass normalized eigenvector matrix that $\mathbf{U}^T\mathbf{K}\mathbf{U} = \mathbf{\Lambda}$. It is seen that the system of differential equations given by Eq. (4–6) is also decoupled with the exception that the right-hand side is not zero. However, as we will soon see, the fact that the right-hand side is nonzero does not pose a computational problem for modally damped systems. In particular, let

$$\mathbf{U}^T\mathbf{f} = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix} \tag{4–7}$$

Then in scalar form Eq. (4–6) can be written as

$$\ddot{q}_k + (\alpha + \beta\lambda_k)\dot{q}_k + \lambda_k q_k = F_k(t), \qquad (k = 1, \ldots, n) \tag{4–8}$$

It is seen that Eq. (4–8) is a system of n scalar uncoupled second-order LTI differential equations. Consequently, for *nonperiodic* inputs the differential equations of Eq. (4–8) can be solved using the techniques in Chapter 2. In order to see more clearly how these differential equations can be solved, suppose that each function $f_k(t),\ (k = 1, \ldots, n)$ is a linear combination of fundamental nonperiodic function (e.g., linear combinations of functions such as the unit impulse function, the unit step function, the unit ramp function, etc.). Then each differential equation in Eq. (4–8) can be written as

$$\ddot{q}_k + (\alpha + \beta\lambda_k)\dot{q}_k + \lambda_k q_k = \sum_{j=0}^{m} a_{kj} h_j(t), \qquad (k = 1, \ldots, n) \tag{4–9}$$

where

$$\begin{aligned} h_0(t) &= \delta(t) \\ h_1(t) &= \int_0^t h_0(\tau)d\tau \\ h_2(t) &= \int_0^t h_1(\tau)d\tau \\ &\vdots \\ h_m(t) &= \int_0^t h_m(\tau)d\tau \end{aligned} \tag{4–10}$$

In other words, each function $h_j(t),\ (j = 0, \ldots, n)$ is a multiple integral of the unit impulse function. Now suppose that we let $g_{kj}(t),\ (j = 0, \ldots, m)$ be the response of the k^{th} differential equation in Eq. (4–8) to the input $h_j(t),\ (j = 0, \ldots, m)$. Then the response of the k^{th} differential equation in Eq. (4–8) is given as

$$q_k(t) = \sum_{j=0}^{m} a_{kj} g_{kj}(t) \tag{4–11}$$

Then, the response of the *original* system defined by $\mathbf{Y}(t)$ is given as

$$\mathbf{Y}(t) = \mathbf{U}\mathbf{q}(t) = \mathbf{U}\begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix} = \mathbf{U}\begin{bmatrix} \sum_{j=0}^{m} a_{1j}g_{1j}(t) \\ \sum_{j=0}^{m} a_{2j}g_{2j}(t) \\ \vdots \\ \sum_{j=0}^{m} a_{nj}g_{nj}(t) \end{bmatrix} \tag{4–12}$$

Now we know that the eigenvector matrix can be written in general form

$$\mathbf{U} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{bmatrix} \tag{4–13}$$

Consequently, Eq. (4–12) can be written as

$$\mathbf{Y}(t) = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{bmatrix}\begin{bmatrix} \sum_{j=0}^{m} a_{1j}g_{1j}(t) \\ \sum_{j=0}^{m} a_{2j}g_{2j}(t) \\ \vdots \\ \sum_{j=0}^{m} a_{nj}g_{nj}(t) \end{bmatrix}\begin{bmatrix} \sum_{p=1}^{n} u_{1p}\sum_{j=0}^{m} a_{pj}g_{pj}(t) \\ \sum_{p=1}^{n} u_{2p}\sum_{j=0}^{m} a_{pj}g_{pj}(t) \\ \vdots \\ \sum_{p=1}^{n} u_{np}\sum_{j=0}^{m} a_{pj}g_{pj}(t) \end{bmatrix} \tag{4–14}$$

Finally, we can rewrite

$$\mathbf{Y}(t) = \begin{bmatrix} \sum_{p=1}^{n}\sum_{j=0}^{m} u_{1p}a_{pj}g_{pj}(t) \\ \sum_{p=1}^{n}\sum_{j=0}^{m} u_{2p}a_{pj}g_{pj}(t) \\ \vdots \\ \sum_{p=1}^{n}\sum_{j=0}^{m} u_{np}a_{pj}g_{pj}(t) \end{bmatrix} \tag{4–15}$$

In other words, not only is the response of the system in the original coordinates $\mathbf{Y}(t)$ a linear combination of the responses to the input functions $h_j(t)$, $(j = 0, \ldots, n)$, but, due to the fact that the elements of $\mathbf{Y}$ are themselves linear combinations of the elements of $\mathbf{q}$ (due to the eigenvector matrix, $\mathbf{U}$), this response is *simultaneously* a linear combination of these linear combinations.

Example 4–1

Consider the forced damped two degree-of-freedom system

$$\mathbf{M}\ddot{\mathbf{Y}} + \mathbf{C}\dot{\mathbf{Y}} + \mathbf{K}\mathbf{Y} = \mathbf{f}(t)$$

where the mass, damping, and stiffness matrices, are given, respectively, as follows

$$\begin{aligned}\mathbf{M} &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\ \mathbf{C} &= \begin{bmatrix} -3 & 2 \\ 2 & -4 \end{bmatrix} \\ \mathbf{K} &= \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\end{aligned}$$

Determine the response of the system to the input $\mathbf{f}(t)$ where $\mathbf{f}(t)$ is given as

$$\mathbf{f}(t) = \begin{bmatrix} \delta(t) \\ u(t) \end{bmatrix}$$

where $\delta(t)$ and $u(t)$ are the unit impulse function and unit step function, respectively.

Solution to Example 4–1

Recall that the mass, damping, and stiffness matrices correspond to those of Example 4–1. Furthermore, the mass-normalized eigenvector matrix of the weighted eigenvalue problem $\mathbf{Ku} = \lambda\mathbf{Mu}$ is given from Eq. (3–270) on page 80

$$\mathbf{U} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \tag{4–16}$$

Furthermore, we know from Example 4–1 that the system for this problem is modally damped. Consequently, the modal coordinate equations are obtained from Eq. (4–6) as

$$\ddot{\mathbf{q}} + (\alpha\mathbf{I} + \beta\mathbf{\Lambda})\dot{\mathbf{q}} + \mathbf{\Lambda}\mathbf{q} = \mathbf{U}^T\mathbf{f}(t) \tag{4–17}$$

Again, from Eqs. (3–272), and (3–273) we have, respectively,

$$\mathbf{U}^T\mathbf{K}\mathbf{U} \equiv \mathbf{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{5}{2} \end{bmatrix} \tag{4–18}$$

$$\alpha\mathbf{I} + \beta\mathbf{\Lambda} \equiv \mathbf{U}^T\mathbf{C}\mathbf{U} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \tag{4–19}$$

Finally, using $\mathbf{U}$ in Eq. (4–16), we have

$$\mathbf{U}^T\mathbf{f} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \delta(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}}\delta(t) + \frac{1}{\sqrt{3}}u(t) \\ \sqrt{\frac{2}{3}}\delta(t) - \frac{1}{\sqrt{6}}u(t) \end{bmatrix} \tag{4–20}$$

Therefore, from Eq. (4–7) we have

$$F_1(t) = \frac{1}{\sqrt{3}}\delta(t) + \frac{1}{\sqrt{3}}u(t) = a_{10}h_0(t) + a_{11}h_1(t) \equiv \sum_{j=0}^{1} a_{1j}h_j(t) \tag{4–21}$$

$$F_2(t) = \sqrt{\frac{2}{3}}\delta(t) - \frac{1}{\sqrt{6}}u(t) = a_{20}h_0(t) + a_{21}h_1(t) \equiv \sum_{j=0}^{1} a_{2j}h_j(t) \tag{4–22}$$

where $h_0(t) = \delta(t)$ and $h_1(t) = u(t)$. Furthermore, the coefficients a_{kj}, $(k = 1, 2), (j = 0, 1)$ are given as

$$\begin{aligned} a_{10} &= \frac{1}{\sqrt{3}} \quad , \quad a_{11} = \frac{1}{\sqrt{3}} \\ a_{20} &= \sqrt{\frac{2}{3}} \quad , \quad a_{21} = -\frac{1}{\sqrt{6}} \end{aligned} \tag{4–23}$$

Suppose now that we let $g_{10}(t)$ and $g_{11}(t)$ be the response of the first modal coordinate, $q_1(t)$, to the functions $h_0(t) = \delta(t)$ and $h_1(t) = u(t)$, respectively. Then, from the principle of superposition, the *total* response of the first modal coordinate is given as

$$q_1(t) = a_{10}g_{10}(t) + a_{11}g_{11}(t) = \frac{1}{\sqrt{3}}g_{10}(t) + \frac{1}{\sqrt{3}}g_{11}(t) \tag{4–24}$$

Similarly, suppose that we let $g_{20}(t)$ and $g_{21}(t)$ be the response of the second modal coordinate, $q_2(t)$, to the functions $h_0(t) = \delta(t)$ and $h_1(t) = u(t)$, respectively. Then, from the principle of superposition, the *total* response of the second modal coordinate is given as

$$q_2(t) = a_{20}g_{20}(t) + a_{21}g_{21}(t) = \sqrt{\frac{2}{3}}g_{20}(t) - \frac{1}{\sqrt{6}}g_{21}(t) \tag{4–25}$$

The modal coordinate response is then given in vector form as

$$\mathbf{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}}g_{10}(t) + \frac{1}{\sqrt{3}}g_{11}(t) \\ \sqrt{\frac{2}{3}}g_{20}(t) - \frac{1}{\sqrt{6}}g_{21}(t) \end{bmatrix} \tag{4–26}$$

Then, using the fact that $\mathbf{Y} = \mathbf{U}\mathbf{q}$, we obtain the response of the original system as

$$\mathbf{Y} = \mathbf{U}\mathbf{q} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}}g_{10}(t) + \frac{1}{\sqrt{3}}g_{11}(t) \\ \sqrt{\frac{2}{3}}g_{20}(t) - \frac{1}{\sqrt{6}}g_{21}(t) \end{bmatrix} \tag{4–27}$$

Multiplying out Eq. (4–27), we obtain

$$\mathbf{Y} = \begin{bmatrix} \frac{1}{3}g_{10}(t) + \frac{1}{3}g_{11}(t) + \frac{2}{3}g_{20}(t) - \frac{1}{3}g_{21}(t) \\ \frac{1}{3}g_{10}(t) + \frac{1}{3}g_{11}(t) - \frac{2}{3}g_{20}(t) - \frac{1}{6}g_{21}(t) \end{bmatrix} \tag{4–28}$$

It is important to observe that the solution obtained in Eq. (4–28) is identical to that given in Eq. (4–15). In particular, for this example we have $m = 1$ and Eq. (4–15) reduces to

$$\mathbf{Y}(t) = \begin{bmatrix} \displaystyle\sum_{p=1}^{2}\sum_{j=0}^{1} u_{1p}a_{pj}g_{pj}(t) \\ \displaystyle\sum_{p=1}^{2}\sum_{j=0}^{1} u_{2p}a_{pj}g_{pj}(t) \end{bmatrix} \tag{4–29}$$

Using the eigenvector matrix $\mathbf{U}$ in Eq. (4–16), we have

$$\begin{matrix} u_{11} & = & \frac{1}{\sqrt{3}} & , & u_{12} & = & \sqrt{\frac{2}{3}} \\ u_{21} & = & \frac{1}{\sqrt{3}} & , & u_{22} & = & -\frac{1}{\sqrt{6}} \end{matrix} \tag{4–30}$$

Then, combining Eq. (4–30) and (4–23) in Eq. (4–29), we have

$$\mathbf{Y}(t) = \begin{bmatrix} \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}g_{10}(t) + \frac{1}{\sqrt{3}}g_{11}(t)\right) + \sqrt{\frac{2}{3}}\left(\sqrt{\frac{2}{3}}g_{20}(t) - \frac{1}{\sqrt{6}}g_{21}(t)\right) \\ \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}g_{10}(t) + \frac{1}{\sqrt{3}}g_{11}(t)\right) - \frac{1}{\sqrt{6}}\left(\sqrt{\frac{2}{3}}g_{20}(t) - \frac{1}{\sqrt{6}}g_{21}(t)\right) \end{bmatrix} \tag{4–31}$$

Equation (4–31) simplifies to

$$\mathbf{Y} = \begin{bmatrix} \frac{1}{3}g_{10}(t) + \frac{1}{3}g_{11}(t) + \frac{2}{3}g_{20}(t) - \frac{1}{3}g_{21}(t) \\ \frac{1}{3}g_{10}(t) + \frac{1}{3}g_{11}(t) - \frac{2}{3}g_{20}(t) - \frac{1}{6}g_{21}(t) \end{bmatrix} \tag{4–32}$$

which is the same result as obtained in Eq. (4–28).

■

4.3 Response of Modally Damped Systems to Periodic Inputs

Suppose now that the forcing function $\mathbf{f}(t)$ has the form

$$\mathbf{f}(t) = \mathbf{F}e^{i\omega t} \tag{4–33}$$

where $\mathbf{F}$ is a constant. It is seen that the form of $\mathbf{f}(t)$ given in Eq. (4–33) is a vector periodic function of time with input frequency ω. Suppose now that the damping matrix $\mathbf{C}$ is assumed to be *modal*, i.e.,

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K} \tag{4–34}$$

Suppose further that we transform the variable $\mathbf{Y}$ via the eigenvector matrix $\mathbf{U}$ as

$$\mathbf{Y} = \mathbf{U}\mathbf{q} \tag{4–35}$$

We then obtain

$$\mathbf{M}\mathbf{U}\ddot{\mathbf{q}} + \mathbf{C}\mathbf{U}\dot{\mathbf{q}} + \mathbf{K}\mathbf{U}\mathbf{q} = \mathbf{F}e^{i\omega t} \tag{4–36}$$

which further implies

$$\mathbf{U}^T\mathbf{M}\mathbf{U}\ddot{\mathbf{q}} + \mathbf{U}^T\mathbf{C}\mathbf{U}\dot{\mathbf{q}} + \mathbf{U}^T\mathbf{K}\mathbf{U}\mathbf{q} = \mathbf{U}^T\mathbf{F}e^{i\omega t} \tag{4–37}$$

Now, assume that $\mathbf{U}$ has been mass-normalized. Then, because we have assumed that the system is modally damped, this last equation can be written as

$$\ddot{\mathbf{q}} + \mathbf{\Gamma}\dot{\mathbf{q}} + \mathbf{\Lambda}\mathbf{q} = \mathbf{U}^T\mathbf{F}e^{i\omega t} \tag{4–38}$$

where $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$ are *diagonal*, i.e.,

$$\mathbf{\Gamma} = \begin{bmatrix} \gamma_1 & 0 & 0 & \cdots & 0 \\ 0 & \gamma_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \gamma_n \end{bmatrix} \tag{4–39}$$

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix} \tag{4–40}$$

Suppose now that the matrix $\mathbf{U}^T\mathbf{F}$ is given as

$$\mathbf{U}^T\mathbf{F} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \tag{4–41}$$

Then each scalar equation can be written as

$$\ddot{q}_k + \gamma_k\dot{q}_k + \lambda_k q_k = f_k e^{i\omega t}, \qquad (k = 1, \ldots, n) \tag{4–42}$$

Now let

$$\gamma_k = 2\zeta_k\omega_k \tag{4–43}$$

$$\lambda_k = \omega_k^2 \tag{4–44}$$

where $\zeta_k,\ (k = 1,\ldots,n)$ and ω_k are the damping ratios and natural frequencies for the coordinates $q_k,\ (k = 1,\ldots,n)$. Then we can write

$$\ddot{q}_k + 2\zeta_k\omega_k\dot{q}_k + \omega_k^2 q_k = f_k e^{i\omega t}, \qquad (k = 1,\ldots,n) \tag{4–45}$$

Finally, let

$$f_k = A_k\omega_k^2, \qquad (k = 1,\ldots,n) \tag{4–46}$$

Then the system of differential equations in modal coordinates is given as

$$\ddot{q}_k + 2\zeta_k\omega_k\dot{q}_k + \omega_k^2 q_k = A_k\omega_k^2 e^{i\omega t}, \qquad (k = 1,\ldots,n) \tag{4–47}$$

It is seen that Eq. (4–47) is a system of decoupled equations (i.e., modal coordinate equations) and are in the standard form as given in Eq. (2–50), i.e., each equation in Eq. (4–47) has the form

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 A e^{i\omega t} \tag{4–48}$$

Now we now that the solution to Eq. (4–48) is given as

$$x(t) = A|G(i\omega)|e^{i(\omega t-\phi)} \tag{4–49}$$

where $G(i\omega)$ is the transfer function

$$G(i\omega) = \frac{1}{1-\left(\dfrac{\omega}{\omega_n}\right)^2 + i2\zeta\dfrac{\omega}{\omega_n}} \tag{4–50}$$

and $|G(i\omega)|$ and $\phi = \phi(\omega)$ are the magnitude and phase of $G(i\omega)$, respectively. Then, for each modal coordinate the solution is given as

$$q_k(t) = A_k|G_k(i\omega)|e^{i(\omega t-\phi_k)} \tag{4–51}$$

where $G_k(i\omega)$ is the transfer function associated with the k^{th} modal coordinate, i.e.,

$$G_k(i\omega) = \frac{1}{1-\left(\dfrac{\omega}{\omega_k}\right)^2 + i2\zeta_k\dfrac{\omega}{\omega_k}} \tag{4–52}$$

and $|G_k(i\omega)|$ and $\phi_k = \phi_k(\omega)$ are the magnitude and phase of $G_k(i\omega)$, respectively. Then the solution $\mathbf{Y}(t)$ is obtained as

$$\mathbf{Y} = \mathbf{U}\mathbf{q} \tag{4–53}$$

where

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} A_1|G_1(i\omega)|e^{i(\omega t-\phi_1)} \\ A_2|G_2(i\omega)|e^{i(\omega t-\phi_2)} \\ \vdots \\ A_n|G_n(i\omega)|e^{i(\omega t-\phi_n)} \end{bmatrix} \tag{4–54}$$

Consequently,

$$\mathbf{Y} = \mathbf{U}\mathbf{q} = \mathbf{U}\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = \mathbf{U}\begin{bmatrix} A_1|G_1(i\omega)|e^{i(\omega t-\phi_1)} \\ A_2|G_2(i\omega)|e^{i(\omega t-\phi_2)} \\ \vdots \\ A_n|G_n(i\omega)|e^{i(\omega t-\phi_n)} \end{bmatrix} \tag{4–55}$$

Example 4–2

Consider the two degree-of-freedom system

$$\mathbf{M}\ddot{\mathbf{Y}} + \mathbf{C}\dot{\mathbf{Y}} + \mathbf{K}\mathbf{Y} = \mathbf{F}$$

where

$$\mathbf{F} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\omega t}$$

where the mass, damping, and stiffness matrices, are given, respectively, as follows

$$\begin{aligned} \mathbf{M} &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\ \mathbf{C} &= \begin{bmatrix} -3 & 2 \\ 2 & -4 \end{bmatrix} \\ \mathbf{K} &= \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \end{aligned}$$

Determine the time response of the above system.

Solution to Example 4–2

Recall that **M**, **C**, and **K** correspond to those given in Example 3–3. Furthermore, recalling the mass-normalized eigenvector matrix **U** from Eq. (3–270) on page 80 of Example 3–3, we have

$$\mathbf{U} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \tag{4–56}$$

from which we obtain

$$\mathbf{U}^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \tag{4–57}$$

Then, using the value of **F** given in the problem statement we have

$$\mathbf{U}^T\mathbf{F} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} - \frac{1}{\sqrt{6}} \end{bmatrix} \tag{4–58}$$

Now, recall that Example 3–3 was modally damped. Consequently, the unforced system can be decoupled as given in Eqs. (3–277) and Eqs. (3–278). Then, using the result of Eq. (4–58), we obtain the forced differential equations in modal coordinates as

$$\ddot{q}_1 + \dot{q}_1 + q_1 = \frac{2}{\sqrt{3}} e^{i\omega t} \tag{4–59}$$

$$\ddot{q}_2 + 4\dot{q}_2 + \frac{5}{2} q_2 = \left(\sqrt{\frac{2}{3}} - \frac{1}{\sqrt{6}}\right) e^{i\omega t} \tag{4–60}$$

Also recall from Example 3–3 that the eigenvalues of the undamped system

$$\lambda_1 = 1 \tag{4–61}$$

$$\lambda_2 = \frac{5}{2} \tag{4–62}$$

which implies that the modal natural frequencies are given as

$$\omega_1 = 1 \tag{4–63}$$

$$\omega_2 = \sqrt{\frac{5}{2}} \tag{4–64}$$

Furthermore, the damping ratios are obtained as

$$2\zeta_1\omega_1 = \gamma_1 = 1 \tag{4–65}$$
$$2\zeta_2\omega_2 = \gamma_2 = 4 \tag{4–66}$$

where we recall that γ_1 and γ_2 are the coefficients associated with the terms involving $\dot{q}_1$ and $\dot{q}_2$, respectively. We then obtain

$$\zeta_1 = \frac{1}{2} \tag{4–67}$$
$$\zeta_2 = \frac{4}{2\sqrt{\frac{5}{2}}} = 2\sqrt{\frac{2}{5}} \tag{4–68}$$

We can now determine the values of A_1 and A_2 as follows:

$$A_1\omega_1^2 = f_1 = \frac{2}{\sqrt{3}} \tag{4–69}$$
$$A_2\omega_2^2 = f_2 = \left(\sqrt{\tfrac{2}{3}} - \tfrac{1}{\sqrt{6}}\right) \tag{4–70}$$

Using the expressions for ω_1 and ω_2, we obtain

$$A_1 = \frac{2}{\sqrt{3}} \tag{4–71}$$
$$A_2 = \tfrac{2}{5}\left(\sqrt{\tfrac{2}{3}} - \tfrac{1}{\sqrt{6}}\right) \tag{4–72}$$

from which we obtain

$$A_1 = \frac{2}{\sqrt{3}} \tag{4–73}$$
$$A_2 = \tfrac{2}{5}\left(\sqrt{\tfrac{2}{3}} - \tfrac{1}{\sqrt{6}}\right) \tag{4–74}$$

Then the modal responses are given as

$$q_1(t) = A_1|G_1(i\omega)|e^{i(\omega t - \phi_1)} \tag{4–75}$$
$$q_2(t) = A_2|G_2(i\omega)|e^{i(\omega t - \phi_2)} \tag{4–76}$$

where $G_1(i\omega)$ and $G_2(i\omega)$ are given as

$$G_1(i\omega) = \frac{1}{1 - \left(\dfrac{\omega}{\omega_1}\right)^2 + i2\zeta_1\dfrac{\omega}{\omega_1}} \tag{4–77}$$
$$G_2(i\omega) = \frac{1}{1 - \left(\dfrac{\omega}{\omega_2}\right)^2 + i2\zeta_2\dfrac{\omega}{\omega_2}} \tag{4–78}$$

and ω_1, ω_2, ζ_1, ζ_2, A_1, and A_2 are as obtained earlier in the solution to this example. Finally, the solution in the original coordinates defined by $\mathbf{Y}$ is given as

$$\mathbf{Y} = \mathbf{Uq} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} \tag{4–79}$$

■

4.4 Response of Systems with General Damping to Periodic Inputs

Suppose now that we consider again a multiple degree-of-freedom LTI system subject to a periodic forcing function, i.e.,

$$\mathbf{M}\ddot{\mathbf{Y}} + \mathbf{C}\dot{\mathbf{Y}} + \mathbf{K}\mathbf{Y} = \mathbf{F}e^{i\omega t} \tag{4–80}$$

Now because the input is periodic, we know that the steady-state output will also be periodic, i.e., $\mathbf{Y}(t)$ will have the form

$$\mathbf{Y}(t) = \bar{\mathbf{Y}}e^{i\omega t} \tag{4–81}$$

which implies that

$$\begin{aligned} \dot{\mathbf{Y}}(t) &= i\omega\bar{\mathbf{Y}}e^{i\omega t} \\ \ddot{\mathbf{Y}}(t) &= -\omega^2\bar{\mathbf{Y}}e^{i\omega t} \end{aligned} \tag{4–82}$$

Substiuting the expression for $\mathbf{Y}$, $\dot{\mathbf{Y}}$, and $\ddot{\mathbf{Y}}$ into Eq. (4–80), we obtain

$$e^{i\omega t}\left[-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}\right]\bar{\mathbf{Y}} = \mathbf{F}e^{i\omega t} \tag{4–83}$$

Observing that $e^{i\omega t}$ is not zero as a function of time, we obtain

$$\left[-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}\right]\bar{\mathbf{Y}} = \mathbf{F} \tag{4–84}$$

Suppose now that we define

$$\mathbf{Z}(i\omega) \equiv \mathbf{Z} = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K} \tag{4–85}$$

The quantity $\mathbf{Z} = \mathbf{Z}(i\omega)$ is called the *impedance matrix*. In terms of the impedance matrix we can write

$$\mathbf{Z}(i\omega)\bar{\mathbf{Y}}(i\omega) = \mathbf{F} \tag{4–86}$$

where we note that $\bar{\mathbf{Y}}$ is also a function of $i\omega$. Then, assuming that $\mathbf{Z}$ is nonsingular (otherwise we would not have a unique solution), we obtain

$$\bar{\mathbf{Y}}(i\omega) = \mathbf{Z}^{-1}(i\omega)\mathbf{F} \tag{4–87}$$

The time response is then given as

$$\mathbf{Y}(t) = \mathbf{Z}^{-1}(i\omega)\mathbf{F}e^{i\omega t} \tag{4–88}$$

4.4.1 Response of Two Degree-of-Freedom System to Periodic Input

Suppose now that we consider the special case of a *two* degree-of-freedom system with mass, damping, and stiffness matrices given, respectively, as

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \tag{4–89}$$

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ m_{12} & c_{22} \end{bmatrix} \tag{4–90}$$

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \tag{4–91}$$

Furthermore, assume that $\mathbf{M}$ is positive definite and that $\mathbf{C}$ and $\mathbf{K}$ are positive semidefinite. Then, the impedance matrix is given as

$$Z(i\omega) = \begin{bmatrix} -\omega^2 m_{11} + i\omega c_{11} + k_{11} & -\omega^2 m_{12} + i\omega c_{12} + k_{12} \\ -\omega^2 m_{12} + i\omega c_{12} + k_{12} & -\omega^2 m_{22} + i\omega c_{22} + k_{22} \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{bmatrix} \tag{4–92}$$

where it is noted that $Z(i\omega)$ is symmetric because **M**, **C**, and **K** are symmetric. The inverse of $Z(i\omega)$ is then given as

$$\mathbf{Z}^{-1}(i\omega) = \frac{1}{\det \mathbf{Z}(i\omega)} = \begin{bmatrix} z_{22} & -z_{12} \\ -z_{12} & z_{11} \end{bmatrix} = \frac{1}{z_{11}z_{22} - z_{12}^2} \begin{bmatrix} z_{22} & -z_{12} \\ -z_{12} & z_{11} \end{bmatrix} \tag{4–93}$$

Multiplying $\mathbf{Z}^{-1}(i\omega)$ by **F** where, we obtain $\bar{\mathbf{Y}}$ as

$$\bar{\mathbf{Y}} = \mathbf{Z}^{-1}(i\omega)\mathbf{F} = \frac{1}{z_{11}z_{22} - z_{12}^2} \begin{bmatrix} z_{22} & -z_{12} \\ -z_{12} & z_{11} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{bmatrix} \tag{4–94}$$

Consequently,

$$\bar{Y}_1 = \frac{z_{22}F_1 - z_{12}F_2}{z_{11}z_{22} - z_{12}^2} \tag{4–95}$$

$$\bar{Y}_2 = \frac{-z_{12}F_1 + z_{11}F_2}{z_{11}z_{22} - z_{12}^2} \tag{4–96}$$

4.4.2 Response of Undamped Two Degree-of-Freedom System to Periodic Input

Suppose now that we specialize further to the case of an *undamped* two degree-of-freedom system. In this case we know that **C** is zero. Then, from Eq. (4–92) we obtain

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{bmatrix} = \begin{bmatrix} -\omega^2 m_{11} + k_{11} & -\omega^2 m_{12} + k_{12} \\ -\omega^2 m_{12} + k_{12} & -\omega^2 m_{22} + k_{22} \end{bmatrix} \tag{4–97}$$

Equations (4–95) and (4–96) then reduce to

$$\bar{Y}_1 = \frac{(k_{22} - \omega^2 m_{22})F_1 - (k_{12} - \omega^2 m_{12})F_2}{(k_{11} - \omega^2 m_{11})(k_{22} - \omega^2 m_{22}) - (k_{12} - \omega^2 m_{12})^2} \tag{4–98}$$

$$\bar{Y}_2 = \frac{-(k_{12} - \omega^2 m_{12})F_1 + (k_{11} - \omega^2 m_{11})F_2}{(k_{11} - \omega^2 m_{11})(k_{22} - \omega^2 m_{22}) - (k_{12} - \omega^2 m_{12})^2} \tag{4–99}$$

The time responses $Y_1(t)$ and $Y_2(t)$ are then given as

$$Y_1(t) = \left[\frac{(k_{22} - \omega^2 m_{22})F_1 - (k_{12} - \omega^2 m_{12})F_2}{(k_{11} - \omega^2 m_{11})(k_{22} - \omega^2 m_{22}) - (k_{12} - \omega^2 m_{12})^2}\right] e^{i\omega t} \tag{4–100}$$

$$Y_2(t) = \left[\frac{-(k_{12} - \omega^2 m_{12})F_1 + (k_{11} - \omega^2 m_{11})F_2}{(k_{11} - \omega^2 m_{11})(k_{22} - \omega^2 m_{22}) - (k_{12} - \omega^2 m_{12})^2}\right] e^{i\omega t} \tag{4–101}$$

4.5 Undamped Vibration Absorbers

Consider now the undamped system shown in Fig. 4–1 of a block of mass M attached to two linear springs with spring constants K_1 and K_2 and a second block of mass m attached in tandem connected to the second spring. Furthermore, assume that a force $F(t)$ is applied to the first block.

Assuming that the unstretched lengths of the springs are ℓ_{10} and ℓ_{20}, the force applied to the block of mass M is given as

$$\mathbf{F}_1 = \mathbf{F} + \mathbf{F}_{s1} + \mathbf{F}_{s2} = F\mathbf{E}_x - K_1(\ell_1 - \ell_{10})\mathbf{u}_{s1} - K_2(\ell_2 - \ell_{20})\mathbf{u}_{s2} \tag{4–102}$$

Now we have

$$\ell_1 = \|\mathbf{r}_1 - \mathbf{r}_O\| = x_1 \tag{4–103}$$

$$\ell_2 = \|\mathbf{r}_1 - \mathbf{r}_2\| = x_2 - x_1 \tag{4–104}$$

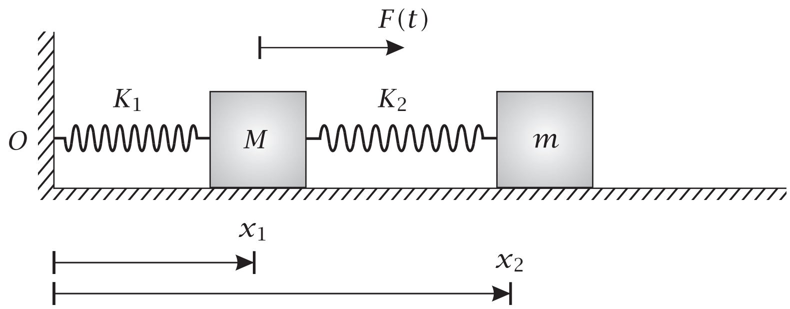

Figure 4–1 Two masses on two springs representing a model for a vibration absorber.

Furthermore,

$$\mathbf{u}_{s1} = \frac{\mathbf{r}_1 - \mathbf{r}_O}{\|\mathbf{r}_1 - \mathbf{r}_O\|} = \mathbf{E}_x \quad (4\text{–}105)$$

$$\mathbf{u}_{s2} = \frac{\mathbf{r}_1 - \mathbf{r}_2}{\|\mathbf{r}_1 - \mathbf{r}_2\|} = -\mathbf{E}_x \quad (4\text{–}106)$$

Therefore,

$$\mathbf{F}_1 = F\mathbf{E}_x - K_1(x_1 - \ell_{10})\mathbf{E}_x - K_2(x_2 - x_1 - \ell_{20})(-\mathbf{E}_x) = [-K_1(x_1 - \ell_{10}) + K_2(x_2 - x_1 - \ell_{20})]\,\mathbf{E}_x \quad (4\text{–}107)$$

Then, because the inertial acceleration of the first block is ${}^{\mathcal{F}}\mathbf{a}_1 = \ddot{x}_1\mathbf{E}_x$, from Newton's second law we obtain

$$[F - K_1(x_1 - \ell_{10}) + K_2(x_2 - x_1 - \ell_{20})]\,\mathbf{E}_x = M\ddot{x}_1\mathbf{E}_x \quad (4\text{–}108)$$

which implies that the first differential equation is given as

$$M\ddot{x}_1 + (K_1 + K_2)x_1 - K_2x_2 = F + K_1\ell_{10} - K_2\ell_{20} \quad (4\text{–}109)$$

Similarly, the force exerted on the block of mass m is given as

$$\mathbf{F}_2 = -\mathbf{F}_{s2} = -K_2(x_2 - x_1 - \ell_{20})\mathbf{E}_x \quad (4\text{–}110)$$

Then, because the inertial acceleration of the second block is ${}^{\mathcal{F}}\mathbf{a}_2 = \ddot{x}_2\mathbf{E}_x$, from Newton's second law we obtain

$$-K_2(x_2 - x_1 - \ell_{20})\mathbf{E}_x = m\ddot{x}_2\mathbf{E}_x \quad (4\text{–}111)$$

which implies that the second differential equation is given as

$$m\ddot{x}_2 + -K_2x_1 + K_2x_2 = K_2\ell_{20} \quad (4\text{–}112)$$

It is noted that the system consisting of the block of mass M together with the first spring is called the *main system* while the second system consisting of the block of mass m together with the second spring is called the *absorber*. The objective of this analysis is to determine the design that enables the absorber to absorb as much of the response of the main system as possible.

Observing the form of the differential equations, it is seen that the differential equations *relative* to the static equilibrium point are given as

$$M\ddot{y}_1(K_1 + K_2)y_1 - K_2y_2 = F(t) \quad (4\text{–}113)$$

$$m\ddot{y}_2 - K_2y_1 + K_2y_2 = 0 \quad (4\text{–}114)$$

where $x_1 = y_1 - y_{1,eq}$ and $x_2 = y_2 - y_{2,eq}$. Suppose now that we consider the case where $F(t)$ is periodic of the form $F(t) = F_1 \sin \omega t$. Then, because the system is undamped, we know that the phase of the output will be zero which implies that

$$y_1(t) = \bar{Y}_1 \sin \omega t \tag{4–115}$$

$$y_2(t) = \bar{Y}_2 \sin \omega t \tag{4–116}$$

Then, using the results of Eqs. (4–100) and (4–101) we obtain $y_1(t)$ and $y_2(t)$ as

$$y_1(t) = \left[\frac{(K_2 - \omega^2 m)F_1}{(K_1 + K_2 - \omega^2 M)(K_2 - \omega^2 m) - K_2^2}\right] \sin \omega t \tag{4–117}$$

$$y_2(t) = \left[\frac{K_2 F_1}{(K_1 + K_2 - \omega^2 M)(K_2 - \omega^2 m) - K_2^2}\right] \sin \omega t \tag{4–118}$$

where

$$\bar{Y}_1 = \left[\frac{(K_2 - \omega^2 m)F_1}{(K_1 + K_2 - \omega^2 M)(K_2 - \omega^2 m) - K_2^2}\right] \tag{4–119}$$

$$\bar{Y}_2 = \left[\frac{K_2 F_1}{(K_1 + K_2 - \omega^2 M)(K_2 - \omega^2 m) - K_2^2}\right] \tag{4–120}$$

Suppose now that we introduce the following notation:

$$\begin{array}{lclcl}
\omega_n & = & \sqrt{K_1/M} & = & \text{natural frequency of main system} \\
\omega_a & = & \sqrt{K_2/m} & = & \text{natural frequency of absorber} \\
y_{\text{st}} & = & F_1/K_1 & = & \text{static deflection of main system} \\
\gamma & = & m/M & = & \text{ratio of absorber to main system}
\end{array}$$

Then Eqs. (4–119) and (4–120) can be written in terms of ω_n, ω_a, y_{st} and γ as

$$\bar{Y}_1 = \frac{\left[1 - (\omega/\omega_a)^2\right] y_{\text{st}}}{\left[1 + \gamma(\omega_a/\omega_n)^2 - (\omega/\omega_n)^2\right]\left[1 - (\omega/\omega_a)^2\right] - \gamma(\omega_a/\omega_n)^2} \tag{4–121}$$

$$\bar{Y}_2 = \frac{y_{\text{st}}}{\left[1 + \gamma(\omega_a/\omega_n)^2 - (\omega/\omega_n)^2\right]\left[1 - (\omega/\omega_a)^2\right] - \gamma(\omega_a/\omega_n)^2} \tag{4–122}$$

It can be seen that

$$\bar{Y}_1(\omega_a) \equiv 0 \tag{4–123}$$

$$\bar{Y}_2(\omega_a) = -\frac{y_{\text{st}}}{\gamma(\omega_a/\omega_n)^2} = -\left(\frac{\omega_n}{\omega_a}\right)^2 \frac{y_{\text{st}}}{\gamma} = -\frac{F_1}{K_2} \tag{4–124}$$

It is seen from Eq. (4–121) that the mass m (i.e., the absorber) will absorb the motion of the main system if $\omega \equiv \omega_a$, i.e., the best natural frequency for the absorber is $\omega_a = \omega$. In other words, the best design for the absorber is one where the natural frequency of the absorber is the same as the frequency of the forcing function $F(t)$.

Appendix A

Review of Linear Algebra

A.1 Row Vectors, Column Vectors, and Matrices

Let $\mathbb{C}$ and $\mathbb{R}$ denote the set of complex and real numbers, respectively. Furthermore, let $q_i \in \mathbb{C}$, $(i = 1, \ldots, n)$ be complex-valued scalars. These scalars can be arranged in either a row or a column as follows. When arranged in a row, we can write $\mathbf{q} = [q_1\ q_2\ \ldots\ q_n]$. Then the quantity $\mathbf{q} = [q_1\ q_2\ \ldots\ q_n]$ is called a *row vector*. Alternatively, a row vector can be written as $\mathbf{q} = (q_1, \ldots, q_n)$,i.e.,

$$\mathbf{q} = (q_1, \ldots, q_n) \equiv [q_1\ q_2\ \ldots\ q_n] \tag{A–1}$$

When arranged as a column, i.e., as

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \tag{A–2}$$

the quantity $\mathbf{q}$ is called a *column vector*. Suppose now that we consider a set of complex-valued coefficients a_{ij} $(i = 1, \ldots, m;\ j = 1, \ldots, n)$. Also, suppose that we arrange these coefficients as follows:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{A–3}$$

The quantity $\mathbf{A} \in \mathbb{C}^{m \times n}$ is called an $m \times n$ *matrix* and the quantities a_{ij} $(i = 1, \ldots, m;\ j = 1, \ldots, n)$ are called the *elements* of $\mathbf{A}$. Furthermore, because a_{ij} $(i = 1, \ldots, m;\ j = 1, \ldots, n)$ are real numbers, the matrix $\mathbf{A}$ is more specifically referred to as a *real* matrix. We note that a special case of an $m \times n$ real-valued matrix is a so called *square matrix*. A square matrix is one where $m = n$, i.e., a square matrix is written in element form as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{A–4}$$

Examining a row vector, a column vector, and a matrix, it is seen that the following are true. First, an n-dimensional row vector is a matrix of size $1 \times n$, i.e., if $\mathbf{q}$ is a row vector, then $\mathbf{q} \in \mathbb{C}^{1 \times n}$. Next, an m-dimensional column vector is a matrix of size $m \times 1$, i.e., if $\mathbf{q}$ is a column vector, then $\mathbf{q} \in \mathbb{C}^{m \times 1}$.

A.2 Types of Matrices

Identity Matrix

The most basic matrix is the *identity matrix*. The $n \times n$ identity matrix, denoted $\mathbf{I}_n$, is defined as

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{A–5}$$

In other words, $\mathbf{I}_n \in \mathbb{R}^{n\times n}$ and is such that its diagonal elements are unity while its off-diagonal elements are zero. In index form, we can write the identity matrix as follows:

$$[\mathbf{I}_n]_{ij} = \delta_{ij} = \begin{cases} 1 & , \quad i = j \\ 0 & , \quad i \neq j \end{cases} \tag{A–6}$$

where the quantity δ is the *Kronecker delta function*. We note that for any column vector $\mathbf{q}$ the identity matrix satisfies the property that

$$\mathbf{Iq} = \mathbf{q} \tag{A–7}$$

Transpose of a Matrix

Let $\mathbf{A}$ be an $m \times n$ complex-valued matrix. Then the *transpose* of $\mathbf{A}$, denoted $\mathbf{A}^T$, is defined as

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \tag{A–8}$$

It is noted that $\mathbf{A}^T$ is obtained from $\mathbf{A}$ by interchanging the elements of $\mathbf{A}$, i.e., the element a_{ij} in $\mathbf{A}$ is equal to the element a_{ji} in $\mathbf{A}^T$.

Complex Conjugate of a Matrix

Let $\mathbf{A}$ be an $m \times n$ complex-valued matrix. Then the *complex conjugate* of $\mathbf{A}$, denoted $\bar{\mathbf{A}}$, is defined as

$$\bar{\mathbf{A}} = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1n} \\ \bar{a}_{21} & \bar{a}_{22} & \cdots & \bar{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{m1} & \bar{a}_{m2} & \cdots & \bar{a}_{mn} \end{bmatrix} \tag{A–9}$$

where $\bar{a}_{ij}$ is the complex conjugate of a_{ij}.

Properties of Square Matrices

Because square matrices arise so frequently in linear algebra, we devote a separate section to defining particular classes of square matrices. The remainder of this section deals specifically with square matrices, i.e., matrices that have the same number of rows and columns.

A.2.1 Hermitian Matrix

Let $\mathbf{A}$ be an $n \times n$ square complex-valued matrix. Then $\mathbf{A}$ is said to be *Hermitian* if $\mathbf{A} = \bar{\mathbf{A}}^T$, i.e., a complex valued square matrix $\mathbf{A}$ is Hermitian if $\mathbf{A}$ is equal to the transpose of its complex conjugate. In element form, a Hermitian matrix is one such that $a_{ij} = \bar{a}_{ji}$, $(i, j = 1, \dots, n)$.

Skew-Hermitian Matrix

Let $\mathbf{A}$ be an $n \times n$ square complex-valued matrix. Then $\mathbf{A}$ is said to be *skew-Hermitian* if $\mathbf{A} = -\bar{\mathbf{A}}^T$, i.e., a complex valued square matrix $\mathbf{A}$ is skew-Hermitian if $\mathbf{A}$ is equal to the negative of the transpose of its complex conjugate. In element form, a Hermitian matrix is one such that $a_{ij} = -\bar{a}_{ji}$, $(i, j = 1, \ldots, n)$.

Symmetric Matrix

Let $\mathbf{A}$ be a square complex-valued $n \times n$ matrix, Then $\mathbf{A}$ is said to be *symmetric* if $\mathbf{A} = \mathbf{A}^T$, i.e., a square matrix is symmetric if it is equal to its transpose. In scalar form, a real-valued square matrix is symmetric if $a_{ij} = a_{ji}$, $(i, j = 1, \ldots, n)$.

Skew-Symmetric Matrix

Let $\mathbf{A}$ be a square complex-valued matrix. Then $\mathbf{A}$ is said to be *skew-symmetric* $\mathbf{A} = -\mathbf{A}^T$, i.e., a square matrix is skew-symmetric if it is equal to the negative of its transpose. In scalar form, a matrix is symmetric if $a_{ij} = a_{ji}$, $(i, j = 1, \ldots, n)$.

Inverse of a Matrix

Let $\mathbf{A}$ be a square complex-valued matrix. Then $\mathbf{A}$ is said to be *invertible* if there exists a matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \tag{A–10}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. Any invertible matrix is said to be *nonsingular*.

Orthogonal Matrix

Let $\mathbf{A}$ be a square complex-valued matrix. Then $\mathbf{A}$ is said to be *orthogonal* if $\mathbf{A}^{-1} = \mathbf{A}^T$, i.e., a matrix is *orthogonal* if its inverse is equal to its transpose. Because of the property of an orthogonal matrix, we know that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^T\mathbf{A} = \mathbf{I} \tag{A–11}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. Suppose we write an orthogonal matrix in column form as

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \tag{A–12}$$

Then,

$$\mathbf{A}^T = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} \tag{A–13}$$

Multiplying $\mathbf{A}^T$ by $\mathbf{A}$ gives

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T\mathbf{a}_1 & \mathbf{a}_1^T\mathbf{a}_2 & \cdots & \mathbf{a}_1^T\mathbf{a}_n \\ \mathbf{a}_2^T\mathbf{a}_1 & \mathbf{a}_2^T\mathbf{a}_2 & \cdots & \mathbf{a}_2^T\mathbf{a}_n \\ \vdots & \cdots & \ddots & \vdots \\ \mathbf{a}_n^T\mathbf{a}_1 & \mathbf{a}_n^T\mathbf{a}_2 & \cdots & \mathbf{a}_n^T\mathbf{a}_n \end{bmatrix} = \mathbf{I} \tag{A–14}$$

Then for an orthogonal matrix we obtain

$$\mathbf{a}_i^T\mathbf{a}_j = \delta_{ij} = \begin{cases} 1 & , \quad i = j \\ 0 & , \quad i \neq j \end{cases} \tag{A–15}$$

where δ_{ij} is the Kronecker delta function.

Determinant of a Matrix

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be an $n \times n$ real-valued matrix. Then the *determinant* of $\mathbf{A}$, denoted $\det \mathbf{A}$ or $|\mathbf{A}|$, is defined as

$$\det \mathbf{A} = \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{A-16}$$

and is computed recursively as

$$\begin{aligned} \det \mathbf{A} = a_{11} \det & \begin{bmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \\ - a_{12} \det & \begin{bmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ a_{22} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \\ \pm a_{1k} \det & \begin{bmatrix} a_{21} & a_{23} & \cdots & a_{2(n-1)} \\ a_{22} & a_{33} & \cdots & a_{3(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n3} & \cdots & a_{n(n-1)} \end{bmatrix} \end{aligned} \tag{A-17}$$

As mentioned, it is seen that the determinant of a matrix is defined in terms of determinants of smaller matrices. The most basic matrix for which a determinant must be computed is a 2×2 matrix. Suppose that $\mathbf{A} \in \mathbb{R}^{2 \times 2}$ matrix. Then the determinant of a 2×2 matrix is given as

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{A-18}$$

Next, let $\mathbf{A} \in \mathbb{R}^{3 \times 3}$ matrix. Then the determinant of a 2×2 matrix is given as

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \tag{A-19}$$

It is noted that the determinants of 2×2 and 3×3 matrices can be used as building blocks to compute the determinant of an $n \times n$ matrix. Also, it is important to understand that a square matrix is invertible if and only if its determinant is nonzero, i.e.,

$$\mathbf{A}^{-1} \text{ exists} \Leftrightarrow \det \mathbf{A} \neq 0$$

A.3 Simple Algebra Associated with Matrices

Sum and Difference of Matrices

Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ real-valued matrices. Furthermore, denote $\mathbf{A}$ and $\mathbf{B}$ in element form, respectively, as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{A-20}$$

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \quad \text{(A–21)}$$

Then the *sum* of **A** and **B** is defined as

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix} \quad \text{(A–22)}$$

It is seen from Eq. (A–22) that $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$. Similar to the sum of two matrices, the difference between **A** and **B** is defined as

$$\mathbf{A}-\mathbf{B} = \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} & \cdots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} & \cdots & a_{2n}-b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} & \cdots & a_{mn}-b_{mn} \end{bmatrix} \quad \text{(A–23)}$$

It is noted that matrices can only be added or subtracted if they are the same size.

Product of a Matrix with a Constant

Let **A** be an $m \times n$ real-valued matrix and let k be a scalar. Then the product of k with **A** is defined as

$$k\mathbf{A} = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix} \quad \text{(A–24)}$$

Product of Two Matrices of Conforming Size

Let **A** and **B** be $m \times p$ and $p \times n$ real-valued matrices, respectively. Then the product of **A** with **B** is defined as

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{k=1}^{p} a_{1k}b_{k1} & \sum_{k=1}^{p} a_{1k}b_{k2} & \cdots & \sum_{k=1}^{p} a_{1k}b_{kn} \\ \sum_{k=1}^{p} a_{2k}b_{k1} & \sum_{k=1}^{p} a_{2k}b_{k2} & \cdots & \sum_{k=1}^{p} a_{2k}b_{kn} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{p} a_{pk}b_{k1} & \sum_{k=1}^{p} a_{pk}b_{k2} & \cdots & \sum_{k=1}^{p} a_{pk}b_{kn} \end{bmatrix} \end{aligned} \quad \text{(A–25)}$$

In other words, the $(l,m)^{th}$ element of **AB** is given as

$$[\mathbf{AB}]_{l,m} = \sum_{k=1}^{p} a_{lk}b_{km} \quad \text{(A–26)}$$

It is very important to note that two matrices can be multiplied only if the number of rows of **A** must be the same as the number of columns of **B** (i.e., if $\mathbf{A} \in \mathbb{R}^{m,p}$ and $\mathbf{B} \in \mathbb{R}^{q,n}$, then **A** and **B** can be multiplied only if $p = q$). This last requirement is called *conformability*, i.e., two matrices can be multiplied only if their sizes are conforming. Due to the conformability requirement, the only case in which *both* **AB** and **BA** are valid operations is if **A** and **B** are both square matrices. Finally, we note that, even in the case of two square matrices **A** and **B** it is generally the case that $\mathbf{AB} \neq \mathbf{BA}$ (i.e., the matrix product is *not* commutative).

Inner Product Between Row and Column Vectors

The *standard inner product* (or *dot product*) between two row vectors or column vectors **p** and **q** is defined as

$$\mathbf{p} \cdot \mathbf{q} = \sum_{k=1}^{n} p_i q_i \tag{A–27}$$

where $p_i,\ (i = 1, \ldots, n)$ and $q_i,\ (i = 1, \ldots, n)$ are the elements of **p** and **q**, respectively. From the definition of the product of two matrices, it is seen that if **p** and **q** are both column vectors then the dot product between **p** and **q** can be written as

$$\mathbf{p} \cdot \mathbf{q} = \mathbf{p}^T\mathbf{q} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}^T \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = \sum_{k=1}^{n} p_i q_i \tag{A–28}$$

Similarly, if **p** and **q** are both row vectors, then the dot product between **p** and **q** can be written as

$$\begin{aligned} \mathbf{p} \cdot \mathbf{q} = \mathbf{p}^T\mathbf{q} &= \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}^T \\ &= \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = \sum_{k=1}^{n} p_i q_i \end{aligned} \tag{A–29}$$

We say that two vectors **p** and **q** are orthogonal with respect to the standard inner product (i.e., dot product) if and only if $\mathbf{p} \cdot \mathbf{q} = 0$, i.e.,

$$\mathbf{p} \text{ and } \mathbf{q} \text{ are orthogonal with respect to the standard inner product} \iff \mathbf{p}^T\mathbf{q} = 0 \tag{A–30}$$

Next, the *weighted inner product* is defined as

$$\mathbf{p} \cdot \mathbf{W}\mathbf{q} = \mathbf{p}^T\mathbf{W}\mathbf{q} \tag{A–31}$$

where $\mathbf{W} \in \mathbb{R}^{n\times n}$ is a *weighting matrix*. It is noted that two vectors **p** and **q** are orthogonal with respect to the weighting matrix **W** if and only if $\mathbf{p}^T\mathbf{W}\mathbf{q} = 0$, i.e.,

$$\mathbf{p} \text{ and } \mathbf{q} \text{ are orthogonal with respect to } \mathbf{W} \iff \mathbf{p}^T\mathbf{W}\mathbf{q} = 0 \tag{A–32}$$

A.4 Null Space and Range Space of a Real Matrix

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ be an $m \times n$ real-valued matrix. It is observed that **A** operates on column vectors of length n and produces column vectors of length m. Then the *null space* of **A** is defined as the set of all column vectors $\mathbf{q} \in \mathbb{R}^n$ (where $\mathbf{q} \neq \mathbf{0}$) such that

$$\mathbf{A}\mathbf{q} = \mathbf{0} \tag{A–33}$$

The null space of $\mathbf{A}$ is denoted $\mathcal{N}(\mathbf{A})$. Any vector $\mathbf{q}$ that satisfies Eq. (A–33) is said to lie in $\mathcal{N}(\mathbf{A})$. Moreover, we say that any vector $\mathbf{q}$ that lies in $\mathcal{N}(\mathbf{A})$ is *annihilated* by $\mathbf{A}$ (i.e., if $\mathbf{q} \in \mathcal{N}(\mathbf{A})$, then $\mathbf{A}$ *annihilates* $\mathbf{q}$). It is noted that the null space of a *square* matrix is nonzero if and only if the determinant of $\mathbf{A}$ is zero.

Next, the *range space* of $\mathbf{A}$ is the set of all vectors $\mathbf{p} \in \mathbb{R}^m$ (where $\mathbf{p} \neq \mathbf{0}$) for which there exists a vector $\mathbf{q} \in \mathbb{R}^n$ such that

$$\mathbf{A}\mathbf{q} = \mathbf{p} \tag{A–34}$$

The range space of a matrix is denoted $\mathcal{R}(\mathbf{A})$. Any vector $\mathbf{p}$ that satisfies Eq. (A–34) is said to lie in $\mathcal{R}(\mathbf{A})$.

A.5 Eigenvalues and Eigenvectors of a Real Square Matrix

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be an $n\times n$ real-valued square matrix. Then the scalar λ is said to be an *eigenvalue* of $\mathbf{A}$ with *eigenvector* $\mathbf{u}$ if

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u} \tag{A–35}$$

Rearranging Eq. (A–35), we obtain

$$\lambda\mathbf{u} - \mathbf{A}\mathbf{u} = \mathbf{0} \tag{A–36}$$

Now we know that $\mathbf{q} = \mathbf{I}\mathbf{q}$ where $\mathbf{I}$ is the $n \times n$ identity matrix. Therefore, Eq. (A–36) can be rewritten as

$$\lambda\mathbf{I}\mathbf{u} - \mathbf{A}\mathbf{u} = \mathbf{0} \tag{A–37}$$

Eq. (A–37) can be rearranged as

$$(\lambda\mathbf{I} - \mathbf{A})\,\mathbf{u} = \mathbf{0} \tag{A–38}$$

Eq. (A–38) implies one of two things. Either $\mathbf{u} = \mathbf{0}$ or the vector $\mathbf{q}$ must lie in the null space of the matrix $\lambda\mathbf{I} - \mathbf{A}$ (i.e., $\lambda\mathbf{I} - \mathbf{A}$ must annihilate $\mathbf{u}$). The former case is the trivial solution and hence is of no interest. Therefore, the latter case must be true. Now, in order for $\mathbf{u}$ to lie in the null space of $\lambda\mathbf{I} - \mathbf{A}$, the matrix $\lambda\mathbf{I} - \mathbf{A}$ must have a nonzero null space. Recall that $\lambda\mathbf{I} - \mathbf{A}$ has a nonzero null space, it must be *singular* (i.e., $\lambda\mathbf{I} - \mathbf{A}$ does not have an inverse) and, therefore, $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$

$$\det(\lambda\mathbf{I} - \mathbf{A}) = 0 \tag{A–39}$$

Therefore, the condition of Eq. (A–38) that leads to a nontrivial value of $\mathbf{u}$ is given by Eq. (A–39).

Examining Eq. (A–39) and using the general form for a determinant from Eq. (A–17), it is seen that $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ is a polynomial in λ, i.e., $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ can be written as

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = \sum_{k=1}^{n} a_{n-k}\lambda^k = 0 \tag{A–40}$$

where we note that $a_0 \equiv 1$. Eq. (A–40) is called the *characteristic equation* of the matrix $\mathbf{A}$. Now, because $\mathbf{A}$ is a real matrix, the coefficients $(a_0, \ldots, a_n)$ must also be real.

Multiplicity of Eigenvalues and Eigenvectors

Because Eq. (A–40) is a polynomial of degree n with real coefficients, from the fundamental theorem of algebra its roots (i.e., the eigenvalues of $\mathbf{A}$) must either be real or occur in complex conjugate pairs. Suppose we let $\lambda_, \ldots, \lambda_n$ be the eigenvalues of $\mathbf{A}$. Then the characteristic equation can be written in factored form as

$$\det(\lambda\mathbf{I} - \mathbf{A}) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) \tag{A–41}$$

In general, the eigenvalues $\lambda_, \ldots, \lambda_n$ will *not* be distinct [i.e., Eq. (A–41) may have repeated roots]. When an eigenvalue λ_i of Eq. (A–41) repeats itself k times (i.e., k roots of Eq. (A–40) are equal to λ_i), we say that the eigenvalue λ_i has *algebraic multiplicity* k. For example, suppose that two of the roots of the characteristic equation are equal to μ. Then in factored form the characteristic equation would have the factor $\lambda - \mu$ appear twice [i.e., we would have a factor $(\lambda - \mu)^2$ in the characteristic equation]. In this case the algebraic multiplicity of the eigenvalue μ would be *two*.

Suppose now that $m \leq n$ is the number of *distinct* eigenvalues of **A** and let $\lambda_1, \ldots, \lambda_m$ be the corresponding eigenvalues of **A**. Next, let $k_1, \ldots, k_m$ be the algebraic multiplicities of $\lambda_1, \ldots, \lambda_m$, respectively. Furthermore, let the set $[1, \ldots, n]$ be partitioned into sets $P_i = [p_{i-1} + 1, \ldots, p_i]$, $(i = 1, \ldots, m)$ such that

$$p_i = \sum_{j=1}^{i} k_j \tag{A–42}$$

and $p_0 = 0$. Then for each distinct eigenvalue λ_i, $(i = 1, \ldots, l)$ we have

$$\mathbf{A}\mathbf{u}_{r_i} = \lambda_i \mathbf{u}_{r_i}, \qquad (r_i \in P_i),\ (i = 1, \ldots, m) \tag{A–43}$$

It is seen that the sum of the algebraic multiplicities must add to n, i.e.,

$$\sum_{j=1}^{m} k_j = n \tag{A–44}$$

Furthermore, the eigenvectors associated with each partition P_i need not be distinct. Suppose that $m_i \leq k_i$ is the number of *linearly independent* eigenvectors associated with each partition P_i. Then we say that the eigenvalue λ_i, $(i = 1, \ldots, m)$ has *geometric multiplicity* $m_i \leq k_i$. In the case where $m_i = k_i$, the algebraic and geometric multiplicities of λ_i are the same.

Diagonalization of Square Matrices and Similarity Transformations

In the case where the geometric and algebraic multiplicities of every distinct eigenvalue $\lambda_1, \ldots, \lambda_m$ of a matrix **A** are the same (i.e., $m_i = k_i$ for all $i = 1, \ldots, m$), we say that the matrix **A** has a *complete* set of eigenvectors. Moreover, when a matrix **A** has a complete set of eigenvectors, it is seen that n linearly independent eigenvectors can be obtained, i.e., the eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ form a basis form a basis for $\mathbb{R}^n$. Stated somewhat more rigorously, we can write the following:

$$\text{Eigenvectors of } \mathbf{A} \text{ complete} \iff \text{The set } \{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \text{ forms a basis for } \mathbb{R}^n$$

Then, for each eigenvector $\mathbf{u}_i$, $(i = 1, \ldots, n)$ we have

$$\mathbf{A}\mathbf{u}_i = \lambda_i \mathbf{u}_i, \qquad (i = 1, \ldots, n) \tag{A–45}$$

Eq. (A–45) implies that

$$\mathbf{A}\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{u}_1 & \lambda_2\mathbf{u}_2 & \cdots & \lambda_n\mathbf{u}_n \end{bmatrix} \tag{A–46}$$

Eq. (A–46) can be rewritten as

$$\mathbf{A}\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{A–47}$$

Now let

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} \tag{A–48}$$

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{A–49}$$

The matrix **U** is called the *eigenvector matrix* of the matrix **A**. In terms of **U**, Eq. (A–47) can then be written as

$$\mathbf{AU} = \mathbf{U\Lambda} \tag{A–50}$$

Now because the eigenvectors of **A** are complete and form a basis for $\mathbb{R}^n$, it is known that the matrix **U** is *nonsingular* (i.e., $\mathbf{U}^{-1}$ exists). Consequently, we can multiply both sides of Eq. (A–50) on the left by $\mathbf{U}^{-1}$ to obtain

$$\mathbf{\Lambda} = \mathbf{U}^{-1}\mathbf{AU} \tag{A–51}$$

The quantity $\mathbf{U}^{-1}\mathbf{AU}$ is a *similarity transformation* of the matrix **A** by the eigenvector matrix **U**. It is seen that, for a matrix **A** that has a complete set of eigenvectors, using the similarity transformation of Eq. (A–51) produces a diagonal matrix $\mathbf{\Lambda}$.

Eigenvectors Associated with Complex Pairs of Eigenvectors

Let λ_i be an eigenvalue of a real-valued matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$. Furthermore, assume that λ_i is complex. Then, because eigenvalues of a real-valued matrix must occur in complex conjugate pairs, there must exist an eigenvalue λ_j such that $\lambda_j = \bar{\lambda}_i$, where $\bar{\lambda}_i$ is the complex conjugate of λ_i. Next, let $\mathbf{u}_i$ be the eigenvector associated with λ_i. Then we have

$$\mathbf{Au}_i = \lambda_i \mathbf{u}_i \tag{A–52}$$

Taking the complex conjugate of Eq. (A–52) gives

$$\overline{\mathbf{Au}_i} = \overline{\lambda_i \mathbf{u}_i} \tag{A–53}$$

Now the right-hand and left-hand sides of Eq. (A–53) can be written, respectively, as

$$\overline{\lambda_i \mathbf{u}_i} = \bar{\lambda}_i \bar{\mathbf{u}}_i \tag{A–54}$$

$$\overline{\mathbf{Au}_i} = \mathbf{A}\bar{\mathbf{u}}_i \tag{A–55}$$

where we note that $\bar{\mathbf{A}} = \mathbf{A}$ because **A** is a real-valued matrix. Therefore,

$$\mathbf{A}\bar{\mathbf{u}}_i = \bar{\lambda}_i \bar{\mathbf{u}}_i \tag{A–56}$$

It is seen that Eq. (A–56) satisfies the eigenvalue equation of Eq. (A–35). Therefore, $\bar{\mathbf{u}}_i$ is an eigenvector of **A** with eigenvalue $\bar{\lambda}_i$

A.6 Eigenvalues and Eigenvectors of a Real Symmetric Matrix

Now consider the special case where **A** is real and symmetric, i.e., $\mathbf{A} = \mathbf{A}^T$. Then for eigenvectors $\mathbf{u}_i$ and $\mathbf{u}_j$ with corresponding eigenvalues λ_i and λ_j, respectively, we have

$$\mathbf{Au}_i = \lambda_i \mathbf{u}_i \tag{A–57}$$

$$\mathbf{Au}_j = \lambda_j \mathbf{u}_j \tag{A–58}$$

Suppose now that we multiply both sides of Eq. (A–57) on the left-hand side by $\mathbf{u}_j^T$ and multiply both sides of Eq. (A–58) on the left-hand side by $\mathbf{u}_i^T$. We then obtain

$$\mathbf{u}_j^T \mathbf{Au}_i = \mathbf{u}_j^T \lambda_i \mathbf{u}_i \tag{A–59}$$

$$\mathbf{u}_i^T \mathbf{Au}_j = \mathbf{u}_j^T \lambda_j \mathbf{u}_j \tag{A–60}$$

Now because λ_i and λ_j are scalars, we have

$$\mathbf{u}_j^T \lambda_i \mathbf{u}_i = \lambda_i \mathbf{u}_j^T \mathbf{u}_i \tag{A–61}$$

$$\mathbf{u}_i^T \lambda_j \mathbf{u}_j = \lambda_j \mathbf{u}_i^T \mathbf{u}_j \tag{A–62}$$

Furthermore, we know that $\mathbf{u}_j^T\mathbf{u}_i = \mathbf{u}_j \cdot \mathbf{u}_i$ and, thus, is a scalar. Consequently, $\mathbf{u}_j^T\mathbf{A}\mathbf{u}_i$ is also a scalar and we have

$$\left[\mathbf{u}_j^T\mathbf{A}\mathbf{u}_i\right]^T = \mathbf{u}_i^T\mathbf{A}^T\mathbf{u}_j \tag{A-63}$$

Now because $\mathbf{A}$ is symmetric, Eq. (A–63) implies

$$\left[\mathbf{u}_j^T\mathbf{A}\mathbf{u}_i\right]^T = \mathbf{u}_i^T\mathbf{A}^T\mathbf{u}_j = \mathbf{u}_i^T\mathbf{A}\mathbf{u}_j \tag{A-64}$$

Substituting the result of Eq. (A–64) into Eq. (A–59), placing it alongside Eq. (A–60), and using the results of Eqs. (A–61) and (A–62) gives

$$\mathbf{u}_i^T\mathbf{A}\mathbf{u}_j = \lambda_i\mathbf{u}_j^T\mathbf{u}_i \tag{A-65}$$
$$\mathbf{u}_i^T\mathbf{A}\mathbf{u}_j = \lambda_j\mathbf{u}_j^T\mathbf{u}_j \tag{A-66}$$

Subtracting Eq. (A–65) from (A–66), we obtain

$$\lambda_j\mathbf{u}_j^T\mathbf{u}_i - \lambda_i\mathbf{u}_j^T\mathbf{u}_i = (\lambda_j - \lambda_i)\mathbf{u}_j^T\mathbf{u}_i = 0 \tag{A-67}$$

Then, because $\lambda_i \neq \lambda_j$, it must be the case that

$$\mathbf{u}_j^T\mathbf{u}_i = 0 \tag{A-68}$$

Using the definition of orthogonality of vectors with respect to the standard inner product, it is seen that Equation (A–68) implies that eigenvectors $\mathbf{u}_i$ and $\mathbf{u}_j$ corresponding to distinct eigenvalues λ_i and λ_j of a real symmetric matrix are *orthogonal*.

Next, assume that two of the eigenvalues of a real symmetric matrix $\mathbf{A}$ are a complex conjugate pair, i.e., we consider two eigenvalues λ_i and λ_j such that $\lambda_j = \bar{\lambda}_i$. Now we know from earlier in this section that the eigenvector $\mathbf{u}_j$ associated with the complex conjugate of eigenvalue $\lambda_j = \bar{\lambda}_i$ is the complex conjugate of $\mathbf{u}_i$ (i.e., $\mathbf{u}_j = \bar{\mathbf{u}}_i$). Furthermore, from the definition of an eigenvalue-eigenvector pair we have

$$\mathbf{A}\mathbf{u}_i = \lambda_i\mathbf{u}_i \tag{A-69}$$
$$\mathbf{A}\bar{\mathbf{u}}_i = \bar{\lambda}_i\bar{\mathbf{u}}_i \tag{A-70}$$

Multiplying Eqs. (A–69) and (A–70) by $\bar{\mathbf{u}}_i^T$ and $\mathbf{u}_i^T$, respectively, we obtain

$$\bar{\mathbf{u}}_i^T\mathbf{A}\mathbf{u}_i = \bar{\mathbf{u}}_i^T\lambda_i\mathbf{u}_i = \lambda_i\bar{\mathbf{u}}_i^T\mathbf{u}_i \tag{A-71}$$
$$\mathbf{u}_i^T\mathbf{A}\bar{\mathbf{u}}_i = \mathbf{u}_i^T\bar{\lambda}_i\bar{\mathbf{u}}_i = \bar{\lambda}_i\mathbf{u}_i^T\bar{\mathbf{u}}_i \tag{A-72}$$

Now because $\mathbf{A}$ is symmetric, we have

$$\left[\bar{\mathbf{u}}_i^T\mathbf{A}\mathbf{u}_i\right]^T = \mathbf{u}_i^T\mathbf{A}^T\bar{\mathbf{u}}_i = \mathbf{u}_i^T\mathbf{A}\bar{\mathbf{u}}_i \tag{A-73}$$
$$\left[\bar{\mathbf{u}}_i^T\mathbf{u}_i\right]^T = \mathbf{u}_i^T\bar{\mathbf{u}}_i \tag{A-74}$$

Furthermore, because $\bar{\mathbf{u}}_i^T\mathbf{A}\mathbf{u}_i$ and $\bar{\mathbf{u}}_i^T\mathbf{u}_i$ are scalars, we can substitute the results of Eqs. (A–73) and (A–74) into Eqs. (A–75) from (A–76), respectively, we obtain

$$\mathbf{u}_i^T\mathbf{A}\bar{\mathbf{u}}_i = \lambda_i\mathbf{u}_i^T\bar{\mathbf{u}}_i \tag{A-75}$$
$$\mathbf{u}_i^T\mathbf{A}\bar{\mathbf{u}}_i = \bar{\lambda}_i\mathbf{u}_i^T\bar{\mathbf{u}}_i \tag{A-76}$$

Subtracting Eq. (A–75) from (A–76) gives

$$(\bar{\lambda}_i - \lambda_i)\mathbf{u}_i^T\bar{\mathbf{u}}_i = \mathbf{0} \tag{A-77}$$

Now because $\mathbf{u}_i^T\bar{\mathbf{u}}_i \neq \mathbf{0}$, we have

$$\bar{\lambda}_i - \lambda_i = 0 \tag{A-78}$$

which implies that

$$\bar{\lambda}_i = \lambda_i \tag{A–79}$$

Equation (A–79) states that a complex eigenvalue of a real symmetric matrix is equal to its complex conjugate. The only possible way for a complex number to equal its complex conjugate is if the number is itself real. Another way of looking at this is as follows. Suppose that

$$\lambda_i = \alpha + i\beta \tag{A–80}$$

It then follows that

$$\bar{\lambda}_i = \alpha - i\beta \tag{A–81}$$

Consequently, the only way for $\bar{\lambda}_i$ and λ_i to be equal is if $\beta \equiv 0$, i.e., $\lambda_i \equiv \alpha \in \mathbb{R}$. The key result is that the eigenvalues of a real symmetric matrix are *real*.

Now suppose we let **U** be the eigenvector matrix associated with a real symmetric matrix. Then we have

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} \tag{A–82}$$

Now because the eigenvectors of a real symmetric matrix are orthogonal, we can normalize each eigenvector to unit magnitude, i.e., we can say that

$$\|\mathbf{u}_i\| = 1 \tag{A–83}$$

Then the eigenvector matrix **U** is such that its columns are *orthonormal* which implies that **U** is an orthogonal matrix, i.e.,

$$\mathbf{U}^{-1} = \mathbf{U}^T \tag{A–84}$$

Then because the eigenvectors are complete (by virtue of the fact that they are orthonormal), we know that

$$\mathbf{AU} = \mathbf{U\Lambda} \tag{A–85}$$

where $\mathbf{\Lambda}$ is a diagonal matrix with the eigenvalues on the diagonal. Finally, using the fact that **U** is orthogonal, we know that

$$\mathbf{U}^{-1}\mathbf{AU} = \mathbf{U}^T\mathbf{AU} \tag{A–86}$$

Consequently,

$$\mathbf{U}^T\mathbf{AU} = \mathbf{\Lambda} \tag{A–87}$$

A.7 Symmetric Weighted Eigenvalue Problem

Consider now the eigenvalue problem

$$\mathbf{Au} = \lambda\mathbf{Bu} \tag{A–88}$$

where the matrices **A** and **B** are both *symmetric*. Eq. (A–88) is called a *symmetric weighted eigenvalue problem* or, simply, a *weighted eigenvalue problem* because the matrix **B** is not the identity matrix. Rearranging Eq. (A–88), we obtain

$$= \lambda\mathbf{Bu} - \mathbf{Au} = (\lambda\mathbf{B} - \mathbf{A})\mathbf{u} \tag{A–89}$$

Orthogonality of Eigenvectors of Symmetric Weighted Eigenvalue Problem

As with the standard eigenvalue problem, consider two eigenvectors $\mathbf{u}_i$ and $\mathbf{u}_j$ corresponding to distinct eigenvalues λ_i and λ_j, i.e., consider

$$\mathbf{Au}_i = \lambda_i\mathbf{Bu}_i \tag{A–90}$$

$$\mathbf{Au}_j = \lambda_j\mathbf{Bu}_j \tag{A–91}$$

Equations (A–90) and (A–91) together imply that

$$\begin{aligned} \mathbf{u}_j^T\mathbf{A}\mathbf{u}_i &= \lambda_i\mathbf{u}_j^T\mathbf{B}\mathbf{u}_i && \text{(A–92)} \\ \mathbf{u}_i^T\mathbf{A}\mathbf{u}_j &= \lambda_j\mathbf{u}_i^T\mathbf{B}\mathbf{u}_j && \text{(A–93)} \end{aligned}$$

Then, using the fact that **A** and **B** are symmetric, we know that

$$\begin{aligned} \left[\mathbf{u}_j^T\mathbf{A}\mathbf{u}_i\right]^T &= \mathbf{u}_i^T\mathbf{A}^T\mathbf{u}_j = \mathbf{u}_i^T\mathbf{A}\mathbf{u}_j && \text{(A–94)} \\ \left[\mathbf{u}_i^T\mathbf{B}\mathbf{u}_j\right]^T &= \mathbf{u}_j^T\mathbf{B}^T\mathbf{u}_i = \mathbf{u}_j^T\mathbf{B}\mathbf{u}_i && \text{(A–95)} \end{aligned}$$

Consequently,

$$\begin{aligned} \mathbf{u}_i^T\mathbf{A}\mathbf{u}_j &= \lambda_i\mathbf{u}_i^T\mathbf{B}\mathbf{u}_j && \text{(A–96)} \\ \mathbf{u}_i^T\mathbf{A}\mathbf{u}_j &= \lambda_j\mathbf{u}_i^T\mathbf{B}\mathbf{u}_j && \text{(A–97)} \end{aligned}$$

Subtracting Eqs. (A–97) and (A–97), we obtain

$$(\lambda_i - \lambda_j)\mathbf{u}_i^T\mathbf{B}\mathbf{u}_j = 0 \tag{A–98}$$

This time, because $\lambda_j \neq \lambda_j$, we have

$$\mathbf{u}_i^T\mathbf{B}\mathbf{u}_j = 0 \tag{A–99}$$

Equation (A–99) implies that the eigenvectors $\mathbf{u}_i$ and $\mathbf{u}_j$ are *orthogonal with respect to* the matrix **B**. Furthermore, by rewriting Eq. (A–88) in the form

$$\mathbf{B}\mathbf{u} = \mu\mathbf{A}\mathbf{u} \tag{A–100}$$

where $\mu = 1/\lambda$, it can be seen that

$$\mathbf{u}_i^T\mathbf{A}\mathbf{u}_j = 0 \tag{A–101}$$

Equation (A–101) implies that the eigenvectors $\mathbf{u}_i$ and $\mathbf{u}_j$ are *orthogonal with respect to* the matrix **A**. In other words, for the weighted eigenvalue problem of Eq. (A–88), the eigenvectors of distinct eigenvalues are orthogonal with respect to both the matrices **A** and **B**.

Eigenvalues of Symmetric Weighted Eigenvalue Problem

Now assume that λ_i and λ_j are two eigenvalues of the weighted eigenvalue problem of Eq. (A–88). Furthermore, suppose that $\lambda_j = \bar{\lambda}_i$. Now we have

$$\begin{aligned} \mathbf{A}\mathbf{u}_i &= \lambda_i\mathbf{B}\mathbf{u}_i && \text{(A–102)} \\ \mathbf{A}\mathbf{u}_j &= \lambda_j\mathbf{B}\mathbf{u}_j && \text{(A–103)} \end{aligned}$$

Taking the complex conjugate of Eq. (A–102) gives

$$\overline{\mathbf{A}\mathbf{u}_i} = \overline{\lambda_i\mathbf{B}\mathbf{u}_i} \tag{A–104}$$

Equation (A–104) implies

$$\mathbf{A}\bar{\mathbf{u}}_i = \bar{\lambda}_i\mathbf{B}\bar{\mathbf{u}}_i \tag{A–105}$$

Observing that $\lambda_j = \bar{\lambda}_i$, we obtain

$$\mathbf{A}\bar{\mathbf{u}}_i = \lambda_j\mathbf{B}\bar{\mathbf{u}}_i \tag{A–106}$$

Consequently, $\bar{\mathbf{u}}_j$ is an eigenvector of the weighted eigenvalue problem with eigenvalue λ_j. In other words, if λ_i is a complex eigenvalue of the weighted eigenvalue problem with eigenvector $\mathbf{u}_i$, then $\bar{\lambda}_i$ is an eigenvalue of the weighted eigenvalue problem with eigenvalue $\bar{\mathbf{u}}_i$.

Now suppose that λ_i and λ_j are eigenvalues of the weighted eigenvalue problem of Eq. (A–88). Furthermore, let $\lambda_j = \bar{\lambda}_i$. Then, from the definition of the weighted eigenvalue problem of Eq. (A–88), we have

$$\begin{aligned} \mathbf{A}\mathbf{u}_i &= \lambda_i \mathbf{B}\mathbf{u}_i && \text{(A–107)} \\ \mathbf{A}\bar{\mathbf{u}}_i &= \bar{\lambda}_i \mathbf{B}\bar{\mathbf{u}}_i && \text{(A–108)} \end{aligned}$$

Multiplying Eqs. (A–107) and (A–108) by $\bar{\mathbf{u}}_i^T$ and $\mathbf{u}_i^T$, respectively, we obtain

$$\begin{aligned} \bar{\mathbf{u}}_i^T\mathbf{A}\mathbf{u}_i &= \bar{\mathbf{u}}_i^T\lambda_i\mathbf{B}\mathbf{u}_i = \lambda_i\bar{\mathbf{u}}_i^T\mathbf{B}\mathbf{u}_i && \text{(A–109)} \\ \mathbf{u}_i^T\mathbf{A}\bar{\mathbf{u}}_i &= \mathbf{u}_i^T\bar{\lambda}_i\mathbf{B}\bar{\mathbf{u}}_i = \bar{\lambda}_i\mathbf{u}_i^T\mathbf{B}\bar{\mathbf{u}}_i && \text{(A–110)} \end{aligned}$$

Now because both **A** and **B** are symmetric and we know that

$$\begin{aligned} \left[\bar{\mathbf{u}}_i^T\mathbf{A}\mathbf{u}_i\right]^T &= \mathbf{u}_i^T\mathbf{A}^T\bar{\mathbf{u}}_i = \mathbf{u}_i^T\mathbf{A}\bar{\mathbf{u}}_i && \text{(A–111)} \\ \left[\bar{\mathbf{u}}_i^T\mathbf{B}\mathbf{u}_i\right]^T &= \mathbf{u}_i^T\mathbf{B}^T\bar{\mathbf{u}}_i = \mathbf{u}_i^T\mathbf{B}\bar{\mathbf{u}}_i && \text{(A–112)} \end{aligned}$$

Then, because $\bar{\mathbf{u}}_i^T\mathbf{A}\mathbf{u}_i$ and $\bar{\mathbf{u}}_i^T\mathbf{B}\mathbf{u}_i$ are scalars, we can substitute the results of Eqs. (A–111) and (A–112) into Eqs. (A–109) and (A–110), respectively, to obtain

$$\begin{aligned} \mathbf{u}_i^T\mathbf{A}\bar{\mathbf{u}}_i &= \lambda_i\mathbf{u}_i^T\mathbf{B}\bar{\mathbf{u}}_i && \text{(A–113)} \\ \mathbf{u}_i^T\mathbf{A}\bar{\mathbf{u}}_i &= \bar{\lambda}_i\mathbf{u}_i^T\mathbf{B}\bar{\mathbf{u}}_i && \text{(A–114)} \end{aligned}$$

Subtracting Eq. (A–113) from (A–114), we obtain

$$(\bar{\lambda}_i - \lambda_i)\mathbf{u}_i^T\mathbf{B}\bar{\mathbf{u}}_i = \mathbf{0} \tag{A–115}$$

Now because $\mathbf{u}_i^T\mathbf{B}\bar{\mathbf{u}}_i \neq \mathbf{0}$, we have

$$\bar{\lambda}_i - \lambda_i = 0 \tag{A–116}$$

which implies that

$$\bar{\lambda}_i = \lambda_i \tag{A–117}$$

Equation (A–117) states that a complex eigenvalue of the weighted eigenvalue problem of Eq. (A–88) is equal to its complex conjugate. The only possible way for a complex number to equal its complex conjugate is if the number is itself real. Another way of looking at this is as follows. Suppose that

$$\lambda_i = \alpha + i\beta \tag{A–118}$$

It then follows that

$$\bar{\lambda}_i = \alpha - i\beta \tag{A–119}$$

Consequently, the only way for $\bar{\lambda}_i$ and λ_i to be equal is if $\beta \equiv 0$, i.e., $\lambda_i \equiv \alpha \in \mathbb{R}$. The key result is that the eigenvalues of the weighted eigenvalue problem of Eq. (A–88) are *real.*

Suppose now that we let

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} \tag{A–120}$$

Then we can write

$$\mathbf{A}\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{B}\mathbf{u}_1 & \lambda_2\mathbf{B}\mathbf{u}_2 & \cdots & \lambda_n\mathbf{B}\mathbf{u}_n \end{bmatrix} \tag{A–121}$$

Using the expression for **U** from Eq. (A–120), it is seen that (A–121) can be rewritten as

$$\mathbf{A}\mathbf{U} = \begin{bmatrix} \mathbf{B}\mathbf{u}_1 & \mathbf{B}\mathbf{u}_2 & \cdots & \mathbf{B}\mathbf{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \tag{A–122}$$

Then, defining

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ +0 & \cdots & 0 & \lambda_n \end{bmatrix} \tag{A–123}$$

Equation (A–122) can be written as

$$\mathbf{AU} = \begin{bmatrix} \mathbf{Bu}_1 & \mathbf{Bu}_2 & \cdots & \mathbf{Bu}_n \end{bmatrix} \mathbf{\Lambda} \tag{A–124}$$

Factoring **B** on the left-hand side of Eq. (A–124), we obtain

$$\mathbf{AU} = \mathbf{B} \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} \mathbf{\Lambda} \tag{A–125}$$

Again, using the expression for **U** from Eq. (A–120) gives

$$\mathbf{AU} = \mathbf{BU\Lambda} \tag{A–126}$$

Multiplying both sides of Eq. (A–126) by $\mathbf{U}^T$ gives

$$\mathbf{U}^T\mathbf{AU} = \mathbf{U}^T\mathbf{BU\Lambda} \tag{A–127}$$

Normalization of Eigenvectors of Symmetric Weighted Eigenvalue Problem

Unlike the symmetric standard eigenvalue problem $\mathbf{Au} = \lambda\mathbf{u}$ (where $\mathbf{A} = \mathbf{A}^T$), the eigenvectors of **A** are orthogonal (i.e., $\mathbf{u}_i^T\mathbf{u}_j = 0$), in the symmetric weighted eigenvalue problem $\mathbf{Au} = \lambda\mathbf{Bu}$ (where $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{B} = \mathbf{B}^T$) it was seen that the eigenvectors are orthogonal *with respect to* the matrices **A** and **B**, i.e.,

$$\begin{array}{rcl} \mathbf{u}_i^T\mathbf{Au}_j & = & 0 \\ \mathbf{u}_i^T\mathbf{Bu}_j & = & 0 \end{array}, \qquad (i = 1, \ldots, n) \tag{A–128}$$

Consequently, in the symmetric weighted eigenvalue problem with $\mathbf{B} \neq \mathbf{I}$ it is *not* possible to find a set of normalized eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ such that the eigenvector matrix **U** is an orthogonal matrix (i.e., for the symmetric weighted eigenvalue problem it is generally the case that $\mathbf{U}^{-1} \neq \mathbf{U}^T$). Instead, due to the fact that the eigenvector matrix is orthogonal with respect to both **A** and **B**, the eigenvector matrix **U** is commonly normalized with respect to *either* $\mathbf{U}^T\mathbf{AU} = \mathbf{I}$ *or* $\mathbf{U}^T\mathbf{BU} = \mathbf{I}$. In the former case we say that **U** is normalized with respect to **A** whereas in the latter case we say that **U** is normalized with respect to **B**. Suppose we choose to normalize **U** with respect to **A**. To this end, let $\mathbf{w}_1, \ldots, \mathbf{w}_n$ be the *unnormalized* eigenvectors of the symmetric weighted eigenvalue problem, i.e.,

$$\mathbf{Aw}_i = \lambda_i\mathbf{Bw}_i \tag{A–129}$$

Then, pre-multiplying by $\mathbf{w}_i^T$, we have

$$\mathbf{w}_i^T\mathbf{Aw}_i = \mathbf{w}_i^T\lambda_i\mathbf{Bw}_i = \lambda_i\mathbf{w}_i^T\mathbf{Bw}_i \tag{A–130}$$

Suppose now that we choose the normalized eigenvectors $\mathbf{u}_i$, $(i = 1, \ldots, n)$ such that

$$\mathbf{u}_i = \frac{\mathbf{w}_i}{\sqrt{\mathbf{w}_i^T\mathbf{Aw}_i}}, \qquad (i = 1, \ldots, n) \tag{A–131}$$

Then it is seen that

$$\mathbf{u}_i^T\mathbf{Au}_i = \left[\frac{\mathbf{w}_i}{\sqrt{\mathbf{w}_i^T\mathbf{Aw}_i}}\right]^T \mathbf{A} \left[\frac{\mathbf{w}_i}{\sqrt{\mathbf{w}_i^T\mathbf{Aw}_i}}\right] = \frac{\mathbf{w}_i^T\mathbf{Aw}_i}{\mathbf{w}_i^T\mathbf{Aw}_i} = 1, \qquad (i = 1, \ldots, n) \tag{A–132}$$

Consequently, normalizing with respect to the matrix **A**, we have

$$\mathbf{u}_i = \frac{\mathbf{w}_i}{\sqrt{\mathbf{w}_i^T \mathbf{A} \mathbf{w}_i}}, \qquad (i = 1, \ldots, n) \tag{A–133}$$

which implies that

$$\mathbf{U}^T \mathbf{A} \mathbf{U} = \mathbf{I} \Longleftrightarrow \text{Eigenvectors Are Normalized With Respect to } \mathbf{A} \tag{A–134}$$

Next, suppose we choose to normalize the eigenvectors respect to the matrix **B**. Then in this case we would choose the normalized eigenvectors such that

$$\mathbf{u}_i = \frac{\mathbf{w}_i}{\sqrt{\mathbf{w}_i^T \mathbf{B} \mathbf{w}_i}}, \qquad (i = 1, \ldots, n) \tag{A–135}$$

Then it is seen that

$$\mathbf{u}_i^T \mathbf{B} \mathbf{u}_i = \left[\frac{\mathbf{w}_i}{\sqrt{\mathbf{w}_i^T \mathbf{B} \mathbf{w}_i}} \right]^T \mathbf{B} \left[\frac{\mathbf{w}_i}{\sqrt{\mathbf{w}_i^T \mathbf{B} \mathbf{w}_i}} \right] = \frac{\mathbf{w}_i^T \mathbf{B} \mathbf{w}_i}{\mathbf{w}_i^T \mathbf{B} \mathbf{w}_i} = 1, \qquad (i = 1, \ldots, n) \tag{A–136}$$

Consequently, using a normalization

$$\mathbf{u}_i = \frac{\mathbf{w}_i}{\sqrt{\mathbf{w}_i^T \mathbf{B} \mathbf{w}_i}}, \qquad (i = 1, \ldots, n) \tag{A–137}$$

we see that

$$\mathbf{U}^T \mathbf{B} \mathbf{U} = \mathbf{I} \Longleftrightarrow \text{Eigenvectors Are Normalized With Respect to } \mathbf{B} \tag{A–138}$$

A.8 Definiteness of Matrices

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be an $n \times n$ real-valued matrix. Then **A** is said to be *positive definite* if and only if $\mathbf{u}^T \mathbf{A} \mathbf{u} > 0$ for all $\mathbf{u} \neq 0$, i.e.,

$$\mathbf{A} \text{ positive definite} \Longleftrightarrow \mathbf{u}^T \mathbf{A} \mathbf{u} > 0 \ \forall \mathbf{u} \neq 0 \tag{A–139}$$

Similarly, the matrix **A** is said to be *positive semi-definite* if and only if $\mathbf{u}^T \mathbf{A} \mathbf{u} \geq 0$ for all $\mathbf{u} \neq 0$, i.e.,

$$\mathbf{A} \text{ positive semi-definite} \Longleftrightarrow \mathbf{u}^T \mathbf{A} \mathbf{u} \geq 0 \ \forall \mathbf{u} \neq 0 \tag{A–140}$$

Finally, the matrix **A** is said to be *negative semi-definite* if and only if $\mathbf{u}^T \mathbf{A} \mathbf{u} \leq 0$ for all $\mathbf{u} \neq 0$, i.e.,

$$\mathbf{A} \text{ negative semi-definite} \Longleftrightarrow \mathbf{u}^T \mathbf{A} \mathbf{u} \leq 0 \ \forall \mathbf{u} \neq 0 \tag{A–141}$$

Any matrix that can neither be classified as positive definite, positive-semi-definite, negative definite, nor negative semi-definite, is called *indefinite*.

Bibliography

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