Lecture Notes in Microeconomics

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1 Foreword

The aim of these notes is to provide a concise introduction to microeconomic modeling at the advanced undergraduate level. No final year undergraduate student in economics is expected to find in these notes any concept or idea he is not already familiar with. These old ideas will however be presented in a way that is likely to be new to most students at that stage in the course of their curriculum. Some familiarity with the language of Mathematics will undoubtly help the reader of these notes. While a number of key mathematical results are briefly presented in the appendix to these notes, these results are intended for immediate reference only. In no way do they substitute an introductory course in real analysis. Sydsaeter's *Mathematical Analysis* is an excellent reference for economists. Those with a stronger background in Mathematics may want to use Rudin's *Principles of Mathematical Analysis* instead.

All introductory textbooks on microeconomics cover most of the material found in these notes, and indeed very often more than that. Students are therefore encouraged to satisfy their curiosity by consulting alternative sources. Rubinstein's outstanding *Lecture Notes in Microeconomics* for instance are freely available online. Bear in mind however that any set of notes derives as much of its added value from what it chooses to leave out as from what it effectively contains. Finally, while in principle usable on a stand-alone basis, these notes are primarily designed to support lectures. Attending lectures should therefore help you improve your understanding of the material covered in the notes.

These notes are organized as follows. Section 2 is devoted to the study of the consumer. Section 2.1 elaborates a general framework in which to study issues related to consumption. Section 2.2 illustrates some of the most important applications of the framework: intertemporal consumption (2.2.1), consumption under uncertainty (2.2.2), as well as labor supply

(2.2.3). We show in section 3 how our approach to consumption can be transferred over to think about production. Section 4 introduces General Equilibrium. Section 5 develops the concept of a financial asset. That section unifies the applications covered in 2.2, and allows us to explicitly deal with temporal aspects of general equilibrium – which constitute the theme of section 6.

Results are divided into lemmas and propositions. The lemmas tend to be purely technical results. They are mere tools in the build-up to the propositions, in which the economic insights really lie. Scattered in the text is also a series of questions. While these questions are meant to give you an opportunity to exert the knowledge you have acquired, the results developed in them are often important complement to the material covered in the lectures. As such they are part and parcel of these notes. Sketch answers to all questions are provided in the Appendix. More detailed answers will be given during the weekly seminars.

2 The Consumer

2.1 General Framework

Wherever possible, we will in these notes confine our analysis to a world containing two goods. All results developed here naturally extend to higher dimensions, but our aim is to keep the analysis simple in order to focus on economic content. Restricting attention to the 2-dimensional case also offers the great advantage of accommodating a complete graphical representation.

The ultimate foundation of our approach is the *utility function*, used to represent preferences[1] of the consumer over vectors of goods $\underline{x} = (x_1, x_2)$. A consumer having utility function u is one who prefers $\underline{x}$ to $\underline{y}$ iff $u(\underline{x}) > u(\underline{y})$.[2] Always bear in mind that utility functions are mere numerical tools used to represent underlying preferences. In particular, if v is strictly increasing and a consumer has utility function u then $v \circ u$ is an equally valid utility function for that consumer. A consumer's utility function is thus determined only up to an increasing transformation.

In principle, utility functions may take a variety of forms. In order to make progress, one is bound to make certain restrictive assumptions regarding consumers' preferences. While these assumptions may to some extent be justified economically, their main asset is to greatly simplify the analysis of the model we build. We will immediately state these assumptions in terms of utility functions. You should convince yourself however that all these assumptions are preserved under any (smooth) increasing transformation. This is critical, given we are claiming to make assumptions concerning consumers' underlying preferences.

Our first assumption embodies the idea that consumers exhibit smooth preferences.

Assumption A.1: *u is C^∞ (u is smooth).*

Our second assumption embodies the idea that consumers always prefer consuming more.

Assumption A.2: $\nabla_i u > 0$, $\forall i$ *(u is strictly increasing).*

[1] We could spend an entire course examining the relationship between consumers' underlying preferences and the utility function representation of these preferences. The interested reader is referred to Rubinstein's outstanding *Lecture Notes in Microeconomics*, freely available online.

[2] Notice that this immediately precludes certain preferences, in particular non-transitive preferences. It is possible to show however that any 'well-behaved' preferences can be represented using a utility function.

Our third assumption embodies the idea that consumers prefer balanced baskets of goods.[3] This is the natural assumption for complementary goods, but is also very compelling in the contexts of *intertemporal consumption* and *consumption under uncertainty*. We explore these topics in detail later in the notes.

Assumption A.3: $u(\lambda \underline{x} + (1-\lambda)\underline{y}) > \min\{u(\underline{x}), u(\underline{y})\}$, $\forall \lambda \in (0,1)$ *(u is strictly quasi-concave).*

We will in these notes develop a set of results concerning utility functions satisfying the former assumptions. Utility functions which fail to satisfy one or more of these assumptions have to be dealt with on an individual basis.

Definition 1 *A consumer with utility function u is one who prefers $\underline{x}$ to $\underline{y}$ iff $u(\underline{x}) > u(\underline{y})$. A function $u : \mathbb{R}^n \to \mathbb{R}$ is a* standard utility function *iff it satisfies assumptions **A.1-A.3**.*

Henceforth, all utility functions we will be dealing with in these notes will be assumed to be standard utility functions, unless stated otherwise.

The following technical lemma records a useful way to check whether a given utility function satisfies the standard assumptions.

Lemma 1 *Consider u such that **A.1-A.2** hold. Then assumption **A.3** holds iff u has convex level curves.*[4].

Proof. Suppose **A.3** holds, let L an arbitrary level curve of u ($u(\underline{x}) = l$, $\forall \underline{x} \in L$), and f such that $L = \{(x, f(x)) : x \in \mathbb{R}\}$. We want to show that f is convex. Let $\underline{x}$ and $\underline{y}$ belong to L. By strict quasi-concavity $u(\lambda \underline{x} + (1-\lambda)\underline{y}) > \min\{u(\underline{x}), u(\underline{y})\} = l$. And, since u is strictly increasing, L lies below the line segment joining $\underline{x}$ and $\underline{y}$. But this implies f convex, since $\underline{x}$ and $\underline{y}$ were chosen arbitrarily.

For the converse, suppose u has convex level curves and let $\underline{x}$ and $\underline{y}$ be two consumption bundles. Suppose moreover, wlog, that $u(\underline{y}) \geq u(\underline{x})$. Let $\underline{y}'$ the point on the same level curve as $\underline{x}$ such that $y_2' = y_2$. Note that $\underline{y}'$ lies to the left of $\underline{y}$, so that if we can show that $u(\lambda \underline{x} + (1-\lambda)\underline{y}') > u(\underline{x})$ then we are done. But the previous inequality indeed holds since the level curve containing $\underline{x}$ and $\underline{y}'$ is decreasing, convex, and $\nabla_i u > 0$, $\forall i$, by **A.2**.

[3] Observe that any concave function is also quasi-concave. However, quasi-concavity is preserved under increasing transformation, while concavity is not. So quasi-concavity is a statement about underlying preferences, while concavity is not.

[4] Recall that a level curve of u represents the set of points such that $u(\underline{x}) = l$, for some fixed l.

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[**Q1**] Let $\alpha \in (0,1)$ and consider the Cobb-Douglas functional form: $u(\underline{x}) = x_1^\alpha x_2^{1-\alpha}$. Show that the Cobb-Douglas is (almost) a standard utility function.

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We next introduce an important class of utility functions:

Definition 2 *A utility function u has the* additive property *iff there exists $U : \mathbb{R} \to \mathbb{R}$ and $(\alpha_i)_{1 \le i \le n}$, $\alpha_i > 0$, $\forall i$, such that:*

$$u(\underline{x}) = \sum_i \alpha_i U(x_i) \tag{1}$$

Utility functions satisfying the additive property are particularly convenient to handle and occupy for that reason a prominent place in microeconomic theory. The following lemma shows for instance that assumptions **A.1-A.3** take a particularly simple form for that class of utility functions.

Lemma 2 *Let u satisfy the additive property, $u(\underline{x}) = \sum_i \alpha_i U(x_i)$. Then u satisfies **A.1-A.3** iff*

1. *U is C^∞*
2. *$U' > 0$*
3. *$U'' < 0$*

Proof. Note first that 1 and 2 are evidently necessary and sufficient for **A.1** and **A.2** to hold, respectively.

We next show that if 3 holds then so does **A.3**. We have

$$u(\lambda \underline{x} + (1-\lambda)\underline{y}) = \sum_i \alpha_i U(\lambda x_i + (1-\lambda) y_i) > \sum_i \alpha \big(\lambda U(x_i) + (1-\lambda) U(y_i)\big) \tag{2}$$

Hence $u(\lambda \underline{x} + (1-\lambda)\underline{y}) > \lambda u(\underline{x}) + (1-\lambda) u(\underline{y})$.

We finally show that concavity of U is in fact a necessary condition for **A.3** to hold. Suppose U not strictly concave. We can thus find x such that $U''(x) \ge 0$. Let $x_2(x_1)$ denote the level curve of u passing through $\underline{x} = (x, x)$, so that

$$\alpha_1 U(x_1) + \alpha_2 U(x_2(x_1)) = \alpha_1 U(x) + \alpha_2 U(x) \tag{3}$$

Differentiating twice with respect to x_1 yields

$$x_2'' = -\left[\frac{\alpha_1 U''(x_1) + \alpha_2 U''(x_2)(x_2')^2}{\alpha_2 U'(x_2)}\right] \tag{4}$$

In particular $x_2''(x) \leq 0$. By Lemma 1 therefore, u violates **A.3**.

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[**Q2**] Using this time Lemma 2, show once more that the Cobb-Douglas is (almost) a standard utility function.

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The problem of the consumer is straightforward to formulate on the basis of Definition 1. Let m denote the budget of a consumer with utility function u, and $\underline{p}$ the given vector of prices. A consumer choosing his most preferred affordable bundle of goods effectively solves

$$\max_{\underline{x} \geq 0} u(\underline{x}) \quad s.t. \quad \underline{p}.\underline{x} \leq m \tag{5}$$

Notice that if $p_i = 0$ for some good i then, by **A.2**, the consumer will demand an infinite amount of that good. We will therefore suppose throughout that $p_i > 0$, $\forall i$. Under this assumption, a solution to (5) always exists.[5] There is moreover a unique optimal consumption bundle $\underline{x}$. Indeed, if there were two we could take a weighted average of them and, owing to the strict quasi-concavity of u, obtain a strictly higher utility level. Note also that $\underline{p}.\underline{x} = m$ at the optimum, by **A.2**. We will say that the solution is *interior* if $x_i > 0$, $\forall i$.

Our first result embodies the following idea: prices establish the rate at which the consumer can trade goods against one another. So unless this rate reflects his own valuation of the goods the consumer always has an incentive to trade goods at the market rate. Thus, at an optimum, the consumer's *marginal rate of substitution* (MRS) must equal the price ratio. The proposition also provides a useful set of sufficient conditions to elicit the optimal consumption bundle.

Proposition 1 *Let $\underline{p} > 0$. Any interior solution to (5) satisfies*

$$\frac{\partial u/\partial x_i}{\partial u/\partial x_j} = \frac{p_i}{p_j} \qquad (i.e.\ MRS = price\ ratio) \tag{6}$$

[5] All so-called 'existence' results of microeconomics follow from standard results in topology. These results are however beyond the scope of these notes.

Moreover, if $\underline{x}$ satisfies (6) and $\underline{p}.\underline{x} = m$ then $\underline{x}$ solves the consumer problem.

Proof. Necessity of condition (6) is an immediate application of the Karush-Kuhn-Tucker Theorem (notice that we are here using the fact that the constraint is binding at an optimum).

Suppose next $\underline{x}$ satisfies (6), and $\underline{p}.\underline{x} = m$. By the Implicit Function Theorem the level curve of u at $\underline{x}$ is then tangent to the hyperplane $\underline{p}.\underline{x} = m$. But utility functions have convex level curves by Lemma 1. The level curve of u at $\underline{x}$ thus lies entirely above its tangent. This shows that $\underline{x}$ is optimal for the consumer problem.

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We record for future reference the following definitions:

Definition 3 *The optimal consumption bundle in (5) is called* Marshallian demand, *and denoted $\underline{x}(\underline{p}, m)$.*

Definition 4 *The maximum utility level attained in (5) is called* indirect utility, *and denoted $v(\underline{p}, m)$.*

Definition 5 *A good is said* normal[6] *iff the Marshallian demand for it satisfies $\partial x_i(\underline{p}, m)/\partial m \geq 0$.*

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[**Q3**] Suppose two goods are perfect complements for a given consumer. What functional form does this imply for this consumer's utility function? Comment. What if the two goods are perfect substitutes instead?

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[**Q4**] Show that Marshallian demand is homogenous degree zero in $(\underline{p}, m)$, and satisfies $\underline{p}.\underline{x} = m$. Show moreover that indirect utility is homogenous degree zero in $(\underline{p}, m)$, increasing in income, decreasing in prices, and quasiconvex in $(\underline{p}, m)$.

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[**Q5**] Let u a standard utility function and $v : \mathbb{R} \to \mathbb{R}$ smooth and strictly increasing. Apply (6) to $v(u(\underline{x}))$ instead of u. What do you observe? Comment.

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[**Q6**] Let $\alpha \in (0, 1)$ and consider a consumer with Cobb-Douglas utility function. Find the Marshallian demand functions and indirect utility function in this case. Show that both goods are normal goods for this consumer.

[6] The definition is slightly misleading since a good may be normal for one consumer but not for another.

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[Q7] Let $u(\underline{x}) = x_1 + \phi(x_2)$. Under what conditions is u a standard utility function? Assuming these conditions to be satisfied, find the Marshallian demand functions. Comment.

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Some careful inspection[7] will bring you to the observation that the solution to the consumer problem (5) is also the solution to a different – closely related – problem. We next develop this intuition.

Consider the following problem and definitions:

$$\min_{\underline{x} \geq 0} \underline{p}.\underline{x} \quad s.t. \quad u(\underline{x}) \geq u \tag{7}$$

Definition 6 *The optimal consumption bundle in (7) is called* Hicksian demand*, and denoted* $\underline{h}(\underline{p}, u)$*.*

Definition 7 *The minimum expenditure attained in (7) is called* expenditure function*, and denoted* $e(\underline{p}, u)$*.*

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[Q8] Show that the expenditure function is homogenous degree one in $\underline{p}$, increasing in u as well as in prices, and concave in $\underline{p}$.

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The *dual* formulation of consumer theory rests on the following, mirror, results:

Lemma 3

$$\underline{x}(\underline{p}, m) = \underline{h}(\underline{p}, v(\underline{p}, m)) \tag{8}$$

Proof. Let $u = v(\underline{p}, m)$. Note first that $e\big(\underline{p}, u\big) \geq m$. Indeed, suppose we can find $\underline{x}$ such that $u(\underline{x}) \geq u$ all the while $\underline{p}.\underline{x} < m$. Since $\nabla_i u > 0$, $\forall i$, we can also find $\underline{y}$ close to $\underline{x}$ such that $u(\underline{y}) \geq u$ and $\underline{p}.\underline{y} < m$. But this contradicts $u = v(\underline{p}, m)$. So $e\big(\underline{p}, u\big) \geq m$.

Note also that $e\big(\underline{p}, v(\underline{p}, m)\big) \leq m$ since we have in particular $\underline{p}.\underline{x}(\underline{p}, m) = m$, while $u(\underline{x}(\underline{p}, m)) = u$. Hence $e\big(\underline{p}, u\big) = m$

The result follows since, once again, $\underline{p}.\underline{x}(\underline{p}, m) = m$. ■

Lemma 4

$$\underline{h}(\underline{p}, u) = \underline{x}(\underline{p}, e(\underline{p}, u)) \tag{9}$$

[7] Though admittedly mixed with a fair amount of intuition.

Proof. Let $m = e(\underline{p}, u)$. Note first that $v(\underline{p}, m) \leq u$. Indeed, suppose we can find $\underline{x}$ such that $u(\underline{x}) > u$ and $\underline{p}.\underline{x} \leq m$ then we could also find $\underline{y}$ close to $\underline{x}$ such that $u(\underline{y}) > u$ and $\underline{p}.\underline{y} < m$. But this implies $e(\underline{p}, u) < m$, a contradiction. So $v(\underline{p}, m) \leq u$.

But we also have $v(\underline{p}, m) \geq u$, since $u\big(\underline{h}(\underline{p}, u)\big) = u$ while $\underline{p}.\underline{h}(\underline{p}, u) = e(\underline{p}, u) = m$. Hence $v(\underline{p}, m) = u$.

The result now follows, since like we said already $u\big(\underline{h}(\underline{p}, u)\big) = u$ and $\underline{p}.\underline{h}(\underline{p}, u) = e(\underline{p}, u) = m$.

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Corollary 1

$$m = e\big(\underline{p}, v(\underline{p}, m)\big) \tag{10}$$

Proof. This was proven within the proof of lemma 3.

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Corollary 2

$$u = v\big(\underline{p}, e(\underline{p}, u)\big) \tag{11}$$

Proof. This was proven within the proof of lemma 4.

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[**Q9**] Consider a consumer with Cobb-Douglas preferences. Compute the Hicksian demand and expenditure functions in this case.

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The dual formulation is somewhat more than a simple theoretical curiosity. It is useful for at least two reasons. It first allows us to illustrate income and substitution effects, and opens the way to comparative statics exercises. Second, and maybe more importantly, it suggests an ingenious way of providing a unified treatment of consumption and production.

Leave production aside for now. Our second result embodies the following idea: An increase in the price of good 1 makes good 1 relatively more expensive compared to other available goods. In addition, it reduces the purchasing power of the consumer. These effects – *substitution effect* on the one hand, *income effect* on the other – are essentially distinct effects. The slutsky equation establishes this distinction formally and confirms that in so far as normal goods are concerned, the total effect on demand of an own-price increase is negative.

Proposition 2 **[Slutsky Equation]** *Suppose all goods are normal goods.*[8] *For all i, j:*

$$\frac{\partial x_i(\underline{p}, m)}{\partial p_j} = \frac{\partial h_i(\underline{p}, v(\underline{p}, m))}{\partial p_j} - x_j(\underline{p}, m).\frac{\partial x_i(\underline{p}, m)}{\partial m} \tag{12}$$

In particular if $i = j$ then both sides in (12) are negative. Moreover, $\partial v(\underline{p}, m)/\partial p_i < 0$.

Proof. By (9), and using the chain rule

$$\frac{\partial h_i(\underline{p}, u)}{\partial p_j} = \frac{\partial x_i(\underline{p}, e(\underline{p}, u))}{\partial p_j} + \frac{\partial x_i(\underline{p}, e(\underline{p}, u))}{\partial m}.\frac{\partial e(\underline{p}, u)}{\partial p_j} \tag{13}$$

By the Envelope Theorem, we also have

$$\frac{\partial e(\underline{p}, u)}{\partial p_j} = h_j(\underline{p}, u) \tag{14}$$

This establishes (12).

Moreover, from (14):

$$\frac{\partial h_i(\underline{p}, u)}{\partial p_i} = \frac{\partial^2 e(\underline{p}, u)}{\partial p_i^2} \tag{15}$$

But we have shown in **[Q3]** that $e(\underline{p}, u)$ is concave in $\underline{p}$. Hence $\frac{\partial h_i(\underline{p}, u)}{\partial p_i} < 0$ and, if goods are normal, both sides in (12) are negative.

That $dv(\underline{p}, m(\underline{p}))/dp_i < 0$ is immediate since an increase in prices strictly reduces the set of goods affordable.

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[Q10] Show that the substitution effect following an own-price increase is always negative.

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The general framework we have developed remains silent on the source of the consumer's budget m. As we will later see when using this framework for practical purposes, a recurrent scenario is one in which[9] $m = m(\underline{p})$, i.e. in which wealth itself is a function of prices. In the canonical example for instance, the consumer is originally endowed with a vector of goods $\underline{e}$,

[8] Observe that the Slutsky Equation holds for all goods, normal or not. The sign however of the total effect of an own-price increase is determined only if the good is normal.

[9] Notice that $m = m(p)$ allows for the possibility that $\partial m/\partial p = 0$. So the truly general framework is the one in which $m = m(\underline{p})$. I follow here however common practice and treat this case as an extension.

so that he is both a seller and buyer of the goods he consumes. Given prices $\underline{p}$ then: $m = \underline{p}.\underline{e}$. The consumer is a net buyer of good i iff $x_i > e_i$, and a net seller otherwise.

Naturally, extending the framework in this way creates important distortions regrading the effect of price changes and we must look beyond the Slutsky equation. It is in particular no longer generally true that the total effect on demand of an own-price increase is negative. If the consumer is a net seller of the good whose price increases he may benefit so much that he actually ends up raising his consumption of that good. In a similar vein, the consumer need not be worse off following a price increase. Indeed if he is a net seller of the good whose price increases then his old consumption bundle remains affordable, and he will even be left with a little cash in hand. That consumer is thus in fact unambiguously better off. Our next proposition formalizes these ideas.

Proposition 3 *Suppose all goods normal and consider a consumer with endowment* $\underline{e}$*, so that* $m = \underline{p}.\underline{e}$*. Then, for all* i*:*

1. $\frac{dx_i(\underline{p},m(\underline{p}))}{dp_i} < 0$ *if* $x_i > e_i$
2. $\frac{dv(\underline{p},m(\underline{p}))}{dp_i} < 0$ *if* $x_i > e_i$*,* $\frac{dv(\underline{p},m(\underline{p}))}{dp_i} > 0$ *if* $e_i > x_i$

Proof. Using the chain rule, we obtain the total derivative:

$$\frac{dx_i(\underline{p},m(\underline{p}))}{dp_i} = \frac{\partial x_i(\underline{p},m(\underline{p}))}{\partial p_i} + \frac{\partial x_i(\underline{p},m(\underline{p}))}{\partial m}\frac{\partial m(\underline{p})}{\partial p_i} \tag{16}$$

We then obtain by (12) and the fact that $\frac{\partial m(\underline{p})}{\partial p_i} = e_i$:

$$\frac{dx_i(\underline{p},m(\underline{p}))}{dp_i} = \left[\frac{\partial h_i(\underline{p},v(\underline{p},m(\underline{p})))}{\partial p_i} - x_i(\underline{p},m(\underline{p})).\frac{\partial x_i(\underline{p},m(\underline{p}))}{\partial m}\right] + \frac{\partial x_i(\underline{p},m(\underline{p}))}{\partial m}.e_i \tag{17}$$

And, rearranging:

$$\frac{dx_i(\underline{p},m(\underline{p}))}{dp_i} = \frac{\partial h_i(\underline{p},v(\underline{p},m(\underline{p})))}{\partial p_i} - \frac{\partial x_i(\underline{p},m(\underline{p}))}{\partial m}\left[x_i(\underline{p},m(\underline{p})). - e_i\right] \tag{18}$$

This concludes the first part of the proof since, as already indicated in the proof of Proposition 2, $\frac{\partial h_i(\underline{p},u)}{\partial p_i} < 0$.

Suppose next $e_i > x_i$. A rise in the price of good i leaves consumption bundle $\underline{x}$ strictly affordable. Thus $\frac{dv(\underline{p},m(\underline{p}))}{dp_i} > 0$ in this case.

Finally, suppose $x_i > e_i$. Let $u = v(\underline{p}, m(\underline{p}))$ and assume for a contradiction that $\frac{dv(\underline{p}, m(\underline{p}))}{dp_i} \geq 0$, i.e. assume that the consumer remains able to reach level curve u following a rise in the price of good i. By revealed preference the new consumption bundle must lie 'to the left' of $\underline{e}$. But $x_i > e_i$, which implies that the demand for good i must be making a discontinuous jump. This is impossible, owing to the continuity of demand (which itself is a consequence of the Maximum Theorem).

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2.2 Applications

2.2.1 Intertemporal Consumption

Consider a consumer with a two periods horizon, x_i his consumption in period i, and $U(x)$ his (ex post) utility from consuming quantity x.[10] We are interested in the consumer's ex ante preferences over profiles of consumption over time $\underline{x}$, represented by the (ex ante) utility function u. While there are in principle infinitely many ways in which to define u, the time-discount assumption is by far the most convenient to work with.[11] The time-discount assumption sets $u(\underline{x}) = U(x_1) + \delta U(x_2)$, where $\delta \leq 1$.

Let y_i denote income received by the consumer in period $i = 1, 2$. If the consumer is able to freely borrow and lend at the market interest rate r, his budget constraint is $x_2 \leq y_2 + (1+r)(y_1 - x_1)$: he can in period 2 consume his income from period 2 plus capital and interest from any saving he has from period 1. Letting $\underline{p} = (1, \frac{1}{1+r})$ we can then formulate the consumer problem as

$$\max_{\underline{x} \geq 0} u(\underline{x}) \quad s.t. \quad \underline{p}.\underline{x} \leq \underline{p}.\underline{y} \tag{19}$$

Note that in case borrowing constraints apply we then have the additional condition that $x_1 \leq y_1$.

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[**Q11**] Consider a consumer with a two periods horizon receiving income y_i in period $i = 1, 2$, and such that $u(\underline{x}) = U(x_1) + \delta U(x_2)$. Show that consumption is greater in period 2 than in period 1 iff $\delta > (1+r)^{-1}$. Can you say anything concerning the effect of changing the interest

[10] We suppose here that this utility is time-independent.

[11] While extremely convenient analytically, bear in mind that the time-discount assumption is far from innocuous. Indeed, it supposes important restrictions on consumers' underlying preferences. The fact for instance that cross-effects are excluded from the framework ($\partial^2 u/\partial x_1 \partial x_2 = 0$) is hard to reconcile with certain aspects of human behavior such as habits.

rate on saving? What if the utility function has the Cobb-Douglas functional form?

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[**Q12**] Consider a consumer with a two periods horizon receiving income y_i in period $i = 1, 2$. Suppose we know that if the consumer can freely borrow and lend he consumes $x_1 > y_1$. Find that consumer's optimal consumption profile if now he can freely lend but cannot borrow money. What if $x_1 < y_1$ instead?

2.2.2 Consumption under Uncertainty

Uncertainty is conveniently illustrated by means of bets, or lotteries. A bet B with n possible outcomes is a vector-pair $(\underline{p}, \underline{b})$ where $\underline{p}$ is the vector of probabilities and $\underline{b}$ the associated vector of contingent gains. In particular $\mu(B)$, the expected gain of bet B, is then given by $\underline{p}.\underline{b}$.

Definition 8 *A bet is said to be* fair *iff it entails an expected gain/loss of* 0*, i.e.* $\mu(B) = \underline{p}.\underline{b} = 0$*.*

Let $\underline{\mathbf{1}}$ denote the vector with all entries equal to 1. It is sometimes convenient to think of B as composed of a sure gain $\mu(B)$ on the one hand and a fair bet B_0 on the other hand, where $\underline{b}_0 = \underline{b} - \mu(b)\underline{\mathbf{1}}$. The attractiveness of this representation lies in the fact that we thereby separate the 'gain component' of B from the 'risk component' of B. A risk-averse consumer for instance is one who would ideally like to part with the 'risk component' of B.

Definition 9 *A consumer is said to be* risk-averse *iff to any bet* $B = (\underline{p}, \underline{b})$ *he prefers the sure gain* $\mu(B)$*.*

The following definition is useful in the context of insurance, a theme we later investigate.

Definition 10 *Let* $B = (\underline{p}, \underline{b})$*. The* hedge *of* B *is the bet* $\overline{B}$ *such that* $\overline{B} = (\underline{p}, -\underline{b})$*,* $\overline{B}$ *yielding* $-b_i$ *whenever* B *yields* b_i*.*

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[**Q13**] Consider a risk-averse consumer facing bet B, where $\mu(B) < 0$. Show that $\overline{B}$ is worth strictly more than $|\mu(\overline{B})|$ for that consumer. Comment.

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We next introduce a very central concept in the study of consumption under uncertainty.

Definition 11 *Consider an uncertain environment with n contingencies and associated probabilities $(p_i)_{1\leq i\leq n}$. Let x_i represent consumption under contingency i and u the utility function representing consumer preferences over vectors of contingent consumption. A consumer is an* expected-utility maximizer *iff*[12] *there exists a function U such that $u(\underline{x}) = \sum_i p_i U(x_i)$.*

The *expected-utility assumption* is quite compelling. However, its greatest credential lies in the extent to which it simplifies the analysis of most problems. [13] In particular, while risk-aversion in no way invokes expected-utility, the latter phenomena takes a remarkably simple form under the expected-utility assumption. Indeed, the following Lemma establishes that, in the expected-utility model, risk-averse consumers are precisely those with concave ex post utility function U.

Lemma 5 *Consider an expected-utility maximizing consumer, and let U denote his ex post utility function. The consumer is risk-averse iff U is concave.*

Proof. If U is concave then the consumer is risk-averse by Jensen's Inequality.

If U is not concave then we can find y, z, and λ such that $U(\lambda y + (1-\lambda)z) < \lambda U(y) + (1-\lambda)U(z)$. But then consider bet $B = (\underline{p}, \underline{b})$ where $\underline{p} = (\lambda, 1-\lambda)$ and $\underline{b} = (y, z)$. We have $u(B) = \lambda U(y) + (1-\lambda)U(z) > U(\lambda y + (1-\lambda)z) = u(I_{\mu(B)})$, where $I_{\mu(B)}$ denotes the bet in which the consumer receives $\mu(B)$ for sure. This shows that the consumer is not risk averse. ■

It is noteworthy to underline in lieu of concluding remark that Lemma 5, together with Lemma 2, show that in the context of expected-utility the general framework we developed in section 2.1 is, in fact, precisely a model of risk-aversion.

.

[**Q14**] Consider an expected-utility maximizing consumer. Let U his ex post utility and m his wealth. Show that the set of 'small' bets he is willing to make expands according to $U''(m)/U'(m)$.

.

[12]In this definition $\underline{p}$ is treated as a parameter. Of course, $\underline{p}$ may be a variable of the problem under study. The interested reader is referred to Section 8.1.

[13]While extremely convenient analytically, bear in mind that the expected-utility assumption is far from innocuous. Indeed, it supposes important restrictions on consumers' underlying preferences. The Allais Paradox in particular indicates that human behavior is to some extent irreconcilable with the expected-utility assumption. Section 8.1 elaborates on this important point.

Application to the market for insurance:

We saw in Q13 that risk-averse consumers are precisely those willing to pay a premium to hedge themselves against uncertain outcomes. An insurance company is just another firm whose business model is to take advantage of this observation. We develop here the simplest possible model of insurance.

Consider a consumer (or insuree), with initial wealth M, facing the risk of an accident occurring with probability q.[14] If an accident occurs the consumer loses $L \leq M$, otherwise nothing happens. An insurance company offers to insure the consumer for a unit price p. In other words, it offers for a price pK a contract specifying that the insurance company commits to pay the consumer amount K in case an accident occurs.

The object of what follows is to show that the problem of the consumer, namely choosing K, is in fact nothing more than a special case of the general problem we examined at length in section 2.1.

Let x_1 denote consumption in case of no accident and x_2 denote consumption in case of accident. We will start by showing that choosing K is equivalent to choosing $\underline{x} = (x_1, x_2)$ satisfying

$$(1-p)x_1 + px_2 \leq M - pL \tag{20}$$

If the consumer purchases K units of insurance he is left with $M - pK$ if no accident takes place, i.e. he can choose $x_1 \leq M - pK$. If an accident occurs on the other hand he will end up with $M - pK - L + K = M - L + (1-p)K$. He can thus choose to consume $x_2 \leq M - L + (1-p)K$. Summing a weighted average of these inequalities yields $(1-p)x_1 + px_2 \leq (1-p)(M - pK) + p(M - L + (1-p)K) = M - pL$. But note that this is exactly (20).

Conversely, suppose that the consumer chooses (x_1^*, x_2^*) satisfying (20) and let us show that he can achieve this outcome with an appropriate choice of K. Let $K^* = \frac{M - x_1^*}{p}$, and check that this choice of K works. In case of no accident the consumer is left with $M - pK^* = M - p\frac{M - x_1^*}{p} = x_1^*$, as desired. In case of an accident he ends up with $M - L + (1-p)K^* = M - L + (1-p)\frac{M - x_1^*}{p}$. He can thus consume $x_2 \leq \frac{1}{p}M - L - \frac{1-p}{p}x_1^*$. But notice that the RHS of this inequality is at most x_2^* since by assumption (x_1^*, x_2^*) satisfies (20).

Since we have now shown that choosing K is equivalent to choosing $\underline{x} = (x_1, x_2)$ satisfying (20), this also means that the insurance problem can be stated as $\max_{\underline{x} \geq 0} u(\underline{x})$ such that (20) holds. But notice that (20) can be written as $\underline{p}.\underline{x} \leq \underline{p}.\underline{e}$, where $\underline{p} = (1-p, p)$ and

[14] We suppose here the accident probability given and, in particular, independent of the behavior of the insuree. This circumvents the (important) topic of *moral hazard*.

$\underline{e} = (M, M-L)$. In other words, an equivalent formulation of the insurance problem is:

$$\max_{\underline{x}\geq 0} u(\underline{x}) \quad s.t. \quad \underline{p}.\underline{x} \leq \underline{p}.\underline{e} \tag{21}$$

The insurance problem can thus be framed as a standard consumer problem in which (i) the consumer's initial 'endowments' are M in good times and $M-L$ in bad times and (ii) the consumer can trade consumption in good times for consumption in bad times at rate $(1-p)/p$.[15]

We conclude with the following useful definitions:

Definition 12 *An insurance policy is* fair *iff it entails a fair bet.*

Definition 13 *A consumer is said to be* fully insured *iff he bears no more risk, i.e.* $x_i = x_j$ *for all* i *and* j*.*

.

[**Q15**] Show that a risk-averse consumer always chooses full insurance whenever insurance is fair, i.e. whenever $p = q$. Show moreover that the converse holds under the expected-utility assumption (i.e. the consumer does not choose full insurance if $p \neq q$).

.

.

[**Q16**] Consider an expected-utility maximizing risk-averse consumer with logarithmic ex-post utility function, and a risk-neutral insurance company. Suppose the consumer's wealth is M but, with probability q he will lose $L < M$. What is the set of pareto optimal contracts between consumer and insurer? Which of these are acceptable to both parties. Which one of these entails a fair insurance policy?

.

2.2.3 Labour Supply

[**Q17**] Suggest a way in which to approach labour supply using the general framework developed in section 2.1. Can you say anything concerning the effect of changing wages on labour supply?

[15] i.e. give up p in good times and receive $1-p$ in bad times.

3 The Producer

We have in section 2 developed a theory of demand based on prices and consumers' preferences. This sections aims to outline the way in which a similar exercise can be carried out in view of developing a theory of supply based on prices and firms' technological constraints. One of the great qualities of the consumer model developed in these notes resides in the ease with which its formalism can be transferred over to production, thus yielding largely isomorphic approaches to consumption on the one hand and production on the other. This section aims to elicit this isomorphism.

The starting point of a theory of production is the *production function* f such that to input vector $\underline{z}$ corresponds output $y = f(\underline{z})$. It will once again prove useful to make certain assumptions concerning production functions. We summarize these below:

Assumption A.1': f *is* C^∞.

Assumption A.2': $\nabla_i f > 0$, $\forall i$.

Assumption A.3': f *is strictly concave.*

The problem of profit maximization given output price p and input prices $\underline{w}$ is

$$\max_{(y,\underline{z})\geq 0} py - \underline{w}.\underline{z} \quad s.t. \quad f(\underline{z}) \geq y \tag{22}$$

Observe that a solution $(y, \underline{z})$ to (22) is such that $\underline{z}$ minimizes the cost of producing y, i.e. solves

$$\min_{\underline{z}\geq 0} \underline{w}.\underline{z} \quad s.t. \quad f(\underline{z}) \geq y \tag{23}$$

But note that (23) is precisely the dual formulation of the consumer problem (7) in which $\underline{x} \leftrightarrow \underline{z}$, $u(.) \leftrightarrow f(.)$, $\underline{p} \leftrightarrow \underline{w}$, and $u \leftrightarrow y$. Formally speaking the problem of the consumer and that of the producer are thus interchangeable. Of course since in the case of the consumer it is the primal formulation which bears economic significance, while in the case of the producer it is the dual formulation, part of the analysis undertaken for the consumer looses relevance when carried over to the producer. Consider Slutsky's equation for instance, eliciting changes in demand for a given level of expenditure. Now, in the case of production, expenditure is a mere variable of the problem: Indeed firms care about profits, not costs. So Slutsky's equation is of very little practical importance when discussing production.

4 General Equilibrium I

We developed in section 2 a theory of demand based on prices and consumers' preferences, and in section 3 a theory of supply based on prices and firms' technological constraints. In both cases prices were taken as parameters of the problem under study. In reality of course prices are endogenously determined, and the only parameters of the economy consist in consumers' preferences and firms' technological constraints. General Equilibrium (GE) is the theory of prices. It seeks to determine which prices will prevail in the economy and, as a consequence, which quantities of each goods will be produced and consumed.

It will be useful to think of an economy as a *dynamical system*. A dynamical system (Ω, f) consists of a state space Ω and a process f describing the evolution of the system, such that if $\underline{x}_t \in \Omega$ denotes the state of the system in period t then

$$\underline{x}_{t+1} = f(\underline{x}_t) \tag{24}$$

One of the most important concepts in the study of dynamical systems is that of an equilibrium. An equilibrium of (Ω, f) is a state which, once reached, the system remains in indefinitely. Observe from (24) that the set of equilibria thus coincides with the set of fixed points of the process f, i.e. the set of points such that $f(\underline{x}) = \underline{x}$.

It is very compelling to think of the economy as a dynamical system in which prices determine the state. Somewhat peculiarly however, standard economic theory remains largely silent regarding the process describing the evolution of the system. Instead, the theory characterizes equilibria *at the outset*, avoiding at that stage any reference to the underlying dynamic process. It only later demonstrates – as a consistency check, in a sense – that the former equilibria in fact coincide with those of a well-defined dynamical process. We will follow this practice here.

We will in these notes limit ourselves to the simplest form of GE models. While supply in general results from production, we will take it to be fixed exogenously. Such models are usually called *pure exchange* models.

A pure exchange model[16] begins with an endowment point $E = ({}^A\underline{e},\ {}^B\underline{e})$. At the outset, agent A (B) is endowed with goods-vector ${}^A\underline{e}$ (${}^B\underline{e}$). Such models have the great advantage of introducing supply and tying it up with consumers' wealth in a very simple and convenient way. Indeed, total supply is ${}^A\underline{e} +\ {}^B\underline{e}$, while the wealth of agent A (B) is given by $\underline{p}.{}^A\underline{e}$ ($\underline{p}.{}^B\underline{e}$).

[16]We limit ourselves here to two consumers in order to shorten and simplify the exposition.

A *market equilibrium* is characterized by (i) prices $\underline{p}$ such that aggregate demand equals aggregate supply, and (ii) a consumption profile for each one of the consumers. Formally:

Definition 14 *Let* $X = ({}^A\underline{x}, {}^B\underline{x})$. *The pair* $(X, \underline{p})$ *is a* market equilibrium *for the pure exchange economy with endowment* $E = ({}^A\underline{e}, {}^B\underline{e})$ *if and only if* ${}^A\underline{x} + {}^B\underline{x} = {}^A\underline{e} + {}^B\underline{e}$ *and each consumer's consumption profile maximizes his own utility function given prices* $\underline{p}$.

Let ${}^j\underline{x}(\underline{p})$ denote the Marshallian demand of consumer j. The vector $\sum_j^j \underline{x}(\underline{p}) - \sum_j {}^j\underline{e}$ has an important economic interpretation. It represents the *aggregate excess demand vector* given prices $\underline{p}$. We will use $\underline{z}(\underline{p})$ for a shorthand, so that

$$\underline{z}(\underline{p}) = \sum_j^j \underline{x}(\underline{p}) - \sum_j {}^j\underline{e} \tag{25}$$

The following property is absolutely key. It states that the value of the aggregate excess demand vector is always 0:

$$\underline{z}(\underline{p}).\underline{p} = 0 \tag{26}$$

.

[**Q18**] Prove (26).

.

Observe, using (25), that a vector of prices $\underline{p}$ is an equilibrium price vector if and only if $\underline{z}(\underline{p}) = 0$. We will now show that the set of prices such that $\underline{z}(\underline{p}) = 0$ in fact corresponds to the equilibrium set of a well-defined dynamical system.

For technical reasons (that we will not dwell onto) we will, for the purposes of our next lemma, truncate the consumers' Marshallian demands and set:

$${}^j\widetilde{x}_i = {}^jx_i I_{\{{}^jx_i \leq {}^je_i + \epsilon\}} + ({}^je_i + \epsilon) I_{\{{}^jx_i > {}^je_i + \epsilon\}} \tag{27}$$

These truncated demands have the great advantage of being well-defined when prices are 0. It is moreover easy to show that a property corresponding to (26) holds, i.e. $\underline{\widetilde{z}}(\underline{p}).\underline{p} = 0$, for all $\underline{p}$.[17] Lastly, notice that $p_i = 0$ implies $\widetilde{z}_i > 0$.[18]

[17]Each consumer must exhaust his budget constraint. If he does not, this implies that we can find a good i such that $p_i > 0$ and $x_i < e_i$. But then, by (27), that consumer is not behaving optimally.

[18]If $p_i = 0$ then by (27) each consumer demands strictly more of that good than he his endowed with. There is thus strict excess demand for that good.

Lemma 6 *Let Σ denote the n-dimensional simplex,*[19] *and $g : \Sigma \to \Sigma$ such that*

$$g_i(\underline{p}) = \frac{p_i + \max\{0, \widetilde{z}_i(\underline{p})\}}{\sum_j (p_j + \max\{0, \widetilde{z}_j(\underline{p})\})} \tag{28}$$

Then, $g(\underline{p}) = \underline{p}$ if and only if $\widetilde{\underline{z}}(\underline{p}) = 0$.

Proof. That $\widetilde{\underline{z}}(\underline{p}) = 0$ implies $g(\underline{p}) = \underline{p}$ is immediate. We now show the converse. Suppose, for a contradiction, that $g(\underline{p}) = \underline{p}$ while $\widetilde{\underline{z}}(\underline{p}) \neq 0$. By (28), this implies that $\max\{0, \widetilde{z}_i(\underline{p})\} = kp_i$ for some $k \geq 0$, and all i. In particular, one of the following must hold:

1. $\widetilde{z}_i \leq 0$, $\forall i$, with strict inequality for some i
2. $\widetilde{z}_i > 0$, $\forall i$, and $p_i > 0$, $\forall i$
3. $\exists i$ such that $\widetilde{z}_i \leq 0$ and $p_i = 0$

We claim that none of the above can hold. Suppose (1.) holds. Given $\widetilde{\underline{z}}.\underline{p} = 0$ this implies that there exists i such that $\widetilde{z}_i < 0$ while $p_i = 0$. But this is impossible, by a former remark. This also shows that (3.) cannot hold. Finally, (2.) cannot hold since $\widetilde{\underline{z}}.\underline{p} = 0$.

■

Lemma 6 is a central result. It shows that a vector of prices $\underline{p}$ is a (market) equilibrium price vector if and only if it is an equilibrium state of the dynamical system (Σ, g). But given that Σ is a compact of $\mathbb{R}^n$, and g is continuous, we know that the dynamical system (Σ, g) has at least one equilibrium.[20] It ensues that a market equilibrium does exist.[21]

.

[**Q19**] Let $X = ({}^A\underline{x},\ {}^B\underline{x})$, $E = ({}^A\underline{e},\ {}^B\underline{e})$, and $\underline{p}$ such that $p_i > 0$ for all i. Suppose ${}^A\underline{x}$ (${}^B\underline{x}$) maximizes agent A's (B's) utility given prices $\underline{p}$ and wealth $\underline{p}.{}^A\underline{e}$ ($\underline{p}.{}^B\underline{e}$). Suppose moreover that ${}^Ax_i +\ {}^Bx_i =^A e_i +\ {}^Be_i$ in all but possibly one market. Show that $(X, \underline{p})$ is a market equilibrium for the pure exchange economy with endowment E.

.

[**Q20**] Consider a pure exchange economy, two consumers (A and B), two goods (1 and 2). Suppose consumer A has Cobb-Douglas utility function, while the two goods are perfect complements for consumer B. Characterize market equilibrium if (i) A initially owns 10 units of

[19] i.e. $\Sigma = \{\underline{x} \in \mathbb{R}^n_+ | \sum_i x_i = 1\}$.
[20] This is a consequence of Brouwer's fixed point theorem.
[21] More precisely, lemma 6 shows that a market equilibrium exists given truncated demands a la (27).

good 1 and B 10 units of good 2 and (ii) A initially owns 10 units of good 2 and B 10 units of good 1. What is the set of pareto optimal allocations?

.

Having formally defined market equilibria, we now aim to explore some of the properties these equilibria have. The welfare theorems are the central results of GE. Schematically, their message is the following. Suppose S denotes the set of possible states of the economy, and $P \subset S$ the subset of states which are also pareto optimal. Let $s_0 \in S$ an arbitrary state, and s_b the social planner's preferred state. No matter how intricate the social planner's objective, the assumption that $s_b \in P$ is certainly compelling. The First Welfare Theorem states that starting at s_0 and from there letting markets operate, the economy will eventually end in P.[22] The Second Welfare Theorem states that if the social planner somehow manages to stir the economy to s_b, he can then let markets operate and rest assured that the economy will remain in that state. Somewhat ironically the welfare theorems are often perceived of as supporting arguments in favor of markets.

The First Welfare Theorem rests on the following observation: Prices fix the universal rate at which goods trade against one another and thereby also 'divide up' the set of feasible allocations among consumers. In equilibrium, each consumer chooses his preferred allocation within his subset. Any improvement for one consumer must therefore lie in another consumer's subset and must, thereby, also be worse for that consumer owing to the fact that he chose to forego that allocation. [23]

The formal proof of the theorem requires some additional notation, which we now introduce. Let $F(E)$ denote the set of feasible consumption points[24] given endowment point E, i.e. $F(E) = \{X = ({}^A\underline{x},\ {}^B\underline{x}) :\ {}^A x_i +\ {}^B x_i =\ {}^A e_i +\ {}^B e_i,\ \forall i\}$. We call $F(E)$ the *Edgeworth box*.

Let also ${}^A F(\underline{p}, E) = \{X \in F(E) : \underline{p}.{}^A\underline{x} \leq \underline{p}.{}^A\underline{e}\}$, so that ${}^A F(\underline{p}, E)$ is the subset of feasible consumption points affordable to consumer A under prices $\underline{p}$. Define ${}^B F(\underline{p}, E)$ similarly.

Lemma 7 *$F(E) =\ {}^A F(\underline{p}, E) \bigcup\ {}^B F(\underline{p}, E)$, for all $\underline{p}$.*

Proof. Let $X \in F(E) -\ {}^A F(\underline{p}, E)$. Then

[22] The Theorem remains silent however on *how* it will get from s_0 to P.

[23] By definition of a market equilibrium, each agent maximizes his own utility given prices. There are thus no additional (mutual) gains from trade to be made at those prices. The first welfare theorem notes that there are in fact no (mutual) gains from trade to be made at any prices.

[24] Strictly speaking, there are other feasible consumption points, namely those such that ${}^A x_i +\ {}^B x_i <\ {}^A e_i +\ {}^B e_i$ for some i. But these are not really interesting as they suppose that some resources go to waste. In particular, it is cleat that none of these can be Pareto optimal.

$\underline{p}.^B\underline{x} = \underline{p}.(^A\underline{e} + \ ^B\underline{e} - \ ^A\underline{x}) = \underline{p}.(^A\underline{e} - \ ^A\underline{x}) + \underline{p}.^B\underline{e} < \underline{p}.^B\underline{e}$, i.e. $X \in \ ^BF(\underline{p}, E)$.

■

Proposition 4 **[First Welfare Theorem]** *A market equilibrium is pareto optimal.*

Proof. Let E the endowment point, $\underline{p}$ the equilibrium price vector, and X the equilibrium consumption point. Suppose a feasible Y pareto dominates X, and suppose wlog that consumer A strictly prefers Y to X. By definition of a market equilibrium $Y \notin \ ^AF(\underline{p}, E)$, and so by Lemma 7: $Y \in \ ^BF(\underline{p}, E)$. But this too is impossible, owing again to the definition of market equilibrium.

■

A last word of wisdom concerning the First Welfare Theorem. Beware the following fallacious reasoning: 'in equilibrium each consumer maximizes his own utility so there are by definition no more gains from trade, and the First Welfare Theorem is a mere tautology'. The fault in this statement lies in the fact that there are by definition no more gains from trade *at the equilibrium prices*. On the other hand what the First Welfare Theorem establishes is that there are in fact no more gains from trade *at any prices*.

.

[**Q21**] Argue that the reasoning used in the proof of the First Welfare Theorem collapses in the presence of externalities.

.

Proposition 5 **[Second Welfare Theorem]**

Let $X \in F(E)$ a pareto-optimal consumption point. There exists $\underline{p}$ such that, in the pure exchange economy with endowment $X, (X, \underline{p})$ is a market equilibrium.

Proof. Let $X = (^A\underline{x}, \ ^B\underline{x})$.

Let $^AL = \{Y = (^A\underline{y}, \ ^B\underline{y}) \in F(E) : \ ^Au(^A\underline{y}) \geq \ ^Au(^A\underline{x})\}$, and $^BL = \{Y = (^A\underline{y}, \ ^B\underline{y}) \in F(E) : \ ^Bu(^B\underline{y}) \geq \ ^Bu(^B\underline{x})\}$.

Note first that by quasi-concavity of utility functions both sets are convex. We also have $^AL \bigcap \ ^BL = \{X\}$. Indeed, if $Y \in F(E) - \{X\}$ belongs to $^AL \bigcap \ ^BL$, we can take a convex combination of X and Y and contradict the pareto-optimality of X. So ^AL and ^BL are closed, convex sets with a singleton intersection. The statement of the proposition is thus an immediate consequence of the Hyperplane Theorem.

■

.

[**Q22**] Suppose $(X, \underline{p})$ is a market equilibrium for the pure exchange economy with endowment E. Can there exist an endowment point E' and a price vector $\underline{p}' \neq \underline{p}$ such that $(X, \underline{p}')$ is a market equilibrium for the pure exchange economy with endowment E'? What is the set of endowment points E' such that $(X, \underline{p})$ is a market equilibrium for the pure exchange economy with endowment E'.

.

5 Financial assets

The analysis of general equilibrium developed in section 4 wholly disregards the issue of time. In real life however, trade and consumption take place over time. At any point in time therefore, individuals must decide whether to borrow or save – all the while choosing what to consume at that point in time. The key concept in this respect is that of a *financial asset*. This section introduces financial assets, and sets the stage for section 6 in which we revisit GE with a view to incorporate temporal aspects.

We will throughout this section and the next consider $T + 1$ time periods, with $t = 1$ representing 'today' and $t > 1$ representing future time periods. We further suppose that for each $t > 1$ there are S possible contingencies, representing uncertainty about the future. A pair (s, t) determines a *state of the world*. Let W denote the set of all states of the world. For expository purposes, it is useful to distinguish today's state of the world from future states. We will let W^{-1} denote the set of all future states.

A financial asset is determined by a vector $\underline{a} \in \mathbb{R}^{ST}$, specifying one payout for each possible future state of the world. We suppose that the set of tradable assets is $\{\underline{a}^i\}_{i=1}^{I}$, so that I denotes the total number of tradable assets. Let A denote the $ST \times I$ matrix whose columns represent all tradable assets. We refer to A as the *asset structure*. We will henceforth make the assumption that $I = ST$. [25]

A *portfolio* $\underline{\varphi} \in \mathbb{R}^I$ is a combination of tradable assets, such that φ_i indicates the quantity of asset i contained in the portfolio (note that we allow $\varphi_i < 0$, $\forall i$). The portfolio $\underline{\varphi}$ therefore induces payout vector $A\underline{\varphi}$ in future states. We let $\underline{q}$ denote the price vector of the tradable assets at $t = 1$, so that, in particular, $\underline{q}.\underline{\varphi}$ is the period 1 price of portfolio $\underline{\varphi}$.

[25] The crucial assumption here is $I \geq ST$. If $I < ST$ then $rank(A) < ST$, which most of the results derived in this section preclude. The assumption that I actually equals ST is for simplification only.

If the number of assets is large, different portfolios may induce identical payout vectors. For a long time, one of financial traders' main role was to seek out arbitrage opportunities between such portfolios. Nowadays this is all computerized. There are as a result, at any point in time, approximately no arbitrage opportunities remaining. Formally:

Definition 15 *The* no arbitrage condition *states:*

$$A\underline{\varphi}^1 = A\underline{\varphi}^2 \Rightarrow \underline{q}.\underline{\varphi}^1 = \underline{q}.\underline{\varphi}^2 \tag{29}$$

Intuitively, arbitrage creates a link between the prices of different assets. We will now formalize this important idea.

Let $\left(\underline{e}^i\right)_{i=1}^n$ denote the canonical basis in $\mathbb{R}^n$. If $rank(A) = ST$ then for each $i \in \{1, ..., ST\}$ we can find a portfolio $\underline{v}^i$ such that

$$A\underline{v}^i = \underline{e}^i \tag{30}$$

Definition 16 *A family of portfolios* $\left(\underline{v}^i\right)_{i=1}^{ST}$ *is a* basis of elementary portfolios *iff* $\underline{v}^i$ *satisfies (30), for each* i.

The following lemma provides a simple method which in practice allows us to find a basis of elementary portfolios.

Lemma 8 *If* $rank(A) = ST$ *and if moreover* A *is invertible then the column vectors of* A^{-1} *constitute a basis of elementary portfolios.*

Proof. Immediate from (30).

■

Our next lemma summarizes the main practical consequence of the no arbitrage condition.

Lemma 9 *Suppose the no arbitrage condition holds and* $rank(A) = ST$. *Let* $\left(\underline{v}^i\right)_{i=1}^{ST}$ *an arbitrary basis of elementary portfolios. Then there exists* $\underline{\pi}$ *such that* $q_i = \underline{\pi}.\underline{a}^i$, $\forall i$, *or, in matrix form*

$$\underline{q} = {}^tA\underline{\pi} \tag{31}$$

Moreover,

$$\pi_i = \underline{q}.\underline{v}^i \quad , \quad \forall i \tag{32}$$

Proof. We have

$$\underline{a}^i = \sum_j a^i_j \underline{e}^j = \sum_j a^i_j (A\underline{v}^j) = A \sum_j a^i_j \underline{v}^j \tag{33}$$

On the other hand

$$\underline{a}^i = A\underline{e}^i \tag{34}$$

Hence, by no arbitrage

$$q_i = \sum_j a^i_j \pi_j = \underline{\pi}.\underline{a}^i \tag{35}$$

where

$$\pi_j = \underline{q}.\underline{v}^j \tag{36}$$

■

Naturally, any time the price of an elementary portfolio is lesser or equal than zero traders will demand an infinite amount of that portfolio, which in turn will push prices up. We will thus henceforth assume $\underline{\pi} \gg 0$.

.

[**Q23**] Let $\underline{\pi} \geq 0$ given by (31). Show that $A\underline{\varphi} \geq 0 \Rightarrow \underline{q}.\underline{\varphi} \geq 0$. Then show the converse, namely that $A\underline{\varphi} \geq 0 \Rightarrow \underline{q}.\underline{\varphi} \geq 0$ implies the existence of a vector $\pi \geq 0$ such that $\underline{q} = {}^t A \underline{\pi}$.

.

We next define an important class of portfolios.

Definition 17 *Let $\tau \in \{2, ..., T+1\}$ and $\underline{b}^\tau$ denote the portfolio paying out 1 if $t = \tau$ and 0 otherwise. We call this portfolio a bond with maturity at date τ. We define r_τ such that $\frac{1}{1+r_\tau}$ indicates the price of this portfolio.*

Notice that if $\left(\underline{v}^i\right)_{i=1}^{ST}$ denotes a basis of elementary portfolios then:

$$A\underline{b}^\tau = \sum_s A\underline{v}^{s\tau} \tag{37}$$

By (31), we then have, using the no arbitrage condition:

$$\frac{1}{1+r_\tau} = \sum_s \pi_{s\tau} \tag{38}$$

In particular, we can define a probability distribution $\underline{\alpha}^\tau$ such that, $\forall s \in \{1, ..., S\}$

$$\pi_{s\tau} = \frac{\alpha_s^\tau}{1 + r_\tau} \tag{39}$$

α_s^τ has a natural economic interpretation: we can view it as the probability which the market assigns to state of the world (s, τ) occurring.[26]

.

[Q24] Consider a world with two periods, today and tomorrow, and S states of the world. Let A denote the asset structure of the economy, and $\underline{q}$ the asset price-vector. Letting $\underline{d} = {}^t(A^{-1})\underline{q}$, show that $r = \frac{1-\sum_i d_i}{\sum_i d_i}$.

.

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[Q25] A call option on a stock is a contract giving one the option to buy the stock at a specific price (the strike price) at a specified time in the future. Consider a world with two future states (a good state and a bad state) and three financial assets. There is first a bond, with interest rate r. There is also a stock, whose price today is 1, and whose price tomorrow is $u > 1$ in the good state but $d < 1$ in the bad state. There is finally a call option on the previous stock, with strike price K. Assuming the no arbitrage condition holds, what is the price of the call option? (The formula you will obtain is a special case of the well-known Black-Scholes formula.) What probability does the market assign to the good state occurring?

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In order to shed some light on financial assets' role in the economy we will start with a simple model in which consumers only care about available wealth in each state of the world. As usual, let u denote a utility function representing the consumer's preferences. We suppose that the consumer receives income y_w in each state of the world $w \in W$. It will be useful to distinguish first-period income from later-periods income; we will write in this case $\underline{y} = (y^1, \underline{y}^{-1})$. The consumer problem in this model is then:[27]

$$\max_{\underline{\varphi}, \underline{x} \geq 0} u(\underline{x}) \quad s.t. \quad x^1 \leq y^1 - \underline{q}.\underline{\varphi} \;\; ; \;\; \underline{x}^{-1} \leq \underline{y}^{-1} + A\underline{\varphi} \tag{40}$$

[26] To be more specific, suppose there exists a risk-neutral speculator who also happens to be fully unconstrained regarding his trading position. If for some τ that speculator's beliefs do not coincide with $\underline{\alpha}^\tau$ then he will take an infinite position.

[27] Notice that φ_i in (40) may be positive or negative. $\varphi_i < 0$ is sometimes referred to as short-selling of asset i. Finally, notice that the consumer is constrained in the speculative positions he may take. Effectively, his income here plays the role of collateral.

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[**Q26**] Using the framework developed in the present section, re-formulate the intertemporal consumption problem as well as the insurance problem from sections 2.2.1 and 2.2.2, respectively. In particular, specify A and $\underline{q}$ in each of these cases.

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The purpose of the subsequent analysis is to show that (40) is in fact equivalent to (19), for an appropriate choice of $\underline{p}$. The idea is the following. Consumers forego consumption today in order to buy assets which deliver future payouts; they then use these payouts to buy goods in the future. Formally, we must therefore be able to by-pass financial assets.

Proposition 6 *Suppose the no arbitrage condition holds and $rank(A) = ST$. Let $\underline{\pi} \gg 0$ given by (31) and $\underline{\xi} = (1, \underline{\pi})$. Then $\underline{x}$ solves*

$$\max_{\underline{x} \geq 0} u(\underline{x}) \quad s.t. \quad \underline{\xi}.\underline{x} \leq \underline{\xi}.\underline{y} \tag{41}$$

iff there exists $\underline{\varphi}$ such that $(\underline{x}, \underline{\varphi})$ solves (40).

Proof. Note first that since (i) the objective functions are the same and (ii) all constraints are binding at any optimum of the two problems, it is enough to show that $\underline{x} \in \Lambda \Leftrightarrow \exists \underline{\varphi}$ s.t. $(\underline{x}, \underline{\varphi}) \in \Gamma$, where

$$\Gamma = \{(\underline{x}, \underline{\varphi}) : x^1 = y^1 - \underline{q}.\underline{\varphi} \; ; \; \underline{x}^{-1} = \underline{y}^{-1} + A\underline{\varphi}\} \quad ; \quad \Lambda = \{\underline{x} : \underline{\xi}.\underline{x} = \underline{\xi}.\underline{y}\} \tag{42}$$

We will first show that $(\underline{x}, \underline{\varphi}) \in \Gamma \Rightarrow \underline{x} \in \Lambda$. We will then show that $\forall \underline{x} \in \Lambda, \exists! \; \underline{\varphi}$ s.t. $(\underline{x}, \underline{\varphi}) \in \Gamma$.

Let thus $(\underline{x}, \underline{\varphi}) \in \Gamma$. Using (31) to substitute yields

$$x^1 = y^1 - \underline{\pi}.A\underline{\varphi} \tag{43}$$

We then have

$$x^1 = y^1 - \underline{\pi}.\left(\underline{x}^{-1} - \underline{y}^{-1}\right) \tag{44}$$

And, by definition of $\underline{\xi}$

$$\underline{\xi}.\underline{x} = \underline{\xi}.\underline{y} \tag{45}$$

Conversely, let $\underline{x} \in \Lambda$. Since $rank(A) = ST$ we can find a unique φ such that $\underline{x}^{-1} =$

$\underline{y}^{-1} + A\underline{\varphi}$. We then have

$$x^1 = y^1 - \underline{\pi}.\left(\underline{x}^{-1} - \underline{y}^{-1}\right) = y^1 - \underline{\pi}.A\underline{\varphi} = y^1 - \underline{q}.\underline{\varphi} \tag{46}$$

■

While possibly not immediately apparent, Proposition 6 embodies a very familiar lesson. To make this clear, suppose that to each time period corresponds a single state of the world. By (39), the constraint in (41) can be rewritten as:

$$\sum_t \frac{x^t}{1+r_t} = \sum_t \frac{y^t}{1+r_t} \tag{47}$$

Proposition 6 thus quite simply states that the consumer maximizes utility such that the present value of his consumption equals the present value of his lifetime income. The intuition is straightforward. If financial markets are complete, then agents can shift resources from one state of the world to any other (at a set rate, determined by financial markets). So the only constraint must be a *global* one.

For completeness, we will now extend problem (40) to consider explicitly the consumption of different goods across states of the world. What follows is unfortunately, but unavoidably, notationally burdensome.

Suppose the total number of goods is L, so that a full consumption profile for the consumer specifies a vector $\underline{x}^w \in \mathbb{R}^L$ for each state of the world $w \in W$. We let $\underline{x}$ denote the vector in $\mathbb{R}^{LW}$ summarizing this profile, so that $x_{wl} = x_l^w$, $\forall w \in W$, $\forall l \in L$. Similarly a full profile of prices specifies prices $\underline{\widetilde{p}}^w \in \mathbb{R}^L$ in each state of the world $w \in W$. We let $\underline{\widetilde{p}}$ denote the vector in $\mathbb{R}^{LW}$ summarizing this profile, so that $\widetilde{p}_{wl} = \widetilde{p}_l^w$, $\forall w \in W$, $\forall l \in L$. We will sometimes say that $\underline{\widetilde{p}}$ represents the vector of *spot prices*. Using this notation, the consumer problem is now:

$$\max_{\underline{\varphi},\underline{x}\geq 0} u(\underline{x}) \quad s.t. \quad \underline{\widetilde{p}}^1.\underline{x}^1 \leq y^1 - \underline{q}.\underline{\varphi} \quad ; \quad \underline{\widetilde{p}}^{st}.\underline{x}^{st} \leq y^{st} + (A\underline{\varphi})_{st} \ \ , \forall st \in W^{-1} \tag{48}$$

Our next proposition is a straightforward generalization of proposition 6.

Proposition 7 **[NPV equivalence]** *Suppose the no arbitrage condition holds and* $rank(A) = ST$. *Let* $\underline{\pi} \gg 0$ *given by (31), and* $\underline{\xi} = (1, \underline{\pi})$. *Define* $\underline{p}$ *such that* $p_{wl} = \xi_w \widetilde{p}_l^w$, $\forall w \in W$, $\forall l \in L$. *Then* $\underline{x}$ *solves*

$$\max_{\underline{x}\geq 0} u(\underline{x}) \quad s.t. \quad \underline{p}.\underline{x} \leq \underline{\xi}.\underline{y} \tag{49}$$

iff there exists $\underline{\varphi}$ *such that* $(\underline{x}, \underline{\varphi})$ *solves (48).*

Proof. The proof follows closely that of proposition 6.

First, notice that since (i) the objective functions are the same and (ii) all constraints are binding at any optimum of the two problems, it is enough to show that the following sets are bijective:

$$\Gamma = \{(\underline{x}, \varphi) : \underline{\widetilde{p}}^1.\underline{x}^1 = y^1 - \underline{q}.\underline{\varphi}\ ;\ \underline{\widetilde{p}}^{st}.\underline{x}^{st} = y^{st} + (A\underline{\varphi})_{st}\ \ ,\ \forall st \in W^{-1}\} \quad ; \quad \Lambda = \{\underline{x} : \underline{p}.\underline{x} = \underline{\xi}.\underline{y}\} \tag{50}$$

We will first show that $(\underline{x}, \varphi) \in \Gamma \Rightarrow \underline{x} \in \Lambda$. We will then show that $\forall \underline{x} \in \Lambda, \exists\ \varphi$ s.t. $(\underline{x}, \varphi) \in \Gamma$.

Let thus $(\underline{x}, \varphi) \in \Gamma$. Using (31), we have:

$$\underline{\widetilde{p}}^1.\underline{x}^1 = y^1 - \underline{\pi}.A\underline{\varphi} \tag{51}$$

Using subsequent budget constraints to substitute then yields

$$\underline{\widetilde{p}}^1.\underline{x}^1 = y^1 - \sum_{st \in W^{-1}} \pi_{st}(\underline{\widetilde{p}}^{st}.\underline{x}^{st} - y^{st}) \tag{52}$$

And, by definition of $\underline{\xi}$ and $\underline{p}$

$$\underline{p}.\underline{x} = \underline{\xi}.\underline{y} \tag{53}$$

Conversely, let $\underline{x} \in \Lambda$. Since $rank(A) = ST$ we can find a unique φ such that

$$\underline{\widetilde{p}}^{st}.\underline{x}^{st} - y^{st} = \frac{1}{\pi_{st}}\left(\underline{p}^{st}.\underline{x}^{st} - \pi_{st} y^{st}\right) = (A\underline{\varphi})_{st}\ \ ,\ \ \forall st \in W^{-1} \tag{54}$$

We then have

$$\underline{\widetilde{p}}^1.\underline{x}^1 = \underline{p}^1.\underline{x}^1 = y^1 + \sum_{st \in W^{-1}} \left(\pi_{st} y_{st} - \underline{p}^{st}.\underline{x}^{st}\right) = y^1 - \sum_{st \in W^{-1}} \pi_{st}(A\underline{\varphi})_{st} = y^1 - \underline{\pi}.A\underline{\varphi} = y^1 - \underline{q}.\underline{\varphi} \tag{55}$$

■

As in (47), suppose that to each time period corresponds a single state of the world. Expanding the constraint in (49) then yields:

$$\sum_t \sum_l \frac{\widetilde{p}_l^t x_l^t}{1 + r_t} = \sum_t \frac{y^t}{1 + r_t} \tag{56}$$

The product $\underline{p}.\underline{x}$ thus gives the present value of consumption bundle $\underline{x}$, while proposition 7, once again, indicates that consumers maximize utility such that the present value of their consumption equals the present value of their lifetime income. By extension, we will sometimes say that $\underline{p} \in \mathbb{R}^{LW}$ defined in (49) represents the vector of *present prices.*

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[**Q27**] Consider a world with 2 time periods (today and tomorrow), 3 states of the world (today, bad tomorrow, and good tomorrow) and two goods. Consider also a consumer earning 100 today, and 100 tomorrow – but only in the good state. The consumer has Cobb-Douglas preferences such that, if x_i^w represents consumption of good i in state of the world w, $u(\underline{x}) = \Pi x_i^w$. The asset structure of the economy is A, where $a_{11} = 1$, $a_{21} = 1$, $a_{12} = 0$, $a_{22} = 1$. It is supposed that today good 2 is twice as expensive as good 1, inflation is uniform and at 8 percent, the interest rate is 3 percent, and the market assigns probability one third to the bad state occurring. Find the consumer's net demand of each tradable asset.

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6 General equilibrium II

We will in this final section build on the tools and results developed in the previous section in order to formulate a theory of general equilibrium over time.

We first define the concept of a market equilibrium when trade and consumption take place over time. The unique but crucial novelty is the addition of a financial market. A such market equilibrium is usually called a Radner equilibrium.

Definition 18 *Let $X = ({}^A\underline{x},\ {}^B\underline{x})$ and $\Phi = ({}^A\underline{\varphi},\ {}^B\underline{\varphi})$. Then $(X, \Phi, \underline{\widetilde{p}}, \underline{q})$ is a* Radner equilibrium *for the pure exchange economy with endowment $E = ({}^A\underline{e},\ {}^B\underline{e})$ and asset structure A if and only if:*

1. ${}^A\underline{x} + {}^B\underline{x} = {}^A\underline{e} + {}^B\underline{e}$
2. ${}^A\underline{\varphi} + {}^B\underline{\varphi} = 0$
3. *for each consumer $i \in \{A, B\}$, $({}^i\underline{x},\ {}^i\underline{\varphi})$ solves*

$$\max_{\underline{\varphi},\underline{x} \geq 0} u(\underline{x}) \quad s.t. \quad \underline{\widetilde{p}}^1.\underline{x}^1 + \underline{q}.\underline{\varphi} \leq \underline{\widetilde{p}}^1.{}^i\underline{e}^1 \quad ; \quad \underline{\widetilde{p}}^{st}.\underline{x}^{st} \leq \underline{\widetilde{p}}^{st}.{}^i\underline{e}^{st} + (A\underline{\varphi})_{st} \ , \ \forall st \in W^{-1} \quad (57)$$

We seek in this section to exploit once again the main idea of the previous section: financial assets are pure practical intermediaries; it is thus compelling to by-pass them formally. The following definition is useful in this view. An Arrow-Debreu equilibrium is a market equilibrium in which nothing but goods are traded (i.e., no assets are traded), but in which goods from different time periods may be traded against one another.

Definition 19 *Let $X = ({}^A\underline{x},\ {}^B\underline{x})$. Then $(X, \underline{p})$ is an* Arrow-Debreu equilibrium *for the pure exchange economy with endowment $E = ({}^A\underline{e},\ {}^B\underline{e})$ if and only if:*

1. ${}^A\underline{x} + {}^B\underline{x} = {}^A\underline{e} + {}^B\underline{e}$
2. *for each consumer $i \in \{A, B\}$, ${}^i\underline{x}$ solves*

$$\max_{\underline{x} \geq 0} u(\underline{x}) \quad s.t. \quad \underline{p}.\underline{x} \leq \underline{p}.{}^i\underline{e} \tag{58}$$

Our next result shows that to any Radner equilibrium corresponds an Arrow-Debreu equilibrium, and that to any Arrow-Debreu equilibrium corresponds an infinity of Radner equilibria, in the sense that asset prices are in fact undetermined by the fundamentals of our framework (i.e. consumers preferences and firms technology) - they are pure nominal objects.

Proposition 8 *Let $X = ({}^A\underline{x},\ {}^B\underline{x})$. If $rank(A) = ST$ then the following are equivalent:*

1. *$\exists\, (\underline{\widetilde{p}}, \underline{q})$ and Φ such that $(X, \Phi, \underline{\widetilde{p}}, \underline{q})$ is a Radner equilibrium*
2. *$\exists\, \underline{p}$ such that $(X, \underline{p})$ is an Arrow-Debreu equilibrium*

Proof. Suppose $(X, \Phi, \underline{\widetilde{p}}, \underline{q})$ is a Radner equilibrium. Let $\underline{\pi}$ given by (31) (notice that we are using here the fact that $rank(A) = ST$; notice also that $\underline{\pi} \gg 0$ since otherwise the demand for some asset would be infinite, which is precluded in equilibrium), and $\underline{\xi} = (1, \underline{\pi})$. Define $\underline{p}$ such that $p_{wl} = \xi_w \widetilde{p}_l^w$, $\forall w \in W$, $\forall l \in L$. Observe then that, by proposition 7, $(X, \underline{p})$ is an Arrow-Debreu equilibrium.

Next, suppose $(X, \underline{p})$ is an Arrow-Debreu equilibrium. Choose an arbitrary $\underline{\pi} \gg 0$ and let $\underline{q}$ given by (31) (again, we are using here the fact that $rank(A) = ST$). Let $\underline{\xi} = (1, \underline{\pi})$ and define $\underline{\widetilde{p}}$ such that $p_{wl} = \xi_w \widetilde{p}_l^w$, $\forall w \in W$, $\forall l \in L$. For each $i \in \{A, B\}$, define then ${}^i\underline{\varphi}$ such that

$$(A\, {}^i\underline{\varphi})_{st} = \underline{\widetilde{p}}^{st}.{}^i\underline{x}^{st} - \underline{\widetilde{p}}^{st}.{}^i\underline{e}^{st} \tag{59}$$

By proposition 7, for each consumer $i \in \{A, B\}$, $({}^i\underline{x}, {}^i\underline{\varphi})$ solves (57). Moreover, the market for each good clears by virtue of the Arrow-Debreu equilibrium. So $(X, \Phi, \underline{\widetilde{p}}, \underline{q})$ is a Radner equilibrium iff ${}^A\underline{\varphi} + {}^B\underline{\varphi} = 0$. Since $rank(A) = ST$, the latter condition is equivalent to $A({}^A\underline{\varphi} + {}^B\underline{\varphi}) = 0$. But using (59) we have, $\forall st \in W_{-1}$,

$$\left(A({}^A\underline{\varphi} + {}^B\underline{\varphi})\right)_{st} = \sum_{i=A,B} (\underline{\widetilde{p}}^{st} . {}^i\underline{x}^{st} - \underline{\widetilde{p}}^{st} . {}^i\underline{e}^{st}) = \underline{\widetilde{p}}^{st} . \sum_{i=A,B} ({}^i\underline{x}^{st} - {}^i\underline{e}^{st}) = 0 \tag{60}$$

So $(X, \Phi, \underline{\widetilde{p}}, \underline{q})$ is a Radner equilibrium.

■

Corollary 3 *$\underline{p}$ is an Arrow-Debreu equilibrium price-vector iff there exists $\underline{\pi} \gg 0$ such that $(\underline{\widetilde{p}}, \underline{q})$ is a Radner equilibrium price-vector where*

1. $\underline{q} = {}^tA\underline{\pi}$
2. $p_{wl} = \xi_w \widetilde{p}_l^w$, $\forall w \in W$, $\forall l \in L$, *where* $\underline{\xi} = (1, \underline{\pi})$

To conclude this section, notice that – once we abstract from the fact that different goods may be consumed in different time periods – Arrow-Debreu equilibria formally coincide with the market equilibria defined in section 4. The results obtained there thus extend to Arrow-Debreu equilibria and, owing to proposition 8, also to Radner equilibria. In particular, Radner equilibria exist, they are pareto optimal, and any pareto optimal outcome may be sustained as a Radner equilibrium. We have thus come full circle.

7 Answers

[**Q1**] That **A1** hold is obvious. **A2** holds if and only $\underline{x} \gg 0$. For **A3** it is enough, by Lemma 1, to show that u has convex level curves. The equation of a level curve is $x_1^\alpha x_2^{1-\alpha} = l$, from which we obtain $x_2 = l^{\frac{1}{1-\alpha}} x_1^{\frac{-\alpha}{1-\alpha}}$, which is convex.

[**Q2**] Recall, u satisfies **A1**-**A.3** iff $v \circ u$ satisfies **A1**-**A.3** wherever v is a (smooth) increasing transformation. Taking $\ln$ on the Cobb-Douglas gives $\ln(u) = \alpha \ln(x_1) + (1-\alpha)\ln(x_2)$. The result follows since $\ln$ is concave.

[**Q3**] The implied functional forms are $u(\underline{x}) = \min\{x_1, x_2\}$ for perfectly complementary goods, and $u(\underline{x}) = x_1 + x_2$ for perfectly substitutable goods. The first of these violates assumption **A.1** as well as **A.2**. This implies in particular that we cannot apply first-order conditions as is done for example in Proposition 1. The second functional form violates **A.3**. This opens the possibility in particular for more than one solution to the consumer problem (5). Marshallian demand can nonetheless be obtained using non-standard approaches. In the first case, we must have $x_1 = x_2$. And given $p_1 x_1 + p_2 x_2 = m$, we obtain $x_1 = x_2 = m/(p_1 + p_2)$. In the second case, the consumer always spends his entire wealth on the cheapest good. So $x_1 = m/p_1$ and $x_2 = 0$ whenever $p_1 < p_2$, and conversely in the alternative. If $p_1 = p_2$ Marshallian demands are undetermined since any consumption bundle exhausting the budget constraint solves the consumer problem.

[**Q4**] Homogeneity degree zero follows from the invariance of (5) under multiplication of $(\underline{p}, m)$. Walras' law and the fact that $v(\underline{p}, m)$ is increasing in income, decreasing in prices all follow from the fact that $\nabla_i u > 0$, $\forall i$. Let next $(\underline{p}'', m'') = \lambda(\underline{p}, m) + (1-\lambda)(\underline{p}', m')$. Any bundle $\underline{x}$ affordable under $(\underline{p}'', m'')$ must also be affordable under one of $(\underline{p}, m)$ and $(\underline{p}', m')$. But then clearly $v(\underline{p}'', m'') \leq \max\{v(\underline{p}, m), v(\underline{p}', m')\}$.

[**Q5**]If v is strictly increasing then u and $v(u)$ represent the same preferences. The solution to the consumer problem must therefore be invariant upon this transformation.

[**Q6**]By (6) we have $\alpha x_1^{\alpha-1} x_2^{1-\alpha} \big((1-\alpha) x_1^\alpha x_2^{-\alpha}\big)^{-1} = p_1/p_2$, and $\alpha x_2/(1-\alpha)x_1 = p_1/p_2$ upon rearrangement. We also have $p_1 x_1 + p_2 x_2 = m$ since at an optimum the budget constraint must be binding. Solving this 2x2 linear system yields $x_1 = \alpha m/p_1$ and $x_2 = (1-\alpha)m/p_2$. Using the fact that $v(\underline{p}, m) = u(\underline{x}(p, m))$ yields $v(\underline{p}, m) = \alpha^\alpha (1-\alpha)^{(1-\alpha)} m p_1^{-\alpha} p_2^{\alpha-1}$. Both goods are normal since $\partial x_1/\partial m = \alpha/p_1$ and $\partial x_2/\partial m = (1-\alpha)/p_2$.

[**Q7**] The set of necessary and sufficient conditions are (i) $\phi \in C^\infty$, (ii) $\phi' > 0$, (iii) $\phi'' < 0$. If an interior solution exists it then solves $\underline{p}.\underline{x} = m$ and $\phi'(x_2) = p_2$. The demand for good 2 is thus independent of income. Graphically, the level curves of u are horizontal translations of one another.

[**Q8**] $\lambda \underline{p}.h(\underline{p}, u) = \lambda e(\underline{p}, u)$. Moreover, by definition, any $\underline{x}$ such that $u(\underline{x}) \geq u$ satisfies $\underline{p}.\underline{x} \geq \underline{p}.h(\underline{p}, u)$, and thus also $\lambda \underline{p}.\underline{x} \geq \lambda \underline{p}.h(\underline{p}, u)$. This shows that $e(\underline{p}, u)$ is homogenous degree one in $\underline{p}$. That $e(\underline{p}, u)$ is increasing in $\underline{p}$ and u is immediate from (7) (set $\Omega(\underline{p}, u) = \{\underline{p}.\underline{x} : u(\underline{x}) \geq u\}$ and note that $e(\underline{p}, u) = \min \Omega(\underline{p}, u)$). Let next $\underline{p}'' = \lambda \underline{p} + (1-\lambda)\underline{p}'$. Then $e(\underline{p}'', u) = \underline{p}''.\underline{h}''$, where $\underline{h}'' = h(\underline{p}'', u)$. But notice that $\underline{p}''.\underline{h}'' = \lambda \underline{p}.\underline{h}'' + (1-\lambda)\underline{p}'.\underline{h}''$ where the first term in the sum is, by definition, more than $\lambda e(\underline{p}, u)$ and the second term more than $(1-\lambda)e(\underline{p}', u)$. This finishes to show that the expenditure function is concave in prices.
[**Q9**] Use KKT to find $\alpha h_2/(1-\alpha)h_1 = p_1/p_2$. Then solve using $u(\underline{h}) = u$. Compute the expenditure function using the fact that $e(\underline{p}, u) = \underline{p}.\underline{h}$.
[**Q10**] We have shown that the expenditure function is concave in prices. Thus, in particular $\frac{\partial^2 e}{\partial p_i^2} < 0$. But, by application of the envelope theorem $\frac{\partial e}{\partial p_i} = h_i$. Hence $\frac{\partial h_i}{\partial p_i} < 0$.
[**Q11**] Immediate application of (6) to the present context, for the first part. For the second part, increasing the interest rate means raising the relative price of period 1 consumption. We know from proposition 3 that it is not in general possible to determine whether period 1 consumption rises or fall as this hinges on whether the consumer is a net buyer or a net seller of period 1 consumption (i.e. a borrower or a saver, respectively). If $x_1 > y_1$ (borrower) then period 1 consumption falls with a rise in r, and the consumer ends up worse off. If $y_1 > x_1$ (saver) the effect on period 1 consumption is indeterminate, but the consumer is better off as a result no matter what. If Cobb-Douglas functional form then $x_1 = \alpha \underline{p}.\underline{y}/p_1$, so that the total effect is $dx_1/dp_1 = -\alpha p_2 y_2/p_1^2 \leq 0$. Substituting using $p_1 = 1 + r$ and $p_2 = 1$ gives $dx_1/dr = (dx_1/dp_1)(dp_1/dr) = -\alpha y_2/(1+r)^2 \leq 0$.
[**Q12**] Argue that at $\underline{x} = \underline{y}$ the consumer's level curve crosses the budget constraint 'from above'. So if there is a borrowing constraint the optimal consumption profile is precisely $\underline{x} = \underline{y}$. If on the other hand $x_1 < y_1$ without borrowing constraint then introducing a constraint does not bind the consumer in any way, and he can consume as if no constraint existed.
[**Q13**]Suppose the consumer faces both B and $\overline{B}$. He neither gains nor looses anything in any contingency. Facing B alone on the other hand is worse for him than losing $|\mu(\overline{B})|$ for sure, since the consumer is risk-averse. This concludes the proof. The existence of a market for insurance is founded upon this result.
[**Q14**] Consider bets of the form $(\underline{p}, \underline{b})$, with $\underline{p} = (p_1, p_2)$ fixed. The frontier $x_2(x_1)$ of acceptable bets is determined by $p_1 U(m+x_1) + p_2 U(m+x_2) = U(m)$. Applying the Implicit Function Theorem gives $x_2'(x_1) = -p_1 U'(m+x_1)/p_2 U'(m+x_2)$. Since $x_2(0) = 0$ we find $x_2'(0) = -p_1/p_2$. Differentiating once more gives $x_2''(x_1)$, from which we find $x_2''(0) = -[p_1/p_2 + (p_1/p_2)^2]U''(m)/U'(m)$.
[**Q15**] The uninsured consumer facing accident probability q has $x_1 = M$, $x_2 = M - L$, and $\mu(\underline{x}) = M - qL$. By fully insuring, he secures $x_1 = M - pL$, $x_2 = M - pL$, and $\mu(\underline{x}) = M - pL$.

A risk-averse consumer thus, by definition, always chooses full insurance whenever $p = q$.[28] If the expected-utility assumption holds observe that (6) yields, $\frac{1-q}{q}\frac{U'(x_1)}{U'(x_2)} = \frac{1-p}{p}$.

[Q16] Argue first that any pareto optimal contract transfers all the risk from consumer to insurer. Let x_1 denote consumption in case no accident and x_2 consumption when accident. If $x_1 \neq x_2$ then write a new contract as follows. Let $\mu = (1-q)x_1 + qx_2$. Consider the contract: 'pay $x_1 - \mu$ to insurance company in case no accident and receive $\mu - x_2$ is case accident'. This contract makes the consumer strictly better off and the insurance company no worse off. The set of pareto optimal contracts is thus the set of contracts of the form ' pay F and receive L in case of accident', where $F \leq M$. Observe however that if he is uninsured then the consumer's utility is $(1-q)\ln(M) + q\ln(M-L) = \ln(M^{1-q}(M-L)^q)$. So the maximum fee the consumer is willing to pay for full insurance is $F = M - M^{1-q}(M-L)^q$. Also, the insurance company won't insure the consumer at a loss. So $F \geq qL$. The set of pareto optimal contracts acceptable to both parties is thus the set of contracts of the form ' pay F and receive L in case of accident', where $qL \leq F \leq M - M^{1-q}(M-L)^q$. To check that $qL \leq M - M^{1-q}(M-L)^q$ rewrite as $M^{1-q}(M-L)^q \leq M - qL$, take $\ln$ on both sides, and apply Jensen's inequality. Observe finally that the contract which entails a fair insurance policy is the one where $F = qL$.

[Q17] The relevant trade-off here is between leisure l and standard consumption of goods x (of which we normalize the price to 1). Each worker is 'endowed' with 365 days of work, valued at $365w$, where w indicates the daily wage. Labour supply is then $L = 365 - l$, and each worker can consume up to his total wage $w(365 - l)$. Let $u(x, l)$ denote utility from consuming x and l leisure days. The consumer problem is $\max u(\underline{x})$ s.t. $\underline{p}\underline{x} \leq \underline{p}.\underline{x}^\dagger$, where $\underline{x} = (x, l)$, $\underline{p} = (1, w)$, and $\underline{x}^\dagger = (0, 365)$. A rise in w corresponds to a rise in the price of leisure. But the consumer is always a net seller of leisure, so by proposition 3 whether the consumer consumes less leisure (i.e. increases labour supply) or not is indeterminate. He will however unambiguously be better off, owing to that same proposition.

[Q18] Follows from the fact that the budget constraint of each consumer is binding at an optimum.

[Q19] Since each consumer maximizes his utility we have $\underline{p}.{}^A\underline{x} = \underline{p}.{}^A\underline{e}$ and $\underline{p}.{}^B\underline{x} = \underline{p}.{}^B\underline{e}$. Thus $\underline{p}.({}^A\underline{x} + {}^B\underline{x} - {}^A\underline{e} - {}^B\underline{e}) = 0$. Suppose ${}^Ax_i + {}^Bx_i = {}^Ae_i + {}^Be_i$ in all but possibly market j. Then $p_j({}^Ax_j + {}^Bx_j - {}^Ae_j + {}^Be_j) = 0$. But $p_j > 0$ by assumption, so ${}^Ax_j + {}^Bx_j = {}^Ae_j + {}^Be_j$ and market j also clears.

[Q20] The Marshallian demands are ${}^Ax_1 = \alpha\, {}^Am/p_1$, ${}^Ax_2 = (1-\alpha)\, {}^Am/p_2$, ${}^Bx_1 = {}^Bm/(p_1 + p_2) = {}^Bx_2$, where ${}^Am = 10p_1$ and ${}^Bm = 10p_2$. Market clearance for good 1 gives $10 =$

[28] A risk-averse consumer dislikes fair bets, while a fair insurance policy is one which offers an 'inverse fair bet'. The question is thus largely tautological.

${}^A x_1 +^B x_1 = 10(a + p_2/(p_1 + p_2))$, from which we obtain $p_1/p_2 = \alpha/(1-\alpha)$. The set of pareto optimal allocations coincides with the median of the Edgeworth box.

[**Q21**] Lemma 7 still holds in the presence of externalities. However now, if A strictly prefers Y to X it does not follow that $Y \notin {}^A F(\underline{p}, E)$. Indeed, in the presence of externalities A may prefer a feasible allocation in his 'own' feasible subset, ${}^A F(\underline{p}, E)$. It is only given B's consumption that he finds ${}^A \underline{x}$ optimal.

[**Q22**] If X is a corner of the Edgeworth box then there may exist $\underline{p}' \neq \underline{p}$ such that $(X, \underline{p}')$ is a market equilibrium for the pure exchange economy with endowment X. However if X is not a boundary point of the Edgeworth box then this is impossible, owing to Proposition 1. The set of endowment points E' such that $(X, \underline{p})$ is a market equilibrium for the pure exchange economy with endowment E' is $\{E' = ({}^A\underline{e}',\ {}^B\underline{e}') \in F(E) : \underline{p}.{}^A\underline{e}' = \underline{p}.{}^A\underline{e}\} = \{E' = ({}^A\underline{e}',\ {}^B\underline{e}') \in F(E) : \underline{p}.{}^B\underline{e}' = \underline{p}.{}^B\underline{e}\} = {}^A F(\underline{p}, E) \bigcap {}^B F(\underline{p}, E)$.

[**Q23**] For the first part, note that $\underline{q}.\underline{\varphi} = {}^t A \underline{\pi}.\underline{\varphi} = \underline{\pi}.A\underline{\varphi} \geq 0$. For the second part note that if $A\underline{\varphi} \geq 0 \Rightarrow \underline{q}.\underline{\varphi} \geq 0$ then there does not exist a hyperplane separating $\underline{q}$ from the cone of the rows of A. But the cone of the rows of A is closed and convex. It thus follows from the separating hyperplane theorem that $\underline{q}$ belongs to the cone of the rows of A.

[**Q24**] Let $\mathbf{\underline{1}}$ denote the vector with all entries equal to 1, and $\underline{b}$ the bond of this economy. By definition, $A\underline{b} = \mathbf{\underline{1}}$, and so $\underline{b} = A^{-1}\mathbf{\underline{1}}$. Hence, $\frac{1}{1+r} = \underline{q}.\underline{b} = \underline{q}.A^{-1}\mathbf{\underline{1}} = {}^t(A^{-1})\underline{q}.\mathbf{\underline{1}}$. Letting $\underline{d} = {}^t(A^{-1})\underline{q}$, we then obtain $\frac{1}{1+r} = \sum_i d_i$.

[**Q25**] The matrix A is such that $a_{11} = 1$, $a_{21} = 1$, $a_{12} = u$, $a_{22} = d$, $a_{13} = \max\{0, u-K\}$, $a_{23} = \max\{0, d-K\}$. The price vector $\underline{q}$ is given by $(\frac{1}{1+r}, 1, p)$, where p indicates the price we are looking for. Using the fact that ${}^t A\underline{\pi} = \underline{q}$, find first $\pi_1 = \frac{(1+r)-d}{(1+r)(u-d)}$ and $\pi_2 = \frac{u-(1+r)}{(1+r)(u-d)}$. We then obtain $p = \pi_1 \max\{0, u-K\} + \pi_2 \max\{0, d-K\}$.

[**Q26**] For intertemporal consumption, $A = I$, the identity matrix, while $q_i = \frac{1}{(1+r)^i}$. Each state of the world represents a different time period. In the case of insurance, there are two future states of the world: with and without accident. An insurance policy is an asset delivering 1 in case of an accident and 0 otherwise. By contrast, a bond delivers 1 in both future states of the world. The matrix A is therefore given by $a_{11} = 1$, $a_{21} = 1$, $a_{12} = 0$, $a_{22} = 1$. We have $\underline{q} = (1, p)$, where p is the unit price of insurance. In the first period, the consumer buys K units of the insurance policy and $M - pk$ bonds.

[**Q27**] The idea is as follows. First, find the Marshallian demands using (49). Deduce from that the total spot value of consumption in each state of the world, and then obtain $\underline{\varphi}$ by equating the previous vector with $\underline{y} + A\underline{\varphi}$. Let M denote the present value of lifetime income, so that

$M = 100 + \frac{\frac{2}{3}100}{1+r}$. Given Cobb-Douglas preferences we know that the consumer will spend, in present value terms, $\frac{1}{6}M$ on each 'good'. Expressed at spot values, the total values of consumption are thus $\frac{2}{6}M\frac{3}{2}(1+r)$ in tomorrow's good state and $\frac{2}{6}M3(1+r)$ in tomorrow's bad state. The consumer therefore needs assets delivering $\frac{2}{6}M\frac{3}{2}(1+r) - 100$ in tomorrow's good state and $\frac{2}{6}M3(1+r)$ in tomorrow's bad state, i.e. $A\underline{\varphi} = (\frac{2}{6}M\frac{3}{2}(1+r) - 100, \frac{2}{6}M3(1+r))$. Use A^{-1} to retrieve $\underline{\varphi}$.

8 Supplementary Material

8.1 Further remarks on expected utility

While very appealing, the expected utility assumption supposes important restrictions on consumers' underlying preferences. To illustrate, let $tB \oplus (1-t)B'$ denote the bet resulting from playing bet B with probability t and bet B' with complementary probability $1-t$. One implication of the expected utility assumption is then that

$$B \sim B' \Rightarrow tB \oplus (1-t)C \sim tB' \oplus (1-t)C \tag{61}$$

It is in fact possible to show that any preferences satisfying property (61) are consistent with the expected utility assumption. This is important as this says that property (61) effectively constitutes the ultimate test of the expected utility assumption: if we can show that it is satisfied by consumers' preferences we will be vindicated in our use of the expected utility assumption; if on the other hand we can show that consumers' preferences violate this property we will then be unable to justify our use of the expected utility assumption. In what follows, we proceed to establish that any preferences satisfying property (61) are consistent with the expected utility assumption.

Some notation is useful. Let $l, h \in \mathbf{R}$, $l < h$. Let next $\mathfrak{B}$ the set of bets $B = (\underline{p}, \underline{b})$ with $b_i \in [l, h]$, $\forall i$. Let $I_b \in \mathfrak{B}$ the bet yielding b for sure. We will suppose throughout the good to be strictly desirable as well as disposable so that for all consumers I_l is the least preferred bet in $\mathfrak{B}$ and I_h the most preferred one. Since I_h is the most preferred bet by all consumers, we will also suppose throughout that for all consumers and $\forall t \in [0, 1)$:

$$tB \oplus (1-t)I_h \succ B \tag{62}$$

Note also that for any $B \in \mathfrak{B}$ we have:

$$B = p_1 I_{b_1} \oplus ... \oplus p_n I_{b_n} \tag{63}$$

Proposition 9 *Consider a consumer with continuous preferences satisfying property (61) for all B, B', and $C \in \mathfrak{B}$, and any $t \in [0, 1]$. There then exists $u : \mathfrak{B} \to \mathbf{R}$ and $U : \mathbf{R} \to \mathbf{R}$, $u(B) = \sum_i p_i U(b_i)$, such that $B \succ C$ iff $u(B) > u(C)$.*

Proof. Let $B \in \mathfrak{B}$. Observe that the sets $\{t \in [0, 1] : tI_l \oplus (1-t)I_h \succeq B\}$ and $\{t \in [0, 1] :$

$tI_l \oplus (1-t)I_h \preceq B\}$ are both non-empty. Moreover their union is $[0,1]$ and, since preferences are continuous, both of them are closed sets. We can thus find t in their intersection. Let us show moreover that t is in fact unique. Suppose not. We then have a $t' \neq t$ and $tI_l \oplus (1-t)I_h \sim t'I_l \oplus (1-t')I_h$. Let $t' > t$, say. Observe that $tI_l \oplus (1-t)I_h = t/t'\big(t'I_l \oplus (1-t')I_h\big) \oplus (1-t/t')I_h$. But then we have found an $s \in [0,1)$ such that $t'I_l \oplus (1-t')I_h \sim s\big(t'I_l \oplus (1-t')I_h\big) + (1-s)I_h$, which is impossible according to (62). So t is unique and we can define $u(B) = t$. Then, By property (61), applied twice:

$$\begin{aligned} sB \oplus (1-s)C &\sim s\big(u(B)I_l \oplus (1-u(B))I_h\big) \oplus (1-s)C \\ &\sim s\big(u(B)I_l \oplus (1-u(B))I_h\big) \oplus (1-s)\big(u(C)I_l \oplus (1-u(C))I_h\big) \end{aligned} \tag{64}$$

And, upon rearrangement:

$$sB \oplus (1-s)C \sim \big((su(B) + (1-s)u(C))\big)I_l \oplus \big(s(1-u(B)) + (1-s)(1-u(C))\big)I_h \tag{65}$$

$$= \big(su(B) + (1-s)u(C)\big)I_l \oplus \big(1 - (su(B) + (1-s)u(C))\big)I_h \tag{66}$$

So $u(sB \oplus (1-s)C) = su(B) + (1-s)u(C)$. Setting $U(b) = u(I_b)$ then concludes the proof since for any $B \in \mathfrak{B}$ we have $B = p_1 I_{b_1} \oplus ... \oplus p_n I_{b_n}$.

■

9 Mathematical Appendix

In this appendix we record the theorems underlying all results in these notes. The Implicit Function Theorem provides a simple way of finding the level curve of a function: it is orthogonal to the gradient of that function. The Karush-Kuhn-Tucker Theorem is the key to solving constrained optimization problems: it notes that at an optimum the level curves of the objective and of the constraint are tangent to one another. Once solved, the Envelope Theorem tells us how the optimal value attained varies as one changes the parameters of the problem, while the Maximum Theorem tells us something about the way in which the optimal point varies as one changes the parameters of the problem. The Hyperplane Theorem establishes that we can always draw a line to separate a convex set from any point not belonging to that set. Finally, Jensen's inequality shows that if a function is convex then the mean of the function is larger than the function of the mean.

Theorem 1 [**Implicit Function Theorem**] *Let $F : \mathbf{R}^2 \to \mathbf{R}$, $F \in C^\infty$ s.t. $\nabla_i F \neq 0$, $\forall i$, and an interval I such that $\forall x \in I$, $\exists y$ s.t. $F(x, y) = 0$. Then there exists a function $f : I \to \mathbf{R}$ such that $\forall x \in I$, $F(x, y) = 0$ iff $y = f(x)$. Moreover $f'(x) = -\frac{\partial F}{\partial x} / \frac{\partial F}{\partial y}$.*

Proof. To each $x \in I$ corresponds a unique y such that $F(x, y) = 0$. This follows from the fact that $\frac{\partial F}{\partial y} \neq 0$. The function f such that $f(x) = y$ iff $F(x, y) = 0$ is thus well-defined. We thus have $F(x, f(x)) = 0$ for all $x \in I$ and, differentiating w.r.t. x gives $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} f'(x) = 0$. Hence $f'(x) = -\frac{\partial F}{\partial x} / \frac{\partial F}{\partial y}$. ■

Let Ω a compact subset of $\mathbf{R}^n$. Let f, $g : \Omega \times \mathbf{R} \to \mathbf{R}$, and $f, g \in C^\infty$. The next Theorems explore the solution to the optimization problem:

$$\max_{\underline{x} \in \Omega} f(\underline{x}, a) \quad s.t. \quad g(\underline{x}, a) = 0 \tag{67}$$

Let $X(a) \subset \Omega$ denote the set of solutions to problem (67), and $M(a)$ the maximum value attained. A solution $\underline{x}^*(a)$ is said to be interior if it belongs to some open set within Ω. In what follows ∇f and ∇g refer to variations in the variables of the vector $\underline{x}$ only.

Theorem 2 [**Karush-Kuhn-Tucker Theorem**] *For all interior solution $\underline{x}^*(a) \in X(a)$, $\exists \lambda(\underline{x}^*(a)) \in \mathbf{R}$ such that $\nabla f(\underline{x}^*(a), a) = \lambda \nabla g(\underline{x}^*(a), a)$.*

Proof. If $\nabla f(\underline{x}^*, a) = 0$ just choose $\lambda = 0$. Hence, assume $\nabla f(\underline{x}^*, a) \neq 0$. If $\nabla f(\underline{x}^*, a)$ and $\nabla g(\underline{x}^*, a)$ are not proportional we can move orthogonally to $\nabla g(\underline{x}^*, a)$, leaving g unchanged, and raise f.

∎

Theorem 3 **[Envelope Theorem]** *Let $a \in \mathbf{R}$, $x^*(a)$ an interior solution to (67), and λ such that $\nabla f(\underline{x}^*(a), a) = \lambda \nabla g(\underline{x}(a)^*, a)$. Define $\mathcal{L} = f - \lambda g$. Then*

$$\frac{dM}{da} = \frac{\partial \mathcal{L}}{\partial a}(\underline{x}^*(a), a) \tag{68}$$

Proof. We have, using the chain rule

$$\frac{dM}{da} = \nabla f(\underline{x}^*(a), a).\frac{dx^*(a)}{da} + \frac{\partial f}{\partial a}(\underline{x}^*(a), a) \tag{69}$$

Thus, by Proposition 2

$$\frac{dM}{da} = \lambda \nabla g(\underline{x}^*(a), a).\frac{dx^*(a)}{da} + \frac{\partial f}{\partial a}(\underline{x}^*(a), a) \tag{70}$$

Now notice that $g(\underline{x}^*(a), a) = 0$ for all a. This implies that

$$\nabla g(\underline{x}^*(a), a).\frac{dx^*(a)}{da} + \frac{\partial g}{\partial a}(\underline{x}^*(a), a) = 0 \tag{71}$$

∎

Theorem 4 **[Maximum Theorem]** *$X(a)$ is upper-hemicontinuous, and in particular continuous if it is single-valued.*

Proof. Suppose for a contradiction that we can find two sequences, $(a_n)_{n\in\mathbf{N}}$ and $(\underline{y}_n)_{n\in\mathbf{N}}$, such that $a_n \to a$, $\underline{y}_n \to y$, $\underline{y}_n \in X(a_n)$ for all n, and $\underline{y} \notin X(a)$.

Let $\underline{x} \in X(a)$ and $f(\underline{x}, a) = M$. We can find $\varepsilon > 0$ such that $f(\underline{y}, a) < M - \varepsilon$ and, by continuity, $f(\underline{y}_n, a_n) \leq M - \varepsilon$, for large enough n. But this contradicts $\underline{y}_n \in X(a_n)$ for all n, since clearly for n large enough we can find $\underline{z}_n$ in the neighbourhood of $\underline{x}$ such that $f(\underline{z}_n, a_n) > M - \varepsilon$. ∎

Theorem 5 **[Hyperplane Theorem]** *Let C a closed and convex set and $\underline{s} \notin C$. There exists $\underline{p}$, and l, such that $\underline{p}.\underline{s} < l$ while $\underline{p}.\underline{x} > l$, $\forall \underline{x} \in C$.*

Proof. Let $\underline{x}$ denote the point in C closest to $\underline{s}$ (such a point exists since C is closed, and it is unique since C is convex). Let $y \in C - \{\underline{x}\}$. By convexity of C, $\lambda \underline{y} + (1-\lambda)\underline{x} \in C$ for all

$\lambda \in [0,1]$. Let $\underline{p} = \underline{x} - \underline{s}$, and $l = \underline{p}.\underline{x}$. We first have

$$\underline{p}.\underline{s} = \underline{p}.(\underline{x} - \underline{p}) < l \tag{72}$$

Observe next that

$$\|\underline{s} - \big(\lambda\underline{y} + (1-\lambda)\underline{x}\big)\|^2 = \|\big(\underline{s}-\underline{x}\big) - \lambda\big(\underline{y}-\underline{x}\big)\|^2 = \|\underline{s}-\underline{x}\|^2 + \lambda^2\|\underline{y}-\underline{x}\|^2 - 2\lambda\big(\underline{s}-\underline{x}\big).\big(\underline{y}-\underline{x}\big) \tag{73}$$

Since by definition $\underline{x}$ is the point in C closest to $\underline{s}$, we must have $\big(\underline{s} - \underline{x}\big).\big(\underline{y} - \underline{x}\big) \leq 0$, i.e. $\underline{p}.\big(\underline{y} - \underline{x}\big) \geq 0$ or equivalently $\underline{p}.\underline{y} \geq l$ for all $y \in C - \{\underline{x}\}$. ■

Theorem 6 **[Jensen's inequality]** *Let X a random variable and u concave. Then $\mathbb{E}(u(X)) \leq u(\mathbb{E}(X))$.*

Proof. Let $c = \mathbb{E}(X)$. Since u is concave we can find b such that $u(x) \leq u(c) + b(x - c)$. Hence

$$\mathbb{E}\big(u(X)\big) \leq \mathbb{E}\big(u(c) + b(X - c)\big) = u(\mathbb{E}(X)) \tag{74}$$

■