LECTURE NOTES

ON

DIGITAL SIGNAL PROCESSING

6th sem DIPLOMA ENGINEERING

DEPARTMENT OF ELECTRONICS AND COMMUNICATIONS ENGINEERING

SKDAV GOVT.POLYTECHNIC

ROURKELA

Introduction

Signal

A signal is any physical quantity that carries information, and that varies with time, space, or any other independent variable or variables. Mathematically, a signal is defined as a function of one or more independent variables.

1 – Dimensional signals mostly have time as the independent variable. For example,

Eg., $S_1 (t) = 20\, t^2$

2 – Dimensional signals have two independent variables. For example, image is a 2 – D signal whose independent variables are the two spatial coordinates (x,y)

Eg., $S_2 (t) = 3x + 2xy + 10y^2$

Video is a 3 – dimensional signal whose independent variables are the two spatial coordinates, (x,y) and time (t).
Similarly, a 3 – D picture is also a 3 – D signal whose independent variables are the three spatial coordinates (x,y,z).

Signals $S_1 (t)$ and $S_2 (t)$ belong to a class that are precisely defined by specifying the functional dependence on the independent variables.
Natural signals like speech signal, ECG, EEG, images, videos, etc. belong to the class which cannot be described functionally by mathematical expressions.

System

A system is a physical device that performs an operation on a signal. For example, natural signals are generated by a system that responds to a stimulus or force.
For eg., speech signals are generated by forcing air through the vocal cords. Here, the vocal cord and the vocal tract constitute the system (also called the vocal cavity). The air is the stimulus.
The stimulus along with the system is called a signal source.

An electronic filter is also a system. Here, the system performs an operation on the signal, which has the effect of reducing the noise and interference from the desired information – bearing signal.

When the signal is passed through a system, the signal is said to have been processed.

Processing

The operation performed on the signal by the system is called **Signal Processing.** The system is characterized by the type of operation that it performs on the signal. For example, if the operation is linear, the system is called linear system, and so on.

Digital Signal Processing

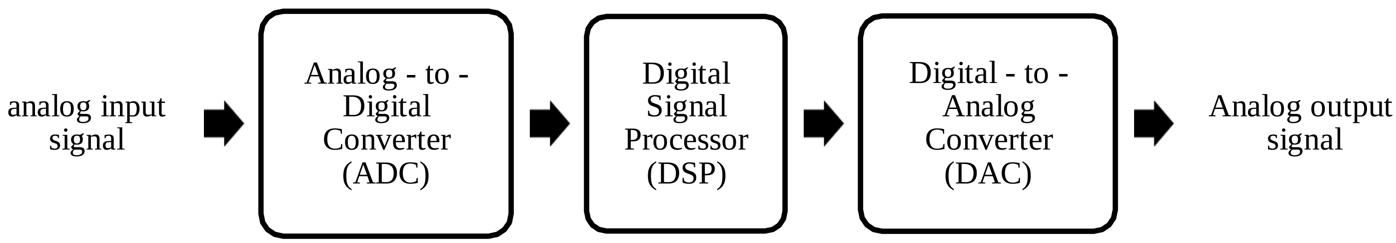
Digital Signal Processing of signals may consist of a number of mathematical operations as specified by a software program, in which case, the program represents an implementation of the system in software. Alternatively, digital processing of signals may also be performed by digital hardware (logic circuits). So, a digital system can be implemented as a combination of digital hardware and software, each of which performs its own set of specified operations.

Basic elements of a Digital Signal Processing System

Most of the signals encountered in real world are analog in nature .i.e., the signal value and the independent variable take on values in a continuous range. Such signals may be processed directly by appropriate analog systems, in which case, the processing is called **analog signal processing**. Here, both the input and output signals are in analog form.

These analog signals can also be processed digitally, in which case, there is a need for an interface between the analog signal and the **Digital Signal Processor**. This interface is called the **Analog – to – Digital Converter (ADC)**, whose output is a digital signal that is appropriate as an input to the digital processor.
In applications such as speech communications, that require the digital output of the digital signal processor to be given to the user in analog form, another interface from digital domain to analog domain is required. This interface is called the **Digital – to – Analog Converter (DAC)**.
In applications like radar signal processing, the information extracted from the radar signal, such as the position of the aircraft and its speed are required in digital format. So, there is no need for a DAC in this case.

Block Diagram Representation of Digital Signal Processing



Advantages of Digital Signal Processing over Analog Signal Processing

1. A digital programmable system allows flexibility in reconfiguring the digital signal processing operations simply by changing the program.
   Reconfiguration of an analog system usually implies a redesign of the hardware followed by testing and verification.
2. Tolerances in analog circuit components and power supply make it extremely difficult to control the accuracy of analog signal processor.
   A digital signal processor provides better control of accuracy requirements in terms of word length, floating – point versus fixed – point arithmetic, and similar factors.
3. Digital signals are easily stored on magnetic tapes and disks without deterioration or loss of signal fidelity beyond that introduced in A/D conversion. So the signals become transportable and can be processed offline.
4. Digital signal processing is cheaper than its analog counterpart.
5. Digital circuits are amenable for full integration. This is not possible for analog circuits because inductances of respectable value (μH or mH) require large space to generate flux.
6. The same digital signal processor can be used to perform two operations by time multiplexing, since digital signals are defined only at finite number of time instants.

7. Different parts of digital signal processor can work at different sampling rates.
8. It is very difficult to perform precise mathematical operations on signals in analog form but these operations can be routinely implemented on a digital computer using software.
9. Several filters need several boards in analog signal processing, whereas in digital signal processing, same DSP processor is used for many filters.

Disadvantages of Digital Signal Processing over Analog Signal Processing

1. Digital signal processors have increased complexity.
2. Signals having extremely wide bandwidths require fast – sampling – rate ADCs. Hence the frequency range of operation of DSPs is limited by the speed of ADC.
3. In analog signal processor, passive elements are used, which dissipate very less power.
   In digital signal processor, active elements like transistors are used, which dissipate more power.

The above are some of the advantages and disadvantages of digital signal processing over analog signal processing.

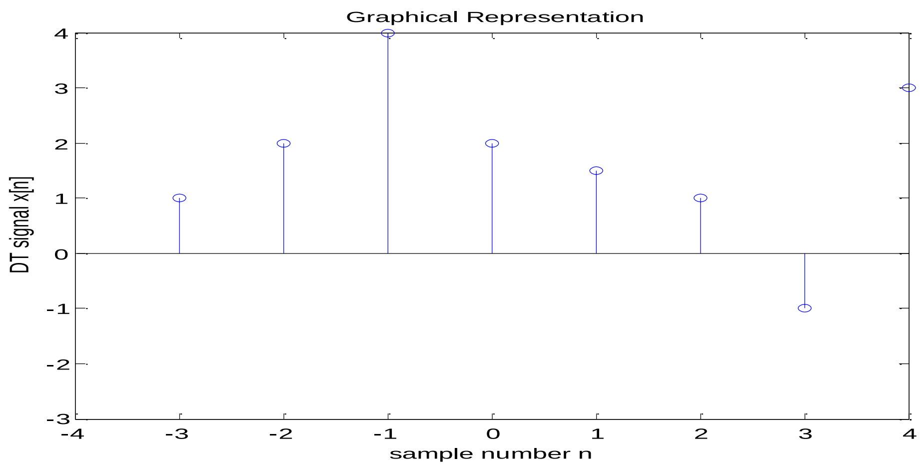
Discrete – time signals

A discrete time signal is a function of an independent variable that is an integer, and is represented by x [n] , where n represents the sample number (**and not the time at which the sample occurs).**
A discrete time signal is not defined at instants between two successive samples, or in other words, for non – integer values of n. (**But, it is not zero, if n is not an integer).**

Discrete time signal representation

The different representations of a discrete time signal are

1. **Graphical Representation**



2. **Functional representation**

$$x[n] = \begin{cases} 1, for\ n = 1, 2\ 3 \\ 4, for\ n = 2 \\ 0, elsewhere \end{cases}$$

3. **Tabular representation**

N	-	- - - - -	-2	-1	0	1	2	3	4	5	- - - - - -
x [n]	-	- - - - -	0	0	1	1	4	1	0	0	- - - - - -

4. **Sequence representation**

x [n] = { - , -, -. -, - , 0, 0, 1, 4, 1, 0, 0, - , - , - , - }
↑

the above is a representation of a two – sided infinite duration sequence, and the symbol ↑ indicates the time origin (n = 0).

If the sequence is zero for n < 0, it can be represented as

x [n] = { 1, 4, 1, 2, - , - , - , - }

Here the leftmost point in the sequence is assumed to be the time origin, and so the symbol ↑ is optional in this case.

A finite duration sequence can be represented as
x[n] = { 3, -1, -2, 5, 0, 4, -1}
↑
This is referred to as a 7 – point sequence.

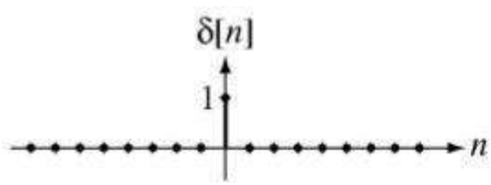
Elementary discrete time sequences

These are the basic sequences that appear often, and play an important role. Any arbitrary sequence can be represented in terms of these elementary sequences.

1. **Unit – Sample sequence** It is denoted by δ [n]. It is defined as

$$\delta[n] = \begin{cases} 1, for\ n = 0 \\ 0, for\ n \neq 0 \end{cases}$$

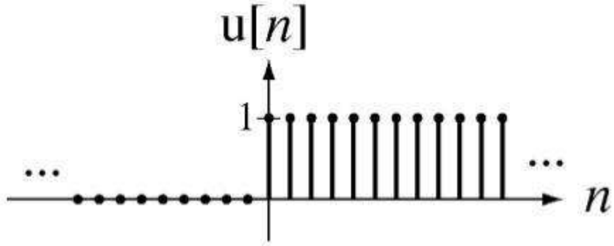
It is also referred as discrete time impulse.
It is mathematically much less complicated than the continuous impulse δ (t), which is zero everywhere except at t = 0. At t = 0, it is defined in terms of its area (unit area), but not by its absolute value.
It is graphically represented as



2. **Unit step sequence**
It is denoted by u [n] and defined as

$$u[n] = \begin{cases} 1, for\ n \geq 0 \\ 0, for\ n < 0 \end{cases}$$

It is graphically represented as

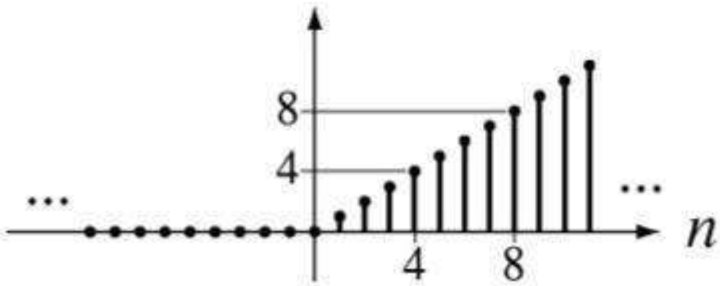


3. <u>Unit ramp sequence</u>

It is denoted by U_r[n], and is defined as

$$u_r[n] = \begin{cases} n, for\ n \geq 0 \\ 0, for\ n < 0 \end{cases}$$

It is graphically represented as



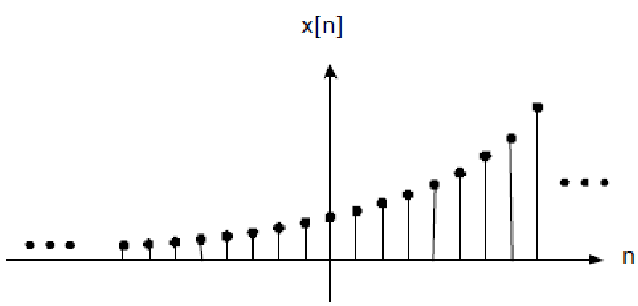
4. <u>Exponential sequence</u>

It is defined as

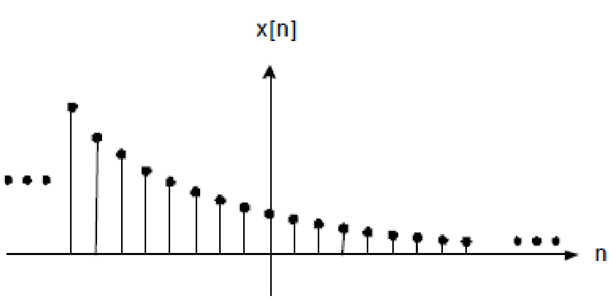
$$x[n] = a^n\ for\ all\ n$$

a. If a is real, x[n] is a real exponential.

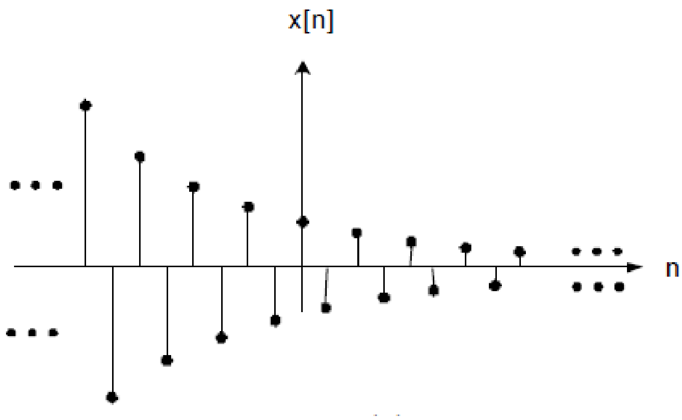
a > 1



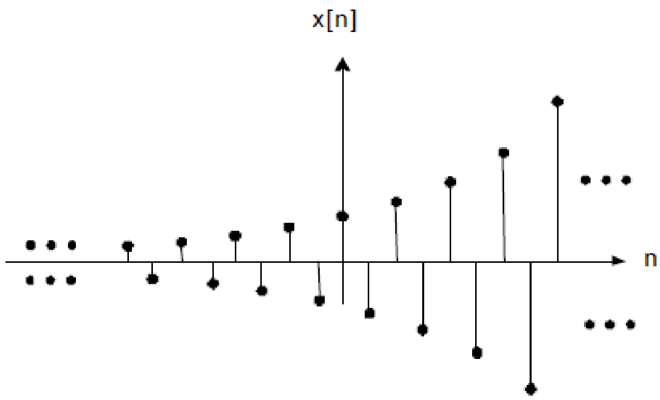
a < 1



-1 < a < 0



a < -1



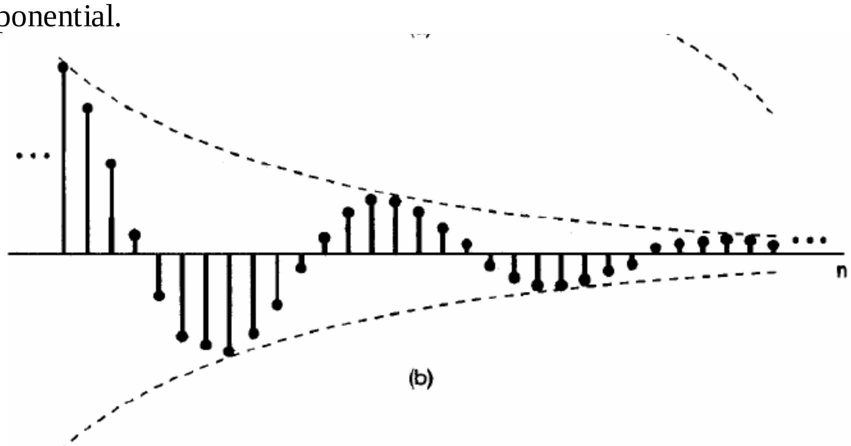
b. If a is complex valued, then a can be expressed as a = re$^{j\theta}$, so that x[n] can be represented as

$$x[n] = r^n e^{jn\theta}$$
$$= r^n[\cos n\theta + j \sin n\theta]$$

So, x [n] is represented graphically by plotting the real part and imaginary parts separately as functions of n, which are

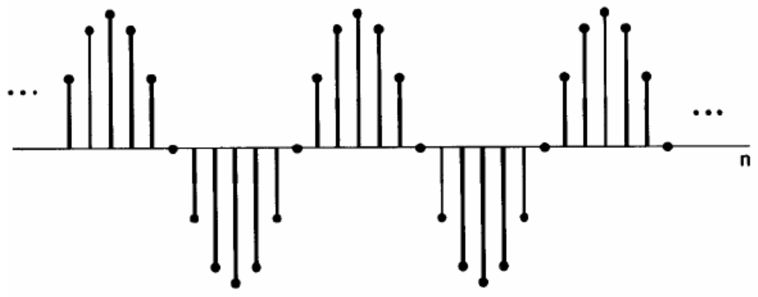
$$x_R[n] = r^n \cos n\theta$$
$$x_I[n] = r^n \sin n\theta$$

If r < 1, the above two functions are damped cosine and sine functions, whose amplitude is a decaying exponential.

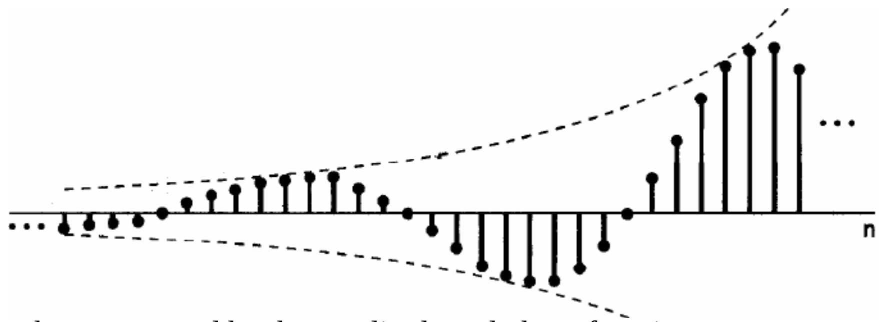


(b)

If r = 1, then both the functions have fixed amplitude of unity.



If r > 1, then they are cosine and sine functions respectively, with exponentially growing amplitudes.

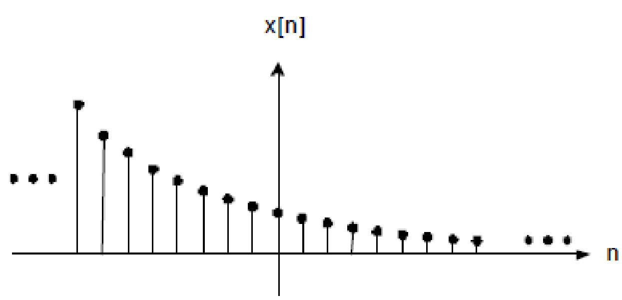


Alternatively, x [n] can be represented by the amplitude and phase functions:

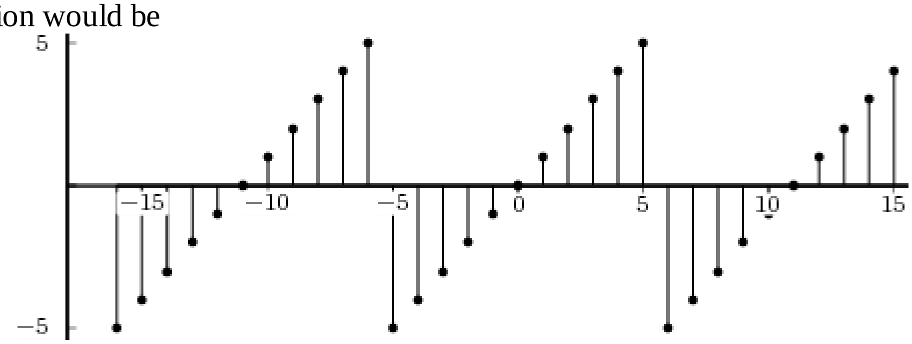
$$\text{Amplitude function, } A[n] = |x[n]| = r^n$$

$$\text{Phase function, } \emptyset[n] = \angle x[n] = n\theta$$

For example, for r < 1, the amplitude function would be



And the phase function would be



Although the phase function $\emptyset[n] = n\theta$ is a linear function of n, it is defined only over an interval of 2π (since it is an angle).i.e., over an interval –π<θ<π or 0<θ<2π.

So we subtract multiples of 2π from $\emptyset[n]$ before plotting .i.e., we plot $\emptyset[n]$ modulo 2π instead of $\emptyset[n]$. This results in a piecewise linear graph for the phase function, instead of a linear graph.

Classification of Discrete – Time Sequences:

1. **Energy Signals and Power Signals**

   The energy of a signal x[n] is defined as

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

   If this energy is finite, i.e., $0 < E < \infty$, then x[n] is called an **Energy Signal**.

   For signals having infinite energy, the average power can be calculated, which is defined as

$$P_{av} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

$$or, P_{av} = \lim_{N\to\infty} \frac{1}{2N+1} E_N , \qquad where$$

   E_N = signal energy of x[n] over the finite interval $-N \leq n \leq N$, .i.e.,

$$E = \lim_{N\to\infty} E_N$$

- For signals with finite energy .i.e., for Energy Signals, E is finite, thus resulting in zero average power. So, for energy signals, $P_{av} = 0$.
- Signals with infinite energy may have finite or infinite average power. If the average power is finite and nonzero, such signals are called **Power Signals**.
- Signals with finite power have infinite energy.
- If both energy, E as well as average power, P_{av} of a signal are infinite, then the signal is neither an energy signal nor a power signal.
- Periodic signals have infinite energy. Their average power is equal to its average power over one period.
- A signal cannot both be an energy signal and a power signal.
- All practical signals are energy signals.

2. **Periodic and aperiodic signals**

   A signal x[n] is periodic with period N if and only if

$$x[n+N] = x[n] \ \forall \ n$$

   The smallest N for which the above relation holds is called the **fundamental period.**
   If no finite value of N satisfies the above relation, the signal is said to be **aperiodic** or **non – periodic.**
   The sum of M periodic Discrete – time sequences with periods N_1, N_2, …, N_M, is always periodic with period N where

$$N = LCM(N_1, N_2, \ldots, N_M)$$

3. **Even and Odd Signals**

   A real – valued discrete – time signal is called an **Even Signal** if it is identical with its reflection about the origin .i.e., it must be symmetrical about the vertical axis.

$$x[n] = x[-n] \ \ \forall n$$

A real – valued discrete – time signal is called an **Odd Signal** if it is antisymmetrical about the vertical axis.

$$x[n] = -x[-n] \ \ \forall n$$

From the above relation, it can be inferred that an odd signal must be zero at time origin, n = 0.

Every signal x[n] can be expressed as the sum of its even and odd components.

$$x[n] = x_e[n] + x_o[n]$$

Where

$$x_e[n] = \frac{x[n] + x[-n]}{2}$$
$$x_o[n] = \frac{x[n] - x[-n]}{2}$$

- Product of even and odd sequences results in an odd sequence.
- Product of two odd sequences results in an even sequence.
- Product of two even sequences results in an even sequence.

4. <u>Conjugate Symmetric and Conjugate Antisymmetric sequences</u>

A complex discrete – time signal is **conjugate – symmetric** if

$$x[n] = x^*[-n] \quad \forall n$$

And **conjugate – antisymmetric** if

$$x[n] = -x^*[-n] \quad \forall n$$

Any complex signal can be expressed as the sum of conjugate – symmetric and conjugate – antisymmetric parts

$$x[n] = x_{cs}[n] + x_{ca}[n]$$

Where

$$x_{cs}[n] = \frac{x[n] + x^*[-n]}{2}$$

And

$$x_{ca}[n] = \frac{x[n] - x^*[-n]}{2}$$

5. <u>Bounded and Unbounded sequences</u>

A discrete – time sequence x[n] is said to be **bounded** if each of its samples is of finite magnitude .i.e.,

$$|x[n]| \leq M_x < \infty \ \ \forall n$$

For example,

The unit step sequence u[n] is a bounded sequence,
but the sequence nu[n] is an unbounded sequence.

6. <u>Absolutely summable and square summable sequences</u>

A discrete – time sequence x[n] is said to be **absolutely summable** if,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

And it is said to be **square summable** if

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \quad (\boldsymbol{Energy\ Signal})$$

Discrete – Time Systems

A system accepts an input such as voltage, displacement, etc. and produces an output in response to this input. A system can be viewed as a process that results in transforming input signals into output signals.

A discrete – time system can be represented as

$$x[n] \;\rightarrow\; y[n]$$

$$or, \qquad y[n] = T\,\{x[n]\}$$

Discrete – Time System Properties

1. **Linearity**

A system is said to be **linear** if it satisfies superposition principle, which in turn is a combination of **additivity** and **homogeneity**.

Additivity implies that

If the response of the DT system to $x_1[n]$ is $y_1[n]$, and the response to $x_2[n]$ is $y_2[n]$, then the response of the system to $\{x_1[n]+x_2[n]\}$ must be $\{y_1[n]+y_2[n]\}$.

Homogeneity implies that

if the response of a DT system to x[n] is y[n], then the response of the system to ax[n] must be ay[n], where a is a constant.

Thus, for a DT system,

If

$$x[n] \;\rightarrow\; y[n]$$

$$x_1[n] \;\rightarrow\; y_1[n]$$

$$and, \qquad x_2[n] \;\rightarrow\; y_2[n]$$

Then according to **additivity principle**

$$x_1[n] + x_2[n] \;\rightarrow\; y_1[n] + y_2[n]$$

And according to **homogeneity principle**

$$ax[n] \;\rightarrow\; ay[n] \;\; (a = constant)$$

- If a = 0, then the above relation implies that a zero input must result in a zero output.

Combining the above two principle to get **superposition principle**, we obtain

A system is Linear if it satisfies the following relation

$$ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n] \; (a, b = constants)$$

2. **Time – Variant and Time – Invariant Systems**

A system is **time – invariant** if its characteristics and behavior are fixed over time .i.e., a time – shift in input signal causes an identical time – shift in output signal.

$$if\ x[n] \ \rightarrow\ y[n]$$
$$then,\ x[n-n_0]\ \rightarrow y[n-n_0]\ \ \forall\ n_0$$

If the above the relation is not satisfied, then the system is **time – variant**.

3. **Causal and Non – causal Systems**

A system is **causal or non – anticipatory or physically realizable**, if the output at any time n_0 depends only on present and past inputs ($n \leq n_0$), but not on future inputs.
In other words, if the inputs are equal upto some time n_0, the corresponding outputs must also be equal upto that time n_0, for a **causal system.**

4. **Stable and unstable systems**

A **stable system** is one in which, a bounded input results in a response that does not diverge. Then the system is said to be **BIBO stable**.
For a system, if the input is bounded .i.e,

$$if\ \ |x[n]| \leq M_x < \infty\ \ \forall n$$

And if the corresponding output is also bounded .i.e.,

$$|y[n]| \leq M_y < \infty\ \ \forall n$$

Then the system is said to be **BIBO stable.**

5. **Memory and memoryless systems**

A system is said to possess **memory**, or is called a **dynamic system**, if its output depends on past or future values of the input.
If the output of the system depends only on the present input, the system is said to be **memoryless.**

6. **Invertible systems**

A system is said to be **invertible** if by observing the output, we can determine its input. i.e., we can construct an inverse system that when cascaded with the given system, yields an output equal to the original input.
A system can have inverse if distinct inputs lead to distinct outputs.

7. **Passive and lossless systems**

A system is said to be **passive** if the output y[n] has at most the same energy as the input.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 \leq \sum_{n=-\infty}^{\infty} |y[n]|^2 < \infty$$

If the energy of the output is equal to the energy of the input, then the system is said to be **lossless.**

Properties of Unit Impulse Sequence

Multiplication property

When a sequence x[n] is multiplied by a unit impulse located at k i.e., δ[n-k], picks out a single value/sample of x[n] at the location of the impulse i.e., x[k].

$$x[n]\delta[n-k] = x[k]\delta[n-k]$$
$$= impulse\ with\ strength\ x[k] located\ at\ n = k$$

Sifting property

The impulse function δ[n-k] "sifts" through the function x[n] and pulls out the value x[k]

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n-k] = x[k]$$

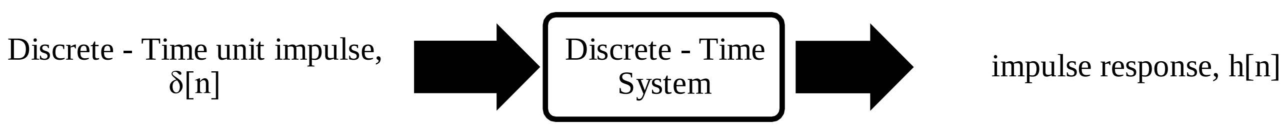
Signal decomposition

Any arbitrary sequence x[n] can be expressed as a weighted sum of shifted impulses.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\ \delta[n-k]$$

Impulse response

Impulse response of a discrete – time system is defined as the output/response of the system to unit impulse input and is represented by **h[n]**.



If for a system,

$$x[n] \rightarrow y[n]$$

Then,

$$\delta[n] \rightarrow h[n]$$

If the DT system satisfies the property of time – invariance, then,

$$\delta[n-k] \rightarrow h[n-k]$$

In addition to being time – invariant, if the system also satisfies linearity (homogeneity and additivity), then,

Homogeneity:

$$x[k]\delta[n-k] \rightarrow x[k]h[n-k]$$

Additivity:

$$\sum_{k=-\infty}^{\infty} \delta[n-k] \rightarrow \sum_{k=-\infty}^{\infty} h[n-k]$$

Combining the above two properties, a **Linear Time – Invariant (LTI)** System can be described by the input – output relation by

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

The Left hand side is the input x[n] expressed as a weighted sum of shifted impulses (from signal decomposition property of impulse function). So, the right hand side must be the output y[n] of the DT system in response to input x[n].

Thus the output of a **Linear Time – Invariant (LTI) system** can be expressed as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$or, y[n] = x[n] * h[n]$$

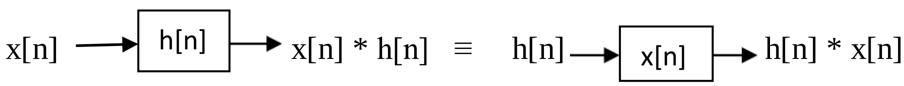
The above relation is called **Convolution Sum.**

So, the impulse response h[n] of an LTI DT system completely characterizes the system .i.e., a knowledge of h[n] is sufficient to obtain the response of an LTI system to any arbitrary input x[n].

Properties of Convolution Sum

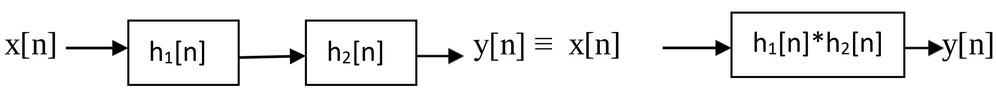
1. **Commutative Property**

$$x[n] \; * \; h[n] = h[n] \; * \; x[n]$$



2. **Associative Property**

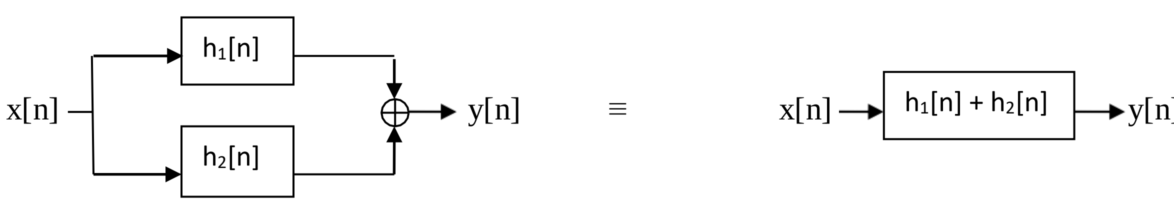
$$x[n] * \{h_1[n] * h_2[n]\} = \{x[n] * h_1[n]\} * h_2[n]$$



From this property it can be inferred that, a cascade combination of LTI systems can be replaced by a single system whose impulse response is the convolution of the individual impulse responses.

3. **Distributive Property**

$$x[n] * \{h_1[n] + h_2[n]\} = \{x[n] * h_1[n]\} + \{x[n] * h_2[n]\}$$



From this property, it can be inferred that, a parallel combination of LTI systems can be replaced by a single system whose impulse response is the sum of individual responses.

Relation between LTI system properties and impulse response

Memory

For an LTI system to be memoryless, the impulse response must be zero for nonzero sample positions.

$$h[n] = 0 \; for \; n \neq 0$$

$$h[n] = k\,\delta[n] \;\; where \; k = constant$$

Causality

For an LTI system to be causal, its impulse response must be zero for negative time instants.

$$h[n] = 0 \; for \; n < 0$$

So, for a causal LTI system the output (from the convolution sum equation) can be expressed as

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

$$or, y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

Stability

An LTI system is BIBO stable if its impulse response is absolutely summable.

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Invertibility

An LTI system with impulse response h[n] is invertible if we can design another LTI system with impulse response $h_I[n]$ such that

$$h[n] * h_I[n] = \delta[n]$$

LTI systems characterized by Linear Constant – Coefficient Difference Equations (LCCDE)

In general, any LTI system with input x[n] and output y[n] can be described by an LCCDE as follows

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \;, \qquad a_0 \equiv 1$$

$$or, y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{l=0}^{M} b_l x[n-l]$$

Where N is called the **order** of the difference equation/ system.

This equation expresses the output of an LTI system at time n in terms of present and past inputs and past outputs.

Solution of LCCDE (Direct Solution – Solution in time domain)

Given an LCCDE, the goal is to determine the output y[n], n≥ 0 given a specific input x[n], n ≥ 0, and a set of initial conditions.

The total solution of the LCCDE is assumed to be the sum of two parts:

Homogeneous/complementary solution, $y_H[n]$ and

Particular solution, $y_P[n]$

Homogeneous Solution

The homogeneous difference equation is obtained by substituting input x[n]=0 in the LCCDE.

$$\sum_{k=0}^{N} a_k y[n-k] = 0 ------Eq.\,1$$

The solution to this homogeneous equation is assumed to be in the form of an exponential .i.e.,

$$y_h[n] = \lambda^n ------Eq.\,2$$

Substituting Eq. 2 in Eq. 1, we obtain

$$\sum_{k=0}^{N} a_k \lambda^{n-k} = 0 \,,\, a_0 = 1$$

Expanding this equation

$$\lambda^{n-N}(\lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \cdots\cdots + a_{N-1}\lambda + a_N) = 0$$

The polynomial in the parenthesis is called the characteristic polynomial of the system.

The characteristic equation is given by

$$\lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \cdots\cdots + a_{N-1}\lambda + a_N = 0$$

Its solution has N roots denoted by $\lambda_1, \lambda_2, \ldots, \lambda_N$, which can be real or complex.

Complex valued roots occur as complex conjugate pairs.

If some roots are identical, then we have multiple order roots.

If all roots are distinct, then the general solution is given by

$$y_H[n] = C_1\lambda_1{}^n + C_2\lambda_2{}^n + \cdots\cdots + C_N\lambda_N{}^n$$

C_1, C_2, … , C_N are weighting coefficients.

For multiple order roots, if λ_1 repeats m times, then the solution is given by

$$y_H[n] = C_1\lambda_1{}^n + C_2 n\lambda_1{}^n + C_3 n^2\lambda_1{}^n \cdots\cdots + C_m n^{m-1}\lambda_1{}^n + C_{m+1}\lambda_2{}^n + C_{m+2}\lambda_3{}^n + \cdots C_N\lambda_N{}^n$$

Particular solution

The particular solution must satisfy the LCCDE for the specific input signal x[n], $n \geq 0$.

We assume a form for $y_P[n]$ that depends on the form of the input x[n] as follows

Input, x[n]	**Particular solution, $y_P[n]$**
Constant, A	Constant, K
$A M^n$	KM^n
An^M	$K_0n^M + K_1n^{M-1}+\ldots+K_M$
$A^n n^M$	$A^n(K_0n^M+K_1n^{M-1}+\ldots+K_M)$
$A \cos \omega_0 n$ $A \sin \omega_0 n$	$K_1 \cos \omega_0 n + K_2 \sin \omega_0 n$

If the particular solution, $y_P[n]$ has the same form as the homogeneous solution $y_H[n]$, we multiply $y_P[n]$ with n or n^2 or n^3 so that it is different from $y_H[n]$.

Total solution $y[n] = y_H[n]+y_P[n]$

The total solution will contain $\{C_i\}$s from the homogeneous solution. They are determined by substituting the given initial conditions in the total solution.

Frequency domain representation of discrete time signals

The concept of frequency is closely related to a specific type of periodic motion called harmonic oscillation, which is described by sinusoidal functions. The CT and DT sinusoidal signals are characterized by the following properties:

1. A continuous time sinusoid $x(t) = \cos(2\pi f_a t)$ is periodic for any value of f_a.

But for DT sinusoid $x[n]=\cos(2\pi f_d n)$ to be periodic with period N (an integer), we require

$$\cos(2\pi f_d n) = \cos[2\pi f_d(n+N)] = \cos(2\pi f_d n + 2\pi f_d N)$$

This is possible only if

$$2\pi f_d N = 2\pi k \;\; (k \text{ is an integer})$$

Or

$$f_d = \frac{k}{N}$$

i.e., the discrete frequency f_d must be a rational number (ratio of two integers).

Similarly, a discrete time exponential $e^{j\omega n}$ is periodic only if $\frac{\omega}{2\pi} = f_d = rational\, number.$

The period is the denominator after $\frac{\omega}{2\pi}$ is simplified such that in $\frac{\omega}{2\pi} = \frac{k}{N}$, k and N are relatively prime.

2. A CT sinusoidal signal x(t) = cos(Ωt) has a unique waveform for every value of Ω, 0 < Ω < ∞. Increasing Ω results in a sinusoidal signal of ever – increasing frequency.
   But, for a DT sinusoidal signal cos (ωn), considering two frequencies separated by an integer multiple of 2π, (ω and ω ± 2πm, m is an integer), we have
   $$\cos[(\omega \pm 2\pi m)n] = \cos(\omega n \pm 2\pi mn)$$
   Since m and n are both integers
   $$\cos(\omega n \pm 2\pi mn) = \cos(\omega n)$$
   So, a DT sinusoidal sequence has unique waveform only for the values of ω over a range of 2π. The range $-\pi \leq \omega \leq \pi$ defines the fundamental range of frequencies or principal range.
3. The highest rate of oscillation in a DT sinusoidal sequence is attained when ω=π or ω= - π . the rate of oscillation increases continually as ω increases from 0 to π, then decreases as ω increases from π to 2π. So low – frequency DT sine waves have ω near 0 or any even multiple of π, while the high – frequency sine waves have ω near ± π or other odd multiples of π.

Frequency domain representation of discrete time systems

The frequency response function completely characterizes a linear time invariant system in the frequency domain. Since, most signals can be expressed in Fourier domain as a weighted sum of harmonically related exponentials, the response of an LTI system to this class of signals can be easily determined.

The response of any relaxed LTI system to an arbitrary input signal x[n] is given by the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Here, the system is characterized in the time domain by its impulse response h[n]. to develop a frequency domain characterization of the system, we excite the system with the complex exponential

$$x[n] = Ae^{j\omega n}, -\infty < n < \infty$$

Where A is the amplitude and ω is any arbitrary frequency confined to the frequency interval [- π, π]. By substituting this in the above convolution sum, we obtain the response as

$$y[n] = \sum_{k=-\infty}^{\infty} h[k][Ae^{j\omega(n-k)}]$$

$$= A \left[\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \right] e^{j\omega n}$$

Here, the term inside the brackets is a function of frequency ω. It is the Fourier Transform of the impulse response h[n], and is denoted by

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

And $y[n] = AH(\omega)\, e^{j\omega n}$

Since the output differs from the input only by a constant multiplicative factor, the exponential input signal is called the **eigen function** of the system, and the multiplicative factor is called the **eigenvalue** of the system.

H(ω) is a complex valued function of the frequency variable ω.

UNIT-3

-

PREREQISTING DISCUSSION ABOUT Z TRANSFORM

For analysis of continuous time LTI system Laplace transform is used. And for analysis of discrete time LTI system z transform is used. Z transform is mathematical tool used for conversion of time domain into frequency domain (z domain) and is a function of the complex valued variable Z. The z transform of a discrete time signal x(n) denoted by X(z) and given as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\ z^{-n} \qquad \textbf{z-Transform……(1)}$$

Z transform is an infinite power series because summation index varies from -∞ to ∞. But it is useful for values of z for which sum is finite. The values of z for which f (z) is finite and lie within the region called as –region of convergence (ROC).

ADVANTAGES OF Z TRANSFORM

1. The DFT can be determined by evaluating z transform.
2. Z transform is widely used for analysis and synthesis of digital filter.
3. Z transform is used for linear filtering. z transform is also used for finding Linear convolution, cross-correlation and auto-correlations of sequences.
4. In z transform user can characterize LTI system (stable/unstable, causal/anti- causal) and its response to various signals by placements of pole and zero plot.

ADVANTAGES OF ROC(REGION OF CONVERGENCE)

1. ROC is going to decide whether system is stable or unstable.
2. ROC decides the type of sequences causal or anti-causal.
3. ROC also decides finite or infinite duration sequences.

Z TRANSFORM PLOT

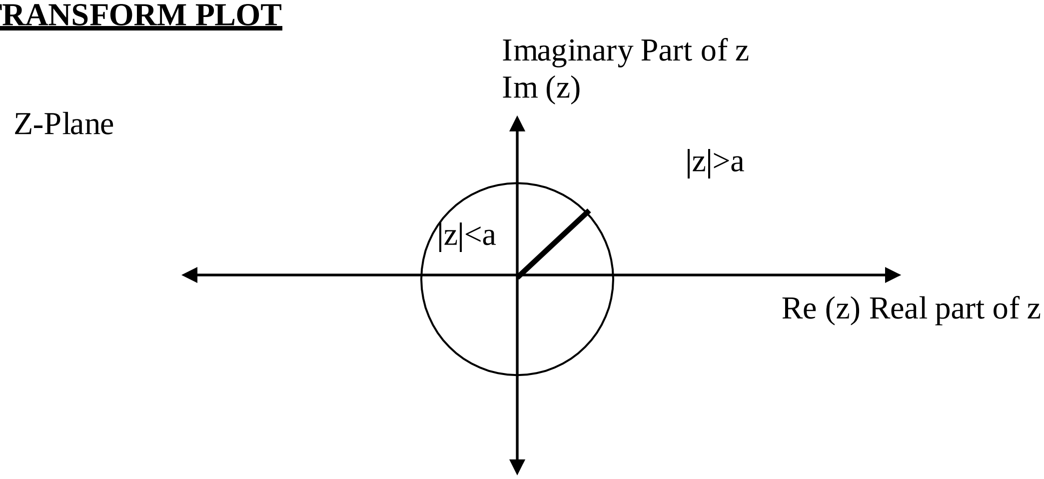


Fig show the plot of z transforms. The z transform has real and imaginary parts. Thus a plot of imaginary part versus real part is called complex z-plane. The radius of circle is 1 called as unit circle.

This complex z plane is used to show ROC, poles and zeros. Complex variable z is also expressed in polar form as $Z = re^{j\omega}$ where r is radius of circle is given by |z| and ω is the frequency of the sequence in radians and given by ∟z.

S.No	Time Domain Sequence	Property	z Transform	ROC
1	δ(n) (Unit sample)		1	complete z plane
2	δ(n-k)	Time shifting	z^{-k}	except z=0
3	δ(n+k)	Time shifting	z^{k}	except z=∞
4	u(n) (Unit step)		$1/1- z^{-1} = z/z-1$	|z| > 1
5	u(-n)	Time reversal	1/1- z	|z| < 1
6	-u(-n-1)	Time reversal	z/z- 1	|z| < 1
7	n u(n) (Unit ramp)	Differentiation	$z^{-1} / (1- z^{-1})^2$	|z| > 1
8	a^n u(n)	Scaling	$1/1- (az^{-1})$	|z| > |a|
9	$-a^n$ u(-n-1)(Left side exponential sequence)		$1/1- (az^{-1})$	|z| < |a|
10	n a^n u(n)	Differentiation	$a z^{-1} / (1- az^{-}1)^2$	|z| > |a|
11	-n a^n u(-n-1)	Differentiation	$a z^{-1} / (1- az^{-}1)^2$	|z| < |a|
12	a^n for 0 < n < N-1		$1- (a z^{-1})^N / 1- az^{-}1$	$\|az^{-}1\| < \infty$ except z=0
13	1 for 0<n<N-1 or u(n) – u(n-N)	Linearity Shifting	$1- z^{-N}/ 1- z^{-}1$	|z| > 1
14	$\cos(\omega_0 n)$ u(n)		$\frac{1- z^{-1}\cos\omega_0}{1- 2z^{-1}\cos\omega_0+z^{-2}}$	|z| > 1
15	$\sin(\omega_0 n)$ u(n)		$\frac{z^{-1}\sin\omega_0}{1- 2z^{-1}\cos\omega_0+z^{-2}}$	|z| > 1
16	$a^n \cos(\omega_0 n)$ u(n)	Time scaling	$\frac{1- (z/a)^{-1}\cos\omega_0}{1- 2(z/a)^{-1}\cos\omega_0+(z/a)^{-2}}$	|z| > |a|
17	$a^n \sin(\omega_0 n)$ u(n)	Time scaling	$\frac{(z/a)^{-1}\sin\omega_0}{1- 2(z/a)^{-1}\cos\omega_0+(z/a)^{-2}}$	|z| > |a|

Tutorial problems:

Q) Determine z transform of following signals. Also draw ROC.

i) x(n) = {<u>1</u>,2,3,4,5}

ii) x(n)={1,2,3,<u>4</u>,5,0,7}

Q) Determine z transform and ROC for $x(n) = (-1/3)^n u(n) - (1/2)^n u(-n-1)$.

Q) Determine z transform and ROC for $x(n) = [3.(4^n)-4(2^n)] u(n)$.

Q) Determine z transform and ROC for $x(n) = (1/2)^n u(-n)$.

Q) Determine z transform and ROC for $x(n) = (1/2)^n \{u(n) - u(n-10)\}$.

Q) Find linear convolution using z transform. X(n)={1,2,3} & h(n)={1,2}

PROPERTIES OF Z TRANSFORM (ZT)

1) Linearity

The linearity property states that if

$$x1(n) \overset{z}{\longleftrightarrow} X1(z) \text{ And}$$

$$x2(n) \overset{z}{\longleftrightarrow} X2(z) \text{ Then}$$

Then

$$a1\, x1(n) + a2\, x2(n) \overset{z}{\longleftrightarrow} a1\, X1(z) + a2\, X2(z)$$

z Transform of linear combination of two or more signals is equal to the same linear combination of z transform of individual signals.

2) Time shifting

The Time shifting property states that if

$$x(n) \overset{z}{\longleftrightarrow} X(z) \text{ And}$$

Then $$x(n-k) \overset{z}{\longleftrightarrow} X(z)\, z^{-k}$$

Thus shifting the sequence circularly by ‗k' samples is equivalent to multiplying its z transform by z^{-k}

3) Scaling in z domain

This property states that if

$$x(n) \overset{z}{\longleftrightarrow} X(z) \text{ And}$$

Then $$a^n\, x(n) \overset{z}{\longleftrightarrow} x(z/a)$$

Thus scaling in z transform is equivalent to multiplying by a^n in time domain.

4) Time reversal Property

The Time reversal property states that if

$$x(n) \overset{z}{\longleftrightarrow} X(z) \text{ And}$$

Then $$x(-n) \overset{z}{\longleftrightarrow} x(z^{-1})$$

It means that if the sequence is folded it is equivalent to replacing z by z^{-1} in z domain.

5) Differentiation in z domain

The Differentiation property states that if

$$x(n) \overset{z}{\longleftrightarrow} X(z) \text{ And}$$

Then $$n\, x(n) \overset{z}{\longleftrightarrow} -z\, d/dz\, (X(z))$$

6) Convolution Theorem

The Circular property states that if

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z) \text{ And}$$

$$x_2(n) \overset{z}{\longleftrightarrow} X_2(z) \text{ Then}$$

$$\text{Then } x_1(n) * x_2(n) \overset{z}{\longleftrightarrow} X_1(z) X_2$$

Convolution of two sequences in time domain corresponds to multiplication of its Z transform sequence in frequency domain.

7) Correlation Property

The Correlation of two sequences states that if

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z) \text{ And}$$

$$x_2(n) \overset{z}{\longleftrightarrow} X_2(z) \text{ Then}$$

$$\text{then } \sum_{n=-\infty}^{\infty} x_1(l)\ x_2(-l) \overset{z}{\longleftrightarrow} X_1(z)\ x_2(z^{-1})$$

8) Initial value Theorem

Initial value theorem states that if

$$x(n) \longleftrightarrow X(z) \text{ And}$$

then

$$x(0) = \lim_{z \to \infty} X(Z)$$

9) Final value Theorem

Final value theorem states that if

$$x(n) \overset{z}{\longleftrightarrow} X(z) \text{ And}$$

then

$$\lim_{z \to \infty} x(n) = \lim_{z \to 1} (z-1)\ X(z)$$

RELATIONSHIP BETWEEN FOURIER TRANSFORM AND Z TRANSFORM.

There is a close relationship between Z transform and Fourier transform. If we replace the complex variable z by $e^{-j\omega}$, then z transform is reduced to Fourier transform.

Z transform of sequence x(n) is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\ z^{-n} \qquad \textbf{(Definition of z-Transform)}$$

Fourier transform of sequence x(n) is given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$ **(Definition of Fourier Transform)**

Complex variable z is expressed in polar form as $Z= re^{j\omega}$ where r= |z| and ω is ∟z. Thus we can be written as

$$X(z) = \sum_{n=-\infty}^{\infty} [\, x(n)\, r^{-n}]\, e^{-j\omega n}$$

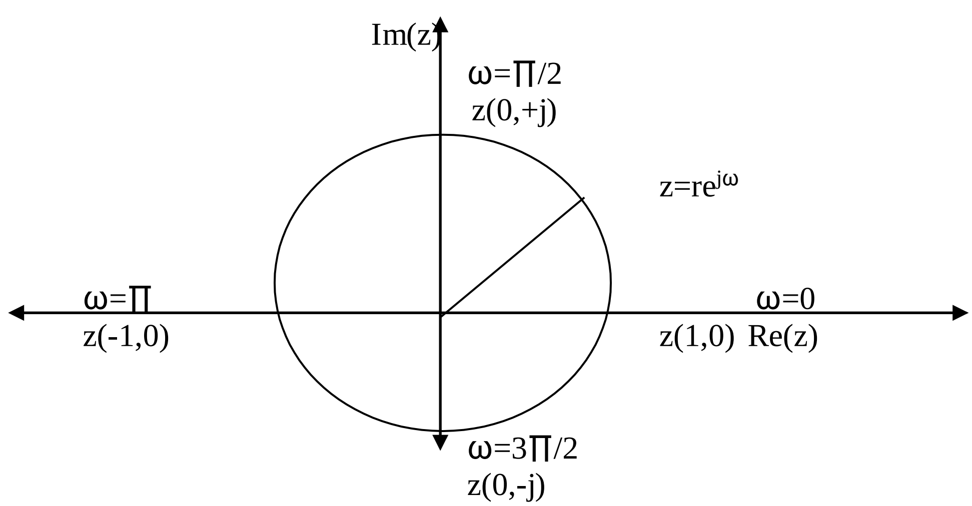
$$X(z)\Big|_{z=e^{jw}} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$X(z)\Big|_{z=e^{jw}} = x(\omega) \qquad \text{at } |z| = \text{unit circle.}$$

Thus, X(z) can be interpreted as Fourier Transform of signal sequence $(x(n)\, r^{-n})$. Here r^{-n} grows with n if r<1 and decays with n if r>1. X(z) converges for |r|= 1. hence Fourier transform may be viewed as Z transform of the sequence evaluated on unit circle. Thus The relationship between DFT and Z transform is given by

$$X(z)_{z=e^{j2\Pi kn}} = x(k)$$

The frequency ω=0 is along the positive Re(z) axis and the frequency ∏/2 is along the positive Im(z) axis. Frequency ∏ is along the negative Re(z) axis and 3∏/2 is along the negative Im(z) axis.



Frequency scale on unit circle X(z)= X(ω) on unit circle

INVERSE Z TRANSFORM (IZT)

The signal can be converted from time domain into z domain with the help of z transform (ZT). Similar way the signal can be converted from z domain to time domain with the help of inverse z transform(IZT). The inverse z transform can be obtained by using two different methods.

1) Partial fraction expansion Method (PFE) / Application of residue theorem
2) Power series expansion Method (PSE)

1. PARTIAL FRACTION EXPANSION METHOD

In this method X(z) is first expanded into sum of simple partial fraction.

$$X(z) = \frac{a_0 z^m + a_1 z^{m-1} + \ldots\ldots + a_m}{b_0 z^n + b_1 zn^{n-1} + \ldots\ldots + b_n} \qquad \text{for } m \leq n$$

First find the roots of the denominator polynomial

$$X(z) = \frac{a_0 z^m + a_1 z^{m-1} + \ldots\ldots + a_m}{(z - p_1)(z - p_2)\ldots\ldots(z - p_n)}$$

The above equation can be written in partial fraction expansion form and find the coefficient A_K and take IZT.

SOLVE USING PARTIAL FRACTION EXPANSION METHOD (PFE)

S.No	Function (ZT)	Time domain sequence	Comment
1	$\frac{1}{1 - a z^{-1}}$	$a^n u(n)$ for $\|z\| > a$	causal sequence
		$-a^n u(-n-1)$ for $\|z\| < a$	anti-causal sequence
2	$\frac{1}{1+z^{-1}}$	$(-1)^n u(n)$ for $\|z\| > 1$	causal sequence
		$-(-1)^n u(-n-1)$ for $\|z\| < a$	anti-causal sequence
3	$\frac{3-4z^{-1}}{1 - 3.5 z^{-1}+1.5z^{-2}}$	$-2(3)^n u(-n-1) + (0.5)^n u(n)$ for $0.5<\|z\|<3$	stable system
		$2(3)^n u(n) + (0.5)^n u(n)$ for $\|z\|>3$	causal system
		$-2(3)^n u(-n-1) - (0.5)^n u(-n-1)$ for $\|z\|<0.5$	anti-causal system
4	$\frac{1}{1 - 1.5 z^{-1}+0.5z^{-2}}$	$-2(1)^n u(-n-1) + (0.5)^n u(n)$ for $0.5<\|z\|<1$	stable system
		$2(1)^n u(n) + (0.5)^n u(n)$ for $\|z\|>1$	causal system
		$-2(1)^n u(-n-1) - (0.5)^n u(-n-1)$ for $\|z\|<0.5$	anti-causal system

5	$\dfrac{1+2z^{-1}+z^{-2}}{1-3/2\,z^{-1}+0.5z^{-2}}$	$2\delta(n)+8(1)^n u(n)-9(0.5)^n u(n)$ for $\|z\|>1$	causal system
6	$\dfrac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$	$(1/2-j3/2)(1/2+j1/2)^n u(n)+$ $(1/2+j3/2)(1/2+j1/2)^n u(n)$	causal system
7	$\dfrac{1-(0.5)z^{-1}}{1-3/4\,z^{-1}+1/8\,z^{-2}}$	$4(-1/2)^n u(n)-3(-1/4)^n u(n)$ for $\|z\|>1/2$	causal system
8	$\dfrac{1-1/2\,z^{-1}}{1-1/4\,z^{-2}}$	$(-1/2)^n u(n)$ for $\|z\|>1/2$	causal system
9	$\dfrac{z+1}{3z^2-4z+1}$	$\delta(n)+u(n)-2(1/3)^n u(n)$ for $\|z\|>1$	causal system
10	$\dfrac{5z}{(z-1)(z-2)}$	$5(2^n-1)$ for $\|z\|>2$	causal system
11	$\dfrac{z^3}{(z-1)(z-1/2)^2}$	$4-(n+3)(1/2)^n$ for $\|z\|>1$	causal system

2. RESIDUE THEOREM METHOD

In this method, first find $G(z)=z^{n-1}X(Z)$ and find the residue of G(z) at various poles of X(z).

SOLVE USING —RESIDUE THEOREM— METHOD

S. No	Function (ZT)	Time domain Sequence
1	$\dfrac{z}{z-a}$	For causal sequence $(a)^n u(n)$
2	$\dfrac{z}{(z-1)(z-2)}$	$(2^n-1)u(n)$
3	$\dfrac{z^2+z}{(z-1)^2}$	$(2n+1)u(n)$
4	$\dfrac{z^3}{(z-1)(z-0.5)^2}$	$4-(n+3)(0.5)^n u(n)$

POLE –ZERO PLOT

1. $X(z)$ is a rational function, that is a ratio of two polynomials in z^{-1} or z.
   The roots of the denominator or the value of z for which X(z) becomes infinite, defines locations of the poles. The roots of the numerator or the value of z for which X(z) becomes zero, defines locations of the zeros.

2. ROC dos not contain any poles of X(z). This is because x(z) becomes infinite at the locations of the poles. Only poles affect the causality and stability of the system.

3. **CASUALTY CRITERIA FOR LSI SYSTEM**
   LSI system is causal if and only if the ROC the system function is exterior to the circle. i. e $|z| > r$. This is the condition for causality of the LSI system in terms of z transform. (The condition for LSI system to be causal is $h(n) = 0$ ….. $n<0$)

4. **STABILITY CRITERIA FOR LSI SYSTEM**
   Bounded input x(n) produces bounded output y(n) in the LSI system only if

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

With this condition satisfied, the system will be stable. The above equation states that the LSI system is stable if its unit sample response is absolutely summable. This is necessary and sufficient condition for the stability of LSI system.

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n}$$

Z-Transform……(1)

Taking magnitude of both the sides

$$|H(z)| = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n} \qquad \ldots\ldots(2)$$

Magnitudes of overall sum is less than the sum of magnitudes of individual sums.

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} h(n)\, z^{-n}$$

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)|\, |z^{-n}| \qquad \ldots.(3)$$

5. If H(z) is evaluated on the unit circle $|z^{-n}|=|z|=1$.
Hence LSI system is stable if and only if the ROC the system function includes the unit circle. i.e $r < 1$. This is the condition for stability of the LSI system in terms of z transform. Thus

For stable system $|z| < 1$
For unstable system $|z| > 1$
Marginally stable system $|z| = 1$

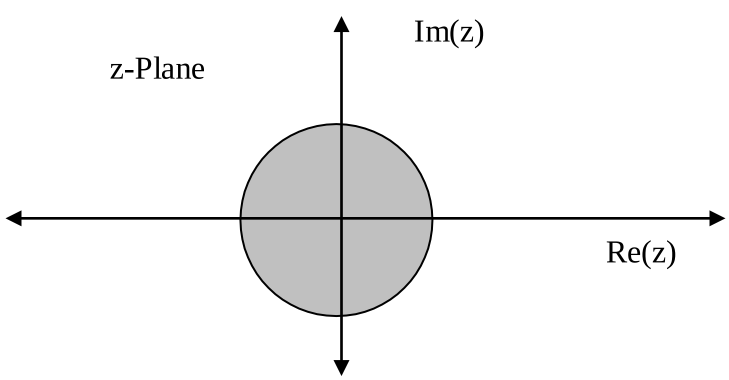


Fig: Stable system

Poles inside unit circle gives stable system. Poles outside unit circle gives unstable system. Poles on unit circle give marginally stable system.

6. A causal and stable system must have a system function that converges for $|z| > r < 1$.

STANDARD INVERSE Z TRANSFORMS

S. No	Function (ZT)	Causal Sequence $\|z\| > \|a\|$	Anti-causal sequence $\|z\| < \|a\|$
1	$\frac{z}{z - a}$	$(a)^n u(n)$	$-(a)^n u(-n-1)$
2	$\frac{z}{z - 1}$	$u(n)$	$u(-n-1)$
3	$\frac{z^2}{(z - a)^2}$	$(n+1)a^n$	$-(n+1)a^n$
4	$\frac{z^k}{(z - a)^k}$	$1/(k-1)!\ (n+1)\ (n+2)\ldots\ldots a^n$	$-1/(k-1)!\ (n+1)\ (n+2)\ldots\ldots\ldots a^n$
5	1	$\delta(n)$	$\delta(n)$
6	Z^k	$\delta(n+k)$	$\delta(n+k)$
7	Z^{-k}	$\delta(n-k)$	$\delta(n-k)$

ONE SIDED Z TRANSFORM

S.No	z Transform (Bilateral)	One sided z Transform (Unilateral)
1	z transform is an infinite power series because summation index varies from ∞ to $-\infty$. Thus Z transform are given by $X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$	One sided z transform summation index varies from 0 to ∞. Thus One sided z transform are given by $X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$
2	z transform is applicable for relaxed systems (having zero initial condition).	One sided z transform is applicable for those systems which are described by differential equations with non zero initial conditions.
3	z transform is also applicable for non-causal systems.	One sided z transform is applicable for causal systems only.
4	ROC of x(z) is exterior or interior to circle hence need to specify with z transform of signals.	ROC of x(z) is always exterior to circle hence need not to be specified.

Properties of one sided z transform are same as that of two sided z transform except shifting property.

1) Time delay

$$x(n) \overset{z+}{\longleftrightarrow} X^+(z) \text{ and}$$

Then $$x(n-k) \overset{z+}{\longleftrightarrow} z^{-k} [\, X^+(z) + \sum_{n=1}^{k} x(-n)\, z^n] \qquad k>0$$

2) Time advance

$$x(n) \overset{z+}{\longleftrightarrow} X^+(z) \text{ and}$$

Then $$x(n+k) \overset{z+}{\longleftrightarrow} z^{k} [\, X^+(z) - \sum_{n=0}^{k-1} x(n)\, z^{-n}] \qquad k>0$$

Examples:

Q) Determine one sided z transform for following signals

1) $x(n)=\{\underline{1},2,3,4,5\}$ 2) $x(n)=\{1,2,\underline{3},4,5\}$

SOLUTION OF DIFFERENTIAL EQUATION

One sided Z transform is very efficient tool for the solution of difference equations with nonzero initial condition. System function of LSI system can be obtained from its difference equation.

$$Z\{x(n-1)\} = \sum_{n=0}^{\infty} x(n-1)\; z^{-n}$$ **(One sided Z transform)**

$$= x(-1) + x(0)\, z^{-1} + x(1)\, z^{-2} + x(2)\, z^{-3} + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= x(-1) + z^{-1} [x(0)\, z^{-1} + x(1)\, z^{-2} + x(2)\, z^{-3} + \ldots\ldots\ldots\ldots\ldots\ldots]$$

$$\mathbf{Z\{ x(n-1) \} = z^{-1} X(z) + x(-1)}$$
$$\mathbf{Z\{ x(n-2) \} = z^{-2} X(z) + z^{-1} x(-1) + x(-2)}$$

Similarly $$\mathbf{Z\{ x(n+1) \} = z\, X(z) - z\, x(0)}$$
$$\mathbf{Z\{ x(n+2) \} = z^{2} X(z) - z^{1} x(0) + x(1)}$$

1. Difference equations are used to find out the relation between input and output sequences. It is also used to relate system function H(z) and Z transform.

2. The transfer function $H(\omega)$ can be obtained from system function H(z) by putting $z=e^{j\omega}$. Magnitude and phase response plot can be obtained by putting various values of ω.

Tutorial problems:

Q) A difference equation of the system is given below

$$Y(n)= 0.5\, y(n-1) + x(n)$$

Determine a) System function
b) Pole zero plot
c) Unit sample response

Q) A difference equation of the system is given below

$Y(n)= 0.7\, y(n-1) – 0.12\, y(n-2) + x(n-1) + x(n-2)$

a) System Function
b) Pole zero plot
c) Response of system to the input $x(n) = nu(n)$
d) Is the system stable? Comment on the result.

Q) A difference equation of the system is given below

$Y(n)= 0.5\, x(n) + 0.5\, x(n-1)$

Determine a) System function
b) Pole zero plot
c) Unit sample response
d) Transfer function
e) Magnitude and phase plot

Q) A difference equation of the system is given below

a. $Y(n)= 0.5\, y(n-1) + x(n) + x(n-1)$
b. $Y(n)= x(n) + 3x(n-1) + 3x(n-2) + x(n-3)$

a) System Function
b) Pole zero plot
c) Unit sample response
d) Find values of y(n) for n=0,1,2,3,4,5 for $x(n)= \delta(n)$ for no initial condition.

Q) Solve second order difference equation

$2x(n-2) – 3x(n-1) + x(n) = 3^{n-2}$ with $x(-2)=-4/9$ and $x(-1)=-1/3$.

Q) Solve second order difference equation

$x(n+2) + 3x(n+1) + 2x(n)$ with $x(0)=0$ and $x(1)=1$.

Q) Find the response of the system by using Z transform

$x(n+2) - 5x(n+1) + 6x(n)= u(n)$ with $x(0)=0$ and $x(1)=1$.

INTRODUCTION TO DFT:

Frequency analysis of discrete time signals is usually performed on digital signal processor, which may be general purpose digital computer or specially designed digital hardware. To perform frequency analysis on discrete time signal, we convert the time domain sequence to an equivalent frequency domain representation. We know that such representation is given by The Fourier transform $X(e^{jw})$ of the sequence x(n). However, $X(e^{jw})$ is a continuous function of frequency and therefore, It is not a computationally convenient representation of the sequence.DFT is a powerful computational tool for performing frequency analysis of discrete time signals. The N-point DFT of discrete time sequence x(n) is denoted by X(k) and is defined as

$$DFT[x(n)] = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}$$ **; k = 0,1,2 .... (N-1)**

Where $W_N = e^{-j\left(\frac{2\pi}{N}\right)}$

IDFT of X(k) is given by

$$IDFT[X(k)] = x(n) = \frac{1}{N} \qquad W_N^{-nk}$$ **; n= 0,1,2… ...... (N-1)**

Where $W_N = e^{-j\left(\frac{2\pi}{N}\right)}$

Example Find the 4-point DFT of the sequence $x(n) = \cos \frac{n\pi}{4}$.

Solution Given $N = 4$,

$$x(n) = [\cos(0), \cos(\pi/4), \cos(\pi/2), \cos(3\pi/4)]$$
$$= \{1, 0.707, 0, -0.707\}$$

The N-point DFT of the sequence $x(n)$ is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N}, \; k = 0, 1, \ldots N-1.$$

The DFT is

$$X(k) = \sum_{n=0}^{3} x(n)\, e^{-j2\pi nk/4}, \; k = 0, 1, 2, 3$$

$$= \sum_{n=0}^{3} x(n)\, e^{-j\pi nk/2},\ k = 0, 1, 2, 3$$

For $k = 0$

$$X(0) = \sum_{n=0}^{3} x(n) = 1$$

For $k = 1$

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{-j\pi(1)n/2}$$

$$= 1 + 0.707\, e^{-j\pi/2} + 0 + (-0.707)e^{-j3\pi/2}$$

$$= 1 + (0.707)(-j) + 0 - (0.707)(j)$$

$$= 1 - j\,1.414$$

For $k = 2$

$$X(2) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\pi(2)n/2} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\pi n}$$

$$= 1 + (0.707)\, e^{-j\pi} + 0 + (-0.707)\, e^{-j3\pi}$$

$$= 1 + (0.707)(-1) + 0 + (-0.707)(-1) = 1$$

For $k = 3$

$$X(3) = \sum_{n=0}^{3} x(n)\, e^{-j\pi(3)n/2}$$

$$= 1 + (0.707)\, e^{-j3\pi/2} + 0 + (-0.707)\, e^{-j9\pi/2}$$

$$= 1 + (0.707)(j) + 0 + (-0.707)(-j) = 1 + j\,1.414$$

$$X(k) = \{1,\ 1 - j\,1.414,\ 1,\ 1 + j\,1.414\}$$

Example Find the N-Point DFT for $x(n) = a^n$ for $0 < a < 1$.

Solution The N-point DFT is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N},\ k = 0, 1, \ldots, N-1.$$

$$= \sum_{n=0}^{N-1} a^n\, e^{-j2\pi nk/N}$$

$$= \sum_{n=0}^{N-1} \left(a\, e^{-j2\pi k/N}\right)^n$$

$$= \frac{1-\left(a\,e^{-j2\pi k/N}\right)^N}{1-a\,e^{-j2\pi k/N}}$$

$$X(k) = \frac{1-a^N}{1-a\,e^{-j2\pi k/N}},\ k = 0, 1, \ldots, N-1$$

Example Derive the DFT of the sample data sequence $x(n) = \{1, 1, 2, 2, 3, 3\}$ and compute the corresponding amplitude and phase spectrum.

Solution The N-point DFT of a finite duration sequence $x(n)$ is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n)\,e^{-j2\pi nk/N},\ k = 0, 1, \ldots, N-1.$$

For $k = 0$

$$X(0) = \sum_{n=0}^{5} x(n)\,e^{-j2\pi(0)n/6} = \sum_{n=0}^{5} x(n) = 1+1+2+2+3+3 = 12$$

For $k = 1$

$$X(1) = \sum_{n=0}^{5} x(n)\,e^{-j2\pi(1)n/6}$$

$$= \sum_{n=0}^{5} x(n)\,e^{-j\pi n/3}$$

$$= 1 + e^{-j\pi/3} + 2\,e^{-j2\pi/3} + 2\,e^{-j\pi} + 3\,e^{-j4\pi/3} + 3\,e^{-j5\pi/3}$$

$$= 1 + 0.5 - j\,0.866 + 2\,(-0.5 - j\,0.866) + 2\,(-1) + 3\,(-0.5 + j\,0.866) + 3(0.5 + j\,0.866)$$

$$= -1.5 + j\,2.598$$

For $k = 2$

$$X(2) = \sum_{n=0}^{5} x(n)\,e^{-j2\pi(2)n/6}$$

$$= \sum_{n=0}^{5} x(n)\,e^{-2j\pi n/3}$$

$$= 1 + e^{-j2\pi/3} + 2e^{-j4\pi/3} + 2e^{-j2\pi} + 3e^{-j8\pi/3} + 3e^{-j10\pi/3}$$

$$= 1 + (-0.5) - j\,0.866 + 2(-0.5 + j\,0.866) + 2(1) + 3\,(-0.5 - j\,0.866) + 3(-0.5 + j\,0.866)$$

$$= -1.5 + j\,0.866$$

For $k = 3$

$$X(3) = \sum_{n=0}^{5} x(n)\,e^{-j2\pi(3)n/6}$$

$$= \sum_{n=0}^{5} x(n)\,e^{-j\pi n}$$

$$= 1 + e^{-j\pi} + 2e^{-j2\pi} + 2e^{-j3\pi} + 3e^{-j4\pi} + 3e^{-j5\pi}$$
$$= 1 - 1 + 2(1) + 2(-1) + 3(1) + 3(-1) = 0$$

For $k = 4$

$$X(4) = \sum_{n=0}^{5} x(n)\, e^{-j2\pi(4)n/6}$$
$$= \sum_{n=0}^{5} x(n)\, e^{-j4\pi n/3}$$
$$= 1 + e^{-j4\pi/3} + 2e^{-j8\pi/3} + 2e^{-j4\pi} + 3e^{-j16\pi/3} + 3e^{-j20\pi/3}$$
$$= 1 + (-0.5 + j\,0.866) + 2(-0.5 - j\,0.866) + 2(1) + 3(-0.5 + j\,0.866) + 3(-0.5 - j\,0.866)$$
$$= -1.5 - j\,0.866$$

For $k = 5$

$$X(5) = \sum_{n=0}^{5} x(n)\, e^{-j2\pi(5)n/6}$$
$$= \sum_{n=0}^{5} x(n)\, e^{-j5\pi n/3}$$
$$= 1 + e^{-j5\pi/3} + 2e^{-j10\pi/3} + 2e^{-j5\pi} + 3e^{-j20\pi/3} + 3e^{-j25\pi/3}$$
$$= 1 + (-0.5 + j\,0.866) + 2(-0.5 + j\,0.866) + 2(-1) + 3(-0.5 - j\,0.866) + 3(0.5 - j\,0.866)$$
$$= -1.5 - j2.598$$

$$X(k) = \{12, -1.5 + j2.598, -1.5 + j0.866, 0, -1.5 - j\,0.866, -1.5 - j\,2.598\}$$

The corresponding amplitude spectrum is given by

$$|X(k)| = \left\{\sqrt{12 \times 12}, \sqrt{(-1.5)^2 + (-2.598)^2}, \sqrt{(-1.5)^2 + (0.866)^2}, 0, \sqrt{(-1.5)^2 + (-0.866)^2}, \sqrt{(-1.5)^2 + (-2.598)^2}\right\}$$
$$= \{12, 2.999, 1.732, 0, 1.732, 2.999\}$$

and the corresponding phase spectrum is given by

$$\angle X(k) = \left\{\tan^{-1}(0), \tan^{-1}\left(\frac{2.598}{-1.5}\right), \tan^{-1}\left(\frac{0.866}{-1.5}\right), \tan^{-1}(0)\ \tan^{-1}\left(\frac{-0.866}{-1.5}\right), \tan^{-1}\left(\frac{-2.598}{-1.5}\right)\right\}$$
$$= \left\{0, -\frac{\pi}{3}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{3}\right\}$$

Example Find the inverse DFT of $X(k) = \{1, 2, 3, 4\}$.

Solution The inverse DFT is defined as

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi nk/N},\ n = 0, 1, 2, 3, \ldots, N-1$$

Given $N = 4$, $x(n) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{j2\pi nk/N}, n = 0, 1, 2, 3$

When $n = 0$

$$x(0) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{j\pi(0)k/2}$$

$$= \frac{1}{4}(1+2+3+4) = \frac{5}{2}$$

When $n = 1$

$$x(1) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{j\pi(1)k/2}$$

$$= \frac{1}{4}\left(1 + 2e^{j\pi/2} + 3e^{j\pi} + 4e^{j3\pi/2}\right)$$

$$= \frac{1}{4}(1 + 2(j) + 3(-1) + 4(-j))$$

$$= \frac{1}{4}(-2 - j2) = -\frac{1}{2} - j\frac{1}{2}$$

When $n = 2$

$$x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{j\pi k}$$

$$= \frac{1}{4}\left(1 + 2e^{j\pi} + 3e^{j2\pi} + 4e^{j3\pi}\right)$$

$$= \frac{1}{4}(1 + 2(-1) + 3(1) + 4(-1))$$

$$= \frac{1}{4}(-2) = -1/2$$

When $n = 3$

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{j3\pi k/2}$$

$$= \frac{1}{4}\left(1 + 2e^{j3\pi/2} + 3e^{j3\pi} + 4e^{j9\pi/2}\right)$$

$$= \frac{1}{4}(1 + 2(-j) + 3(-1) + 4j)$$

$$= \frac{1}{4}(-2 + 2j) = -\frac{1}{2} + j\frac{1}{2}$$

$$x(n) = \left\{\frac{5}{2}, -\frac{1}{2} - j\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} + j\frac{1}{2}\right\}$$

Properties of the DFT

The properties of the DFT are useful in the practical techniques for processing signals. The various properties are given below.

Periodicity

If $X(k)$ is an N-point DFT of $x(n)$, then

$$x(n + N) = x(n) \text{ for all } n$$
$$X(k + N) = X(k) \text{ for all } k$$

Linearity

If $X_1(k)$ and $X_2(k)$ are the N-point DFTs of $x_1(n)$ and $x_2(n)$ respectively, and a and b are arbitrary constants either real or complex-valued, then

$$a x_1(n) + b x_2(n) \xleftrightarrow[N]{DFT} aX_1(k) + bX_2(k)$$

Time Reversal of a Sequence

If $\quad x(n) \xleftrightarrow[N]{DFT} X(k)$, then

$$x(-n, (\text{mod } N)) = x(N-n) \xleftrightarrow[N]{DFT} X(-k, (\text{mod } N)) = X(N-k)$$

Hence, when the N-point sequence in time is reversed, it is equivalent to reversing the DFT values.

Circular Time Shift

If $\quad x(n) \xleftrightarrow[N]{DFT} X(k)$, then

$$x(n - l, (\text{mod } N)) \xleftrightarrow[N]{DFT} X(k)e^{-j2\pi kl/N}$$

Shifting of the sequence by l units in the time-domain is equivalent to multiplication of $e^{-j2\pi kl/N}$ in the frequency-domain.

Circular Frequency Shift

If $\quad x(n) \xleftrightarrow[N]{DFT} X(k)$, then

$$x(n)e^{j2\pi ln/N} \xleftrightarrow[N]{DFT} X(k - l, (\text{mod } N))$$

Hence, when the sequence $x(n)$ is multiplied by the complex exponential sequence $e^{j2\pi ln/N}$, it is equivalent to circular shift of the DFT by l units in the frequency domain.

Complex Conjugate Property

If $$x(n) \xleftrightarrow[N]{DFT} X(k), \text{ then}$$

$$x^*(n) \xleftrightarrow[N]{DFT} X^*(-k, (\text{mod } N)) = X^*(N-k)$$

Circular Convolution

If $$x_1(n) \xleftrightarrow[N]{DFT} X_1(k) \quad \text{and} \quad x_2(n) \xleftrightarrow[N]{DFT} X_2(k), \text{ then}$$

$$x_1(n) \textcircled{N} x_2(n) \xleftrightarrow[N]{DFT} X_1(k)\, X_2(k)$$

where $x_1(n) \textcircled{N} x_2(n)$ denotes the circular convolution of the sequence $x_1(n)$ and $x_2(n)$ defined as

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2(n-m, (\text{mod} N))$$

$$= \sum_{m=0}^{N-1} x_2(m)\, x_1(n-m, (\text{mod} N))$$

Multiplication of Two Sequences

If $$x_1(n) \xleftrightarrow[N]{DFT} X_1(k) \quad \text{and} \quad x_2(n) \xleftrightarrow[N]{DFT} X_2(k), \text{ then}$$

$$x_1(n)\, x_2(n) \xleftrightarrow[N]{DFT} \frac{1}{N} X_1(k) \textcircled{N} X_2(k)$$

Parseval's Theorem

For complex-valued sequences $x(n)$ and $y(n)$,

if $$x(n) \xleftrightarrow[N]{DFT} X(k) \quad \text{and} \quad y(n) \xleftrightarrow[N]{DFT} Y(k), \text{ then}$$

$$\sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k)$$

If $y(n) = x(n)$, then the above equation reduces to

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

This expression relates the energy in the finite duration sequence $x(n)$ to the power in the frequency components $X(k)$.

Methods of Circular Convolution:

Generally, there are two methods, which are adopted to perform circular convolution and they are –

(1) Concentric circle method **(2) Matrix multiplication method**.

Concentric Circle Method:

Let $x_1(n)$ and $x_2(n)$ be two given sequences. The steps followed for circular convolution of $x_1(n)$ and $x_2(n)$ are

- **Take two concentric circles. Plot N samples of $x_1(n)$ on the circumference of the outer circle (maintaining equal distance successive points) in anti-clockwise direction.**
- **For plotting $x_2(n)$,plot N samples of $x_2(n)$ in clockwise direction on the inner circle, starting sample placed at the same point as 0^{th} sample of $x_1(n)$**
- **Multiply corresponding samples on the two circles and add them to get output.**
- **Rotate the inner circle anti-clockwise with one sample at a time.**

Matrix Multiplication Method:

Matrix method represents the two given sequence $x_1(n)$ and $x_2(n)$ in matrix form.

- **One of the given sequences is repeated via circular shift of one sample at a time to form a N X N matrix.**
- **The other sequence is represented as column matrix.**

The multiplication of two matrices gives the result of circular convolution

SECTIONED CONVOLUTION:

Suppose, the input sequence x(n) of long duration is to be processed with a system having finite duration impulse response by convolving the two sequences. Since, the linear filtering performed via DFT involves operation on a fixed size data block, the input sequence is divided into different fixed size data block before processing. The successive blocks are then processed one at a time and the results are combined to produce the net result. As the convolution is performed by dividing the long input sequence into different fixed size sections, it is called sectioned convolution. A long input sequence is segmented to fixed size blocks, prior to FIR filter processing. Two methods are used to evaluate the discrete convolution.

(1) Overlap-save method (2) Overlap-add method

Overlap Save Method:

Overlap–save is the traditional name for an efficient way to evaluate the discrete convolution between a very long signal x(n) and a finite impulse response FIR filter h(n).

1. Insert M - 1 zeros at the beginning of the input sequence x(n).

2. Break the padded input signal into overlapping blocks $x_m(n)$ of length N = L + M - 1 where the overlap length is M -1.

3. Zero pad h(n) to be of length N = L + M - 1.

4. Take N-DFT of h(n) to give H(k), k = 0, 1,2, ... N - 1.

5. For each block m:

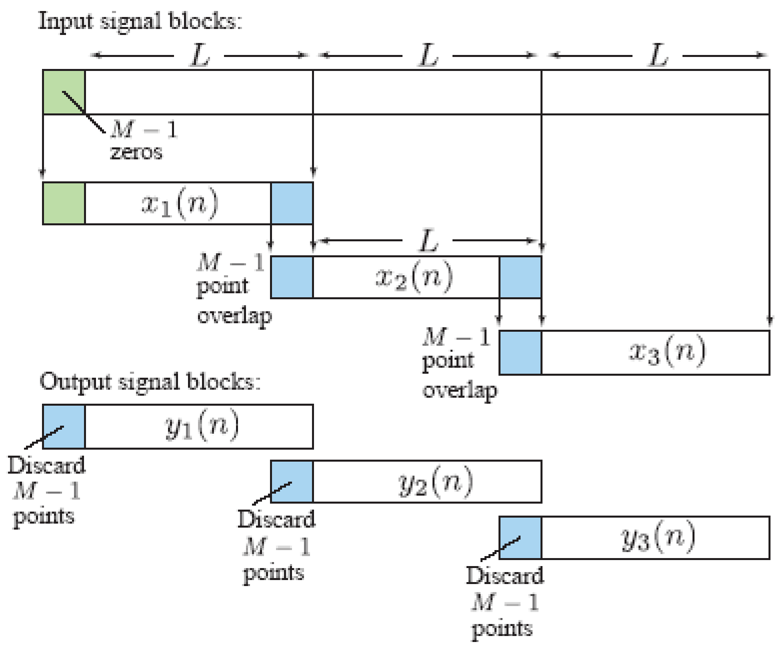
Take N-DFT of $x_m(n)$ to give Xm(k), k = 0, 1,2,…............N - 1.

5.2 Multiply: Ym(k) = Xm(k) . H(k), k = 0, 1,2,……………..N - 1.

Take N-IDFT of Ym(k) to give $y_m(n)$,n = 0, 1,2,……………..N - 1.

Discard the first M - 1 points of each output block $y_m(n)$

6. Form y(n) by appending the remaining (i.e., last) L samples of each block



Overlap Add Method:

Given below are the steps to find out the discrete convolution using Overlap method:

1. Break the input signal x(n) into non-overlapping blocks $x_m(n)$ of length L.

2. Zero pad h(n) to be of length N = L + M - 1.

3. Take N-DFT of h(n) to give H(k), k = 0, 1,2, ... N - 1.

4. For each block m:

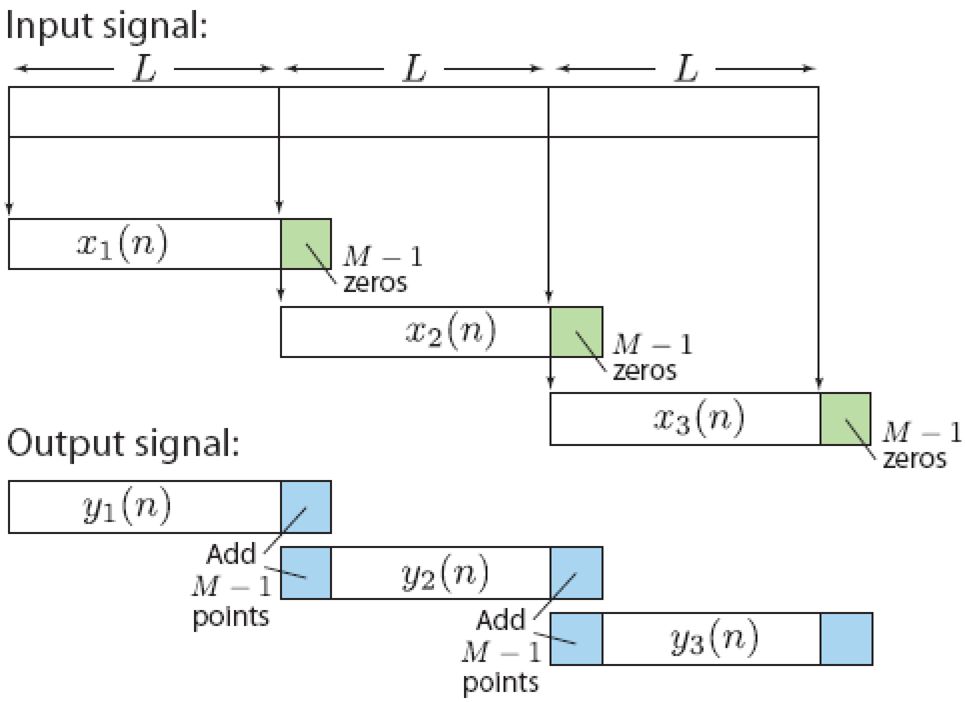
Zero pad $x_m(n)$ to be of length N = L + M - 1.

Take N-DFT of $x_m(n)$to give $X_m(k)$, k = 0, 1,2,……………..N - 1. 4.3

Multiply: $Y_m(k) = X_m(k).H(k)$, k = 0, 1,2,……………..N - 1.

4.4 Take N-IDFT of $Y_m(k)$ to give $y_m(n)$, n = 0, 1,2,……………..N - 1.

5. Form y(n) by overlapping the last M - 1 samples of $y_m(n)$ with the first M -1 samples of $y_{m+1}(n)$ and adding the result.



FAST FOURIER TRANSFORM (FFT)

The fast Fourier transform (FFT) is an algorithm that efficiently computes the discrete Fourier transform (DFT). The DFT of a sequence $\{x(n)\}$ of length N is given by a complex-valued sequence $\{X(k)\}$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N}, \quad 0 \le k \le N-1.$$

Let W_N be the complex-valued phase factor, which is an N th root of unity expressed by

$$W_N = e^{-j2\pi/N}$$

Hence $X(k)$ becomes

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}, \quad 0 \le k \le N-1$$

Similarly, IDFT becomes

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-nk}, \quad 0 \le n \le N-1$$

From the above equations, it is evident that for each value of k, the direct computation of $X(k)$ involves N complex multiplications ($4N$ real multiplications) and $N-1$ complex additions ($4N-2$ real additions). Hence, to compute all N values of DFT, N^2 complex multiplications and $N(N-1)$ complex additions are required. The DFT and IDFT involve the same type of computations.

Decimation-in-Time (DIT) Algorithm

In this case, let us assume that $x(n)$ represents a sequence of N values, where N is an integer power of 2, that is, $N = 2^L$. The given sequence is decimated (broken) into two $\frac{N}{2}$ point sequences consisting of the even numbered values of $x(n)$ and the odd numbered values of $x(n)$.

The N-point DFT of sequence $x(n)$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}, \quad 0 \le k \le N-1$$

Breaking $x(n)$ into its even and odd numbered values, we obtain

$$X(k) = \sum_{n=0,\, n \text{ even}}^{N-1} x(n)\, W_N^{nk} + \sum_{n=0,\, n \text{ odd}}^{N-1} x(n)\, W_N^{nk}$$

Substituting $n = 2r$ for n even and $n = 2r + 1$ for n odd , we have

$$X(k) = \sum_{r=0}^{(N/2-1)} x(2r)\, W_N^{2rk} + \sum_{r=0}^{(N/2-1)} x(2r+1)\, W_N^{(2r+1)k}$$

$$= \sum_{r=0}^{(N/2-1)} x(2r)\, (W_N^2)^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x(2r+1)\, (W_N^2)^{rk}$$

Here, $W_N^2 = [e^{-j(2\pi/N)}]^2 = e^{-j(2\pi/(N/2))} = W_{N/2}$

Therefore, Eq. can be written as

$$X(k) = \sum_{r=0}^{(N/2-1)} x(2r)\, W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2-1)} x(2r+1)\, W_{N/2}^{rk}$$

$$= G(k) + W_N^k \cdot H(k), \quad k = 0, 1, \ldots \frac{N}{2} - 1$$

where $G(k)$ and $H(k)$ are the $N/2$-point DFTs of the even and odd numbered sequences respectively. Here, each of the sums is computed for $0 \le k \le \frac{N}{2} - 1$ since $G(k)$ and $H(k)$ are considered periodic with period $N/2$.

Therefore,

$$X(k) = \begin{cases} G(k) + W_N^k\, H(k), & 0 \le k \le \frac{N}{2} - 1 \\ G\left(k + \frac{N}{2}\right) + W_N^{(k+N/2)}\, H\left(k + \frac{N}{2}\right), & \frac{N}{2} \le k \le N-1 \end{cases}$$

Using the symmetry property of $W_N^{k+N/2} = -W_N^k$,

$$X(k) = \begin{cases} G(k) + W_N^k\, H(k), & 0 \le k \le \frac{N}{2} - 1 \\ G(k + N/2) - W_N^k\, H(k + N/2), & \frac{N}{2} \le k \le N-1 \end{cases}$$

Figure shows the flow graph of the decimation-in-time decomposition of an 8-point ($N = 8$) DFT computation into two 4-point DFT computations. Here the branches entering a node are added to produce the node variable. If no coefficient is indicated, it means that the branch transmittance is equal to one. For other branches, the transmittance is an integer power of W_N.

Then $X(0)$ is obtained by multiplying $H(0)$ by W_N^0 and adding the product to $G(0)$. $X(1)$ is obtained by multiplying $H(1)$ by W_N^1 and adding that result to $G(1)$. For $X(4)$, $H(4)$ is multiplied by W_N^4 and the result is added to $G(4)$. But, since $G(k)$ and $H(k)$ are both periodic in k with period 4, $H(4) = H(0)$ and $G(4) = G(0)$. Therefore, $X(4)$ is obtained by multiplying $H(0)$ by W_N^4 and adding the result to $G(0)$.

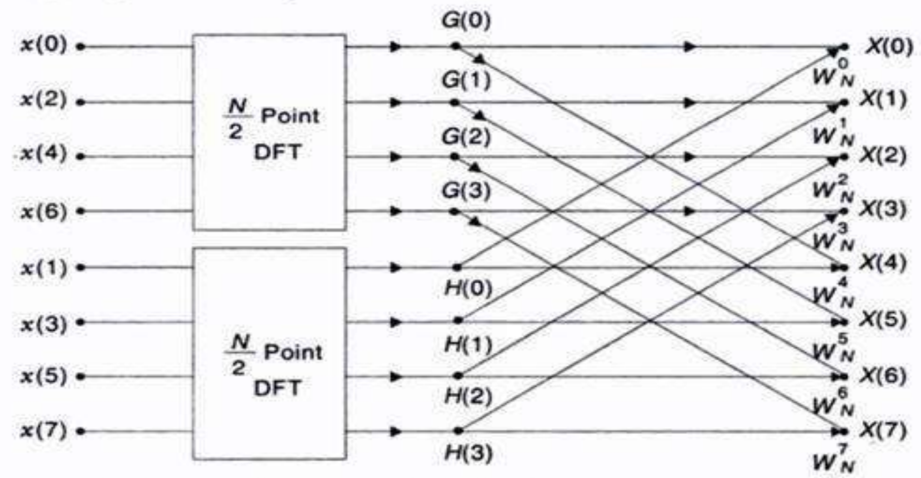


Fig. *Flow Graph of the First Stage Decimation-In-Time FFT Algorithm for N = 8*

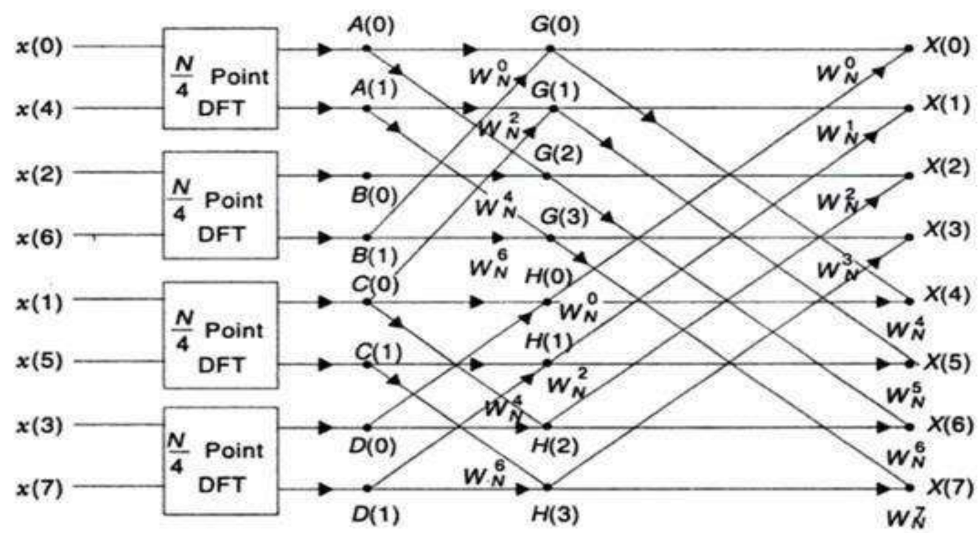


Fig. *Flow Graph of the Second Stage Decimation-in-time FFT Algorithm for N = 8*

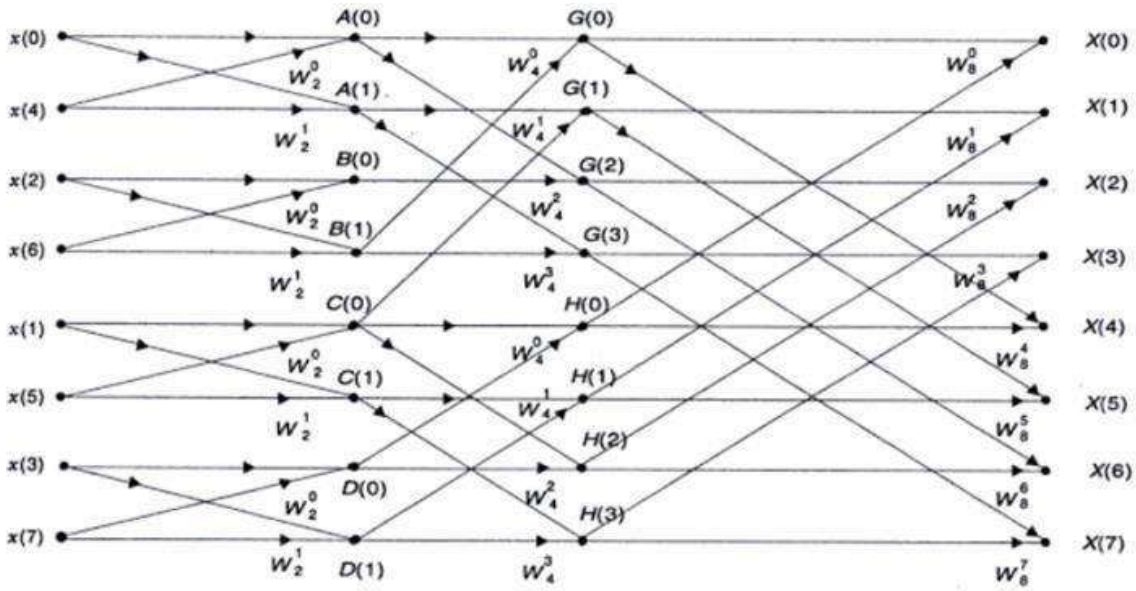


Fig. *The Flow-Graph of the Decimation-in-time FFT Algorithm for N = 8*

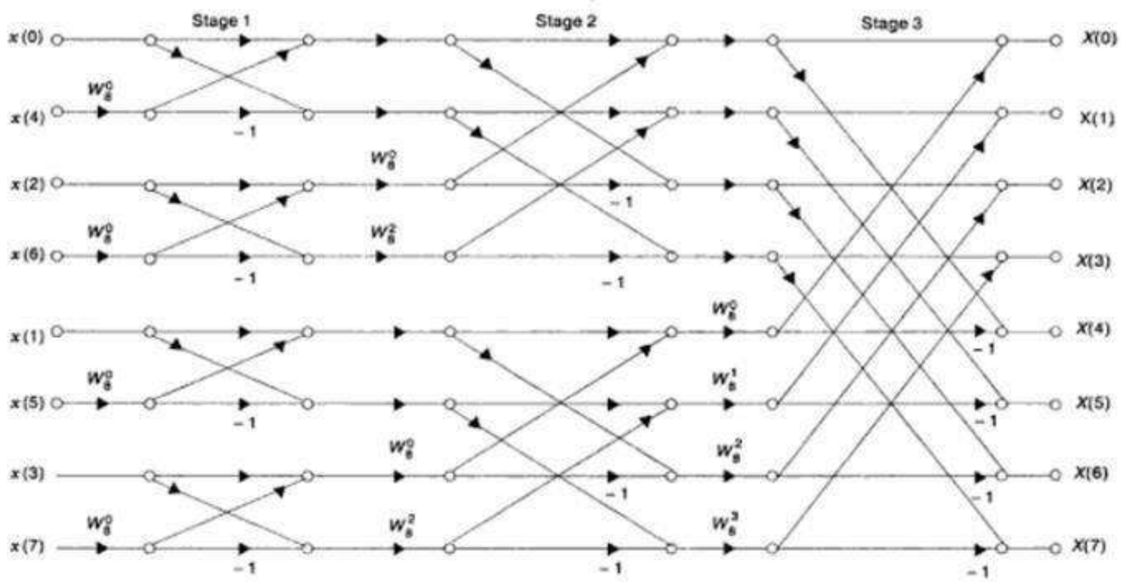


Fig. *Reduced Flow-Graph for an 8-Point DIT FFT*

Example Given $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$, find $X(k)$ using DIT FFT algorithm.

Solution We know that $W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$. Given $N = 8$.

Hence, $W_8^0 = e^{-j\left(\frac{2\pi}{8}\right)0} = 1$

$$W_8^1 = e^{-j\left(\frac{2\pi}{8}\right)} = \cos \pi/4 - j \sin \pi/4 = 0.707 - j\,0.707$$
$$W_8^2 = e^{-j\left(\frac{2\pi}{8}\right)2} = \cos \pi/2 - j \sin \pi/2 = -j$$
$$W_8^3 = e^{-j\left(\frac{2\pi}{8}\right)3} = \cos 3\pi/4 - j \sin 3\pi/4 = -0.707 - j\,0.707$$

Using DIT FFT algorithm, we can find $X(k)$ from the given sequence $x(n)$ as shown in Fig.

Therefore, $X(k)= \{20, -5.828 - j\,2.414, 0, 0.172 - j\,0.414, 0, -0.172 + j\,0.414, 0, -5.828 + j\,2.414\}$

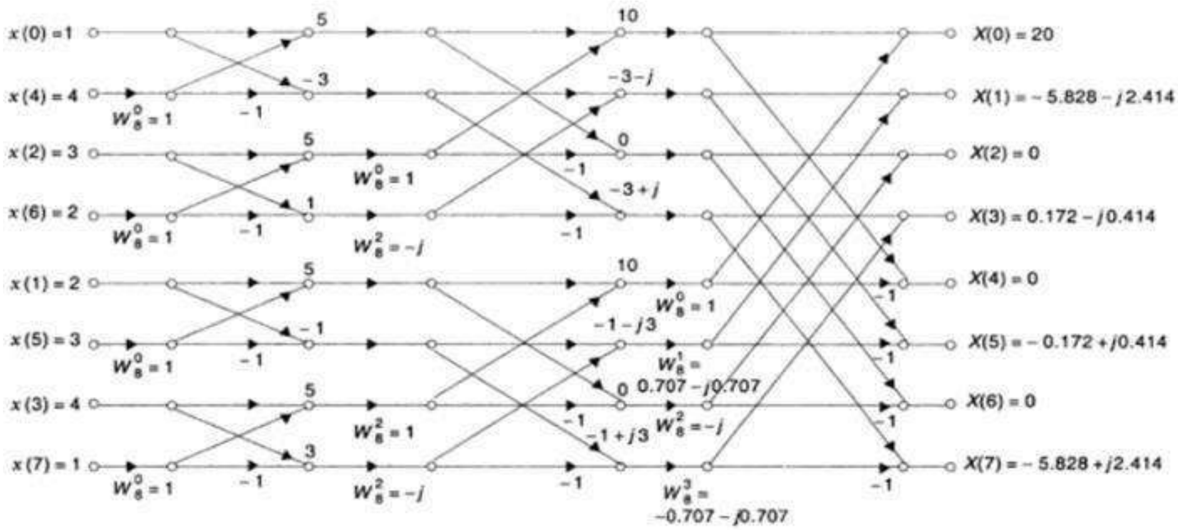


Fig.

Example **Given $x(n) = \{0, 1, 2, 3\}$, find $X(k)$ using DIT FFT algorithm.**

Solution **Given $N = 4$**

$$W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$$
$$W_4^0 = 1 \text{ and } W_4^1 = e^{-j\pi/2} = -j$$

Using DIT FFT algorithm, we can find $X(k)$ from the given sequence $x(n)$ as shown in Fig.

Therefore, $X(k) = \{6, -2 + j2, -2, -2 - j2\}$

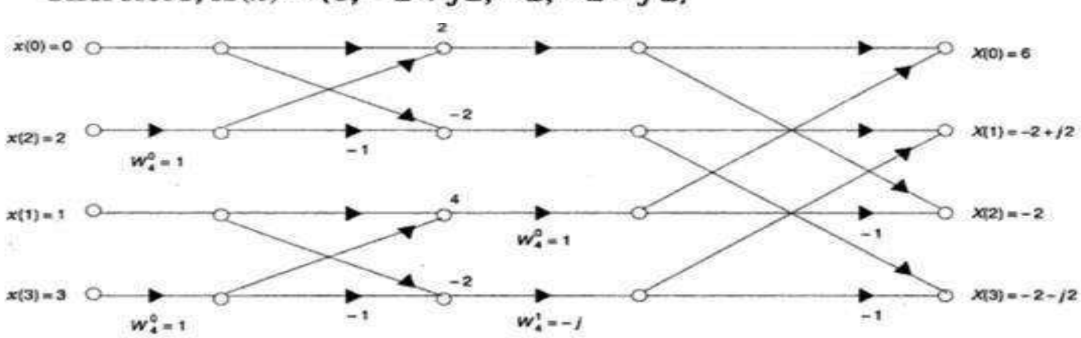


Fig.

Decimation-in-Frequency (DIF) Algorithms

The decimation-in-time FFT algorithm decomposes the DFT by sequentially splitting input samples $x(n)$ in the time domain into sets of smaller and smaller subsequences and then forms a weighted combination of the DFTs of these subsequences. Another algorithm called decimation-in-frequency FFT decomposes the DFT by recursively splitting the sequence elements $X(k)$ in the frequency domain into sets of smaller and smaller subsequences. To derive the decimation-in-frequency FFT algorithm for N, a power of 2, the input sequence $x(n)$ is divided into the first half and the last half of the points

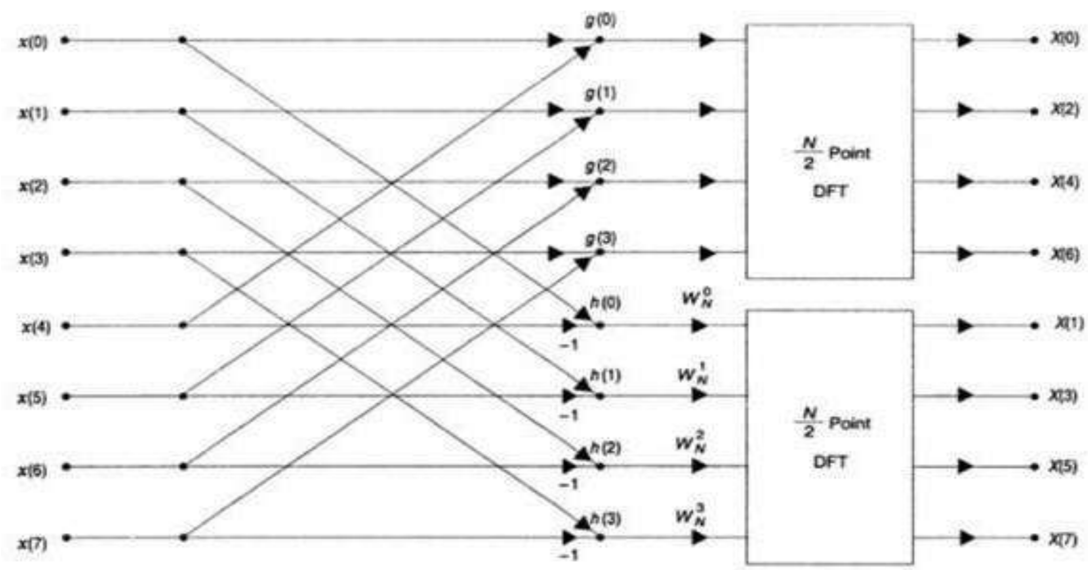


Fig. *Flow Graph of the First Stage of Decimation-In-Frequency FFT for N = 8*

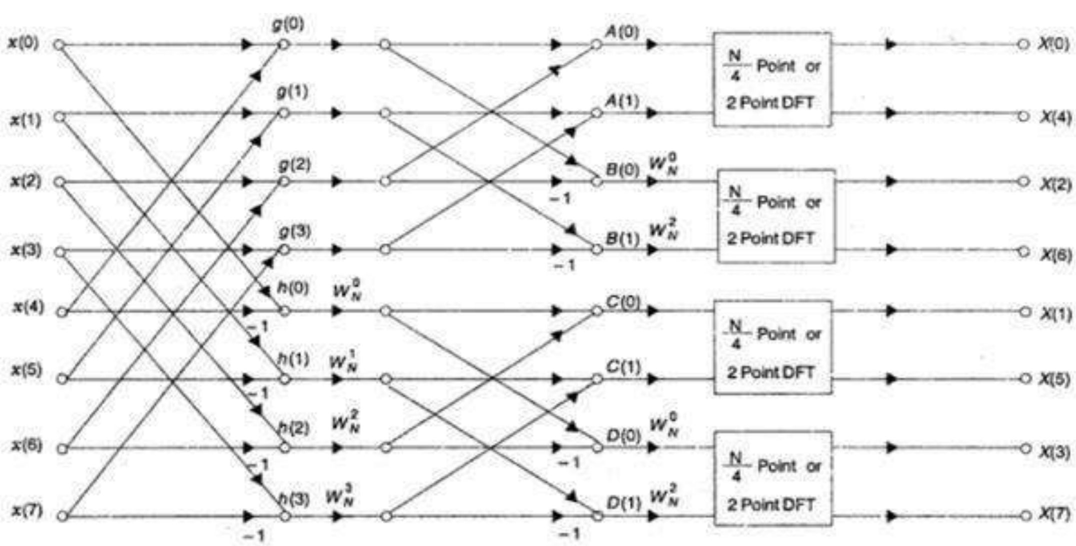


Fig. *Flow Graph of the Second Stage of Decimation-In-Frequency FFT for N = 8*

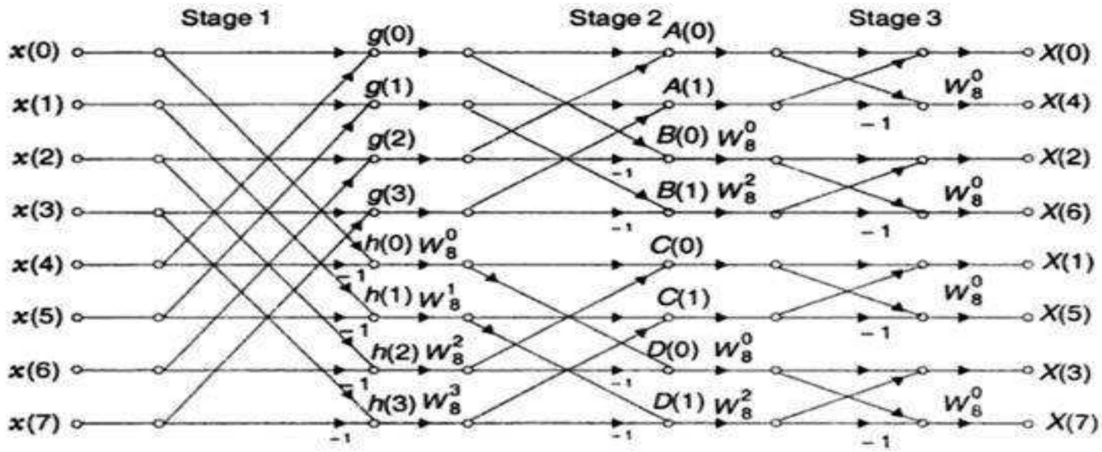


Fig. *Reduced Flow Graph of Final Stage DIF FFT for N = 8*

Example Compute the DFTs of the sequence $x(n) = \cos \frac{n\pi}{2}$, where $N = 4$, using DIF FFT algorithm.

Solution Given $N = 4$ and $x(n) = \{1, 0, -1, 0\}$

$$W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$$

$$W_4^0 = 1 \text{ and } W_4^1 = e^{-j\pi/2} = -j$$

Using DIF FFT algorithm, we can find $X(k)$ from the given sequence $x(n)$ as shown in Fig.

Therefore, $X(k) = \{0, 2, 0, 2\}$

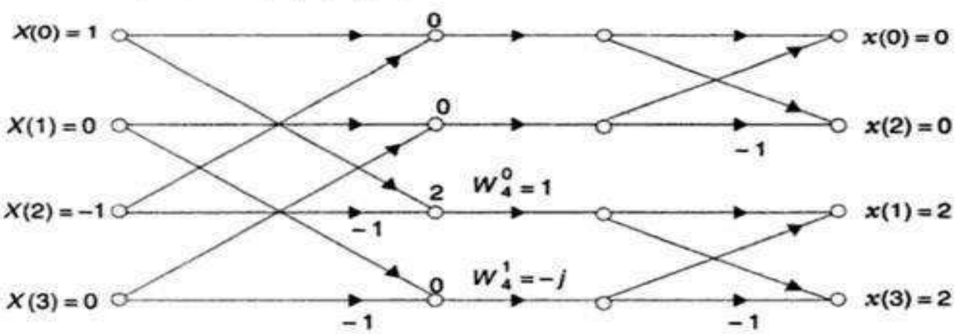


Fig.

Example Given $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$, find $X(k)$ using DIF FFT algorithm.

Solution Given $N = 8$.

We know that $W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$.

Hence, $W_8^0 = 1$, $W_8^1 = 0.707 - j\,0.707$

$W_8^2 = -j$, $W_8^3 = -0.707 - j\,0.707$

Using DIF FFT algorithm, we can find $X(k)$ from the given sequence $x(n)$ as shown in Fig.

Hence, $X(k) = \{20, -5.828 - j\,2.414, 0, -0.172 - j\,0.414, 0, -0.172 + j\,0.414, 0, -5.828 + j\,2.414\}$

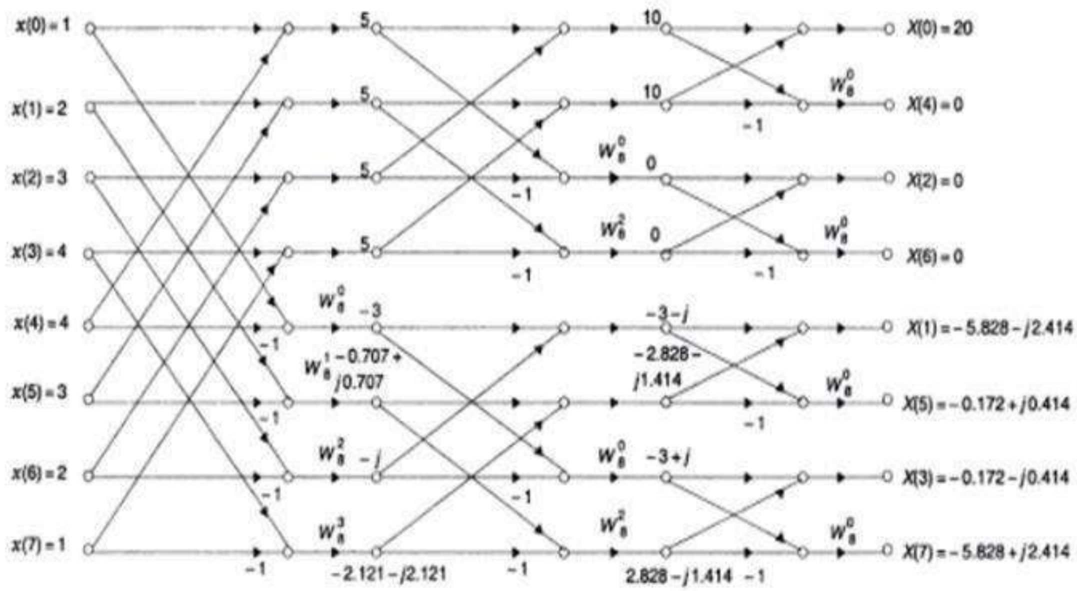


Fig.

INVERSE FFT:

An FFT algorithm can be used to compute the IDFT if the output is divided by N and the "twiddle factors are negative powers of W_N, i.e. powers of W_N^{-1} is used instead of powers of W_N. Therefore, an IFFT flow graph can be obtained from an FFT flow graph by replacing all the $x(n)$ by $X(k)$, dividing the input data by N, or dividing each stage by 2 when N is a power of 2, and changing the exponents of W_N to negative values.

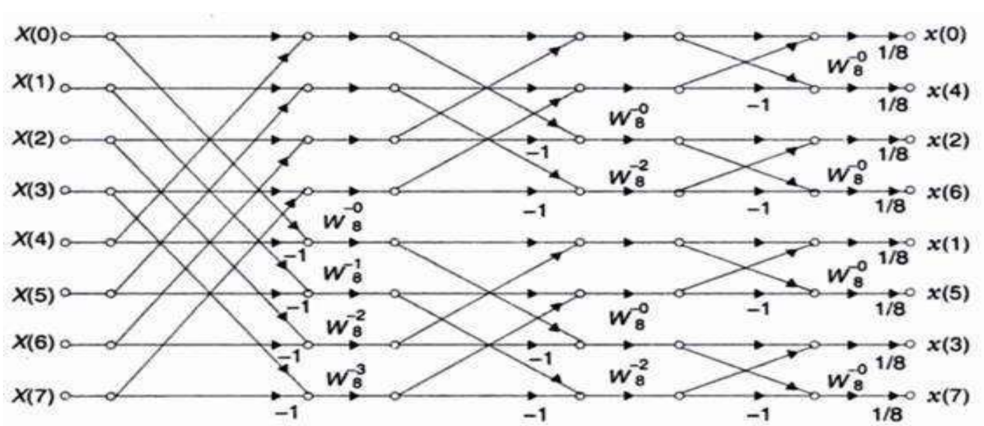


Fig. *Flow Graph of an IDFT Computation*

Example Use the 4-point inverse FFT and verify the DFT results {6, −2 + j2, −2, −2 − j2} obtained in Example 6.18 for the given input sequence {0, 1, 2, 3}.

Solution We know that $W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$. Hence,

$$W_4^{-0} = 1 \text{ and } W_4^{-1} = e^{j\pi/2} = j$$

Using IFFT algorithm, we can find the input sequence $x(n)$ from the given DFT sequence $X(k)$ as shown in Fig.

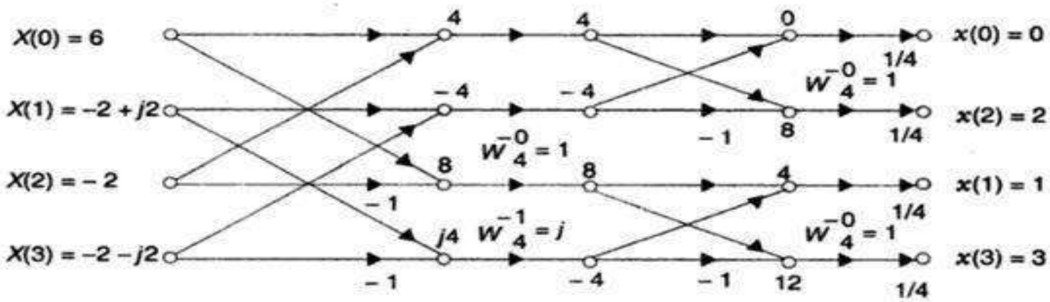


Fig.

Hence, $x(n) = \{0, 1, 2, 3\}$

SELECTION OF THE FILTER TYPE

The selection of the digital filter type *i.e.*, whether an IIR and FIR digital filter to be employed ; depends on the nature of the problem and on the specification of the desired frequency response. For example, FIR filters are used in filtering problems where there is a requirement for a linear phase characteristic within the passband of the filter. When linear phase is not a requirement, either an IIR or FIR filter can be used. However, in most cases, the order (N_{FIR}) of an FIR filter is considerably higher than the order (N_{IIR}) of an equivalent IIR filter meeting the same magnitude specifications. It has been shown that for most practical filter specifications, the ratio N_{FIR}/N_{IIR} is typically of order of ten or more and as a result, IIR filter is usually computationally more efficient.

In this chapter we shall discuss techniques for designing IIR filters from the analog filters, with the restriction that the filters be realizable and, of course, stable. There are four different methods which are available under IIR filter design, these are,

1. Impulse invariance method
2. Bilinear transformation method
3. Matched z-transform technique
4. Approximation of derivatives.

We shall concentrate only the first two methods.

IIR Filter Design by Impulse Invariance

A technique for digitizing an analog filter is called impulse invariance transformation. The objective of this method is to develop an IIR filter transfer function whose impulse response is the sampled version of the impulse response of the analog filter. The main idea behind this technique is to preserve the frequency response characteristics of the analog filter. In the consequence of the result, the frequency response of the digital filter is an aliased version of the frequency response of the corresponding analog filter.

To develop the necessary design formula for impulse invariance method, consider a causal and stable "analog" transfer function $H_a(s)$. Its impulse response $h_a(t)$ is given by inverse Laplace transform of $H_a(s)$, *i.e.*,

$$h_a(t) = L^{-1}\{H_a(s)\} \quad \ldots(1)$$

In this method, we require that unit sample response $h(n)$ of the desired causal digital transfer function H(z) be given by the sampled version of $h_a(t)$ sampled at uniform interval of T seconds.

i.e.,
$$h(n) = h_a(nT) \quad n = 0, 1, 2 \quad \ldots(2)$$

where T is the sampling period

To investigate the mapping of points between z-plane and s-plane implied by the sampling process, the z-transform is related to the Laplace transform of $h_a(t)$ as

$$H(z)\big|_{z=e^{sT}} = Z\{h(n)\} = Z\{h_a(nT)\} \quad \ldots(3)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left(s + j\frac{2\pi k}{T}\right) \quad \ldots(4)$$

where,
$$H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n} \quad \ldots(5)$$

and
$$H(z)\big|_{z=e^{sT}} = \sum_{n=0}^{\infty} h(n) e^{-sTn} \quad \ldots(6)$$

where $s = \sigma + j\Omega$

Let us examine the transform $z = e^{ST}$ of eqn. (4) which can be written alternatively as,

$$z = e^{sT}$$

For $s = \sigma + j\Omega$

$$z = \gamma e^{j\omega} = e^{sT} = e^{\sigma T} = e^{j\Omega T}.$$

This then implies $\Omega = e^{\sigma T}$

$$\omega = \Omega T$$

where Ω is analog frequency and

ω is frequency in digital domain.

9.2.1.1 Development of the transformation

To explore the effect of the impulse invariance design method on the characteristics of resultant filter, let us consider the system function of the analog filter in the partial fraction form. Assume that the poles of the analog filter are distinct *i.e.*,

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - p_k} \qquad \text{...(7)}$$

where $\{A_k\}$ are the co-efficients in the partial fraction expansion and

p_k are the poles of the analog filter. The impulse response $h_a(t)$ corresponding to eqn. (7) has the form

$$h_a(t) = \sum_{k=1}^{N} A_k e^{p_k t} u_a(t) \qquad \text{...(8)}$$

where $u_a(t)$ is the continuous time step function.

If we sample $h_a(t)$ periodically at $t = nT$, we have

$$h(n) = h_a(nT) = \sum_{k=1}^{N} A_k e^{p_k nT} u_a(nT) \qquad \text{...(9)}$$

Now, the system function H(z) of the digital filter is the z-transform of this sequence and is defined by

$$H(z) = z\{h(n)\}.$$

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \qquad \text{...(10)}$$

Using eqn. (10) the system function becomes

$$H(z) = \sum_{n=0}^{\infty} \sum_{k=1}^{N} A_k e^{p_k nT} z^{-n} \qquad \text{...(11)}$$

$$= \sum_{k=1}^{N} \sum_{k=0}^{\infty} (e^{p_k T} z^{-1})^n \qquad \text{...(12)}$$

$$H(z) = \sum_{k=1}^{N} A_k \cdot \frac{1}{1 - e^{p_k T} z^{-1}} \qquad \text{...(13)}$$

provided that $|e^{p_k T}| < 1$, which is always satisfied if $p_k < 0$, indicating that $H_a(s)$ is a stable transfer function. From the eqn. (13) we observe that the digital filter has poles at

$$z_k = e^{p_k T} \quad k = 1, 2, \ldots\ldots\ldots N$$

Comparing the expression (13) and (7), we see that the impulse invariance transformation is accomplished by the mapping.

$$\frac{1}{s-p_k} \longrightarrow \frac{1}{1-e^{p_k T} z^{-1}} \qquad \ldots(14)$$
$$\frac{1}{s+p_k} \longrightarrow \frac{1}{1-e^{-p_k T} z^{-1}}.$$

Problem 1. *For the analog transfer function* $H_a(s) = \dfrac{2}{(s+1)(s+2)}$ *determine the H(z) using impulse invariance method.*

Sol. $$H_a(s) = \frac{2}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{2}{s+2}$$

Using the impulse invariance transformation of eqn. (14), the digital filter transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1-e^{-T}z^{-1}} - \frac{2}{1-e^{-2T}z^{-1}} = \frac{2e^{-T}(1-e^{-T})z^{-1}}{(1-e^{-T}z^{-1})(1-e^{-2T}z^{-1})}.$$

Problem 2. *Convert the analog filter with system function*

$$H_a(s) = \frac{s+2}{(s+1)(s+3)}$$

into the digital IIR filter by means of the impulse invariance method.

Sol. The partial-fraction expansion of $H_a(s)$ is given as

$$H_a(s) = \frac{s+2}{(s+1)(s+3)} = \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s+3}$$

Using eqn. (14) the corresponding digital filter is then

$$H(z) = \frac{1}{2}\left[\frac{1}{1-e^{-T}z^{-1}} + \frac{1}{1-e^{-3T}z^{-1}}\right]$$
$$= \frac{1}{2}\,\frac{2 - z^{-1}(e^{-3T}+e^{-T})}{(1-e^{-T}z^{-1})(1-e^{-3T}z^{-1})}$$
$$= \frac{1}{2}\,\frac{[2 - z^{-1}e^{-2T}(e^{-T}+e^{+T})]}{(1-e^{-T}z^{-1})(1-e^{-3T}z^{-1})}$$
$$H(z) = \frac{1-z^{-1}e^{-2T}\cosh T}{(1-e^{-T}z^{-1})(1-e^{-3T}z^{-1})}$$

It should be noted that zero of H(z) at $z = e^{-2T}\cosh T$ is not obtained by transforming the zero at $s = -z$ into a zero at $z = e^{-2T}$.

Problem *Apply the impulse invariant method to obtain the digital filter from the second order analog filter*

$$H_A(s) = \frac{s+a}{(s+a)^2+b^2}.$$

Sol. The analog filter transfer function is

$$H_A(s) = \frac{s+a}{(s+a+jb)(s+a-jb)}$$

Inverse Laplace transforming,

$$h_A(t) = \begin{cases} e^{-at}\cos(bt), & t \geq 0. \\ 0, & \text{otherwise.} \end{cases}$$

Sampling this function produces

$$h(nT_s) = \begin{cases} e^{-anT_s}\cos(bnT_s), & n \geq 0. \\ 0, & \text{otherwise.} \end{cases}$$

The z-transform of $h(nT_s)$, is equal to

$$H(z) = \sum_{n=0}^{\infty} e^{-anT_s}\cos(b_n T_s)\, z^{-n}$$

$$H(z) = \sum_{n=0}^{\infty} [e^{-aT_s}\cos(bT_s)\, z^{-1}]^n$$

$$H(z) = \frac{1 - e^{-aT}\cos(bT_s)\, z^{-1}}{(1 - e^{-(a+jb)T_s} z^{-1})(1 - e^{-(a-jb)T_s} z^{-1})}.$$

Problem *Using impulse invariance method with T = 1 sec determine*

$$H(z) \text{ if } H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}.$$

Sol. Given that

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

$$h(t) = L^{-1}(H(s)) = L^{-1}\left[\frac{1}{s^2 + \sqrt{2}s + 1}\right]$$

$$= L^{-1}\left[\frac{1}{\left(s + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\right]$$

$$= L^{-1}\left[\sqrt{2}\,.\,\frac{\frac{1}{\sqrt{2}}}{\left(s + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\right]$$

$$= \sqrt{2}\, L^{-1}\left[\frac{\frac{1}{\sqrt{2}}}{\left(s + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\right]$$

$$= \sqrt{2}\, e^{-t/\sqrt{2}} \sin t/\sqrt{2}.$$

Let $\qquad t = nT$

$$h(nT) = \sqrt{2}\, e^{-nT/\sqrt{2}} \sin nT/\sqrt{2}.$$

If $\qquad T = 1$ sec.

$$h(n) = \sqrt{2}\, e^{-n/\sqrt{2}} \sin n/\sqrt{2}.$$

$$H(z) = z[h(n)] = \sqrt{2}\left[\frac{e^{-1/\sqrt{2}} z^{-1} \sin 1/\sqrt{2}}{1 - 2e^{-1/\sqrt{2}} z^{-1} \cos 1/\sqrt{2} + e^{-\sqrt{2}} z^{-2}}\right]$$

$$= \frac{0.453z^{-1}}{1 - 0.7497z^{-1} + 0.2432z^{-2}}.$$

IIR FILTER DESIGN BY THE BILINEAR TRANSFORMATION

The IIR filter design using (i) approximation of derivatives method and (ii) the impulse invariant method are appropriate for the design of low-pass filters and bandpass filters whose resonant frequencies are low. These techniques are not suitable for high-pass or band-reject filters. This limitation is overcome in the mapping technique called the ***bilinear transformation***. This transformation is a one-to-one mapping from the s-domain to the z-domain. That is, the bilinear transformation is a conformal mapping that transforms the $j\,\Omega$-axis into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components. Also, the transformation of a stable analog filter results in a stable digital filter as all the poles in the left half of the s-plane are mapped onto points inside the unit circle of the z-domain. The bilinear transformation is obtained by using the trapezoidal formula for numerical integration. Let the system function of the analog filter be

$$H(s) = \frac{b}{s + a} \qquad (2)$$

The differential equation describing the analog filter can be obtained from Eq. 2 as shown below.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b}{s + a}$$

$$sY(s) + aY(s) = bX(s)$$

Taking inverse Laplace transform,

$$\frac{dy(t)}{dt} + a\,y(t) = b\,x(t) \qquad (3)$$

Eq. 3 is integrated between the limits $(nT - T)$ and nT

$$\int_{nT-T}^{nT} \frac{\mathrm{d}y(t)}{\mathrm{d}t}\,\mathrm{d}t + a\int_{nT-T}^{nT} y(t)\,\mathrm{d}t = b\int_{nT-T}^{nT} x(t)\,\mathrm{d}t \qquad 4$$

The trapezoidal rule for numeric integration is given by

$$\int_{nT-T}^{nT} a(t)\,\mathrm{d}t = \frac{T}{2}\left[a(nT) + a(nT - T)\right] \qquad 5$$

Applying Eq. 5 in Eq. 4 we get

$$y(nT) - y(nT-T) + \frac{aT}{2}y(nT) + \frac{aT}{2}y(nT-T) = \frac{bT}{2}x(nT) + \frac{bT}{2}x(nT-T)$$

Taking z-transform, the system function of the digital filter is,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a} \qquad 6$$

Comparing Eqs. 2 and 6 we get,

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right) \qquad 7$$

The general characteristic of the mapping $z = e^{sT}$ can be obtained by substituting $s = \sigma + j\,\Omega$ and expressing the complex variable z in the polar form as $z = re^{j\omega}$ in Eq. 7

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{T}\left(\frac{re^{j\omega} - 1}{re^{j\omega} + 1}\right)$$

$$= \frac{2}{T}\left(\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega} + j\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right)$$

Therefore,

$$\sigma = \frac{2}{T}\left(\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega}\right) \qquad 8$$

$$\Omega = \frac{2}{T}\left(\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right) \qquad 9$$

From Eq. 8 , it can be noted that if $r < 1$, then $\sigma < 0$, and if $r > 1$, then $\sigma > 0$. Thus, the left-half of the s-plane maps onto the points inside the unit circle in the z-plane and the transformation results in a stable digital system. Consider Eq. 9 for unity magnitude ($r = 1$), σ is zero. In this case,

$$\Omega = \frac{2}{T}\left(\frac{\sin\omega}{1+\cos\omega}\right)$$

$$= \frac{2}{T}\left(\frac{2\sin\omega/2\cos\omega/2}{\cos^2\omega/2+\sin^2\omega/2+\cos^2\omega/2-\sin^2\omega/2}\right)$$

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2} \qquad 10$$

or equivalently,

$$\omega = 2\tan^{-1}\frac{\Omega T}{2} \qquad 11$$

Equation 11 gives the relationship between the frequencies in the two domains and this is shown in Fig. It can be noted that the entire range in Ω is mapped only once into the range $-\pi \le \omega \le \pi$. However, as seen in Fig. , the mapping is non-linear and the lower frequencies in analog domain are expanded in the digital domain, whereas the higher frequencies are compressed. This is due to the non-linearity of the arc tangent function and usually called as ***frequency warping***.

The warping effect can be eliminated by prewarping the analog filter. The effect of non-linear compression at high frequencies can be compensated by prewarping. When the desired magnitude response is piece-wise constant over frequency, this compression can be compensated by introducing a suitable prescaling or prewarping the critical frequencies by using the formula,

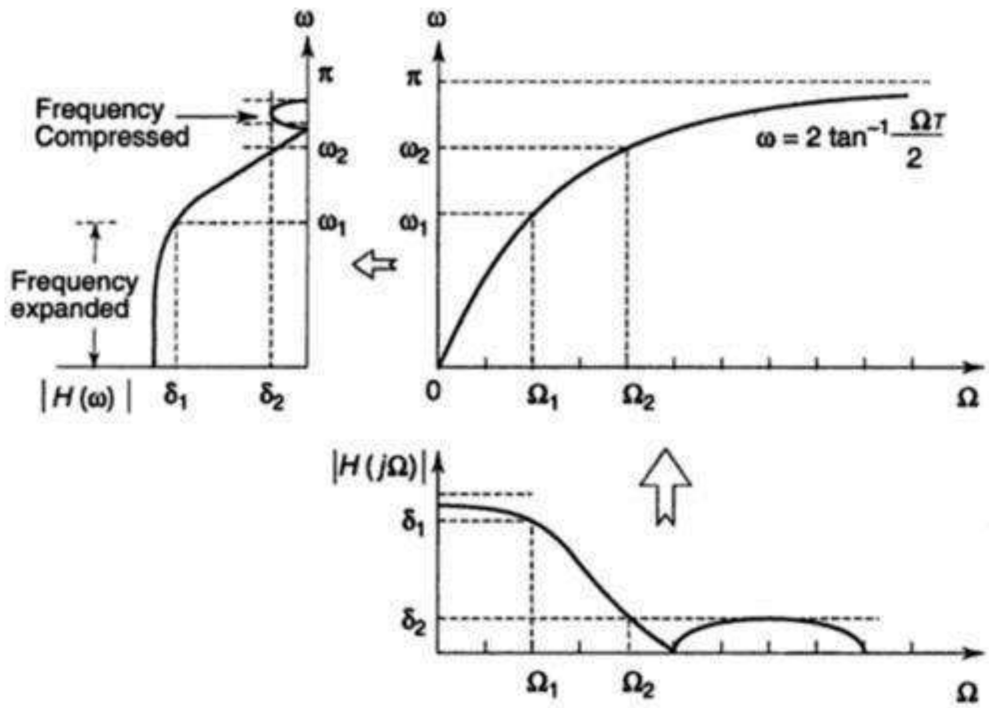


Fig. *Relationship Between ω and Ω as Given in Eq.* 11

Problem *Convert the analog filter with system function.*

$H_a(s) = \dfrac{s+0.1}{(s+0.1)^2+16}$ *into a digital IIR filter by means of bilinear transformation. Resonant frequency of a digital filter is given as* $w_r = \dfrac{\pi}{2}$.

Sol. (*i*) We first note that the analog filter $H_a(s)$ has a resonant frequency.

$$\Omega_r = \sqrt{16} = 4.$$

(*ii*) Let us find T

$$\Omega = \frac{2}{T}\tan\frac{w}{2}$$

$$4 = \frac{2}{T}\tan\frac{\pi}{4}$$

$$T = \frac{1}{2}.$$

(*iii*) Now map

$$S = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

By substituting values of s into H(s), we have,

$$H(z) = H_a(s)\Big|_{s=4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{4\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)+0.01}{\left[4\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)+0.01\right]^2+16}$$

$$= \frac{0.128 + 0.006z^{-1} - 0.122z^{-1}}{1 + 0.0006z^{-1} + 0.975z^{-2}} = \frac{0.128 + 0.006z^{-1} - 0.122z^{-1}}{1 + 0.975z^{-2}}$$

$$= \frac{(z+1)(z-0.95)}{(z - 0.987e^{-j\pi/2})(z - 0.987e^{j\pi/2})}$$

This filter has a pole $P_{1,2} = 0.98\, ze^{\pm j\pi/2}$ and zeros at $z_{1,2} = -1, 0.95$.

Problem *A first order Butterworth low pass transfer function with a 3dB cut off frequency at Ω_c is given by*

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c}.$$

Design a single pole low pass with 3dB bandwidth of 0.2 π using the bilinear transformation.

Sol. $\Omega_c = \frac{2}{T} \tan \frac{\omega c}{2}$.

Given that $\omega_c = 0.2\,\pi$

$$\Omega_c = \frac{2}{T} \tan \frac{0.2\,\pi}{2} = \frac{2}{T} \tan 0.1\,\pi = \frac{0.65}{T}.$$

The analog filter has a system function,

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c} = \frac{0.65/T}{s + \frac{0.65}{T}}$$

Now $$H(z) = H_a(s)\Big|_{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{0.65/T}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + \frac{0.65}{T}}$$

$$H(z) = \frac{0.65(1+z^{-1})}{2 - 2z^{-1} + 0.65} = \frac{0.245(1+z^{-1})}{(1 - 0.509z^{-1})}$$

The frequency response of the digital filter is

$$H(\omega) = \frac{0.245(1 + e^{-j\omega})}{1 - 0.509\, e^{-j\omega}}.$$

Thus at $\omega = 0$, $H(0) = 1$ and at $\omega = 0.2\,\pi$,

$|H(0.2\,\omega)| = 0.707$, which is a desired response.

Problem *Obtain H(z) from $H_a(s)$ when T = 1 sec and $H_a(s) = \frac{s^3}{(s+1)(s^2+s+1)}$.*

Sol. Given that, $H_a(s) = \frac{s^3}{(s+1)(s^2+s+1)}$.

Put $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ in $H_a(s)$ to get H(z).

$$H(z) = \frac{\left[\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^3}{\left[\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 1\right]\left[\left(\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^2 + \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 1\right]}$$

$$= \frac{8(1-z^{-1})^3}{\left[2\left(1-z^{-1}\right)+T(1+z^{-1})\right]\left[4(1-z^{-1})^2+2T(1-z^{-1})(1+z^{-1})+T^2(1+z^{-1})^2\right]}$$

But T = 1 sec.

$$H(z) = \frac{8(1-z^{-1})^3}{(3-z^{-1})(7-6z^{-1}+3z^{-2})}$$

$$H(z) = \frac{2.67(z^{-1}-1)^3}{(z^{-2}-2z^{-1}+2.33)(z^{-1}-3)}.$$

Problem *Design a digital Butterworth filter satisfying the constraints*

$$0.707 \le |H(e^{j\omega})| \le 1 \quad for\ 0 \le \omega \le \frac{\pi}{2}$$

$$|H(e^{j\omega})| \le 0.2 \quad for \quad \frac{3\pi}{4} \le \omega \le \pi.$$

with T = 1 sec using The bilinear transformation

Sol. Bilinear transformation

Given that $\frac{1}{\sqrt{1+\varepsilon^2}} = 0.707$; $\frac{1}{\sqrt{1+\lambda^2}} = 0.2$, $\omega_p = \frac{\pi}{2}$; $\omega_s = \frac{3\pi}{4}$

The analog frequency ratio is

$$\frac{\Omega_s}{\Omega_p} = \frac{\frac{2}{T}\tan\frac{\omega_s}{2}}{\frac{2}{T}\tan\frac{\omega_p}{2}} = \frac{\tan\frac{3\pi}{8}}{\tan\frac{\pi}{4}} = 2.414$$

The order of the filter,

$$N \ge \frac{\log \lambda/\varepsilon}{\log\frac{\Omega_s}{\Omega_p}}.$$

From the given data $\lambda = 4.898$, $\varepsilon = 1$,

So, $$N \ge \frac{\log 4.898}{\log 2.414} = 1.803.$$

Rounding N to nearest higher value we get N = 2.

We know $$\Omega_c = \frac{\Omega_p}{(\varepsilon)^{1/N}} = \Omega_p \qquad (\because \ \varepsilon = 1)$$

$$= \frac{2}{T}\tan\frac{\omega_p}{2} = 2\tan\frac{\pi}{4} = 2 \text{ rad/sec}.$$

The transfer function of second order normalised Butterworth filter is,

$$H(s) = \frac{1}{s^2+\sqrt{2}s+1}.$$

$H_a(s)$ for $\Omega_c = 2$ rad/sec can be obtained by substituting

$s \rightarrow s/2$ in $H(s)$

i.e., $$H_a(s) = \frac{1}{(s/2)^2 + \sqrt{2}(s/2) + 1} = \frac{4}{s^2 + 2.828s + 4}.$$

By using bilinear transformation H(z) can be obtained as

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}.$$

Thus $$H(z) = \frac{4}{s^2 + 2.828s + 4}\Bigg|_{s = 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \qquad (\because \ T = 1 \text{ sec})$$

$$= \frac{4(1+z^{-1})^2}{4(1-z^{-1})^2 + 2.828\,(1-z^{-2}) + 4(1+z^{-1})^2}$$

$$= \frac{0.2929(1+z^{-1})^2}{1 + 0.1716z^{-2}}.$$