Lecture Notes on Precalculus

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Trigonometric identities

$$\boxed{a\pm b} \qquad\qquad\qquad\qquad \boxed{2a}$$

$$\left.\begin{aligned}
\sin(a\pm b) &= \sin a\cos b \pm \sin b\cos a\\
\cos(a\pm b) &= \cos a\cos b \mp \sin a\sin b\\
\tan(a\pm b) &= \frac{\tan a \pm \tan b}{1\mp \tan a\tan b}\\
\cot(a\pm b) &= \frac{\cot a\cot b \mp 1}{\cot b \pm \cot a} \quad \textbf{(!!)}
\end{aligned}\right\} \Longrightarrow
\begin{aligned}
\sin(2a) &= 2\sin a\cos a\\
\cos(2a) &= \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a\\
\tan(2a) &= \frac{2\tan a}{1-\tan^2 a}\\
\cot(2a) &= \frac{\cot^2 a - 1}{2\cot a}
\end{aligned}$$

▶ $\sin(a+b)\sin(a-b) = \sin^2 a - \sin^2 b$
▶ $\cos(a+b)\cos(a-b) = \cos^2 a - \sin^2 b$

$$\boxed{3a} \Longrightarrow \quad \begin{aligned}\sin(3a) &= -4\sin^3 a + 3\sin a\\ \cos(3a) &= +4\cos^3 a - 3\cos a\end{aligned} \qquad \tan(3a) = \frac{3\tan a - \tan^3 a}{1 - 3\tan^2 a}$$

In terms of

$$\boxed{\cos 2a} \qquad\qquad\qquad\qquad \boxed{\tan(a/2)}$$

$$\sin^2 a = \frac{1-\cos(2a)}{2} \quad \cos^2 a = \frac{1+\cos(2a)}{2} \quad \sin a = \frac{2\tan(a/2)}{1+\tan^2(a/2)} \quad \cos a = \frac{1-\tan^2(a/2)}{1+\tan^2(a/2)}$$

$$\tan^2 a = \frac{1-\cos(2a)}{1+\cos(2a)} \quad \cot^2 a = \frac{1+\cos(2a)}{1-\cos(2a)} \quad \tan a = \frac{2\tan(a/2)}{1-\tan^2(a/2)} \quad \cot a = \frac{1-\tan^2(a/2)}{2\tan(a/2)}$$

Transformation to

$$\boxed{\text{sum}} \qquad\qquad\qquad\qquad \boxed{\text{product}}$$

$$\left.\begin{aligned}
2\sin a\cos b &= \sin(a-b) + \sin(a+b)\\
2\cos a\cos b &= \cos(a-b) + \cos(a+b)\\
2\sin a\sin b &= \cos(a-b) - \cos(a+b)
\end{aligned}\right\} \Longrightarrow
\begin{aligned}
\sin a \pm \sin b &= 2\sin\frac{a\pm b}{2}\cos\frac{a\mp b}{2}\\
\cos a + \cos b &= 2\cos\frac{a+b}{2}\cos\frac{a-b}{2}\\
\cos a - \cos b &= 2\sin\frac{a+b}{2}\sin\frac{b-a}{2} \quad \textbf{(!!)}\\
\tan a \pm \tan b &= \frac{\sin(a\pm b)}{\cos a\cos b}\\
\cot a \pm \cot b &= \frac{\sin(b\mp a)}{\sin a\sin b} \quad \textbf{(!!)}
\end{aligned}$$

Also note the factorizations:

▶ $1\pm\sin a = \sin(\pi/2)\pm\sin a = 2\sin\dfrac{(\pi/2)\pm a}{2}\cos\dfrac{(\pi/2)\mp a}{2}$
▶ $\sin a \pm \cos b = \sin a \pm \sin(\pi/2 - b) = 2\sin\dfrac{a\pm(\pi/2-b)}{2}\cos\dfrac{a\mp(\pi/2-b)}{2}$
▶ $1+\cos a = 2\cos^2(a/2)$
▶ $1-\cos a = 2\sin^2(a/2)$

PRE1: Review of geometry

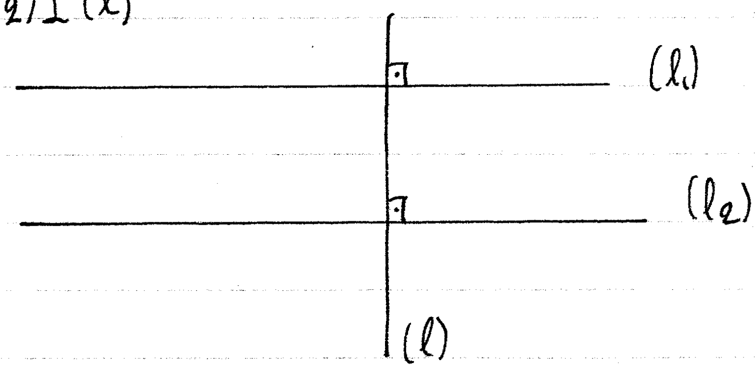
REVIEW OF GEOMETRY

▼ Parallel and perpendicular lines

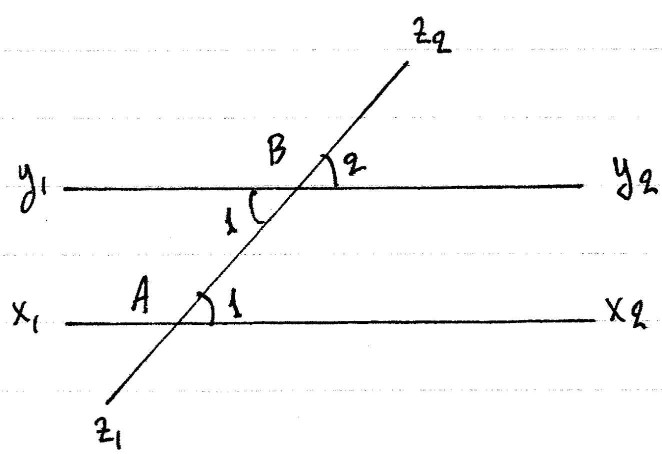
We give the following two results without proof.

1) Given 3 lines $(\ell_1), (\ell_2), (\ell)$

$$\begin{cases} (\ell_1) \perp (\ell) \\ (\ell_2) \perp (\ell) \end{cases} \Rightarrow (\ell_1) // (\ell_2)$$



2) Given two lines $(\ell_1), (\ell_2)$ and a line (ℓ) such that $(\ell_1) // (\ell_2)$ and $(\ell) \cap (\ell_1) = \{A\}$ and $(\ell) \cap (\ell_2) = \{B\}$, we define the angles $\hat{A}_1 = x_2 \hat{A} B$ and $\hat{B}_1 = y_1 \hat{B} A$ and $\hat{B}_2 = y_2 \hat{B} z_2$

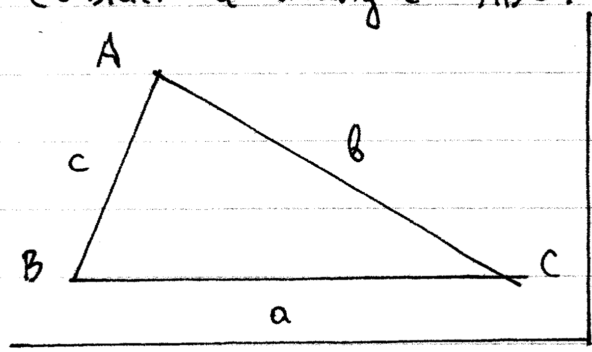


Then: $\boxed{\hat{A}_1 = \hat{B}_1 = \hat{B}_2}$

Terminology: A_1, B_1 : interior alternate angles
A_1, B_2 : interior–exterior corresponding angles.
B_1, B_2 : vertical angles.

▼ Basic properties of triangles

Consider a triangle $\overset{\Delta}{ABC}$. We define:



1) Three angles

$\hat{A} = B\hat{A}C$

$\hat{B} = C\hat{B}A$

$\hat{C} = A\hat{C}B$

2) Three sides

$a = BC$

$b = CA$

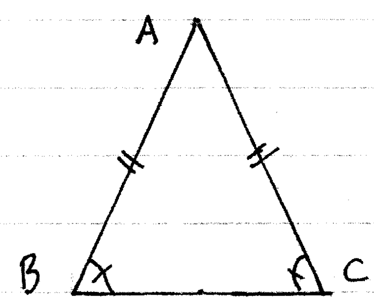
$c = AB$

1 → Isosceles property

$$\boxed{\begin{array}{l} a = b \Leftrightarrow \hat{A} = \hat{B} \\ b = c \Leftrightarrow \hat{B} = \hat{C} \\ c = a \Leftrightarrow \hat{C} = \hat{A} \end{array}}$$

This property can be shown via equality of triangles. We omit the proof.

example



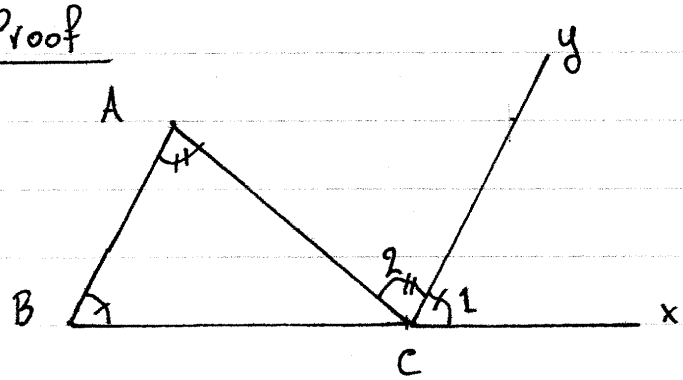
The case:
$b = c \Leftrightarrow \hat{B} = \hat{C}$

We say that
$\overset{\Delta}{ABC}$ is isosceles $\Leftrightarrow a = b \vee b = c \vee c = a$

(2) → Angle sum

$$\boxed{\hat{A} + \hat{B} + \hat{C} = 180^\circ}$$

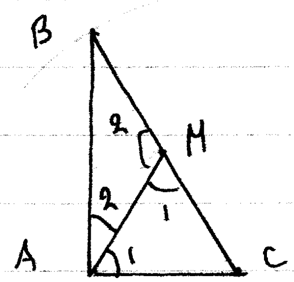
Proof



Extend BC to the side of C with the half line Cx such that $B\hat{C}x = 180^\circ$. Bring $Cy \parallel AB$ on the same half-plane as A.

Define $\hat{C}_1 = x\hat{C}y$ and $\hat{C}_2 = A\hat{C}y$. Then
$\hat{C}_1 = \hat{B}$, as interior-exterior corresponding angles
$\hat{C}_2 = \hat{A}$, as interior-alternate angles
It follows that
$\hat{A} + \hat{B} + \hat{C} = \hat{C}_2 + \hat{C}_1 + \hat{C} = B\hat{C}x = 180^\circ$

(3) → 30-60 triangle



$$\boxed{\left.\begin{array}{l}\hat{A} = 90^\circ \\ \hat{B} = 30^\circ\end{array}\right\} \Rightarrow b = \frac{a}{2}}$$

Proof

Choose M on BC such that $M\hat{A}C = 60^\circ$. Define $\hat{A}_1 = M\hat{A}C$ and $\hat{A}_2 = B\hat{A}M$ and $\hat{M}_1 = A\hat{M}C$ and $\hat{M}_2 = A\hat{M}B$. Note that given $\hat{A}_1 = M\hat{A}C = 60^\circ$ we have:

$$\hat{C} = 180^\circ - \hat{A} - \hat{B} = 180^\circ - 90^\circ - 30^\circ = 60^\circ$$

$$\hat{M}_1 = 180^\circ - \hat{C} - \hat{A}_1 = 180^\circ - 60^\circ - 60^\circ = 60^\circ$$

It follows that $\hat{A}_1 = \hat{C} = \hat{M}_1 = 60^\circ \Rightarrow CM = AM = AC$ (1)

We also have

$\hat{A}_2 = \hat{A} - \hat{A}_1 = 90^\circ - 60^\circ = 30^\circ = \hat{B} \Rightarrow AM = BM$ (2)

and therefore:

$$\begin{aligned} a = BC &= BM + MC \\ &= AM + AC && [\text{via } BM = AM \text{ and } MC = AC] \\ &= AC + AC && [\text{via } AM = AC] \\ &= 2AC = 2b \Rightarrow b = a/2. \end{aligned}$$

□

Similar triangles and the Pythagorean theorem

Def: Consider two triangles $\overset{\Delta}{A_1B_1C_1}$ and $\overset{\Delta}{A_2B_2C_2}$.
We define:

$$\overset{\Delta}{A_1B_1C_1} \sim \overset{\Delta}{A_2B_2C_2} \Longleftrightarrow \begin{cases} \hat{A}_1 = \hat{A}_2 \wedge \hat{B}_1 = \hat{B}_2 \wedge \hat{C}_1 = \hat{C}_2 \\ \dfrac{A_1B_1}{A_2B_2} = \dfrac{B_1C_1}{B_2C_2} = \dfrac{C_1A_1}{C_2A_2} \end{cases}$$

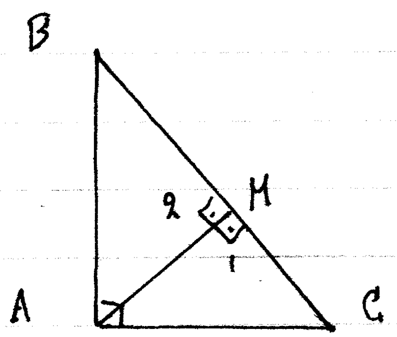
- If $\overset{\Delta}{A_1B_1C_1} \sim \overset{\Delta}{A_2B_2C_2}$, then we say that the triangles $\overset{\Delta}{A_1B_1C_1}$ and $\overset{\Delta}{A_2B_2C_2}$ are <u>similar</u>.
- We can show (proof omitted) that

$$\boxed{\begin{cases} \hat{A}_1 = \hat{A}_2 \\ \hat{B}_1 = \hat{B}_2 \end{cases} \Longrightarrow \overset{\Delta}{A_1B_1C_1} \sim \overset{\Delta}{A_2B_2C_2}}$$

- This result can be used to establish the Pythagorean theorem:

$$\boxed{\hat{A} = 90^\circ \Longrightarrow a^2 = b^2 + c^2}$$

Proof



Choose M on BC such that $AM \perp BC$.
Define $\hat{M}_1 = A\hat{M}C$ and $\hat{M}_2 = A\hat{M}B$
and note that
$AM \perp BC \Rightarrow \hat{M}_1 = \hat{M}_2 = 90^\circ$
Compare $\overset{\Delta}{ABC}$ with $\overset{\Delta}{AMC}$. Both share $\hat{C}$. Also $\hat{A} = \hat{M}_1$. It follows that

$$\overset{\Delta}{ABC} \sim \overset{\Delta}{MAC} \Rightarrow \frac{CM}{AC} = \frac{AC}{BC} \Rightarrow \frac{CM}{b} = \frac{b}{a} \Rightarrow CM = \frac{b^2}{a} \quad (1)$$

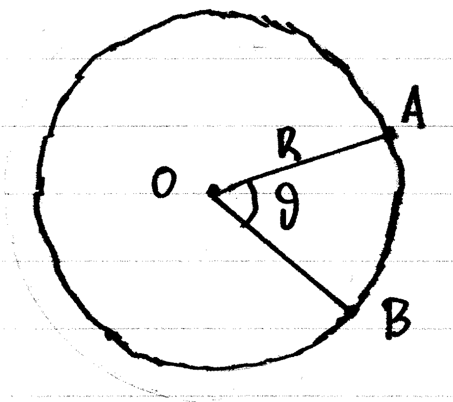
Compare $\overset{\Delta}{ABC}$ with $\overset{\Delta}{AMB}$. Both share $\hat{B}$ and also $\hat{A} = \hat{M}_2$.
It follows that

$$\overset{\Delta}{ABC} \sim \overset{\Delta}{MBA} \Rightarrow \frac{BM}{AB} = \frac{AB}{BC} \Rightarrow \frac{BM}{c} = \frac{c}{a} \Rightarrow BM = \frac{c^2}{a} \quad (2)$$

From Eq.(1) and Eq.(2):

$$BM + CM = BC \Rightarrow \frac{c^2}{a} + \frac{b^2}{a} = a \Rightarrow \underline{a^2 = b^2 + c^2} \qquad \square$$

V Circles



Circumference:
$\ell = 2\pi R$
Area:
$A = \pi R^2$

- Consider the arc $\widehat{AB}$.
- We say that the angle $A\hat{O}B$ subtends the arc $\widehat{AB}$.
- Length of arc:

$$\boxed{\ell(\widehat{AB}) = 2\pi R \cdot \frac{(A\hat{O}B)}{360}}$$

with $(A\hat{O}B)$ given in degrees.

- Angles in radians

The measure ϑ of the angle $A\hat{O}B$ is defineds as

$$\boxed{\vartheta = \frac{2\pi}{360}(A\hat{O}B)} \Rightarrow \boxed{\ell(AB) = R\vartheta}$$

Some commonly used angles in degrees and in radians:

$A\hat{O}B$	30°	45°	60°	90°	180°	360°
ϑ	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	2π

To convert:

$$\boxed{(\text{radians}) = \frac{2\pi}{360}\,(\text{degrees})}$$

$$\boxed{(\text{degrees}) = \frac{360}{2\pi}\,(\text{radians})}$$

- Area of a sector

The area (OAB) of the sector defined by the arc $\widehat{AB}$ is:

$$\boxed{(OAB) = \frac{1}{2} R^2 \vartheta}$$

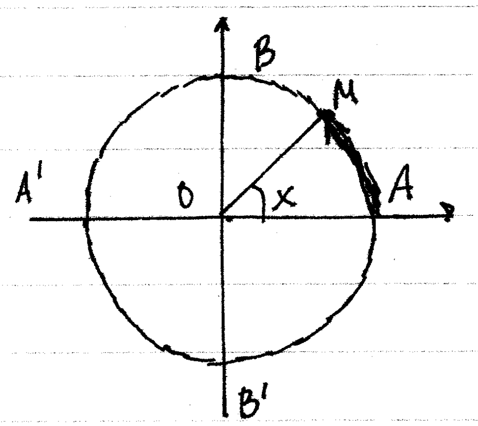
For $\vartheta = 2\pi$, this gives the area of of the whole circle $A = \pi R^2$.

PRE2: Trigonometric functions

TRIGONOMETRIC FUNCTIONS

▼ The trigonometric circle

- The trigonometric circle is an oriented circle with radius 1. An oriented circle is a circle with a well-defined initial point A and a positive (counterclockwise) and negative (clockwise) direction.



$OA = OM = 1$

- Let $x \in \mathbb{R}$ be given. Starting from the point A, we traverse the trigonometric circle counterclockwise (if $x > 0$) or clockwise (if $x < 0$) over an arc with total length x. We stop at the point M. Note that we could go around the whole circle multiple times.

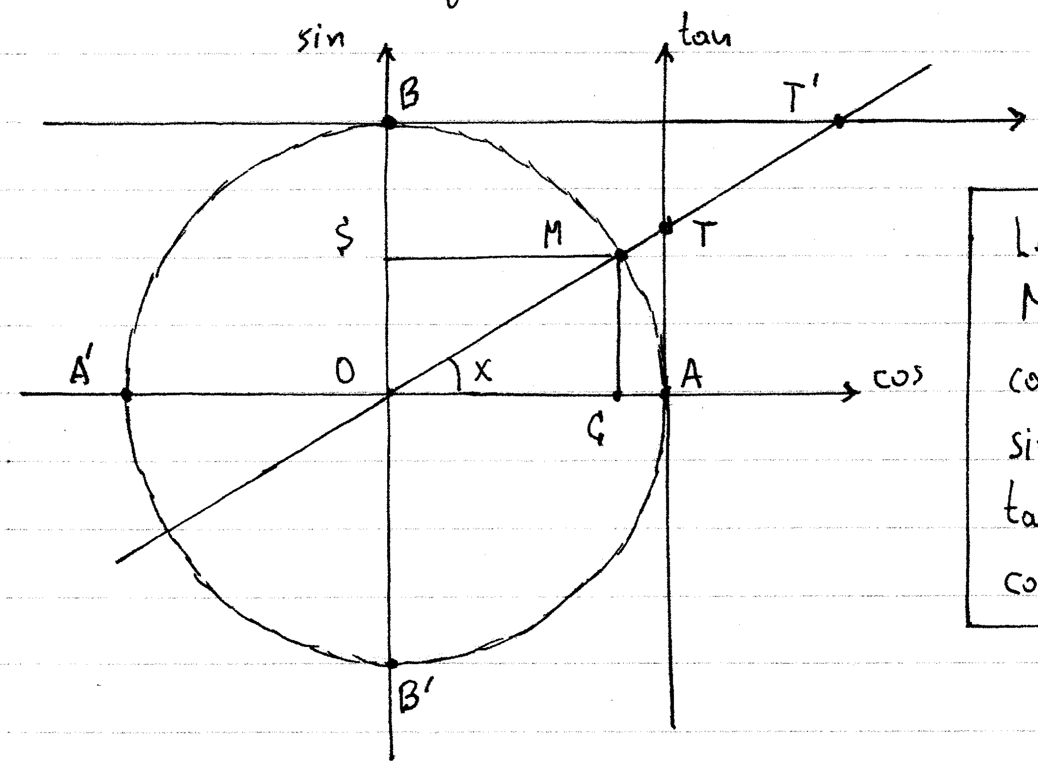
- We say that M is the terminal point of the arc x and define the winding function $\mathcal{C} : \mathbb{R} \to \mathbb{R}^2$ such that $\mathcal{C}(x) = M$.
- Consider the set $\mathbb{Z} = \{0, +1, -1, +2, -2, \ldots\}$. Two arcs $x_1, x_2 \in \mathbb{R}$ have the same terminal point if and only if there exists a $k \in \mathbb{Z}$ such that $x_1 = x_2 + 2k\pi$. Symbolically, we write:

$$\boxed{\mathcal{C}(x_1) = \mathcal{C}(x_2) \Leftrightarrow \exists k \in \mathbb{Z} : x_1 = x_2 + 2k\pi}$$

- It is good to know the general form of arcs with terminal points at A, A', B, B', etc:

Terminal points	Arcs
A	$x = 2k\pi$
A'	$x = (2k+1)\pi$
B	$x = 2k\pi + \pi/2$
B'	$x = (2k+1)\pi + \pi/2$
A or A'	$x = k\pi$
B or B'	$x = k\pi + \pi/2$

Definition of trigonometric functions



Let $x \in \mathbb{R}$

$M = \mathcal{E}(x)$

$\cos x = \overline{OC}$

$\sin x = \overline{OS}$

$\tan x = \overline{AT}$

$\cot x = \overline{BT'}$

•1 On the trigonometric circle we define:

sin-axis: From A' to A

cos-axis: From B' to B

tan-axis: $\begin{cases}\text{Tangent to circle at } A\\ \text{Same direction as sin-axis}\end{cases}$

cot-axis: $\begin{cases}\text{Tangent to circle at } B\\ \text{Same direction as cos-axis.}\end{cases}$

•2 Let $x \in \mathbb{R}$ be an arc with terminal point $M = \mathcal{C}(x)$.

•3 We construct the following points:

C: projection of M to cos-axis

S: projection of M to sin-axis

T: intersection of line (OM) with tan-axis

T': intersection of line (OM) with cot-axis

•4 Now we define the trigonometric functions geometrically as follows:

$\forall x \in \mathbb{R}: \sin(x) = \overline{OS}$ = coordinate of S on sin-axis

$\forall x \in \mathbb{R}: \cos(x) = \overline{OC}$ = coordinate of C on cos-axis.

↳ (in both cases, O is the origin.)

$\forall x \in \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}: \tan(x) = \overline{AT}$ = coordinate of T on tan-axis

↳ (A is the origin on tan-axis, and tanx is not defined at M = B or M = B' because then OM is parallel to tan-axis.)

$\forall x \in \mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\}: \cot(x) = \overline{BT'}$ = coordinate of T' on cot-axis

↳ (B is the origin, and cotx is not defined at M = A or M = A' because then OM // cot-axis).

Basic properties of trigonometric functions

Trigonometric Identities

$$\boxed{\sin^2 x + \cos^2 x = 1} \longrightarrow \begin{cases} \sin^2 x = 1 - \cos^2 x \\ \cos^2 x = 1 - \sin^2 x \end{cases}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

$$\boxed{(\tan x)(\cot x) = 1}$$

$$1 + \tan^2 x = \frac{1}{\cos^2 x}$$

$$1 + \cot^2 x = \frac{1}{\sin^2 x}$$

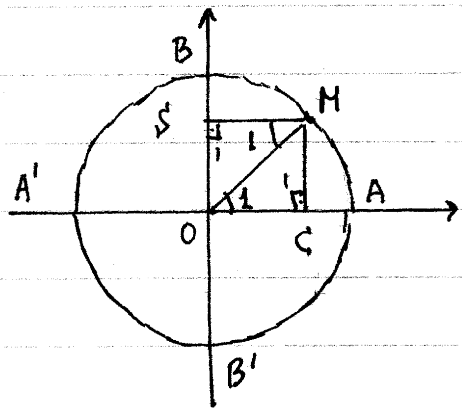
$$\cos^2 x = \frac{1}{1 + \tan^2 x} \qquad \sin^2 x = \frac{1}{1 + \cot^2 x}$$

Evaluation at standard angles

ϑ (radians)	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
ϑ (degrees)	0	30°	45°	60°	90°
$\sin\vartheta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\vartheta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\tan\vartheta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	?
$\cot\vartheta$	?	$\sqrt{3}$	1	$\sqrt{3}/3$	0

"?" corresponds to "undefined".

Proof of $\sin^2 x + \cos^2 x = 1$



With no loss of generality, assume that the terminal point M is in the first quadrant. Then $\sin x = OC$ and $\cos x = OS$.

Define:

$\hat{O}_1 = C\hat{O}M \wedge \hat{C}_1 = O\hat{C}M \wedge \hat{M}_1 = S\hat{M}O$
$\wedge\ \hat{S}_1 = O\hat{S}M$.

Since $\begin{cases} OC \perp BB' \\ MS \perp BB' \end{cases} \Rightarrow OC \parallel MS \Rightarrow \hat{O}_1 = \hat{M}_1$. (1)

Also have $\hat{C}_1 = \hat{S}_1 = 90^\circ$. (2)

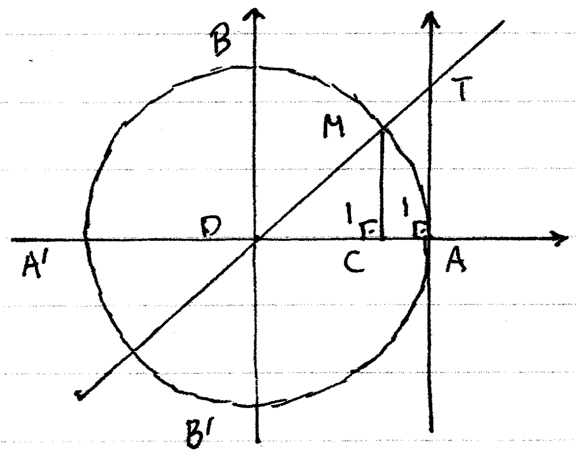
From Eq. (1) and Eq. (2):

$$\overset{\triangle}{OCM} \sim \overset{\triangle}{MSO} \Rightarrow \frac{CM}{OM} = \frac{OS}{OM} \Rightarrow CM = OS$$

and therefore, via the pythagorean theorem:

$$\begin{aligned} \sin^2 x + \cos^2 x &= (OS)^2 + (OC)^2 \\ &= (CM)^2 + (OC)^2 && [\text{via } CM = OS] \\ &= OM^2 && [\text{pythagorean on } \overset{\triangle}{OCM}] \\ &= 1 && [\text{trig-circle radius}]. \end{aligned}$$

Proof of $\tan x = \dfrac{\sin x}{\cos x}$



With no loss of generality assume that the terminal point M is in the first quadrant.

We previously showed that

$$CM = OS \qquad (1)$$

Compare $\overset{\triangle}{OCM}$ with $\overset{\triangle}{OAT}$.

Both share $\hat{O}$ and $\hat{A}_1 = \hat{C}_1 = 90^0$.

It follows that $\overset{\triangle}{OCM} \sim \overset{\triangle}{OAT}$ and therefore:

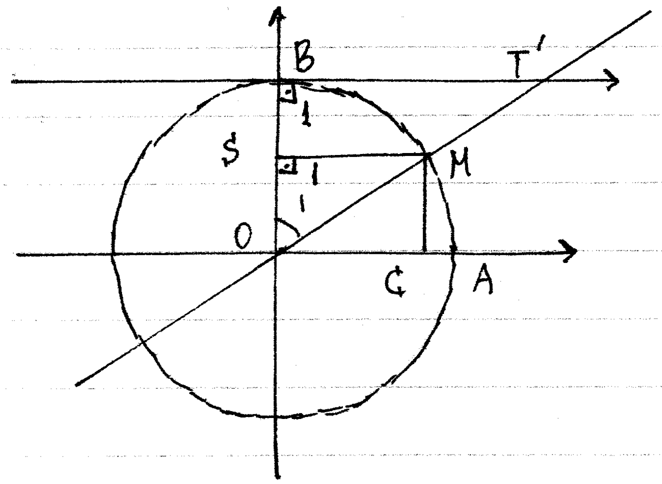
$$\tan x = AT = \frac{AT}{1} = \frac{AT}{OA} \qquad [\text{because } OA = 1]$$

$$= \frac{CM}{OC} \qquad [\text{because } \overset{\triangle}{OCM} \sim \overset{\triangle}{OAT}]$$

$$= \frac{OS}{OC} \qquad [\text{because } CM = OS]$$

$$= \frac{\sin x}{\cos x}$$

Proof of $\cot x = \dfrac{\cos x}{\sin x}$



With no loss of generality assume that the terminal point M is in the first quadrant. Define $\hat{O}_1 = S\hat{O}M$. We have already shown that

$$\overset{\Delta}{OSM} \sim \overset{\Delta}{MCO} \Rightarrow \frac{MS}{OM} = \frac{OC}{OM} \Rightarrow$$

$$\Rightarrow MS = OC. \quad (1)$$

Compare $\overset{\Delta}{OMS}$ with $\overset{\Delta}{OT'B}$. Both share $\hat{O}_1$ and $\hat{S}_1 = \hat{B}_1 = 90^\circ$ (with $\hat{S}_1 = O\hat{S}M$ and $\hat{B}_1 = O\hat{B}T'$), thus $\overset{\Delta}{OSM} \sim \overset{\Delta}{OBT'}$.

It follows that

$$\cot x = BT' = \frac{BT'}{1} = \frac{BT'}{OB} \quad [\text{via } OB = 1]$$

$$= \frac{MS}{OS} \quad [\text{via } \overset{\Delta}{OSM} \sim \overset{\Delta}{OBT'}]$$

$$= \frac{OC}{OS} \quad [\text{via } MS = OC]$$

$$= \frac{\cos x}{\sin x}$$

Proof of other identities.

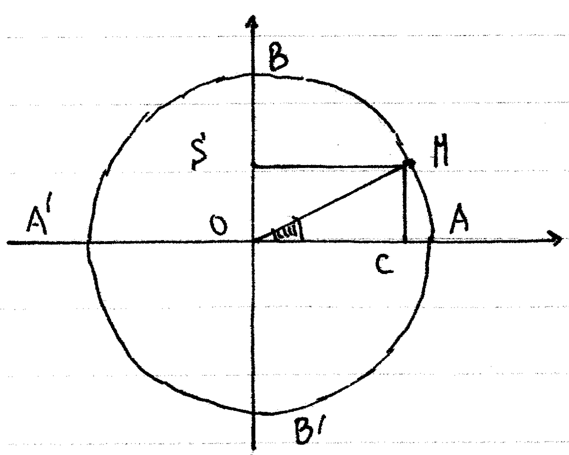
We have:

$$1+\tan^2 x = 1+\frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

and

$$1+\cot^2 x = 1+\frac{\cos^2 x}{\sin^2 x} = \frac{\cos^2 x+\sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

Angle 30° - proof



Assume that $M\hat{O}C = 30^\circ$. Then:

$$\begin{cases} M\hat{O}C = 30^\circ \\ M\hat{C}O = 90^\circ \end{cases} \Rightarrow CM = \frac{OM}{2} = \frac{1}{2}$$

$$\Rightarrow \sin(30^\circ) = OS' = CM = \frac{1}{2}$$

Since $0 < 30^\circ < 90^\circ \Rightarrow \cos 30^\circ > 0$ and it follows that

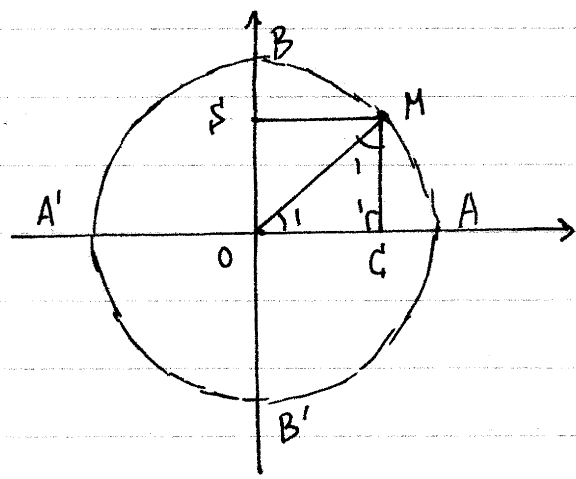
$$\cos^2(30^\circ) = 1-\sin^2(30^\circ) = 1-\left(\frac{1}{2}\right)^2 = 1-\frac{1}{4} = \frac{4-1}{4} = \frac{3}{4} \Rightarrow$$

$$\Rightarrow \cos(30^\circ) = \frac{\sqrt{3}}{2}$$, and also:

$$\tan(30^\circ) = \frac{\sin(30^\circ)}{\cos(30^\circ)} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cot(30^\circ) = \frac{\cos(30^\circ)}{\sin(30^\circ)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

Angle 45^0 - Proof



Assume that $M\hat{O}C = 45^0$.
Define $\hat{O}_1 = M\hat{O}C$ and $\hat{M}_1 = O\hat{M}C$ and $\hat{C}_1 = O\hat{C}M$.

$\hat{M}_1 = 180^0 - \hat{C}_1 - \hat{M}_1 =$
$= 180^0 - 90^0 - 45^0 =$
$= 45^0 = \hat{O}_1 \Rightarrow OC = CM$
$\Rightarrow OC = OS \Rightarrow$
$\Rightarrow \sin(45^0) = \cos(45^0)$

Since

$$\sin^2(45^0) + \cos^2(45^0) = 1 \Rightarrow \sin^2(45^0) + \sin^2(45^0) = 1 \Rightarrow$$

$$\Rightarrow 2\sin^2(45^0) = 1 \Rightarrow \sin^2(45^0) = \frac{1}{2} \Rightarrow$$

$$\Rightarrow \sin(45^0) = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \cos(45^0)$$

we also have:

$$\tan(45^0) = \frac{\sin(45^0)}{\cos(45^0)} = \frac{\cos(45^0)}{\cos(45^0)} = 1$$

$$\cot(45^0) = \frac{\cos(45^0)}{\sin(45^0)} = \frac{\cos(45^0)}{\cos(45^0)} = 1$$

EXAMPLES

a) Simplify the following expression:

$$A = [\sin(\pi/4) \cdot \cos(\pi/6) + \cot(\pi/3)] \tan(\pi/6)$$

Solution

$$A = [\sin(\pi/4)\cos(\pi/6) + \cot(\pi/3)]\tan(\pi/6) =$$

$$= \left[\frac{\sqrt{2}}{2}\,\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3}\right]\frac{\sqrt{3}}{3} = \frac{\sqrt{2}(\sqrt{3})^2}{2\cdot 2\cdot 3} + \frac{(\sqrt{3})^2}{3^2} =$$

$$= \frac{3\sqrt{2}}{3\cdot 4} + \frac{3}{3^2} = \frac{\sqrt{2}}{4} + \frac{1}{3} = \frac{3\sqrt{2}+4}{4\cdot 3} = \frac{3\sqrt{2}+4}{12}$$

b) Simplify the following expression

$$A = \frac{\sin(\pi/6) + \sin(\pi/3)}{\sin(\pi/6) - \sin(\pi/3)}$$

Solution

$$A = \frac{\sin(\pi/6) + \sin(\pi/3)}{\sin(\pi/6) - \sin(\pi/3)} = \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}}{\frac{1}{2} - \frac{\sqrt{3}}{2}} = \frac{1+\sqrt{3}}{1-\sqrt{3}} =$$

$$= \frac{(1+\sqrt{3})^2}{(1-\sqrt{3})(1+\sqrt{3})} = \frac{1^2 + 2\cdot 1\cdot\sqrt{3} + (\sqrt{3})^2}{1^2 - (\sqrt{3})^2} = \frac{1+2\sqrt{3}+3}{1-3} =$$

$$= \frac{4+2\sqrt{3}}{-2} = -2-\sqrt{3}$$

→ When simplifying an arithmetic expression, we should remove radicals from the denominator.

a) To remove $\sqrt{a}$, multiply numerator and denominator with another $\sqrt{a}$

b) To remove $\sqrt{a} \pm \sqrt{b}$, multiply numerator and denominator with the conjugate $\sqrt{a} \mp \sqrt{b}$.

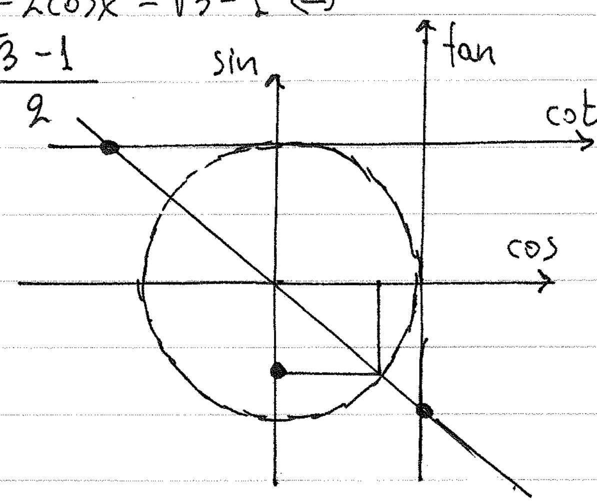
c) If $4\cos x + 1 = 2\cos x + \sqrt{3}$ and $3\pi/2 \leq x \leq 2\pi$, evaluate the expression $A = 2\sin x + 6\tan x$.

Solution

We note that

$4\cos x + 1 = 2\cos x + \sqrt{3} \Leftrightarrow 4\cos x - 2\cos x = \sqrt{3} - 1 \Leftrightarrow$

$\Leftrightarrow 2\cos x = \sqrt{3} - 1 \Leftrightarrow \cos x = \dfrac{\sqrt{3}-1}{2}$



and

$$3\pi/2 \leq x \leq 2\pi \Rightarrow \begin{cases} \sin x \leq 0 \\ \cot x \leq 0 \\ \tan x \leq 0 \end{cases}$$

so it follows that

$\sin^2 x = 1 - \cos^2 x = 1 - \left(\dfrac{\sqrt{3}-1}{2}\right)^2 =$

$$= 1 - \frac{(\sqrt{3}-1)^2}{4} = 1 - \frac{(\sqrt{3})^2 - 2\sqrt{3} + 1^2}{4} =$$

$$= 1 - \frac{3 - 2\sqrt{3} + 1}{4} = 1 - \frac{4 - 2\sqrt{3}}{4} = 1 - 1 + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \Rightarrow$$

$$\Rightarrow \sin x = \left(\frac{\sqrt{3}}{2}\right)^{1/2} \vee \sin x = -\left(\frac{\sqrt{3}}{2}\right)^{1/2} \overset{*}{\Rightarrow}$$

$$\Rightarrow \sin x = -\left(\frac{\sqrt{3}}{2}\right)^{1/2} = -\frac{\sqrt[4]{3}}{\sqrt{2}} = -\frac{\sqrt{2}\sqrt[4]{3}}{\sqrt{2}\sqrt{2}} = \frac{-\sqrt{2}\sqrt[4]{3}}{2}$$

and

$$\tan x = \frac{\sin x}{\cos x} = \frac{\frac{-\sqrt{2}\sqrt[4]{3}}{2}}{\frac{\sqrt{3}-1}{2}} = \frac{-\sqrt{2}\sqrt[4]{3}}{\sqrt{3}-1} = \frac{-\sqrt{2}\sqrt[4]{3}(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} =$$

$$= \frac{-\sqrt{2}\sqrt[4]{3}(\sqrt{3}+1)}{(\sqrt{3})^2-1^2} = \frac{-\sqrt{2}\sqrt[4]{3}(\sqrt{3}+1)}{3-1} = \frac{-\sqrt{2}\sqrt[4]{3}(\sqrt{3}+1)}{2}$$

and therefore

$$A = 2\sin x + 6\tan x = 2\left[\frac{-\sqrt{2}\sqrt[4]{3}}{2}\right] + 6\left[\frac{-\sqrt{2}\sqrt[4]{3}(\sqrt{3}+1)}{2}\right]$$

$$= -\sqrt{2}\sqrt[4]{3} - 3\sqrt{2}\sqrt[4]{3}(\sqrt{3}+1) = -\sqrt{2}\sqrt[4]{3}\left[1+3(\sqrt{3}+1)\right] =$$

$$= -\sqrt{2}\sqrt[4]{3}\left[1+3\sqrt{3}+3\right] = -\sqrt{2}\sqrt[4]{3}\left[4+3\sqrt{3}\right].$$

$\hookrightarrow$ Recall the following identities from algebra:

$$\boxed{\begin{aligned} a^2-b^2 &= (a-b)(a+b) \\ (a+b)^2 &= a^2+2ab+b^2 \\ (a-b)^2 &= a^2-2ab+b^2 \end{aligned}}$$

EXERCISES

① Simplify the following expressions.

a) $5\cot^2(\pi/4) - \sin(\pi/6) - \dfrac{1}{\cos^2(\pi/3)}$

b) $\sin(\pi/6)\cos(\pi/3)\tan(\pi/4)$

c) $\tan(\pi/6)\sin^2(\pi/3) + \cos(\pi/4)$

d) $\tan(\pi/3)\left[\cos(\pi/6)\sin(\pi/3) + \cot(\pi/3)\right]$

② a) If $\sin x = 1/3$ and $\pi/2 < x < \pi$, then evaluate $A = \dfrac{5\tan x + 4\cos^2 x}{3\sin x}$

b) If $\cos x = 12/13$ and $3\pi/2 < x < 2\pi$, then evaluate $A = 5\cos x - 8\sin x + \tan^2 x$

c) If $\tan x = 3$ and $\pi < x < 3\pi/2$, then evaluate $A = 6\sin^2 x + \cos x - 3\tan x$

d) If $\cot x = 2$ and $0 < x < \pi/2$, then evaluate $A = 2\tan x - 4\sin x + \cos x$.

e) If $5\sin x + 4 = 2\sin x + 3$ and $3\pi/2 < x < 2\pi$ then evaluate $A = 2\sin x + 3\cos x - 5\tan x + \cot x$

③ Show that $\tan^2\left(\dfrac{\pi}{6}\right) + \tan^2\left(\dfrac{\pi}{4}\right) + \tan^2\left(\dfrac{\pi}{3}\right) = \dfrac{13}{3}$

Reduction to 1st quadrant

• The challenge is to rewrite a trigonometric function of the arc $k\pi/2 \pm x$ in terms of a trigonometric function of x. To do that we use the following properties:

1) Odd / Even property

$$\boxed{\begin{aligned} \sin(-x) &= -\sin x \\ \cos(-x) &= +\cos x \\ \tan(-x) &= -\tan x \\ \cot(-x) &= -\cot x \end{aligned}}$$

2) Periodicity

$$\boxed{\begin{aligned} \sin(x+2\pi) &= \sin x \\ \cos(x+2\pi) &= \cos x \\ \tan(x+\pi) &= \tan x \\ \cot(x+\pi) &= \cot x \end{aligned}}$$

3) Cofunction identities

$$\boxed{\begin{aligned} \sin(\pi/2-x) &= \cos x \\ \cos(\pi/2-x) &= \sin x \\ \tan(\pi/2-x) &= \cot x \\ \cot(\pi/2-x) &= \tan x \end{aligned}}$$

4) Angle $\pi+x$

$$\boxed{\begin{aligned} \sin(\pi+x) &= -\sin x \\ \cos(\pi+x) &= -\cos x \\ \tan(\pi+x) &= \tan x \\ \cot(\pi+x) &= \cot x \end{aligned}}$$

In general:

$$\boxed{\begin{aligned} \sin(k\pi+x) &= (-1)^k \sin x \\ \cos(k\pi+x) &= (-1)^k \cos x \end{aligned}}$$

EXAMPLES

a) Simplify the expression

$$A = \sin\left(\frac{19\pi}{6}\right)\cos\left(\frac{5\pi}{3}\right)\tan\left(\frac{14\pi}{3}\right)$$

Solution

Since

$$\sin\left(\frac{19\pi}{6}\right) = \sin\left(3\pi + \frac{\pi}{6}\right) = (-1)^3 \sin(\pi/6) = -\sin(\pi/6) = -1/2$$

$$\cos\left(\frac{5\pi}{3}\right) = \cos\left(\pi + \frac{2\pi}{3}\right) = -\cos\left(\frac{2\pi}{3}\right) = -\cos\left(\pi - \frac{\pi}{3}\right) =$$

$$= +\cos(-\pi/3) = \cos(\pi/3) = 1/2$$

$$\tan\left(\frac{14\pi}{3}\right) = \tan\left(\frac{(15-1)\pi}{3}\right) = \tan(5\pi - \pi/3) = \tan(-\pi/3)$$

$$= -\tan(\pi/3) = -\sqrt{3}.$$

it follows that

$$A = \sin\left(\frac{19\pi}{6}\right)\cos\left(\frac{5\pi}{3}\right)\tan\left(\frac{14\pi}{3}\right) =$$

$$= \left(\frac{-1}{2}\right)\left(\frac{1}{2}\right)(-\sqrt{3}) = \frac{\sqrt{3}}{4}$$

b) Simplify the expression

$$A = \frac{\cos(x-5\pi/2)\sin(x+3\pi/2)}{\sin(x-3\pi)\cos(5\pi-x)}$$

<u>Solution</u>

Since

$$\cos(x-5\pi/2) = (-1)^3\cos(3\pi+x-5\pi/2) = -\cos(x+\pi/2) =$$

$$= -\cos(\pi/2-(-x)) = -\sin(-x) = \sin x$$

$$\sin(x+3\pi/2) = \sin(x+2\pi-\pi/2) = \sin(x-\pi/2) = -\sin(\pi/2-x)$$

$$= -\cos x$$

$$\sin(x-3\pi) = (-1)^3\sin(3\pi+x-3\pi) = -\sin x$$

$$\cos(5\pi-x) = (-1)^5\cos(-x) = -\cos(-x) = -\cos x$$

it follows that

$$A = \frac{\cos(x-\pi/2)\sin(x+3\pi/2)}{\sin(x-3\pi)\cos(5\pi-x)} = \frac{\sin x\,[-\cos x]}{[-\sin x][-\cos x]} = -1$$

EXERCISES

④ Simplify the following expressions

a) $A = \sin\left(\frac{7\pi}{3}\right)\cos\left(\frac{13\pi}{6}\right)\cos\left(-\frac{5\pi}{3}\right)\sin\left(\frac{11\pi}{6}\right)$

b) $A = \sin\left(-\frac{2\pi}{3}\right)\tan\left(\frac{5\pi}{3}\right)\cot\left(-\frac{4\pi}{3}\right)\cos\left(\frac{5\pi}{6}\right)$

c) $A = \sin\left(\frac{3\pi}{2}+x\right)\sin(\pi+x) + \sin\left(\frac{3\pi}{2}-x\right)\sin(\pi-x)$

d) $A = \sin\left(\frac{3\pi}{2}+x\right) + \cos\left(\frac{3\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}+x\right)$

e) $A = \dfrac{\sin(a-3\pi/2)\tan(b-\pi)}{\cot(3\pi/2-b)\sin(a+\pi/2)}$

f) $A = \dfrac{\sin(a+\pi/2)\tan(9\pi+a)\cos(a-\pi/2)}{\cos(11\pi-a)\sin(3\pi/2+a)\tan(2\pi+a)}$

g) $A = \dfrac{\sin(5\pi+a)\tan(3\pi+a)\cos(4\pi+a)}{\cos(7\pi-a)\tan(8\pi+a)\sin a}$

▼ Simple trigonometric identities

• Recall that

$$\boxed{\sin^2 x + \cos^2 x = 1} \longrightarrow \begin{cases} \sin^2 x = 1 - \cos^2 x \\ \cos^2 x = 1 - \sin^2 x \end{cases}$$

$$\boxed{\begin{array}{l} \tan x = \dfrac{\sin x}{\cos x} \\ \hline \cot x = \dfrac{\cos x}{\sin x} \end{array}} \longrightarrow \tan x \cot x = 1 \longrightarrow \begin{cases} \tan x = \dfrac{1}{\cot x} \\ \cot x = \dfrac{1}{\tan x} \end{cases}$$

$$1 + \tan^2 x = \frac{1}{\cos^2 x} \qquad 1 + \cot^2 x = \frac{1}{\sin^2 x}$$

• Method: To show $A = B$

a) Direct Method

$A = \dots = \dots = B$

b) Indirect Method

$A = \dots = \dots = C$

$B = \dots = \dots = C$

It follows that $A = B$.

c) Method of Desperation

$A - B = \dots = \dots = 0 \Rightarrow$

$\Rightarrow A = B$

• Recall identities from intermediate algebra

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$a^2 - b^2 = (a-b)(a+b)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$$
$$(a-b)^3 = a^3 - 3a^2 b + 3ab^2 - b^3$$

EXAMPLES

a) Show that:

$$\sin^6 x - \cos^6 x = (1-2\cos^2 x)(1-\sin^2 x\cos^2 x)$$

Solution

We have:

$$\sin^6 x - \cos^6 x = (\sin^3 x - \cos^3 x)(\sin^3 x + \cos^3 x) =$$
$$= (\sin x - \cos x)(\sin^2 x + \sin x\cos x + \cos^2 x)(\sin x + \cos x)$$
$$\times (\sin^2 x - \sin x \cos x + \cos^2 x) =$$
$$= [(\sin x - \cos x)(\sin x + \cos x)][(1+\sin x\cos x)(1-\sin x\cos x)] =$$
$$= (\sin^2 x - \cos^2 x)(1-\sin^2 x\cos^2 x) =$$
$$= [(1-\cos^2 x) - \cos^2 x](1-\sin^2 x\cos^2 x) =$$
$$= (1-2\cos^2 x)(1-\sin^2 x\cos^2 x).$$

→ Recall the following factorizations from algebra:

$$\boxed{\begin{aligned} a^2 - b^2 &= (a-b)(a+b) \\ a^3 - b^3 &= (a-b)(a^2+ab+b^2) \\ a^3 + b^3 &= (a+b)(a^2-ab+b^2) \end{aligned}}$$

b) Show that $\dfrac{1-\sin x}{1+\sin x} = \left(\dfrac{1}{\cos x} - \tan x\right)^2$

Solution

We have:

$$\frac{1-\sin x}{1+\sin x} = \frac{(1-\sin x)^2}{(1+\sin x)(1-\sin x)} = \frac{1-2\sin x+\sin^2 x}{1-\sin^2 x} =$$

$$= \frac{1-2\sin x+\sin^2 x}{\cos^2 x} =$$

$$= \frac{1}{\cos^2 x} - 2\,\frac{\sin x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} =$$

$$= \left(\frac{1}{\cos x}\right)^2 - 2\frac{\sin x}{\cos x}\,\frac{1}{\cos x} + \left(\frac{\sin x}{\cos x}\right)^2 =$$

$$= \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)^2 = \left(\frac{1}{\cos x} - \tan x\right)^2$$

c) Show that: $\dfrac{\tan x+\tan y}{\cot x+\cot y} = \tan x\cdot\tan y$

Solution

We have:

$$\frac{\tan x+\tan y}{\cot x+\cot y} = \frac{\tan x+\tan y}{\dfrac{1}{\tan x}+\dfrac{1}{\tan y}} = \frac{\tan x+\tan y}{\left(\dfrac{\tan y+\tan x}{\tan x\tan y}\right)} =$$

$$= \frac{\tan x\tan y\,(\tan x+\tan y)}{\tan x+\tan y} = \tan x\tan y$$

EXERCISES

⑤ Show that

a) $\tan^2 a - \sin^2 a = \tan^2 a \cdot \sin^2 a$

b) $\cot^2 x - \cos^2 x = \cot^2 x \cdot \cos^2 x$

c) $(\sin\vartheta + \cos\vartheta)^4 - (\sin\vartheta - \cos\vartheta)^4 = 8\sin\vartheta\cos\vartheta$

d) $\tan\vartheta(1 - \cot^2\vartheta) + \cot\vartheta(1 - \tan^2\vartheta) = 0$

e) $\sin^2 x \tan x - \cos^2 x \cot x = \tan x - \cot x$

f) $(\sin x + \cos x + 1)(\sin x + \cos x - 1) = 2\sin x \cos x$

g) $\sin^2 a(1 + \cot^2 a) + \cos^2 a(1 + \tan^2 a) = 2$

h) $\sin^2 x \tan x + \cos^2 x \cot x + 2\sin x \cos x = \tan x + \cot x$

i) $4(\sin^6 x + \cos^6 x) - 3(\cos^4 x - \sin^4 x)^2 = 1$

⑥ Show that

a) $\dfrac{1 + \tan^2 x}{1 + \cot^2 x} = \tan^2 x$ b) $\dfrac{1 - \tan^2 x}{1 + \tan^2 x} = 1 - 2\sin^2 x$

c) $\dfrac{1 - \sin\vartheta}{1 + \sin\vartheta} - \dfrac{1 + \sin\vartheta}{1 - \sin\vartheta} = -4\dfrac{\tan\vartheta}{\cos\vartheta}$

d) $\dfrac{\sin x}{1 - \cot x} + \dfrac{\cos x}{1 - \tan x} = \sin x + \cos x$

e) $\dfrac{\cos^3 a - \cos a + \sin a}{\cos a} = \tan a - \sin^2 a$

f) $\dfrac{\sin a + \sin b}{\cos a + \cos b} + \dfrac{\cos a - \cos b}{\sin a - \sin b} = 0$

(7) Show that

a) $\dfrac{\sin^2 b - \sin^2 a}{\sin^2 a \sin^2 b} = \cot^2 a - \cot^2 b$

b) $\dfrac{(\sin a + \cos b)^2 + (\cos a + \sin b)(\cos a - \sin b)}{\sin a + \cos b} = 2\cos b$

c) $\dfrac{\cos^2 a - \sin^2 b}{\sin^2 a \sin^2 b} = \dfrac{1}{\tan^2 a}\left(\dfrac{1}{\sin^2 b} - \dfrac{1}{\cos^2 a}\right)$

d) $\dfrac{\tan x - \sin x}{\sin^3 x} = \dfrac{1}{\cos^2 x + \cos x}$

e) $\dfrac{\tan a}{1 + \tan^2 a} + \dfrac{\cos^3 a}{\sin a} = \cot a$

f) $2\cos^2 x - \sin^2 x = \dfrac{2 - \tan^2 x}{1 + \tan^2 x}$

g) $\dfrac{1}{(\sin x \cos x)^2} - 4 = (\tan x - \cot x)^2$

h) $(\tan a - \sin a)^2 + (1 - \cos a)^2 = \left(\dfrac{1}{\cos a} - 1\right)^2$.

(8) a) If $\dfrac{1 + \cos^2 b}{1 + 2\sin^2 b} = \sin^2 \alpha$, then show that $\sin^2 b = \dfrac{1 + \cos^2 a}{1 + 2\sin^2 a}$

b) If $\left.\begin{array}{l} \alpha = x\cos\vartheta + y\sin\vartheta \\ b = x\sin\vartheta - y\cos\vartheta \end{array}\right\} \Rightarrow a^2 + b^2 = x^2 + y^2$.

c) Show that $A + B = \pi/2 \Rightarrow \cos^2 A + \cos^2 B = 1$.

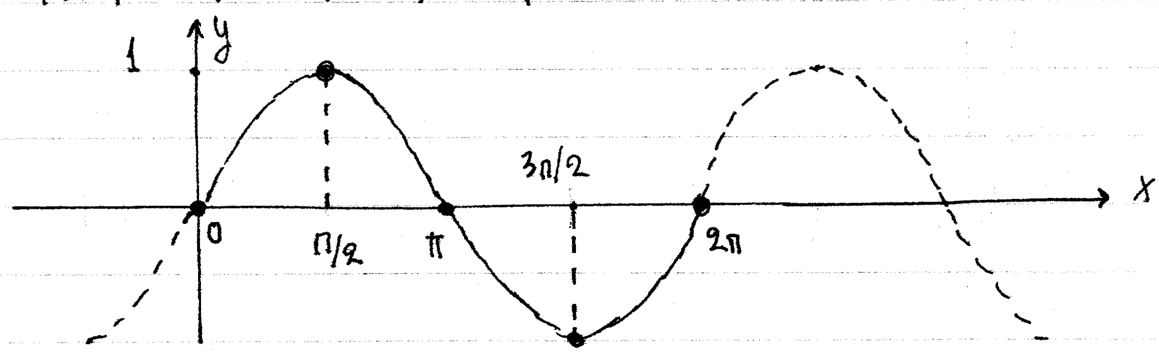
Graphs of sin and cos

1) $\boxed{f(x) = \sin(x)}$

Domain : $A = \mathbb{R}$

Range: $f(A) = [-1, 1]$

x	0	$\pi/2$	π	$3\pi/2$	2π
$f(x)$	0	1	0	-1	0

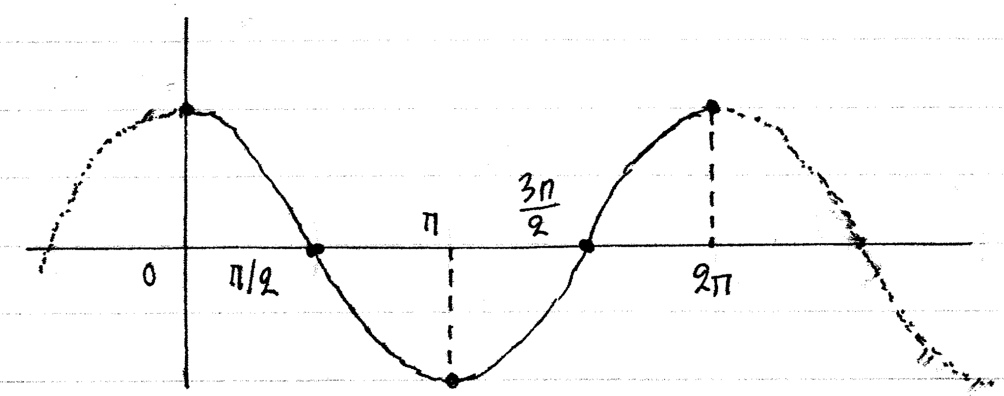


2) $\boxed{f(x) = \cos x}$

Domain : $A = \mathbb{R}$

Range: $f(A) = [-1, 1]$

x	0	$\pi/2$	π	$3\pi/2$	2π
$f(x)$	1	0	-1	0	1



Methodology: The problem is to graph the functions

$$f(t) = a\sin(\omega t + b) + c$$
$$f(t) = a\cos(\omega t + b) + c$$

▸ **Terminology**

ω = angular velocity (if t is time)
(we use kx instead of ωt for spacial dependence; k is the wavenumber)
b = phase shift
$\varphi = \omega t + b$ = phase
a = amplitude
c = vertical shift.

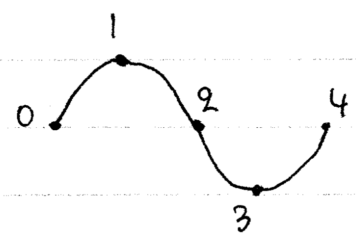
▸ To graph these functions.

•1 Solve the equation $\omega t + b = k\pi/2$ with respect to t.

•2 For $k = 0, 1, 2, 3, 4$ find the corresponding t_0, t_1, t_2, t_3, t_4 and note that:

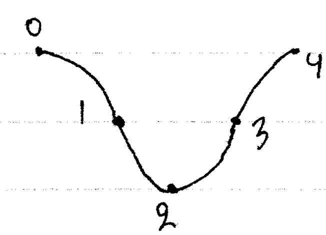
sin:

k	0	1	2	3	4
$\varphi = \omega t + b$	0	$\pi/2$	π	$3\pi/2$	2π
$f(t)$	c	$c+a$	c	$c-a$	c



cos:

k	0	1	2	3	4
$\varphi = \omega t + b$	0	$\pi/2$	π	$3\pi/2$	2π
$f(t)$	$c+a$	c	$c-a$	c	$c+a$



•3 Given the points $(t_0, f(t_0))$, $(t_1, f(t_1))$, $(t_2, f(t_2))$, $(t_3, f(t_3))$, $(t_4, f(t_4))$ we construct the graph.

EXAMPLES

a) Graph the function $f(x) = 2\sin\left(\frac{2x+\pi}{5}\right) - 1$

Solution

Solve:

$$\frac{2x+\pi}{5} = \frac{k\pi}{2} \Leftrightarrow 2(2x+\pi) = 5k\pi \Leftrightarrow 4x + 2\pi = 5k\pi \Leftrightarrow$$

$$\Leftrightarrow 4x = (5k-2)\pi \Leftrightarrow x = \frac{(5k-2)\pi}{4}$$

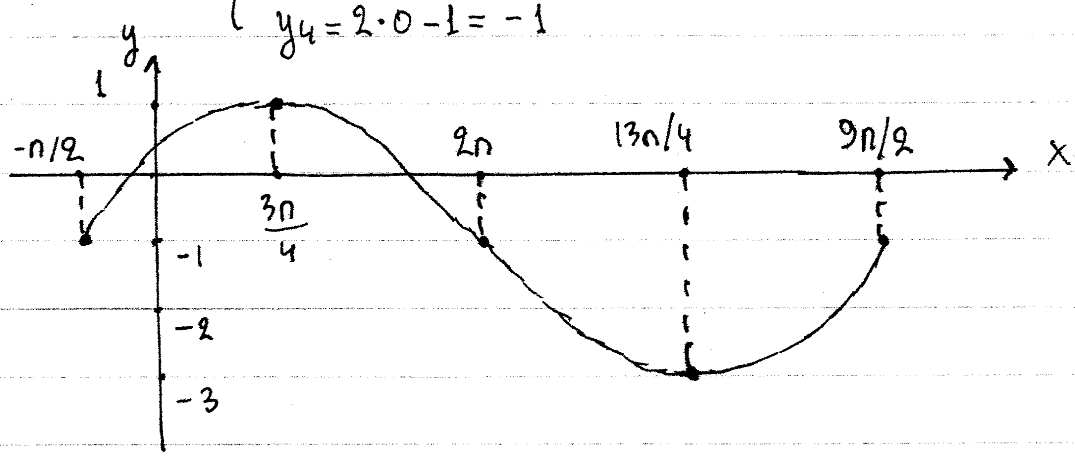
For $k=0$: $\begin{cases} x_0 = (0-2)\pi/4 = -\pi/2 \\ y_0 = 2\cdot 0 - 1 = -1 \end{cases}$

For $k=1$: $\begin{cases} x_1 = (5\cdot 1 - 2)\pi/4 = 3\pi/4 \\ y_1 = 2\cdot 1 - 1 = 1 \end{cases}$

For $k=2$: $\begin{cases} x_2 = (5\cdot 2 - 2)\pi/4 = 8\pi/4 = 2\pi \\ y_2 = 2\cdot 0 - 1 = -1 \end{cases}$

For $k=3$: $\begin{cases} x_3 = (5\cdot 3 - 2)\pi/4 = 13\pi/4 \\ y_3 = 2\cdot(-1) - 1 = -3 \end{cases}$

For $k=4$: $\begin{cases} x_4 = (5\cdot 4 - 2)\pi/4 = 18\pi/4 = 9\pi/2 \\ y_4 = 2\cdot 0 - 1 = -1 \end{cases}$



b) Graph the function $f(x) = \frac{1}{2} - \cos\left(2x + \frac{\pi + x}{3}\right)$

<u>Solution</u>

Solve: $2x + \frac{\pi + x}{3} = \frac{k\pi}{2} \Leftrightarrow 6\left[2x + \frac{\pi + x}{3}\right] = 6 \cdot \frac{k\pi}{2} \Leftrightarrow$

$\Leftrightarrow 12x + 2(\pi + x) = 3k\pi \Leftrightarrow 12x + 2\pi + 2x = 3k\pi \Leftrightarrow 14x = (3k-2)\pi$

$\Leftrightarrow x = \frac{(3k-2)\pi}{14}$

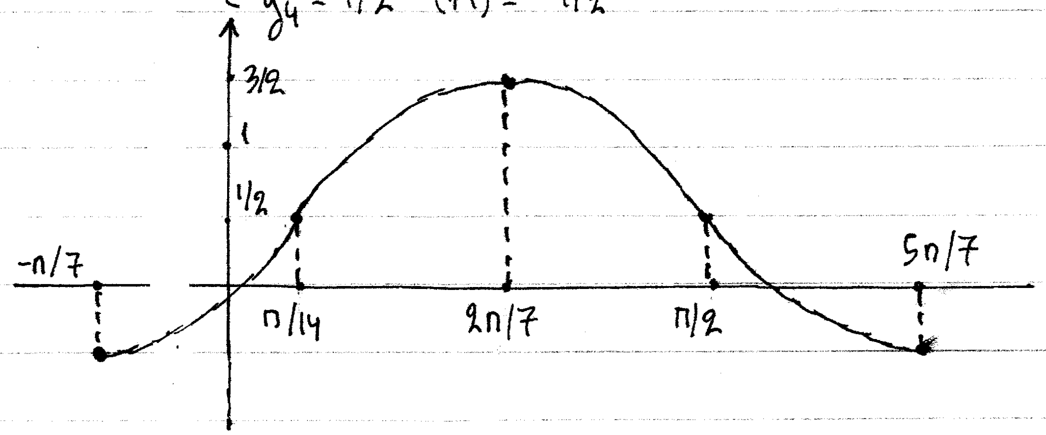
For $k=0$: $\begin{cases} x_0 = (3 \cdot 0 - 2)\pi/14 = -2\pi/14 = -\pi/7 \\ y_0 = 1/2 - (+1) = -1/2 \end{cases}$

For $k=1$: $\begin{cases} x_1 = (3 \cdot 1 - 2)\pi/14 = \pi/14 \\ y_1 = 1/2 - 0 = 1/2 \end{cases}$

For $k=2$: $\begin{cases} x_2 = (3 \cdot 2 - 2)\pi/14 = 4\pi/14 = 2\pi/7 \\ y_2 = 1/2 - (-1) = 3/2 \end{cases}$

For $k=3$: $\begin{cases} x_3 = (3 \cdot 3 - 2)\pi/14 = 7\pi/14 = \pi/2 \\ y_3 = 1/2 - 0 = 1/2 \end{cases}$

For $k=4$: $\begin{cases} x_4 = (3 \cdot 4 - 2)\pi/14 = 10\pi/14 = 5\pi/7 \\ y_4 = 1/2 - (+1) = -1/2 \end{cases}$



EXERCISES

⑨ Graph the following functions

a) $f(x) = \sin\left(\frac{2x+\pi}{3}\right)$

b) $f(x) = \frac{1}{2} + \sin\left(x - \frac{\pi+3x}{4}\right)$

c) $f(x) = 1 - \sin\left(\frac{\pi+x}{3} - \frac{x+4\pi}{6}\right)$

d) $f(x) = 2 - 3\cos\left(2x + \frac{x+\pi}{2} + \frac{2x-\pi}{4}\right)$

e) $f(x) = 1 + \cos\left(x - \frac{x+8\pi}{2} + \frac{\pi+3x}{6}\right)$

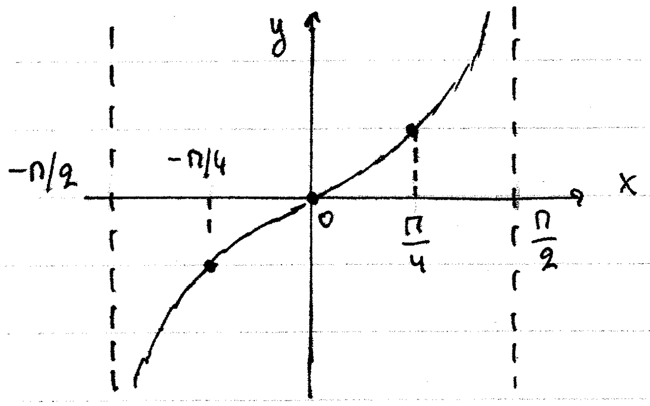
f) $f(x) = -1 + 2\cos\left(x(\pi x+1) - \pi(x+1)(x-1)\right)$

Graphs of tan and cot

1) $\boxed{f(x) = \tan(x)}$ Domain: $A = \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}$

Range: $f(A) = \mathbb{R}$ Period: $T = \pi$

x	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$
$f(x)$	$+\infty // -\infty$	-1	0	$+1$	$+\infty // -\infty$

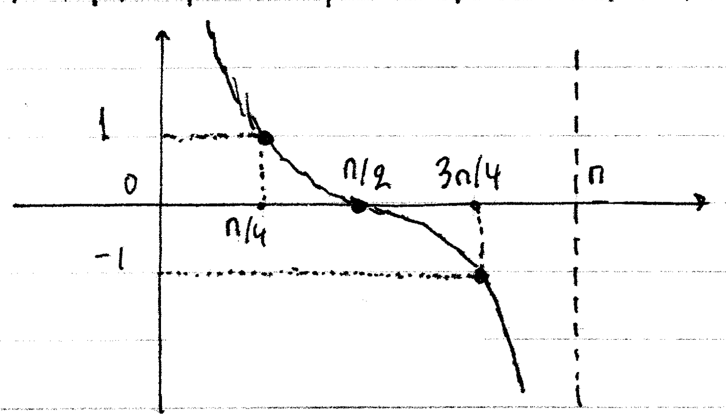


There are vertical asymptotes at $x = k\pi + \pi/2$

2) $\boxed{f(x) = \cot(x)}$ Domain: $A = \mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\}$

Range: $f(A) = \mathbb{R}$. Period $T = \pi$

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
$f(x)$	$-\infty // +\infty$	1	0	-1	$-\infty // +\infty$



There are vertical asymptotes at $x = k\pi$

Methodology: The problem is to graph:

$$\boxed{f(t) = a\tan(\omega t + b) + c}$$

Solve $\boxed{\omega t + b = k\pi/4 - \pi/2}$

k	$\omega t + b$	$f(t)$
0	$-\pi/2^{+}$	$(-\infty)a$
1	$-\pi/4$	$-a+c$
2	0	c
3	$\pi/4$	$a+c$
4	$\pi/2^{-}$	$(+\infty)a$

$$\boxed{f(t) = a\cot(\omega t + b) + c}$$

Solve $\boxed{\omega t + b = k\pi/4}$

k	$\omega t + b$	$f(t)$
0	0^{+}	$(+\infty)a$
1	$\pi/4$	$a+c$
2	$\pi/2$	c
3	$3\pi/4$	$-a+c$
4	π^{-}	$(-\infty)a$

Then, graph the function!

EXAMPLES

a) Graph $f(x) = 2\tan(2x + \pi/2) - 1$

Solution

Solve: $2x + \pi/2 = k\pi/4 - \pi/2 \Leftrightarrow 4(2x+\pi/2) = 4(k\pi/4 - \pi/2) \Leftrightarrow$

$\Leftrightarrow 8x + 2\pi = k\pi - 2\pi \Leftrightarrow 8x = k\pi - 2\pi - 2\pi \Leftrightarrow 8x = (k-4)\pi \Leftrightarrow$

$$\Leftrightarrow x = \frac{(k-4)\pi}{8}$$

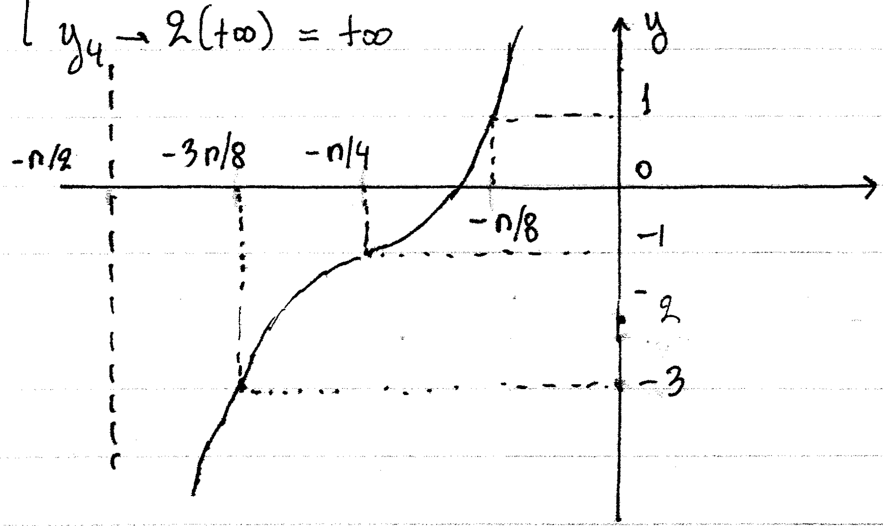
For $k=0$: $\begin{cases} x_0 = (0-4)\pi/8 = -\pi/2 \\ y_0 \to 2(-\infty) = -\infty \end{cases}$

For $k=1$: $\begin{cases} x_1 = (1-4)\pi/8 = -3\pi/8 \\ y_1 = 2(-1) - 1 = -3 \end{cases}$

For $k=2$: $\begin{cases} x_2 = (2-4)\pi/8 = -2\pi/8 = -\pi/4 \\ y_2 = 2 \cdot 0 - 1 = -1 \end{cases}$

For $k=3$: $\begin{cases} x_3 = (3-4)\pi/8 = -\pi/8 \\ y_3 = 2 \cdot (+1) - 1 = 2 - 1 = 1 \end{cases}$

For $k=4$: $\begin{cases} x_4 = (4-4)\pi/8 = 0 \\ y_4 \to 2(+\infty) = +\infty \end{cases}$



b) Graph the function $f(x) = 2\cot(3x - \pi/3) + 1$

Solution

Solve : $3x - \pi/3 = k\pi/4 \Leftrightarrow 12(3x - \pi/3) = 12(k\pi/4) \Leftrightarrow$

$\Leftrightarrow \quad 36x - 4\pi = 3k\pi \Leftrightarrow 36x = (3k+4)\pi \Leftrightarrow$

$\Leftrightarrow \quad x = (3k+4)\pi/36$

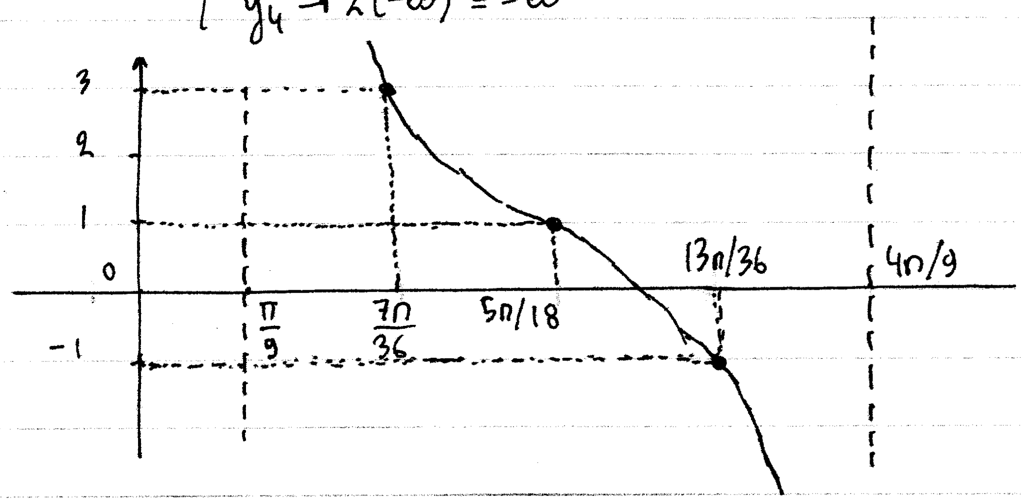
For $k=0$: $\begin{cases} x_0 = (3\cdot 0 + 4)\pi/36 = 4\pi/36 = \pi/9 \\ y_0 \to 2(+\infty) = +\infty \end{cases}$

For $k=1$: $\begin{cases} x_1 = (3\cdot 1 + 4)\pi/36 = 7\pi/36 \\ y_1 = 2\cdot 1 + 1 = 3 \end{cases}$

For $k=2$: $\begin{cases} x_2 = (3\cdot 2 + 4)\pi/36 = 10\pi/36 = 5\pi/18 \\ y_2 = 2\cdot 0 + 1 = 1 \end{cases}$

For $k=3$: $\begin{cases} x_3 = (3\cdot 3 + 4)\pi/36 = 13\pi/36 \\ y_3 = 2(-1) + 1 = -1 \end{cases}$

For $k=4$: $\begin{cases} x_4 = (3\cdot 4 + 4)\pi/36 = 16\pi/36 = 4\pi/9 \\ y_4 \to 2(-\infty) = -\infty \end{cases}$



EXERCISES

(10) Graph the following functions:

a) $f(x) = \tan\left(\frac{\pi - x}{4}\right)$

b) $f(x) = 2 - \tan\left(2x + \frac{\pi}{3}\right)$

c) $f(x) = 1 + \tan\left(\frac{\pi + x}{2} - \frac{\pi - 3x}{3}\right)$

d) $f(x) = \cot\left(x + 3(\pi - 2x)\right)$

e) $f(x) = 1 - \cot\left(2(x+\pi) - 3(2\pi - 3x)\right)$

f) $f(x) = 1 + \cot\left(x - \frac{\pi + 3x}{6}\right)$

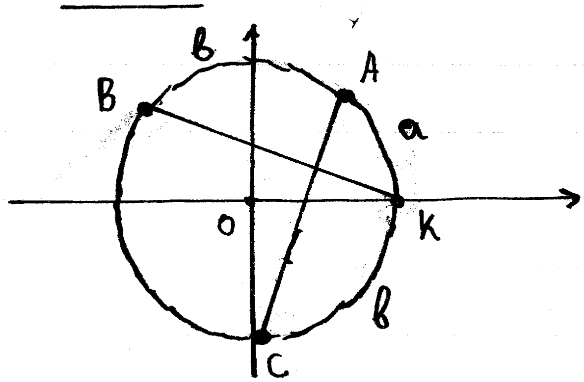
PRE3: Trigonometric identities

TRIGONOMETRIC IDENTITIES

▼ Addition a±b identities

① $\boxed{\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b}$

Proof



Let $O(0,0)$ and $K(1,0)$. Choose A such that $A\hat{O}K = a$, and choose B such that $B\hat{O}A = b$. Also choose C such that $K\hat{O}C = b$ on the other side of the circle. It follows that:

$$x_A = \cos a, \quad y_A = \sin a$$
$$x_B = \cos(a+b), \quad y_B = \sin(a+b)$$
$$x_C = \cos(-b), \quad y_C = \sin(-b)$$
$$x_K = 1, \quad y_K = 0$$

Since

$$\left.\begin{array}{l} B\hat{O}K = B\hat{O}A + A\hat{O}K = b + a = a+b \\ A\hat{O}C = A\hat{O}K + K\hat{O}C = a+b \end{array}\right\} \Rightarrow B\hat{O}K = A\hat{O}C \Rightarrow$$

$$\Rightarrow BK = AC \Rightarrow \underline{BK^2 = AC^2} \quad (1)$$

We note that

$$BK^2 = (x_B - x_K)^2 + (y_B - y_K)^2 =$$
$$= (\cos(a+b) - 1)^2 + (\sin(a+b) - 0)^2 =$$

$$= \cos^2(a+b) - 2\cos(a+b) + 1 + \sin^2(a+b) =$$
$$= 1 - 2\cos(a+b) + [\cos^2(a+b) + \sin^2(a+b)] =$$
$$= 1 - 2\cos(a+b) + 1 = 2 - 2\cos(a+b)$$

and

$$AC^2 = (x_A - x_C)^2 + (y_A - y_C)^2 =$$
$$= (\cos a - \cos b)^2 + (\sin a + \sin b)^2 =$$
$$= \cos^2 a - 2\cos a \cos b + \cos^2 b + \sin^2 a + 2\sin a \sin b + \sin^2 b$$
$$= (\sin^2 a + \cos^2 a) + (\sin^2 b + \cos^2 b) - 2(\cos a \cos b - \sin a \sin b)$$
$$= 1 + 1 - 2(\cos a \cos b - \sin a \sin b) =$$
$$= 2 - 2(\cos a \cos b - \sin a . \sin b)$$

and from (1) it follows that:

$$BK^2 = AC^2 \Rightarrow 2 - 2\cos(a+b) = 2 - 2(\cos a \cos b - \sin a \sin b) \Rightarrow$$
$$\Rightarrow \cos(a+b) = \cos a \cos b - \sin a \sin b.$$

It follows that

$$\cos(a-b) = \cos a \cos b + \sin a \sin b. \qquad \square$$

2 $\boxed{\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a}$

Proof

$$\sin(a+b) = \cos\left(\frac{\pi}{2} - (a+b)\right) = \cos\left(\left(\frac{\pi}{2} - a\right) + (-b)\right) =$$
$$= \cos\left(\frac{\pi}{2} - a\right)\cos(-b) - \sin\left(\frac{\pi}{2} - a\right)\sin(-b)$$
$$= \sin a \cos b - \cos a[-\sin b] =$$

$$= \sin a \cos b + \sin b \cos a$$

It follows that

$$\sin(a-b) = \sin a \cos b - \sin b \cos a. \qquad \square$$

③ $$\boxed{\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \cdot \tan b}}$$

Proof

$$\tan(a+b) = \frac{\sin(a+b)}{\cos(a+b)} = \frac{\sin a \cos b + \sin b \cos a}{\cos a \cos b - \sin a \cdot \sin b} =$$

$$= \frac{\cos a \cos b \left[\frac{\sin a}{\cos a} + \frac{\sin b}{\cos b} \right]}{\cos a \cos b \left[1 - \frac{\sin a}{\cos a} \frac{\sin b}{\cos b} \right]} =$$

$$= \frac{\tan a + \tan b}{1 - \tan a \tan b}.$$

It follows that

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b} \qquad \square$$

④ $$\boxed{\cot(a \pm b) = \frac{\cot a \cot b \mp 1}{\cot b \pm \cot a}} \quad (!!)$$

Proof

$$\cot(a+b) = \frac{1}{\tan(a+b)} = \frac{1-\tan a\tan b}{\tan a+\tan b} =$$

$$= \frac{1-\frac{1}{\cot a}\frac{1}{\cot b}}{\frac{1}{\cot a}+\frac{1}{\cot b}} = \frac{\frac{\cot a\cot b-1}{\cot a\cot b}}{\frac{\cot b+\cot a}{\cot a\cot b}} =$$

$$= \frac{\cot a\cot b-1}{\cot b+\cot a}$$

It follows that

$$\cot(a-b) = \frac{\cot a\cot(-b)-1}{\cot(-b)+\cot a} = \frac{-\cot a\cot b-1}{-\cot b+\cot a} =$$

$$= \frac{\cot a\cot b+1}{\cot b-\cot a} \qquad \square$$

EXAMPLES

a) Evaluate $\sin(7\pi/12)$, $\cos(7\pi/12)$, $\tan(7\pi/12)$

Solution

We have:

$$\sin(7\pi/12) = \sin(3\pi/12 + 4\pi/12) = \sin(\pi/4 + \pi/3) =$$
$$= \sin(\pi/4)\cos(\pi/3) + \sin(\pi/3)\cos(\pi/4) =$$
$$= \frac{\sqrt{2}}{2}\,\frac{1}{2} + \frac{\sqrt{3}}{2}\,\frac{\sqrt{2}}{2} = \frac{\sqrt{2}\,(1+\sqrt{3})}{4}$$

$$\cos(7\pi/12) = \cos(3\pi/12 + 4\pi/12) = \cos(\pi/4 + \pi/3) =$$
$$= \cos(\pi/4)\cos(\pi/3) - \sin(\pi/4)\sin(\pi/3) =$$
$$= \frac{\sqrt{2}}{2}\,\frac{1}{2} - \frac{\sqrt{2}}{2}\,\frac{\sqrt{3}}{2} = \frac{\sqrt{2}\,(1-\sqrt{3})}{4}$$

$$\tan(7\pi/12) = \frac{\sin(7\pi/12)}{\cos(7\pi/12)} = \frac{\frac{\sqrt{2}(1+\sqrt{3})}{4}}{\frac{\sqrt{2}(1-\sqrt{3})}{4}} = \frac{1+\sqrt{3}}{1-\sqrt{3}} =$$
$$= \frac{(1+\sqrt{3})^2}{(1-\sqrt{3})(1+\sqrt{3})} =$$
$$= \frac{1^2 + 2\sqrt{3} + (\sqrt{3})^2}{1^2 - (\sqrt{3})^2} = \frac{1+2\sqrt{3}+3}{1-3} = \frac{4+2\sqrt{3}}{-2}$$
$$= -2 - \sqrt{3}.$$

b) Show that

$$\frac{\sin(a-b)}{\sin a \sin b} + \frac{\sin(b-c)}{\sin b \sin c} + \frac{\sin(c-a)}{\sin c \sin a} = 0$$

Solution

We note that

$$A = \frac{\sin(a-b)}{\sin a \sin b} + \frac{\sin(b-c)}{\sin b \sin c} + \frac{\sin(c-a)}{\sin c \sin a} =$$

$$= \frac{\sin(a-b)\sin c + \sin(b-c)\sin a + \sin(c-a)\sin b}{\sin a \sin b \sin c} \quad (1)$$

and also that:

$$\sin(a-b)\sin c + \sin(b-c)\sin a =$$

$$= [\sin a \cos b - \sin b \cos a]\sin c + [\sin b \cos c - \sin c \cos b]\sin a =$$

$$= \underline{\sin a \cos b \sin c} - \sin b \cos a \sin c + \sin b \cos c \sin a - \underline{\sin c \cos b \sin a} =$$

$$= \sin b \cos c \sin a - \sin b \cos a \sin c =$$

$$= \sin b [\sin a \cos c - \sin c \cos a] = \sin b \sin(a-c)$$

$$= \sin b [-\sin(c-a)] = -\sin(c-a)\sin b \Rightarrow$$

$$\Rightarrow \sin(a-b)\sin c + \sin(b-c)\sin a + \sin(c-a)\sin b = 0 \overset{(1)}{\Rightarrow}$$

$$\Rightarrow A = \frac{0}{\sin a \sin b \sin c} = 0.$$

c) Show that

$$\tan(a-b)+\tan(b-c)+\tan(c-a)=\tan(a-b)\tan(b-c)\tan(c-a)$$

Solution

Define $x=a-b \land y=b-c \land z=c-a$ and note that

$$x+y+z=(a-b)+(b-c)+(c-a)=a+b+b-c+c-a=0 \Rightarrow$$

$$\Rightarrow z=-x-y$$

and therefore

$$A=\tan(a-b)+\tan(b-c)+\tan(c-a)=\tan x+\tan y+\tan z$$

$$=\tan x+\tan y+\tan(-x-y)=\tan x+\tan y-\tan(x+y)=$$

$$=\tan x+\tan y-\frac{\tan x+\tan y}{1-\tan x\tan y}=$$

$$=\frac{(\tan x+\tan y)(1-\tan x\tan y)-(\tan x+\tan y)}{1-\tan x\tan y}=$$

$$=\frac{(\tan x+\tan y)(1-\tan x\tan y-1)}{1-\tan x\tan y}=$$

$$=-(\tan x\tan y)\frac{\tan x+\tan y}{1-\tan x\tan y}=-\tan x\tan y\tan(x+y)$$

$$=\tan x\tan y\tan(-x-y)=\tan x\tan y\tan z=$$

$$=\tan(a-b)\tan(b-c)\tan(c-a)=B$$

d) Show that

$$\sin^2 x + \sin^2(x+2\pi/3) + \sin^2(x+4\pi/3) = 3/2$$

<u>Solution</u>

We note that

$$\sin(x+2\pi/3) = \sin(x+\pi-\pi/3) = -\sin(x-\pi/3) =$$
$$= -[\sin x\cos(\pi/3) - \cos x\sin(\pi/3)] =$$
$$= \cos x\sin(\pi/3) - \sin x\cos(\pi/3) =$$
$$= (\sqrt{3}/2)\cos x - (1/2)\sin x$$

and

$$\sin(x+4\pi/3) = \sin(x+\pi+\pi/3) = -\sin(x+\pi/3) =$$
$$= -[\sin x\cos(\pi/3) + \cos x\sin(\pi/3)] =$$
$$= -\sin x\cos(\pi/3) - \cos x\sin(\pi/3) =$$
$$= -(1/2)\sin x - (\sqrt{3}/2)\cos x$$

so it follows that

$$A = \sin^2 x + \sin^2(x+2\pi/3) + \sin^2(x+4\pi/3) =$$
$$= \sin^2 x + [(\sqrt{3}/2)\cos x - (1/2)\sin x]^2 + [-(1/2)\sin x - (\sqrt{3}/2)\cos x]^2$$
$$= \sin^2 x + [(\sqrt{3}/2)\cos x]^2 - 2[(\sqrt{3}/2)\cos x][(1/2)\sin x] + [(1/2)\sin x]^2$$
$$+ [(\sqrt{3}/2)\cos x]^2 + 2[(\sqrt{3}/2)\cos x][(1/2)\sin x] + [(1/2)\sin x]^2$$
$$= \sin^2 x + 2[(\sqrt{3}/2)\cos x]^2 + 2[(1/2)\sin x]^2$$
$$= \sin^2 x + 2(3/4)\cos^2 x + 2\cdot(1/4)\sin^2 x$$
$$= [1+1/2]\sin^2 x + (3/2)\cos^2 x = (3/2)\sin^2 x + (3/2)\cos^2 x$$
$$= (3/2)(\sin^2 x + \cos^2 x) = 3/2.$$

EXERCISES

① Show that

a) $\sin(a+b)\sin(a-b) = \sin^2 a - \sin^2 b$

b) $\cos(a+b)\cos(a-b) = \cos^2 a - \sin^2 b$

c) $\sin(a-b)\cos b + \sin b \cos(a-b) = \sin a$

d) $\cos(a+b)\cos(a-b) - \sin(a+b)\sin(a-b) = \cos(2a)$

e) $\dfrac{2\sin(a+b)}{\cos(a+b)+\cos(a-b)} = \tan a + \tan b$

f) $\dfrac{\sin(a-b)}{\cos a\cos b} + \dfrac{\sin(b-c)}{\cos b\cos c} + \dfrac{\sin(c-a)}{\cos c \cdot \cos a} = 0$

g) $\dfrac{\sin(a-b)}{\sin a\sin b} + \dfrac{\sin(b-c)}{\sin b\sin c} + \dfrac{\sin(c-a)}{\sin c \cdot \sin a} = 0$

h) $\dfrac{\tan^2(2a)-\tan^2(a)}{1-\tan^2(2a)\tan^2(a)} = \tan(a)\tan(3a)$

i) $\cos x + \cos\left(x+\dfrac{2\pi}{3}\right) + \cos\left(x+\dfrac{4\pi}{3}\right) = 0$

j) $\cos^2 x + \cos^2\left(\dfrac{\pi}{3}+x\right) + \cos^2\left(\dfrac{\pi}{3}-x\right) = \dfrac{3}{2}$

② Calculate the trigonometric numbers $\sin x$, $\cos x$, $\tan x$, $\cot x$ for

a) $x = \pi/12$ b) $x = 5\pi/12$

③ If $\cos(a+b) = \cos a\cos b$, show that $\sin^2(a+b) = (\sin a + \sin b)^2$.

▼ $2a/3a$ identities

- The trigonometric numbers of $2a$ in terms of the trigonometric numbers of a

$\sin(2a) = 2\sin a \cos a$	$\tan(2a) = \dfrac{2\tan a}{1-\tan^2 a}$
$\cos(2a) = \cos^2 a - \sin^2 a$ $= 2\cos^2 a - 1$ $= 1 - 2\sin^2 a$	$\cot(2a) = \dfrac{\cot^2 a - 1}{2\cot a}$

- In terms of $\cos(2a)$:

$\sin^2 a = \dfrac{1-\cos(2a)}{2}$	$\tan^2 a = \dfrac{1-\cos(2a)}{1+\cos(2a)}$
$\cos^2 a = \dfrac{1+\cos(2a)}{2}$	$\cot^2 a = \dfrac{1+\cos(2a)}{1-\cos(2a)}$

Immediate consequence of $\cos(2a) = 2\cos^2 a - 1 = 1 - 2\sin^2 a$.

- In terms of $\tan(a/2)$

$\sin a = \dfrac{2\tan(a/2)}{1+\tan^2(a/2)}$	$\tan a = \dfrac{2\tan(a/2)}{1-\tan^2(a/2)}$
$\cos a = \dfrac{1-\tan^2(a/2)}{1+\tan^2(a/2)}$	$\cot a = \dfrac{1-\tan^2(a/2)}{2\tan(a/2)}$

Proof of tan(a/2) identities

Since $\dfrac{1}{\cos^2 a} = 1+\tan^2 a \Rightarrow \cos^2 a = \dfrac{1}{1+\tan^2 a}$

it follows that:

$$\sin a = 2\sin(a/2)\cos(a/2) = 2\,\frac{\sin(a/2)}{\cos(a/2)}\cos^2(a/2) =$$

$$= 2\tan(a/2)\,\frac{1}{1+\tan^2(a/2)} = \frac{2\tan(a/2)}{1+\tan^2(a/2)}$$

and

$$\cos a = 2\cos^2(a/2) - 1 = 2\,\frac{1}{1+\tan^2(a/2)} - 1 =$$

$$= \frac{2-(1+\tan^2(a/2))}{1+\tan^2(a/2)} = \frac{2-1-\tan^2(a/2)}{1+\tan^2(a/2)}$$

$$= \frac{1-\tan^2(a/2)}{1+\tan^2(a/2)}$$

and

$$\tan a = \frac{\sin a}{\cos a} = \frac{\left(\dfrac{2\tan(a/2)}{1+\tan^2(a/2)}\right)}{\left(\dfrac{1-\tan^2(a/2)}{1+\tan^2(a/2)}\right)} = \frac{2\tan(a/2)}{1-\tan^2(a/2)}$$

$$\cot a = \frac{\cos a}{\sin a} = \frac{\left(\dfrac{1-\tan^2(a/2)}{1+\tan^2(a/2)}\right)}{\left(\dfrac{2\tan(a/2)}{1+\tan^2(a/2)}\right)} = \frac{1-\tan^2(a/2)}{2\tan(a/2)}$$

• 3a identities: $\boxed{\begin{aligned}\sin(3a) &= -4\sin^3 a + 3\sin a \\ \cos(3a) &= 4\cos^3 a - 3\cos a\end{aligned}}$

Proof

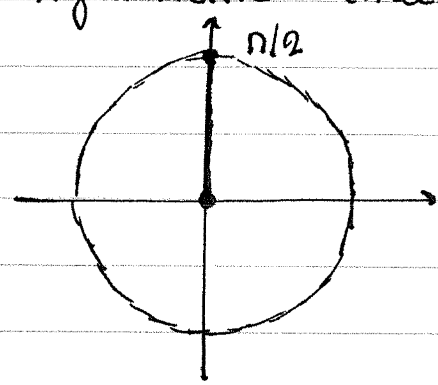
$$\sin(3a) = \sin(a+2a) = \sin a \cos(2a) + \sin(2a)\cos a =$$
$$= \sin a\,(1-2\sin^2 a) + (2\sin a\cos a)\cos a =$$
$$= \sin a\,(1-2\sin^2 a) + 2\sin a\,(1-\sin^2 a)$$
$$= \sin a - 2\sin^3 a + 2\sin a - 2\sin^3 a =$$
$$= (-2-2)\sin^3 a + (1+2)\sin a = -4\sin^3 a + 3\sin a$$

and

$$\cos(3a) = \sin(\pi/2 - 3a) = (-1)\sin(\pi+\pi/2-3a) = -\sin(3\pi/2-3a)$$
$$= -\sin(3(\pi/2-a)) = -[-4\sin^3(\pi/2-a) + 3\sin(\pi/2-a)]$$
$$= -[-4\cos^3 a + 3\cos a] = 4\cos^3 a - 3\cos a \quad \square$$

APPLICATION

These identities can be used to find the trigonometric identities for various angles using $\cos(\pi/2)=0$ as a starting point, which is shown geometrically via the trigonometric circle.



a) $\boxed{\text{Angle } \pi/4}$

$$\cos^2(\pi/4) = \frac{1+\cos(\pi/2)}{2} =$$
$$= \frac{1+0}{2} = \frac{1}{2} \quad (1)$$

and $\cos(\pi/4) > 0$ (2)

From Eq.(1) and Eq.(2):

$$\cos(\pi/4) = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Likewise we can show $\sin(\pi/4) = \sqrt{2}/2$.

b) $\boxed{\text{Angle } \pi/6}$

Let $x = \cos(\pi/6)$ and note that

$4\cos^3(\pi/6) - 3\cos(\pi/6) = \cos(\pi/2) \Leftrightarrow 4x^3 - 3x = 0 \Leftrightarrow$

$\Leftrightarrow x(4x^2-3) = 0 \Leftrightarrow x(2x-\sqrt{3})(2x+\sqrt{3}) = 0 \Leftrightarrow$

$\Leftrightarrow x = 0 \vee 2x - \sqrt{3} = 0 \vee 2x+\sqrt{3} = 0 \Leftrightarrow$

$\Leftrightarrow x = 0 \vee x = \sqrt{3}/2 \vee x = -\sqrt{3}/2$ (1)

Since $x = \cos(\pi/6) > 0$, it follows from Eq.(1) that $\cos(\pi/6) = \sqrt{3}/2$.

Also:

$\sin^2(\pi/6) = 1 - \cos^2(\pi/6) = 1 - (\sqrt{3}/2)^2 = 1 - (3/4) = 1/4 \Rightarrow$

$\Rightarrow \sin(\pi/6) = 1/2 \vee \sin(\pi/6) = -1/2$

$\Rightarrow \sin(\pi/6) = 1/2$ (because $\sin(\pi/6) > 0$)

c) $\boxed{\text{Angle } \pi/3}$

$\sin(\pi/3) = 2\sin(\pi/6)\cos(\pi/6) = 2(1/2)(\sqrt{3}/2) = \sqrt{3}/2$

$\cos(\pi/3) = \cos^2(\pi/6) - \sin^2(\pi/6) = (\sqrt{3}/2)^2 - (1/2)^2 =$

$= (3/4) - (1/4) = 2/4 = 1/2$

EXAMPLES

a) Evaluate $\sin(17\pi/12)$ and $\tan(3\pi/8)$.

Solution

We have

$$\sin(17\pi/12) = \sin((12+5)\pi/12) = \sin(\pi+5\pi/12) = -\sin(5\pi/12)$$

$$\underset{5\pi/12\in[0,\pi/2]}{=} -\sqrt{\frac{1-\cos(5\pi/6)}{2}} = -\sqrt{\frac{1-\cos(\pi-\pi/6)}{2}}$$

$$= -\sqrt{\frac{1+\cos(-\pi/6)}{2}} = -\sqrt{\frac{1+\cos(\pi/6)}{2}} =$$

$$= -\sqrt{\frac{1+(\sqrt{3}/2)}{2}} = -\sqrt{\frac{2+\sqrt{3}}{4}} = \frac{-\sqrt{2+\sqrt{3}}}{2}$$

and

$$\tan(3\pi/8) = \sqrt{\frac{1-\cos(3\pi/4)}{1+\cos(3\pi/4)}} = \sqrt{\frac{1-\cos(\pi-\pi/4)}{1+\cos(\pi-\pi/4)}} =$$

$$= \sqrt{\frac{1+\cos(-\pi/4)}{1-\cos(-\pi/4)}} = \sqrt{\frac{1+\cos(\pi/4)}{1-\cos(\pi/4)}} =$$

$$= \sqrt{\frac{1+\sqrt{2}/2}{1-\sqrt{2}/2}} = \sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} = \sqrt{\frac{(2+\sqrt{2})^2}{(2-\sqrt{2})(2+\sqrt{2})}}$$

$$= \frac{2+\sqrt{2}}{\sqrt{2^2-(\sqrt{2})^2}} = \frac{2+\sqrt{2}}{\sqrt{4-2}} = \frac{2+\sqrt{2}}{\sqrt{2}} =$$

$$= \frac{(2+\sqrt{2})\sqrt{2}}{2}$$

b) Show that $\cos^4(\pi/8) + \cos^4(3\pi/8) = 3/4$

Solution

Since

$$\cos^2(\pi/8) = \frac{1+\cos(\pi/4)}{2} = \frac{1+(\sqrt{2}/2)}{2} = \frac{2+\sqrt{2}}{4} \Rightarrow$$

$$\Rightarrow \cos^4(\pi/8) = \frac{(2+\sqrt{2})^2}{4^2} = \frac{2^2+2\cdot 2\sqrt{2}+(\sqrt{2})^2}{16} = \frac{4+4\sqrt{2}+2}{16}$$

$$= \frac{6+4\sqrt{2}}{16} = \frac{3+2\sqrt{2}}{8}$$

and

$$\cos^2(3\pi/8) = \frac{1+\cos(3\pi/4)}{2} = \frac{1+\cos(\pi-\pi/4)}{2} = \frac{1-\cos(-\pi/4)}{2} =$$

$$= \frac{1-\cos(\pi/4)}{2} = \frac{1-(\sqrt{2}/2)}{2} = \frac{2-\sqrt{2}}{4} \Rightarrow$$

$$\Rightarrow \cos^4(3\pi/8) = \frac{(2-\sqrt{2})^2}{4^2} = \frac{2^2-2\cdot 2\sqrt{2}+(\sqrt{2})^2}{16} = \frac{4-4\sqrt{2}+2}{16}$$

$$= \frac{6-4\sqrt{2}}{16} = \frac{3-2\sqrt{2}}{8}$$

it follows that

$$A = \cos^4(\pi/8) + \cos^4(3\pi/8) = \frac{3+2\sqrt{2}}{8} + \frac{3-2\sqrt{2}}{8} =$$

$$= \frac{3+2\sqrt{2}+3-2\sqrt{2}}{8} = \frac{6}{8} = \frac{3}{4} = B.$$

c) Show that $\dfrac{\sin(3a)}{\sin a} - \dfrac{\cos(3a)}{\cos a} = 2$

Solution

We have:

$$A = \frac{\sin(3a)}{\sin a} - \frac{\cos(3a)}{\cos a} = \frac{\sin(3a)\cos a - \sin a\cos(3a)}{\sin a\cos a} =$$

$$= \frac{\sin(3a-a)}{\sin a\cos a} = \frac{\sin(2a)}{\sin a\cos a} = \frac{2\sin a\cos a}{\sin a\cos a} = 2 = B$$

d) Show that $\tan(\pi/6 + a)\tan(\pi/6 - a) = \dfrac{2\cos(2a) - 1}{2\cos(2a) + 1}$

Solution

We have:

$$A = \tan(\pi/6 + a)\tan(\pi/6 - a) =$$

$$= \frac{\tan(\pi/6) + \tan a}{1 - \tan(\pi/6)\tan a} \cdot \frac{\tan(\pi/6) - \tan a}{1 + \tan(\pi/6)\tan a} =$$

$$= \frac{\tan^2(\pi/6) - \tan^2 a}{1 - \tan^2(\pi/6)\tan^2 a} = \frac{(1/\sqrt{3})^2 - \tan^2 a}{1 - (1/\sqrt{3})^2\tan^2 a} =$$

$$= \frac{(1/3) - \tan^2 a}{1 - (1/3)\tan^2 a} = \frac{1 - 3\tan^2 a}{3 - \tan^2 a} =$$

$$= \frac{1 - 3\,\dfrac{1 - \cos(2a)}{1 + \cos(2a)}}{3 - \dfrac{1 - \cos(2a)}{1 + \cos(2a)}} = \frac{(1 + \cos(2a)) - 3(1 - \cos(2a))}{3(1 + \cos(2a)) - (1 - \cos(2a))}$$

$$= \frac{1+\cos(2a)-3+3\cos(2a)}{3+3\cos(2a)-1+\cos(2a)} = \frac{4\cos(2a)-2}{4\cos(2a)+2} =$$

$$= \frac{2[2\cos(2a)-1]}{2[2\cos(2a)+1]} = \frac{2\cos(2a)-1}{2\cos(2a)+1}$$

EXERCISES

(4) Find the trigonometric numbers for the following angles:

a) $x = \pi/8 = 22.5°$

b) $x = \pi/12 = 15°$

c) $x = 5\pi/12 = 75°$

(5) Use the previous results to show that

a) $\cos^4(\pi/8) + \cos^4(3\pi/8) = 3/4$

b) $$(1+\cos(\pi/8))(1+\cos(3\pi/8))(1+\cos(5\pi/8)) \times (1+\cos(7\pi/8)) = 1/8$$

(6) Show that:

a) $\cos(5a) = 16\cos^5 a - 20\cos^3 a + 5\cos a$

b) $\cos(\pi/10) = \frac{1}{4}\sqrt{10+2\sqrt{5}}$

c) $\cos(\pi/5) = \frac{1}{4}(\sqrt{5}+1)$

(7) Show that

a) $\dfrac{\sin(2a)}{1+\cos(2a)} = \tan a$

b) $\dfrac{\sin(2a)}{1-\cos(2a)} = \cot a$

c) $\cos^4 a - \sin^4 a = \cos(2a)$

d) $\cot a - \tan a = 2\cot(2a)$

e) $\dfrac{1+\cot^2 a}{2\cot a} = \dfrac{1}{\sin(2a)}$

f) $\dfrac{\cot^2 a+1}{\cot^2 a-1} = \dfrac{1}{\cos(2a)}$

(8) Show that

a) $\tan\left(\frac{\pi}{4}-a\right)=\dfrac{\cos(2a)}{1+\sin(2a)}$

b) $\cos^2\left(\frac{\pi}{4}-a\right)-\sin^2\left(\frac{\pi}{4}-a\right)=\sin(2a)$

c) $\tan\left(\frac{\pi}{4}+a\right)-\tan\left(\frac{\pi}{4}-a\right)=2\tan(2a)$

d) $\dfrac{\cos a+\sin a}{\cos a-\sin a}-\dfrac{\cos a-\sin a}{\cos a+\sin a}=2\tan(2a)$

e) $\dfrac{1-\cos(2a)+\sin(2a)}{1+\cos(2a)+\sin(2a)}=\tan a$

f) $\dfrac{\cot a+1}{\cot a-1}=\dfrac{\cos(2a)}{1-\sin(2a)}$

(9) Show that

a) $3-4\cos 2a+\cos 4a=8\sin^4 a$

b) $\dfrac{2}{(1+\tan a)(1+\cot a)}=\dfrac{\sin(2a)}{1+\sin(2a)}$

c) $\tan x+\dfrac{1}{\cos x}=\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$

d) $\tan\left(\dfrac{a+b}{2}\right)=\dfrac{\sin a+\sin b}{\cos a+\cos b}$

e) $\dfrac{\sin(2a)}{1-\cos(2a)}\;\dfrac{1-\cos(a)}{\cos a}=\tan\left(\frac{a}{2}\right)$

f) $\tan\left(\frac{\pi}{6}+a\right)\tan\left(\frac{\pi}{6}-a\right)=\dfrac{2\cos(2a)-1}{2\cos(2a)+1}$

(10) Show that

a) $\dfrac{\sin(3a)}{\sin a} - \dfrac{\cos(3a)}{\cos a} = 2$

b) $\dfrac{3\cos a + \cos(3a)}{3\sin a - \sin(3a)} = \cot^3 a$

c) $4\sin a \cdot \sin\left(\frac{\pi}{3} + a\right)\sin\left(\frac{\pi}{3} - a\right) = \sin(3a)$

d) $\dfrac{\sin(3a) + \sin^3 a}{\cos^3 a - \cos(3a)} = \cot a$

e) $4\sin^3 a \cos 3a + 4\cos^3 a \sin 3a = 3\sin(4a)$

f) $\dfrac{\cos^3 a - \cos(3a)}{\cos a} + \dfrac{\sin^3 a + \sin(3a)}{\sin a} = 3.$

(11) Show that: $\cos(20^\circ)\cos(40^\circ)\cos(60^\circ)\cos(80^\circ) = \dfrac{1}{16}$
(Hint: Use $\sin(2x) = 2\sin x\cos x$)

(12) Show that

a) $\sin\left(\dfrac{\pi}{10}\right) = \dfrac{-1+\sqrt{5}}{4}$ (Hint: For $a = \pi/10$, solve $\sin(2a) = \sin(\pi/2 - 3a)$)

b) $\sin\left(\dfrac{3\pi}{10}\right) = \dfrac{1+\sqrt{5}}{4}$

c) $\tan\left(\dfrac{\pi}{20}\right) - \tan\left(\dfrac{3\pi}{20}\right) - \tan\left(\dfrac{7\pi}{20}\right) + \tan\left(\dfrac{9\pi}{20}\right) = 4$

(Hint: Switch to sin, cos and reduce to
$2/\sin(\pi/10) - 2/\sin(3\pi/10)$
which can be evaluated via (a) (b)).

Product-Sum identities

Product to sum

$$\boxed{\begin{aligned} 2\sin a\cos b &= \sin(a-b)+\sin(a+b)\\ 2\cos a\cos b &= \cos(a-b)+\cos(a+b)\\ 2\sin a\sin b &= \cos(a-b)-\cos(a+b) \end{aligned}} \quad (!!)$$

↳ These are immediate consequences of the $a \pm b$ identities.

Sum to product

$$\sin a \pm \sin b = 2\sin\left(\frac{a\pm b}{2}\right)\cos\left(\frac{a\mp b}{2}\right)$$

$$\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\cos a - \cos b = 2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right) \quad (!!)$$

$$\tan a \pm \tan b = \frac{\sin(a\pm b)}{\cos a\cos b}$$

$$\cot a \pm \cot b = \frac{\sin(b\pm a)}{\sin a\sin b} \quad (!!)$$

• Note that:

$$1 \pm \sin a = \sin(\pi/2) \pm \sin a = \dots$$

$$\sin a \pm \cos b = \sin a \pm \sin(\pi/2 - b) = \dots$$

$$1+\cos a = 2\cos^2(a/2), \quad 1-\cos a = 2\sin^2(a/2)$$

EXAMPLES

a) Show that $\dfrac{\sin x + \sin(3x) + \sin(5x)}{\cos x + \cos(3x) + \cos(5x)} = \tan(3x)$

Solution

We have:

$$A = \frac{\sin x + \sin(3x) + \sin(5x)}{\cos x + \cos(3x) + \cos(5x)} = \frac{[\sin x + \sin(5x)] + \sin(3x)}{[\cos x + \cos(5x)] + \cos(3x)}$$

$$= \frac{2\sin\left(\frac{x+5x}{2}\right)\cos\left(\frac{x-5x}{2}\right) + \sin(3x)}{2\cos\left(\frac{x+5x}{2}\right)\cos\left(\frac{x-5x}{2}\right) + \cos(3x)} =$$

$$= \frac{2\sin(3x)\cos(2x) + \sin(3x)}{2\cos(3x)\cos(2x) + \cos(3x)} = \frac{\sin(3x)[2\cos(2x)+1]}{\cos(3x)[2\cos(2x)+1]}$$

$$= \frac{\sin(3x)}{\cos(3x)} = \tan(3x). = B$$

b) Show that $\sin(2x) + \cos(5x) = 2\sin\left(\frac{\pi}{4} - \frac{3x}{2}\right)\cos\left(\frac{\pi}{4} - \frac{7x}{2}\right)$

Solution

We have:

$$A = \sin(2x) + \cos(5x) = \sin(2x) + \sin(\pi/2 - 5x) =$$

$$= 2\sin\left(\frac{2x + (\pi/2 - 5x)}{2}\right)\cos\left(\frac{2x - (\pi/2 - 5x)}{2}\right) =$$

$$= 2\sin\left(\frac{2x + \pi/2 - 5x}{2}\right)\cos\left(\frac{2x - \pi/2 + 5x}{2}\right) =$$

$$= 2\sin\left(\frac{\pi}{4} - \frac{3x}{2}\right)\cos\left(\frac{7x}{2} - \frac{\pi}{4}\right) =$$

$$= 2\sin\left(\frac{\pi}{4} - \frac{3x}{2}\right)\cos\left(\frac{\pi}{4} - \frac{7x}{2}\right) = B$$

c) Show that $\sin(3x)\cos(8x) - \sin(5x)\cos(6x) = -\sin(2x)\cos(3x)$

Solution

We have:

$A = \sin(3x)\cos(8x) - \sin(5x)\cos(6x) =$

$= (1/2)[\sin(3x+8x) + \sin(3x-8x)] - (1/2)[\sin(5x+6x) + \sin(5x-6x)]$

$= (1/2)[\sin(11x) - \sin(5x)] - (1/2)[\sin(11x) - \sin x] =$

$= (1/2)[\sin(11x) - \sin(5x) - \sin(11x) + \sin x]$

$= (1/2)[\sin x - \sin(5x)] = (1/2)[\sin x + \sin(-5x)] =$

$$= (1/2)\, 2\sin\left(\frac{x+(-5x)}{2}\right)\cos\left(\frac{x-(-5x)}{2}\right) =$$

$= \sin(-2x)\cos(3x) = -\sin(2x)\cos(3x) = B$

EXERCISES

(13) Write the following expressions as a sum or difference:

a) $2\sin(2a)\cos a$ c) $\cos(5a)\cos(7a)$

b) $2\sin a\cos(4a)$ d) $\sin a\cdot\sin(3a)$

(14) Evaluate the following expressions:

a) $2\cos 60^\circ\cdot\sin 30^\circ$ c) $\cos(150^\circ)\cos(30^\circ)$

b) $\sin 45^\circ\cos 75^\circ$ d) $2\sin(36^\circ)\cos(54^\circ)$

(15) Factor the following expressions

a) $\sin(4a)+\sin a$

b) $\sin(7a)-\sin(5a)$

c) $\cos(5a)-\cos(a)$

d) $\cos(3x)+\cos(5x)$

e) $\sin x-\sin 2x+\sin(3x)$

f) $\sin(3x)+\sin(7x)+\sin(10x)$

g) $\cos a+2\cos(2a)+\cos(3a)$

h) $\cos(7a)-\cos(5a)+\cos(3a)-\cos a$

(16) Show that

a) $$\frac{\cos(3a)-\cos(5a)}{\sin(5a)-\sin(3a)}=\tan(4a)$$

b) $$\frac{\sin(2a)+\sin(3a)}{\cos(2a)-\cos(3a)}=\cot\left(\frac{a}{2}\right)$$

c) $$\frac{\cos(2a)-\cos(4a)}{\sin(4a)-\sin(2a)}=\tan(3a)$$

d) $\dfrac{\cos(4a) - \cos a}{\sin a - \sin(4a)} = \tan\left(\dfrac{5a}{2}\right)$

e) $\dfrac{\sin(2a) + \sin(5a) - \sin a}{\cos(2a) + \cos(5a) + \cos a} = \tan(2a)$

f) $\dfrac{\sin a + \sin 3a + \sin 5a + \sin 7a}{\cos a + \cos 3a + \cos 5a + \cos 7a} = \tan(4a)$

g) $\dfrac{\sin a + \sin b}{\cos a + \cos b} = \tan\left(\dfrac{a+b}{2}\right)$

h) $\cos(5a)\cos(2a) - \cos(4a)\cos(3a) = -\sin 2a \sin a$

i) $\sin(4a)\cos a - \sin(3a)\cos(2a) = \sin a \cos 2a$

17 Show that

a) $(\cos a + \cos b)^2 + (\sin a - \sin b)^2 = 4\cos^2\left(\dfrac{a+b}{2}\right)$

b) $(\cos a + \cos b)^2 + (\sin a + \sin b)^2 = 4\cos^2\left(\dfrac{a-b}{2}\right)$

c) $(\cos a - \cos b)^2 + (\sin a - \sin b)^2 = 4\sin^2\left(\dfrac{a-b}{2}\right)$

d) $\dfrac{\sin(a+b)\sin(a-b)}{\cos^2 a \cos^2 b} = \tan^2 a - \tan^2 b$

e) $\cos a + \cos 2a + \cos 3a = \dfrac{\cos(2a)\sin(3a/2)}{\sin(a/2)}$

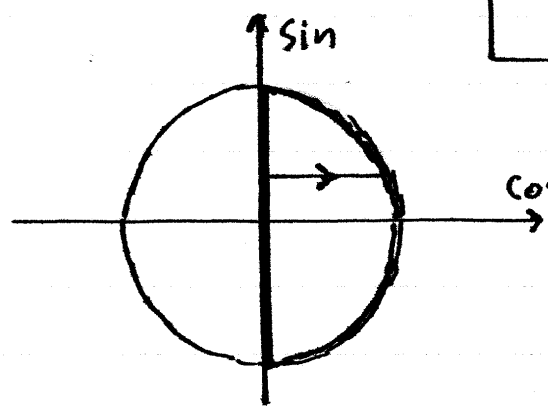
PRE4: Trigonometric equations and inequalities

TRIGONOMETRIC EQUATIONS AND INEQUALITIES

Inverse trigonometric functions

1) Inverse sine :

$$y = \operatorname{Arcsin} x \Leftrightarrow \begin{cases} x = \sin y \\ -\pi/2 \leq y \leq \pi/2 \end{cases}$$

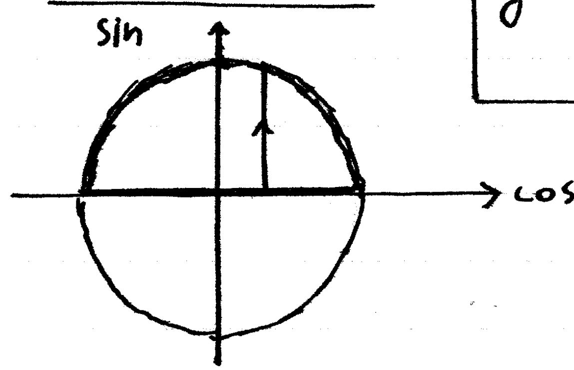


Domain: $A = [-1, 1]$

Range: $f(A) = [-\pi/2, \pi/2]$

2) Inverse cosine :

$$y = \operatorname{Arccos} x \Leftrightarrow \begin{cases} x = \cos y \\ 0 \leq y \leq \pi \end{cases}$$

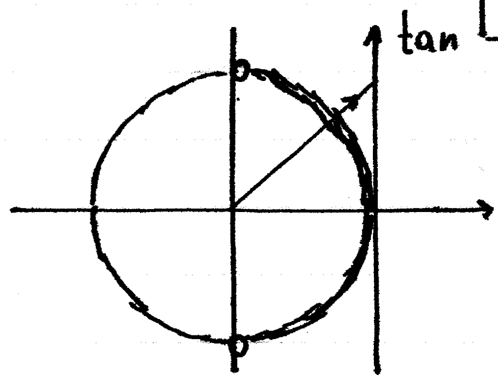


Domain: $A = [-1, 1]$

Range: $f(A) = [0, \pi]$

3) Inverse tangent

$$y = \operatorname{Arctan} x \Leftrightarrow \begin{cases} x = \tan y \\ -\pi/2 \leq y \leq \pi/2 \end{cases}$$

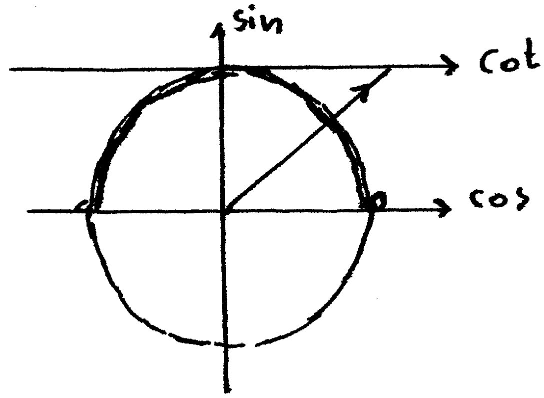


Domain : $A = (-\infty, +\infty)$

Range: $f(A) = [-\pi/2, \pi/2)$

4) Inverse cotangent :

$$y = \mathrm{Arccot}\,x \Leftrightarrow \begin{cases} x = \cot y \\ 0 < y < \pi \end{cases}$$



Domain: $A = (-\infty, +\infty)$
Range: $f(A) = (0, \pi)$

By definition, it follows that

$$\begin{aligned}
&\sin(\mathrm{Arcsin}\,x) = x, && \forall x \in [-1,1] \\
&\cos(\mathrm{Arccos}\,x) = x, && \forall x \in [-1,1] \\
&\tan(\mathrm{Arctan}\,x) = x, && \forall x \in \mathbb{R} \\
&\cot(\mathrm{Arccot}\,x) = x, && \forall x \in \mathbb{R}
\end{aligned}$$

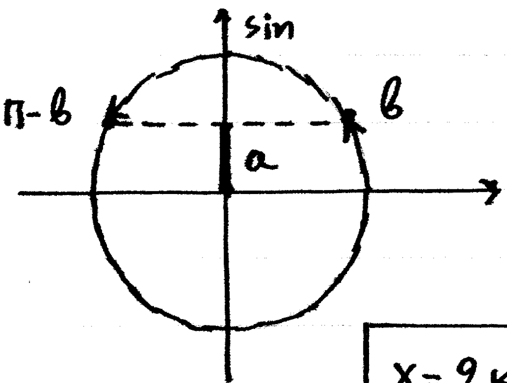
and

$$\begin{aligned}
&\mathrm{Arcsin}(\sin x) = x, && \forall x \in [-\pi/2, \pi/2] \\
&\mathrm{Arccos}(\cos x) = x, && \forall x \in [0, \pi] \\
&\mathrm{Arctan}(\tan x) = x, && \forall x \in (-\pi/2, \pi/2) \\
&\mathrm{Arccot}(\cot x) = x, && \forall x \in (0, \pi)
\end{aligned}$$

V Fundamental Trigonometric Equations

① $\boxed{\sin x = a} \Leftrightarrow \sin x = \sin b$ with $b = \text{Arcsin}(a)$

▶ We assume $|a| \leqslant 1$.



Solutions:

$b \quad , \quad \pi - b$

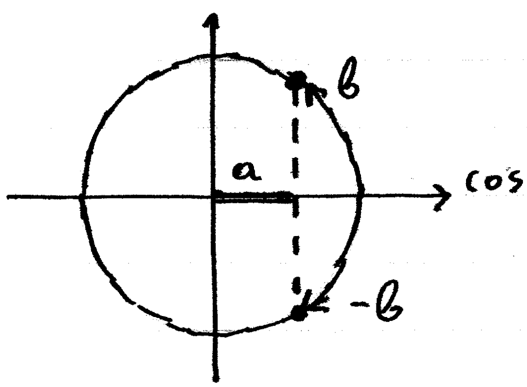
$2\pi + b \quad , \quad 3\pi - b$

$4\pi + b \quad , \quad 5\pi - b$

$$\boxed{x = 2k\pi + b \vee x = (2k+1)\pi - b}$$

② $\boxed{\cos x = a} \Leftrightarrow \cos x = \cos b$ with $b = \text{Arccos}\, b$

▶ We assume $|a| \leqslant 1$



Solutions:

$b \qquad -b$

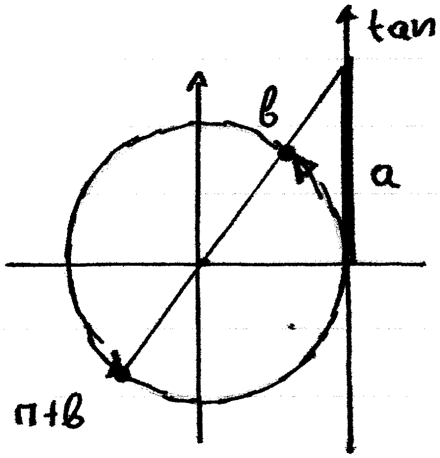
$2\pi + b \qquad 2\pi - b$

$4\pi + b \qquad 4\pi - b$

$$\boxed{x = 2k\pi \pm b}$$

(3) $\boxed{\tan x = a} \iff \tan x = \tan b$ with $b = \mathrm{Arctan}(a)$

- Assume $a \in \mathbb{R}$.



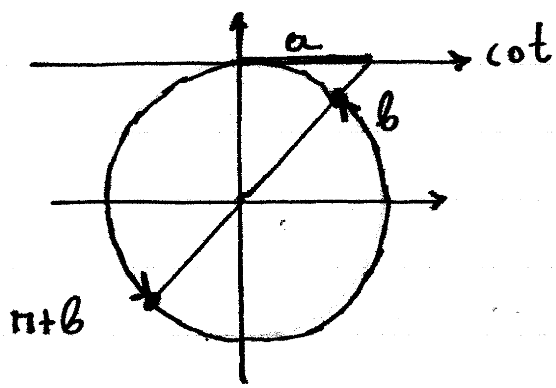
Solutions

b	$\pi + b$
$2\pi + b$	$3\pi + b$
$4\pi + b$	$5\pi + b$

$$\boxed{x = k\pi + b}$$

(4) $\boxed{\cot x = a} \iff \cot x = \cot b$ with $b = \mathrm{Arccot}(a)$

- Assume $a \in \mathbb{R}$.



Solutions

b	$\pi + b$
$2\pi + b$	$3\pi + b$
$4\pi + b$	$5\pi + b$

$$\boxed{x = k\pi + b}$$

Special cases

1) $a=0$

$$\boxed{\begin{array}{l} \sin x = 0 \Leftrightarrow x = k\pi \\ \cos x = 0 \Leftrightarrow x = k\pi + \pi/2 \end{array}}$$

2) $a=1$

$$\boxed{\begin{array}{l} \sin x = 1 \Leftrightarrow x = 2k\pi + \pi/2 \\ \cos x = 1 \Leftrightarrow x = 2k\pi \end{array}}$$

3) $a=-1$

$$\boxed{\begin{array}{l} \sin x = -1 \Leftrightarrow x = 2k\pi - \pi/2 \\ \cos x = -1 \Leftrightarrow x = (2k+1)\pi \end{array}}$$

Forms reducible to fundamental trigonometric equations

1) Forms:

$\sin f(x) = \sin g(x)$ $\cos f(x) = \cos g(x)$	$\tan f(x) = \tan g(x)$ $\cot f(x) = \cot g(x)$

EXAMPLES

a) $2\sin\left(3x+\frac{\pi}{3}\right) - 1 = 0 \Leftrightarrow \sin\left(3x+\frac{\pi}{3}\right) = \frac{1}{2} = \sin\frac{\pi}{6} \Leftrightarrow$

$\Leftrightarrow 3x + \frac{\pi}{3} = 2k\pi + \frac{\pi}{6} \vee 3x + \frac{\pi}{3} = (2k+1)\pi - \frac{\pi}{6} \Leftrightarrow$

$\Leftrightarrow 3x = 2k\pi + \frac{\pi}{6} - \frac{\pi}{3} \vee 3x = (2k+1)\pi - \frac{\pi}{6} - \frac{\pi}{3} \Leftrightarrow$

$$\Leftrightarrow 3x = 2k\pi - \frac{\pi}{6} \vee 3x = (2k+1)\pi - \frac{\pi}{2} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{2k\pi}{3} - \frac{\pi}{18} \vee x = \frac{(2k+1)\pi}{3} - \frac{\pi}{6}$$

2) Forms:

$$\boxed{\begin{aligned} \sin(f(x)) &= \cos(g(x)) \\ \tan(f(x)) &= \cot(g(x)) \end{aligned}}$$

We use the cofunction identities to reduce to the previous form:

$$\boxed{\begin{aligned} \sin(x) &= \cos(\pi/2 - x) \\ \cos(x) &= \sin(\pi/2 - x) \end{aligned}}$$

EXAMPLE

$$\sin(\pi - 2x) - \cos(x + \pi/4) = 0 \Leftrightarrow$$

$$\Leftrightarrow \sin(\pi - 2x) = \cos(x + \pi/4) \Leftrightarrow$$

$$\Leftrightarrow \cos(\pi/2 - \pi + 2x) = \cos(x + \pi/4) \Leftrightarrow$$

$$\Leftrightarrow \cos(2x - \pi/2) = \cos(x + \pi/4) \Leftrightarrow$$

$$\Leftrightarrow 2x - \pi/2 = 2k\pi + x + \pi/4 \vee 2x - \pi/2 = 2k\pi - x - \pi/4 \Leftrightarrow$$

$$\Leftrightarrow 8x - 2\pi = 8k\pi + 4x + \pi \vee 8x - 2\pi = 8k\pi - 4x - \pi$$

$$\Leftrightarrow 4x = 8k\pi + 3\pi \vee 12x = 8k\pi + \pi$$

$$\Leftrightarrow x = 2k\pi + \frac{3\pi}{4} \vee x = \frac{2k\pi}{3} + \frac{\pi}{12}$$

3) <u>Form</u> :

$$\boxed{\begin{array}{l} \sin(f(x)) = -\sin(g(x)) \\ \tan(f(x)) = -\tan(g(x)) \\ \cot(f(x)) = -\cot(g(x)) \end{array}}$$

We use the fact that sin, tan, cot are odd functions.
i.e. $\sin(-x) = -\sin x$ and $\tan(-x) = -\tan x$ and $\cot(-x) = \cot x$.

▶ <u>Remark</u>

For equations containing terms of the form $\tan(f(x))$ or $\cot(f(x))$ we introduce the following restrictions and need to reject solutions that violate these restrictions

For $\tan(g(x)) \longleftrightarrow$ Require $g(x) \neq k\pi + \pi/2$ with $k \in \mathbb{Z}$.

For $\cot(g(x)) \longleftrightarrow$ Require $g(x) \neq k\pi$ with $k \in \mathbb{Z}$

The process for enforcing such restrictions is as follows:

•$_1$ Solving the original equation gives
$x = f_1(k) \vee x_2 = f_2(k) \vee \cdots \vee x = f_n(k)$ with $k \in \mathbb{Z}$.
which may include solutions that need to be rejected.
With no loss of generality, consider the case $n=1$ where we have
$x = f(k)$ with $k \in \mathbb{Z}$.

•$_2$ Given a restriction $g(x) \neq k\pi + \pi/2$, solve:
$g(x) = k\pi + \pi/2 \iff x = G(k)$
To accept $x = f(k)$ with $k \in \mathbb{Z}$, we require
$\forall \lambda \in \mathbb{Z} : f(k) \neq G(\lambda)$
therefore, we solve:
$f(k) = G(\lambda) \iff \cdots \iff \lambda = \Lambda(k)$

We reject all solutions $x = f(k)$ for which $\lambda(k) \in \mathbb{Z}$
We accept all solutions $x = f(k)$ for which $\lambda(k) \notin \mathbb{Z}$.

- $_3$ We work similarly for any restrictions of the form $g(x) \neq k\pi$ and process all restrictions, rejecting solutions as needed.

EXAMPLE

$\tan(3x) + \tan x = 0$

Solution

Require $\begin{cases} x \neq k\pi + \pi/2 \\ 3x \neq k\pi + \pi/2 \end{cases} \Leftrightarrow \begin{cases} x \neq k\pi + \pi/2 \\ x \neq k\pi/3 + \pi/6 \end{cases}$

We have:

$\tan(3x) + \tan(x) = 0 \Leftrightarrow \tan(3x) = -\tan x \Leftrightarrow \tan(3x) = \tan(-x)$

$\Leftrightarrow 3x = k\pi - x \Leftrightarrow 3x + x = k\pi \Leftrightarrow 4x = k\pi \Leftrightarrow x = k\pi/4$.

a) We apply $x \neq k\pi + \pi/2$

Solve $\frac{k\pi}{4} = \lambda\pi + \frac{\pi}{2} \Leftrightarrow \frac{k}{4} = \lambda + \frac{1}{2} \Leftrightarrow$

$$\Leftrightarrow \lambda = \frac{k}{4} - \frac{1}{2} = \frac{k-2}{4}$$

We reject $x = k\pi/4$ for $k \in \mathbb{Z}$ such that $k-2$ is multiple of 4

Thus we remove: $S_1 = \{k\pi/4 \mid \lambda \in \mathbb{Z} \wedge k = 4\lambda + 2\}$.

b) We apply $x \neq k\pi/3 + \pi/6$

Solve: $\frac{k\pi}{4} = \frac{\lambda\pi}{3} + \frac{\pi}{6} \Leftrightarrow 12\frac{k\pi}{4} = 12\left(\frac{\lambda\pi}{3} + \frac{\pi}{6}\right) \Leftrightarrow$

$3k\pi = 4\lambda\pi + 2\pi \Leftrightarrow 3k = 4\lambda + 2 \Leftrightarrow 4\lambda = 3k - 2 \Leftrightarrow$

$\Leftrightarrow \lambda = \dfrac{3k-2}{4}$

We thus reject solutions $x = k\pi/4$ when $3k-2$ is a multiple of 4.

▶ Consider the possibilities $k = 4\lambda$, $k = 4\lambda+1$, $k = 4\lambda+2$, $k = 4\lambda+3$ with $\lambda \in \mathbb{Z}$. Note that $k = 4\lambda+2$ with $\lambda \in \mathbb{Z}$ solutions are already rejected.

For $k = 4\lambda$:

$3k-2 = 3(4\lambda) - 2 = 4(3\lambda) - 4 + 2 = 4(3\lambda - 1) + 2 \Rightarrow$

$\Rightarrow 3k-2$ not multiple of 4.

For $k = 4\lambda+1$:

$3k-2 = 3(4\lambda+1) - 2 = 4(3\lambda) + 3 - 2 = 4(3\lambda) + 1 \Rightarrow$

$\Rightarrow 3k-2$ not multiple of 4.

For $k = 4\lambda+3$:

$3k-2 = 3(4\lambda+3) - 2 = 4(3\lambda) + 9 - 2 = 4(3\lambda) + 7 = 4(3\lambda+1) + 3 \Rightarrow$

$\Rightarrow 3k-2$ not multiple of 4.

It follows that no additional solutions need to be rejected

The solutions that are accepted are:

$S = \{k\pi/4 \mid \lambda \in \mathbb{Z} \wedge (k = 4\lambda \vee k = 4\lambda+1 \vee k = 4\lambda+3)\}$.

4) Form: $\boxed{\cos(f(x)) = -\cos(g(x))} \Leftrightarrow$

$$\Leftrightarrow \cos(f(x)) = \cos(\pi + g(x)) \Leftrightarrow \dots \text{ etc.}$$

EXAMPLE

$$\cos(3x - \pi/4) + \cos(2\pi/3 - 2x) = 0 \Leftrightarrow \cos(3x - \pi/4) = -\cos(2\pi/3 - 2x)$$

$$\Leftrightarrow \cos(3x - \pi/4) = \cos(\pi + 2\pi/3 - 2x) \Leftrightarrow$$

$$\Leftrightarrow \cos(3x - \pi/4) = \cos(5\pi/3 - 2x) \Leftrightarrow$$

$$\Leftrightarrow 3x - \pi/4 = 2k\pi + (5\pi/3 - 2x) \vee 3x - \pi/4 = 2k\pi - (5\pi/3 - 2x) \Leftrightarrow$$

$$\Leftrightarrow 3x + 2x = 2k\pi + 5\pi/3 + \pi/4 \vee 3x - \pi/4 = 2k\pi - 5\pi/3 + 2x \Leftrightarrow$$

$$\Leftrightarrow 5x = 2k\pi + \frac{(5\cdot 4 + 3\cdot 1)\pi}{12} \vee 3x - 2x = 2k\pi + \pi/4 - 5\pi/3 \Leftrightarrow$$

$$\Leftrightarrow 5x = 2k\pi + \frac{23\pi}{12} \vee x = 2k\pi + \frac{3\pi - 4\cdot 5\pi}{12} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{2k\pi}{5} + \frac{23\pi}{60} \vee x = 2k\pi - \frac{17\pi}{12}$$

It is possible to have equations that require a combination of techniques from the forms above.

EXAMPLE

$$\tan(3x) + \cot(2x) = 0 \quad (1)$$

Require: $\begin{cases} 3x \neq k\pi + \pi/2 \\ 2x \neq k\pi \end{cases} \Leftrightarrow \begin{cases} x \neq \frac{k\pi}{3} + \frac{\pi}{2} & (2) \\ x \neq \frac{k\pi}{2} & (3) \end{cases}$

$$(1) \Leftrightarrow \tan(3x) = -\cot(2x) \Leftrightarrow \tan(3x) = \cot(-2x) \Leftrightarrow$$

$$\Leftrightarrow \tan(3x) = \tan\left(\frac{\pi}{2} - (-2x)\right) \Leftrightarrow \tan(3x) = \tan\left(\frac{\pi}{2} + 2x\right)$$

$$\Leftrightarrow 3x = k\pi + \frac{\pi}{2} + 2x \Leftrightarrow x = k\pi + \frac{\pi}{2} = \frac{(2k+1)\pi}{2}$$

This violates condition (3) thus the equation does not have any solution.

EXERCISES

① Solve the following equations

a) $\sin\left(\frac{x}{3}+\frac{\pi}{4}\right)=\sin\left(x-\frac{\pi}{4}\right)$

b) $\tan 3x = \tan\left(7x+\frac{\pi}{8}\right)$

c) $\cos 2x - \cos(x/2) = 0$

d) $\tan\left(2x+\frac{\pi}{3}\right) = \cot(\pi-3x)$

e) $\sin(\pi-2x) - \cos(x+\pi/4) = 0$

f) $\cos(\pi/6+5x) + \sin(-3x) = 0$

g) $\cos(3x-\pi/4) + \cos(2\pi/3-2x) = 0$

h) $\tan(x-\pi/3) = \cot(2x)$

i) $\sin 3x + \sin 2x = 0$.

Trigonometric Equations – 1 unknown

- These are equations of the form

$$f(\sin x)=0 \qquad f(\tan x)=0$$
$$f(\cos x)=0 \qquad f(\cot x)=0$$

and they can be solved by auxiliary substitution.

EXAMPLES

a) $(3+\cot x)^2 = 5(3+\cot x)$. (1)

Require $x \neq k\pi$.

Let $y=3+\cot x$. Then

$(1) \Leftrightarrow y^2 = 5y \Leftrightarrow y^2-5y=0 \Leftrightarrow y(y-5)=0 \Leftrightarrow$

$\Leftrightarrow y=0 \vee y=5$ (2)

Note that

$y=0 \Leftrightarrow 3+\cot x=0 \Leftrightarrow \cot x=-3 \Leftrightarrow x=k\pi+\text{Arccot}(-3)$

$y=5 \Leftrightarrow 3+\cot x=5 \Leftrightarrow \cot x=2 \Leftrightarrow x=k\pi+\text{Arccot}(2)$.

Both solutions are accepted. Thus

$(2) \Leftrightarrow x=k\pi+\text{Arccot}(-3) \vee x=k\pi+\text{Arccot}(2)$.

b) $\sin^3 x - 4\sin x = 0$ (1).

Let $y=\sin x$. Then

$(1) \Leftrightarrow y^3-4y=0 \Leftrightarrow y(y^2-4)=0 \Leftrightarrow y(y-2)(y+2)=0$

$\Leftrightarrow y=0 \vee y=2 \vee y=-2$. (2)

We note that:

$y=0 \Leftrightarrow \sin x = 0 \Leftrightarrow x = k\pi$

$y=2 \Leftrightarrow \sin x = 2 \Rightarrow \leftarrow$ no solutions

$y=-2 \Leftrightarrow \sin x = -2 \leftarrow$ no solutions.

Thus

$(2) \Leftrightarrow x = k\pi$.

$\hookrightarrow$ Note that since $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$ $\sin x = a$ has no solution when $a > 1$ or $a < -1$. Likewise, $\cos x = a$ has no solution when $a > 1$ or $a < -1$.

EXERCISE

2 Solve the equations:

a) $3\tan^2 x - 4\tan x + 1 = 0$

b) $2\cos^2 x = \sqrt{2}\cos x + 2$

c) $2\sin^2 x + \sqrt{3} = (2+\sqrt{3})\sin x$

d) $\tan^2 x - (1+\sqrt{3})\tan x + \sqrt{3} = 0$

e) $4\cos^4 x - 37\cos^2 x + 9 = 0$

▼ Trigonometric Equations – Multiple unknowns

- If possible, we use trigonometric identities to convert all terms into the same angle and the same trigonometric function.

EXAMPLE

a) $\cos 2x - \sin 3x = 1 \Leftrightarrow$

$\Leftrightarrow (1-2\sin^2 x) - (-4\sin^3 x + 3\sin x) = 1 \Leftrightarrow$

$\Leftrightarrow -2\sin^2 x + 4\sin^3 x - 3\sin x = 0 \quad (1)$

Let $y = \sin x$. Then

$(1) \Leftrightarrow -2y^2 + 4y^3 - 3y = 0 \Leftrightarrow 4y^3 - 2y^2 - 3y = 0 \Leftrightarrow$

$\Leftrightarrow y(4y^2 - 2y - 3) = 0 \Leftrightarrow$

$\Leftrightarrow y = 0 \vee 4y^2 - 2y - 3 = 0 \quad (2)$

Solve $4y^2 - 2y - 3 = 0$:

$\Delta = (-2)^2 - 4\cdot 4\cdot(-3) = 4 + 48 = 52 = 4\cdot 13 \Rightarrow$

$$\Rightarrow x_{1,2} = \frac{-(-2) \pm 2\sqrt{13}}{2\cdot 4} = \frac{1 \pm \sqrt{13}}{4}$$

Thus

$$(2) \Leftrightarrow y = 0 \vee y = \frac{1+\sqrt{13}}{4} \vee y = \frac{1-\sqrt{13}}{4} \quad (3)$$

Note that

$y = 0 \Leftrightarrow \sin x = 0 \Leftrightarrow x = k\pi$

$y = \frac{1+\sqrt{13}}{4} \Leftrightarrow \sin x = \frac{1+\sqrt{13}}{4} > 1 \leftarrow$ no solutions.

$$y = \frac{1-\sqrt{13}}{4} = \sin x \Leftrightarrow x = 2k\pi + \mathrm{Arcsin}\left(\frac{1-\sqrt{13}}{4}\right) \vee x = (2k+1)\pi - \mathrm{Arcsin}\left(\frac{1-\sqrt{13}}{4}\right)$$

b) $2\sin x + \tan x = 0 \quad (1)$

Require: $x \neq k\pi + \pi/2$

$(1) \Leftrightarrow 2\sin x + \dfrac{\sin x}{\cos x} = 0 \Leftrightarrow \sin x\left(2 + \dfrac{1}{\cos x}\right) = 0$

$\Leftrightarrow \sin x \cdot \dfrac{2\cos x + 1}{\cos x} = 0 \Leftrightarrow \sin x(2\cos x + 1) = 0$

$\Leftrightarrow \sin x = 0 \vee 2\cos x + 1 = 0 \quad (2)$

We note that:

$\sin x = 0 \Leftrightarrow x = k\pi \longleftarrow$ accepted

$2\cos x + 1 = 0 \Leftrightarrow \cos x = -\dfrac{1}{2} = -\cos\left(\dfrac{\pi}{3}\right) = \cos\left(\pi - \dfrac{\pi}{3}\right)$

$\Leftrightarrow \cos x = \cos\left(\dfrac{2\pi}{3}\right) \Leftrightarrow x = 2k\pi \pm \dfrac{2\pi}{3}$

accepted.

therefore:

$(2) \Leftrightarrow x = k\pi \vee x = 2k\pi \pm \dfrac{\pi}{3}$.

→ <u>Turning sums to products</u>

c) $\sin 5x - \sin 3x = \sin x \Leftrightarrow 2\sin\left(\dfrac{5x-3x}{2}\right)\cos\left(\dfrac{5x+3x}{2}\right) = \sin x$

$\Leftrightarrow 2\sin x\cos 4x - \sin x = 0 \Leftrightarrow \sin x(2\cos 4x - 1) = 0 \Leftrightarrow$

$\Leftrightarrow \sin x = 0 \vee \cos 4x = \dfrac{1}{2} = \cos\dfrac{\pi}{3} \Leftrightarrow$

$\Leftrightarrow x = k\pi \vee 4x = 2k\pi \pm \dfrac{\pi}{3} \Leftrightarrow x = k\pi \vee x = \dfrac{k\pi}{2} \pm \dfrac{\pi}{12}$.

Turn products to sums

d) $\sin(3x)\sin x = \dfrac{1}{2} \Leftrightarrow$

$$\Leftrightarrow \frac{1}{2}\left[\cos(3x-x) - \cos(3x+x)\right] = \frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow \cos 2x - \cos 4x = 1 \Leftrightarrow \cos 2x - (2\cos^2 2x - 1) = 1$$

$$\Leftrightarrow \cos 2x - 2\cos^2 2x = 0 \Leftrightarrow \cos 2x\,(1 - 2\cos 2x) = 0$$

$$\Leftrightarrow \cos 2x = 0 \vee \cos 2x = \frac{1}{2} = \cos\frac{\pi}{3} \Leftrightarrow$$

$$\Leftrightarrow 2x = k\pi + \frac{\pi}{2} \vee 2x = 2k\pi \pm \frac{\pi}{3} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{k\pi}{2} \pm \frac{\pi}{4} \vee x = k\pi \pm \frac{\pi}{6}$$

EXERCISES

③ Solve the following equations

a) $2\sin^2 x + \sqrt{3}\cos x + 1 = 0$

b) $\sin^2 2x - \sin^2 x = 1/2$

c) $\sin 2x = \sin^3 x$

d) $\cos 4x + 2\cos^2 x = 0$

e) $\sin 3x - \cos 2x = 0$

f) $\tan\left(\frac{\pi}{4} + x\right) + \tan x - 2 = 0$

g) $\sqrt{3}\tan x = 2\sin x$

④ Solve the following equations:
(Hint: turn sums to products or vice versa)

a) $\cos 2x + \cos x = \sin x + \sin 2x$

b) $\sin x + \sin 2x + \sin 3x = 0$

c) $2\cos x + \cos 3x + \cos 5x = 0$

d) $\cos 6x + \sin 5x = \sin 3x - \cos 2x$

e) $\cos x \cdot \cos 7x = \cos 3x \cos 5x$

f) $2\sin x \sin 3x = 1$

V Special types of trigonometric equations

① → $\boxed{a\sin x + b\cos x = c}$ (Linear Trigonometric)

These equations have solutions when $a^2+b^2 \geq c^2$ which can be obtained as follows:

$$a\sin x + b\cos x = c \Leftrightarrow \sin x + \frac{b}{a}\cos x = \frac{c}{a} \quad (1)$$

Let $\tan w = \frac{b}{a}$. Then

$$(1) \Leftrightarrow \sin x + \tan w \cos x = \frac{c}{a} \Leftrightarrow$$

$$\Leftrightarrow \sin x + \frac{\sin w}{\cos w}\cos x = \frac{c}{a} \Leftrightarrow$$

$$\Leftrightarrow \sin x \cos w + \sin w \cos x = \frac{c}{a}\cos w \Leftrightarrow$$

$$\Leftrightarrow \sin(x+w) = \frac{c}{a}\cos w \quad (2).$$

Let $\sin\vartheta = \frac{c}{a}\cos w$. Then $(2) \Leftrightarrow \sin(x+w) = \sin\vartheta$
$\Leftrightarrow \ldots$ etc.

To define ϑ we require $|(c/a)\cos w| \leq 1$.

Note that:

$$\left|\frac{c}{a}\cos w\right|^2 = \frac{c^2}{a^2}\cos^2 w = \frac{c^2}{a^2}\frac{1}{1+\tan^2 w} =$$

$$= \frac{c^2}{a^2}\frac{1}{1+(b/a)^2} = \frac{c^2}{a^2+b^2} \leq 1 \Leftrightarrow$$

$$\Leftrightarrow a^2+b^2 \geq c^2.$$

EXAMPLE

$$\sin 4x + \sqrt{3}\cos 4x = \sqrt{2} \Leftrightarrow \sin 4x + \tan(\pi/3)\cos 4x = \sqrt{2}$$

$$\Leftrightarrow \sin 4x \cos(\pi/3) + \sin(\pi/3)\cos 4x = \sqrt{2}\cos(\pi/3) \Leftrightarrow$$

$$\Leftrightarrow \sin(4x + \pi/3) = \sqrt{2}\cdot(1/2) = \frac{\sqrt{2}}{2} = \sin\left(\frac{\pi}{4}\right) \Leftrightarrow$$

$$\Leftrightarrow 4x + \pi/3 = 2k\pi + \frac{\pi}{4} \vee 4x + \pi/3 = (2k+1)\pi - \frac{\pi}{4} \Leftrightarrow$$

$$\Leftrightarrow 4x = 2k\pi - \frac{\pi}{12} \vee 4x = (2k+1)\pi - \frac{7\pi}{12}$$

$$\Leftrightarrow x = \frac{k\pi}{2} - \frac{\pi}{48} \vee x = \frac{(2k+1)\pi}{4} - \frac{7\pi}{48} .$$

② $\rightarrow$ $\boxed{a\sin^2 x + b\sin x\cos x + c\cos^2 x = 0}$ (Homogeneous)

If $\cos x = 0$, then the equation gives:

$a\sin^2 x = 0 \Leftrightarrow \sin x = 0$

which implies that $\sin^2 x + \cos^2 x = 0 \neq 1 \leftarrow$ Contradiction.
We may therefore assume that $\cos x \neq 0$ and divide the equation with $\cos^2 x$:

$$a\frac{\sin^2 x}{\cos^2 x} + b\frac{\sin x\cos x}{\cos^2 x} + c\frac{\cos^2 x}{\cos^2 x} = 0 \Leftrightarrow$$

$\Leftrightarrow a\tan^2 x + b\tan x + c = 0 \Leftrightarrow \dots$ etc.

③ $\rightarrow$ $\boxed{a\sin^2 x + b\sin x\cos x + c\cos^2 x = d}$ (Pseudohomogeneous)

Can be reduced to homogeneous as follows:

$a\sin^2 x + b\sin x\cos x + c\cos^2 x = d(\sin^2 x + \cos^2 x) \Leftrightarrow$

$\Leftrightarrow (a-d)\sin^2 x + b\sin x\cos x + (c-d)\cos^2 x = 0 \Leftrightarrow$

$\Leftrightarrow \dots$ etc.

EXAMPLE

$\sin^2 x + \sin 2x + 2\cos^2 x = \frac{1}{2} \Leftrightarrow$

$\Leftrightarrow \sin^2 x + 2\sin x\cos x + 2\cos^2 x = (1/2)(\sin^2 x + \cos^2 x)$

$\Leftrightarrow 2\sin^2 x + 4\sin x\cos x + 4\cos^2 x = \sin^2 x + \cos^2 x$

$\Leftrightarrow \sin^2 x + 4\sin x\cos x + 3\cos^2 x = 0 \Leftrightarrow$

$$\Leftrightarrow \left.\begin{array}{l} \tan^2 x + 4\tan x + 3 = 0 \\ \Delta = 16 - 4\cdot 3 = 16 - 12 = 4 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \tan x = \frac{-4 \pm 2}{2} = \begin{cases} -3 \\ -1 = -\tan(\pi/4) = \tan(-\pi/4) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \; x = k\pi + \mathrm{Arctan}(-3) \vee x = k\pi - \pi/4 .$$

④ → $\boxed{F(\sin x+\cos x,\ \sin x\cos x)=0}$

Let $y=\sin x+\cos x$. Then

$$y^2=\sin^2x+2\sin x\cos x+\cos^2x=$$
$$=1+2\sin x\cos x \Rightarrow \sin x\cos x=\frac{y^2-1}{2}$$

It follows that $F\left(y,\frac{y^2-1}{2}\right)=0 \Leftrightarrow \dots$ etc.

EXAMPLE

$\sin x+\cos x=\sin x\cos x+1$ (1).

Let $y=\sin x+\cos x$. Then

$$y^2=(\sin x+\cos x)^2=\sin^2x+2\sin x\cos x+\cos^2x=$$
$$=1+2\sin x\cos x \Rightarrow \sin x\cos x=\frac{y^2-1}{2}.$$

$$(1)\Leftrightarrow y=\frac{y^2-1}{2}+1 \Leftrightarrow 2y=y^2-1+2 \Leftrightarrow$$

$$\Leftrightarrow y^2-2y+1=0 \Leftrightarrow (y-1)^2=0 \Leftrightarrow y-1=0 \Leftrightarrow y=1$$

$$\Leftrightarrow \sin x+\cos x=1 \Leftrightarrow \sin x+\tan(\pi/4)\cos x=1 \Leftrightarrow$$
$$\Leftrightarrow \sin x\cos(\pi/4)+\sin(\pi/4)\cos x=\cos(\pi/4)$$
$$\Leftrightarrow \sin(x+\pi/4)=\sin(\pi/2-\pi/4) \Leftrightarrow \sin(x+\pi/4)=\sin(\pi/4)$$
$$\Leftrightarrow x+\pi/4=2k\pi+\pi/4 \vee x+\pi/4=(2k+1)\pi-\pi/4$$
$$\Leftrightarrow x=2k\pi \vee x=(2k+1)\pi-\pi/2 .$$

EXERCISES

(5) Solve the following equations:

a) $3\sin x - \sqrt{3}\cos x = 3$
b) $\sin 4x + \sqrt{3}\cos 4x = \sqrt{2}$
c) $\sin x + \cos x = 1$
d) $2\sin x + 3\cos x = 1$
e) $5\sin^2 x - 3\sin x\cos x - 2\cos^2 x = 0$
f) $\cos^2 x + 4\sin 2x + 3 = 0$
g) $\sin^2 x + \sin 2x - 2\cos^2 x = 1/2$
h) $\sin x + \cos x = 1 + \sin x\cos x$
i) $2\sin x + 2\cos x - 4\sin x\cos x = 1$
j) $\dfrac{1}{\sin x} + \dfrac{1}{\cos x} = 2\sqrt{2}$
k) $\sin x - \cos x + \sin x\cos x = 1.$

Solving trigonometric equations in an interval

To solve a trigonometric equation in an interval (a,b) or $(a,b]$ or $[a,b)$ or $[a,b]$, we work as follows:

- 1 Find the general solutions in terms of $k \in \mathbb{Z}$.
- 2 Require that x belongs to the interval and derive a corresponding inequality for k
- 3 List the solutions that satisfy the inequality for k.

EXAMPLE

$$\tan\left(\frac{\pi}{4}+x\right)-\tan\left(\frac{\pi}{4}-x\right)=2\sqrt{3} \qquad (1)$$

Find all solutions in the interval $[0,\pi]$.

Solution

We require

$$\begin{cases} \frac{\pi}{4}+x \neq k\pi+\frac{\pi}{2} \\ \frac{\pi}{4}-x \neq k\pi+\frac{\pi}{2} \end{cases} \Longleftrightarrow \begin{cases} x \neq k\pi+\pi/4 \\ x \neq k\pi-\pi/4 \end{cases}$$

Let $y=\tan x$. We note that

$$\tan\left(\frac{\pi}{4}+x\right)=\frac{\tan(\pi/4)+\tan x}{1-\tan(\pi/4)\tan x}=\frac{1+\tan x}{1-\tan x}=\frac{1+y}{1-y}$$

$$\tan\left(\frac{\pi}{4}-x\right)=\frac{\tan(\pi/4)-\tan x}{1+\tan(\pi/4)\tan x}=\frac{1-\tan x}{1+\tan x}=\frac{1-y}{1+y}$$

$$(1)\Leftrightarrow \frac{1+y}{1-y}-\frac{1-y}{1+y}=2\sqrt{3}\Leftrightarrow$$

$$\Leftrightarrow (1+y)^2-(1-y)^2=2\sqrt{3}\,(1-y)(1+y)$$

$$\Leftrightarrow 1+2y+y^2-1+2y-y^2=2\sqrt{3}-y^2\cdot 2\sqrt{3}$$

$$\Leftrightarrow 4y=2\sqrt{3}-(2\sqrt{3})y^2\Leftrightarrow$$

$$\Leftrightarrow (2\sqrt{3})y^2+4y-2\sqrt{3}=0\Leftrightarrow$$

$$\left.\begin{array}{l}\Leftrightarrow \sqrt{3}\,y^2+2y-\sqrt{3}=0\\ \Delta=4-4\sqrt{3}\cdot(-\sqrt{3})=\\ =4+12=16=4^2\end{array}\right\}\Rightarrow y_{1,2}=\frac{-2\pm 4}{2\sqrt{3}}\Rightarrow$$

$$\Rightarrow y_1=\frac{-6}{2\sqrt{3}}=\frac{-3}{\sqrt{3}}=-\sqrt{3}\quad\text{or}$$

$$y_2=\frac{2}{2\sqrt{3}}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$$

We note that

$$y=-\sqrt{3}\Leftrightarrow \tan x=-\sqrt{3}=-\tan\left(\frac{\pi}{3}\right)=\tan\left(\frac{-\pi}{3}\right)\Leftrightarrow$$

$$\Leftrightarrow x=k\pi-\frac{\pi}{3}\leftarrow \text{accepted}$$

and

$$y=\frac{\sqrt{3}}{3}\Leftrightarrow \tan x=\frac{\sqrt{3}}{3}=\tan\left(\frac{\pi}{6}\right)\Leftrightarrow x=k\pi+\frac{\pi}{6}$$

accepted.

Now we require that $x \in [0, \pi]$:

a) For $x = k\pi + \pi/3$:

$$0 \leq k\pi - \pi/3 \leq \pi \Leftrightarrow 0 \leq k - 1/3 \leq 1 \Leftrightarrow$$

$$\Leftrightarrow 1/3 \leq k \leq 4/3 \Leftrightarrow k = 1$$

$* \longrightarrow$ (k is an integer).

Thus: $x = \pi - \pi/3 = 2\pi/3$

b) For $K = k\pi + \pi/6$

$$0 \leq k\pi + \pi/6 \leq \pi \Leftrightarrow 0 \leq k + 1/6 \leq 1 \Leftrightarrow$$

$$\Leftrightarrow -1/6 \leq k \leq 5/6 \Leftrightarrow k = 0$$

Thus $x = 0\pi + \pi/6 = \pi/6$.

Thus solution set in $[0, \pi]$ is: $S = \{\pi/6, 2\pi/3\}$

EXERCISES

(6) Solve the following equation in $[-\pi, \pi]$

$$\cos(2x) + 3\cos x = 0$$

(7) Solve $\sin(3x) + \sin(5x) = \sin(8x)$ in $[0, 2\pi)$.

(8) Solve $4\cos^4 x - 37\cos^2 x + 9 = 0$ in $(\pi/2, 3\pi/2]$.

(9) Solve $\cos^2 x + 4\sin 2x + 3 = 0$ in $(2\pi, 3\pi)$

(10) Solve $\sqrt{3}\cos x - 3\sin x = 3$ in $[\pi, 3\pi]$

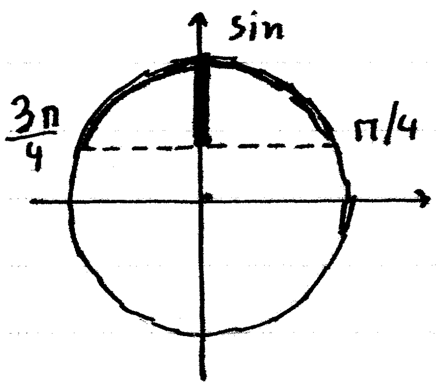
Trigonometric Inequalities

The solution of trigonometric inequalities in the $[0, 2\pi]$ interval can be visualized on the trigonometric circle. These solutions can then be generalized by adding $2k\pi$ for sin or cos and $k\pi$ for tan or cot.

EXAMPLES

a) $2\sin(3x-1) - \sqrt{2} \geq 0 \Leftrightarrow \sin(3x-1) \geq \frac{\sqrt{2}}{2} \Leftrightarrow$

$\Leftrightarrow \sin(3x-1) \geq \sin\left(\frac{\pi}{4}\right) \Leftrightarrow$



$\Leftrightarrow 2k\pi + \pi/4 \leq 3x - 1 \leq 2k\pi + 3\pi/4$

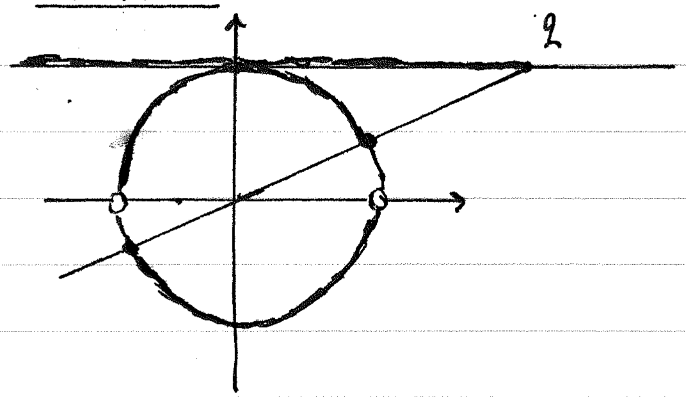
$\Leftrightarrow 2k\pi + \pi/4 + 1 \leq 3x \leq 2k\pi + 3\pi/4 + 1$

$\Leftrightarrow \frac{2k\pi}{3} + \frac{\pi}{12} + \frac{1}{3} \leq x \leq \frac{2k\pi}{3} + \frac{3\pi}{12} + \frac{1}{3}$

$\Leftrightarrow \frac{2k\pi}{3} + \frac{\pi + 4}{12} \leq x \leq \frac{2k\pi}{3} + \frac{3\pi + 4}{12}.$

B) $\cot(x+\pi/3) \leqslant 2$

<u>Solution</u>



$$\cot(x+\pi/3) \leqslant 2 \Leftrightarrow \cot(x+\pi/3) \leqslant \cot(\operatorname{Arccot}(2))$$

$$\Leftrightarrow \operatorname{Arccot}(2) + k\pi \leqslant x + \frac{\pi}{3} < \pi + k\pi \Leftrightarrow$$

$$\Leftrightarrow \operatorname{Arccot}(2) + k\pi - \frac{\pi}{3} \leqslant x < \pi + k\pi - \frac{\pi}{3} \Leftrightarrow$$

$$\Leftrightarrow (\operatorname{Arccot}(2) - \pi/3) + k\pi \leqslant x < \frac{2\pi}{3} + k\pi$$

EXERCISES

(11) Solve the following inequalities

a) $\sin(3x) > \sqrt{2}/2$

b) $\cos(2x) \leq \sqrt{3}/2$

c) $\tan x \geq \sqrt{3}/3$

d) $\cot 4x < \sqrt{3}$

e) $\cos 4x \leq -\sqrt{3}/2$

f) $\sqrt{2}\cos x \leq 1$

(12) Solve the following inequalities

a) $\sin(x-\pi/6) > 0$

b) $\cos(2x+\pi/3) \leq 1/2$

c) $\tan(3x-\pi/4) < 0$

d) $\cot(x+\pi/3) \leq 1$

e) $\tan(x/3) > \sqrt{3}/3$

f) $\sin(x+2\pi/3) > -1/2$

g) $-1/2 < \sin 3x < \sqrt{2}/2$

h) $-\sqrt{2}/2 < \cos 2x < 1/2$

i) $2\sin x + 1 > 0$

j) $2\cos(2x/5) < 1$

k) $2\cos(x+\pi/6) - \sqrt{2} > 0$

l) $2\cos(x-\pi/3) > -\sqrt{3}$

m) $3\tan 2x - \sqrt{3} \leq 0$

n) $\cot(3x) + \sqrt{3} > 0$

o) $\tan(x-\pi/4) - 1 > 0$

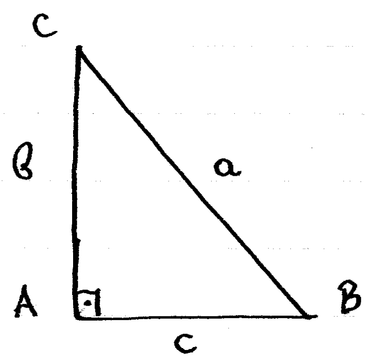
p) $-\sqrt{3}/3 < \tan 5x < 1$

q) $-\sqrt{3} < \cot 3x < 1$

PRE5: Application to Triangles

APPLICATION TO TRIANGLES

Right Triangles

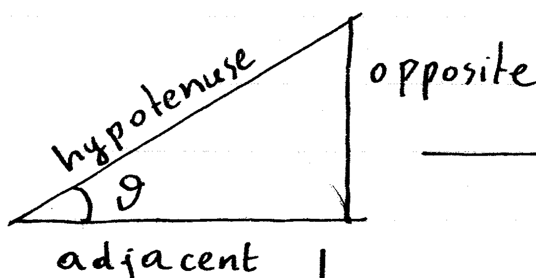


- $\boxed{\hat{A} = 90^\circ \Leftrightarrow \hat{B} + \hat{C} = 90^\circ}$
- $A = 90^\circ \Leftrightarrow a^2 = b^2 + c^2$

  OR

  $A = 90^\circ \Leftrightarrow BC^2 = AC^2 + BC^2$

Mnemonic Rule for trig. relations



$\sin\vartheta = \dfrac{opp}{hyp}$	$\tan\vartheta = \dfrac{opp}{adj}$
$\cos\vartheta = \dfrac{adj}{hyp}$	$\cot\vartheta = \dfrac{adj}{opp}$

$\sin B = \dfrac{b}{a} = \cos C$
$\cos B = \dfrac{c}{a} = \sin C$
$\tan B = \dfrac{b}{c} = \cot C$
$\cot B = \dfrac{c}{b} = \tan C$

$b = a\sin B = a\cos C$

$c = a\cos B = a\sin C$

$b = c\tan B = c\cot C$

$c = b\cot B = b\tan C$

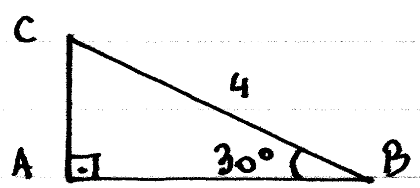
Solving right triangles

Given $A = 90^\circ$ and two other élements, with one of them being a side, it is possible to calculate all other elements of the triangle.
By elements we mean:
a) the angles: A, B, C
b) The sides: a, b, c

EXAMPLES

1) Hypotenuse + Angle:

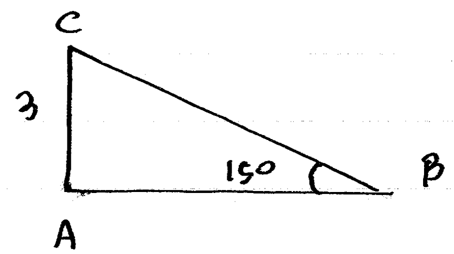
Given: $B = 30^\circ$, $A = 90^\circ$, $a = 4$.



$b = AC = BC \sin B = 4 \sin 30^\circ = 4 \cdot (1/2) = 2$
$c^2 = a^2 - b^2 = 4^2 - 2^2 = 16 - 4 = 12 = 4 \cdot 3 \Rightarrow c = 2\sqrt{3}$.
$C = 90 - B = 90^\circ - 30^\circ = 60^\circ$

2) Side + Angle

Given: $B = 15^\circ$, $A = 90^\circ$, $b = 3$



$$C = 90 - B = 90^\circ - 15^\circ = 75^\circ$$

$$\sin B = \frac{AC}{BC} = \frac{3}{a} \Rightarrow a = \frac{3}{\sin 15^\circ} \quad (1)$$

Note that

$$\sin^2 15^\circ = \frac{1 - \cos 30^\circ}{2} = \frac{1 - \sqrt{3}/2}{2} = \frac{2-\sqrt{3}}{4} \Rightarrow$$

$$\Rightarrow \sin 15^\circ = \frac{\sqrt{2-\sqrt{3}}}{2} \quad (2)$$

From (1) and (2):

$$a = \frac{3}{\frac{\sqrt{2-\sqrt{3}}}{2}} = \frac{6}{\sqrt{2-\sqrt{3}}} = \frac{6\sqrt{2-\sqrt{3}}}{2-\sqrt{3}} =$$

$$= \frac{6(2+\sqrt{3})\sqrt{2-\sqrt{3}}}{(2+\sqrt{3})(2-\sqrt{3})} = \frac{6(2+\sqrt{3})\sqrt{2-\sqrt{3}}}{4-3} =$$

$$= 6(2+\sqrt{3})\sqrt{2-\sqrt{3}} \quad (3)$$

From (3) and $b = 3$:

$$c^2 = a^2 - b^2 = \left[6(2+\sqrt{3})\sqrt{2-\sqrt{3}}\right]^2 - 3^2 =$$

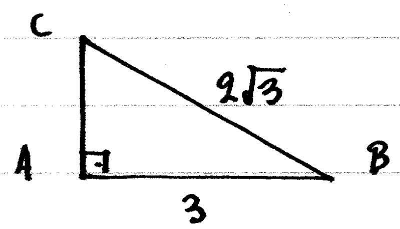
$$= 36(2+\sqrt{3})^2(2-\sqrt{3}) - 9 = 36(2+\sqrt{3})(4-3) - 9$$

$$= 36(2+\sqrt{3}) - 9 = 9\left[4(2+\sqrt{3}) - 1\right] = 9\left[7 + 4\sqrt{3}\right] \Rightarrow$$

$$\Rightarrow c = 3\sqrt{7+4\sqrt{3}}$$

3) Side + Hypotenuse

Given: $A = 90°$, $a = 2\sqrt{3}$, $c = 3$



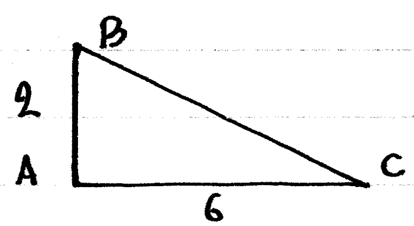
$b^2 = a^2 - c^2 = (2\sqrt{3})^2 - 3^2 = 4 \cdot 3 - 9 =$
$= 12 - 9 = 3 \Rightarrow \underline{b = \sqrt{3}}$

$\cos B = \dfrac{AB}{BC} = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2} = \cos 30° \Rightarrow \underline{B = 30°}$

$\underline{C} = 90 - B = 90 - 30 = \underline{60°}$

4) Side + Side

Given: $A = 90°$, $b = 6$, $c = 2$



$a^2 = b^2 + c^2 = 6^2 + 2^2 = 36 + 4 = 40 \Rightarrow \underline{a = 2\sqrt{10}}$

$\tan B = \dfrac{AC}{AB} = \dfrac{b}{c} = \dfrac{6}{2} = 3 \Rightarrow \underline{B = \text{Arctan}(3).}$

$\underline{C} = 90 - B = \underline{90 - \text{Arctan}(3).}$

EXERCISES

(1) Solve the following right triangles with $A = 90°$:

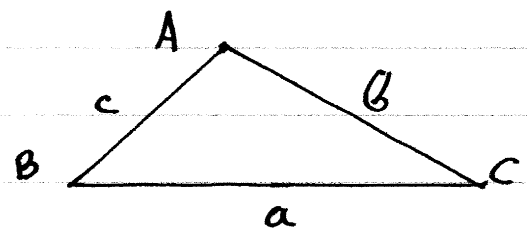
a) $b = 3$, $c = 4$
b) $B = 60°$, $a = 2$
c) $B = 45°$, $b = 3$
d) $C = 15°$, $a = 1$
e) $a = 2\sqrt{3}$, $b = 3$
f) $b = 1+\sqrt{2}$, $a = \sqrt{6}$

$\hookrightarrow$ To check your answers, use a calculator to confirm that your results satisfy Mollweide's identity:

$$\boxed{\frac{b-c}{a}\cos\left(\frac{A}{2}\right) = \sin\left(\frac{B-C}{2}\right)}$$

General Triangles

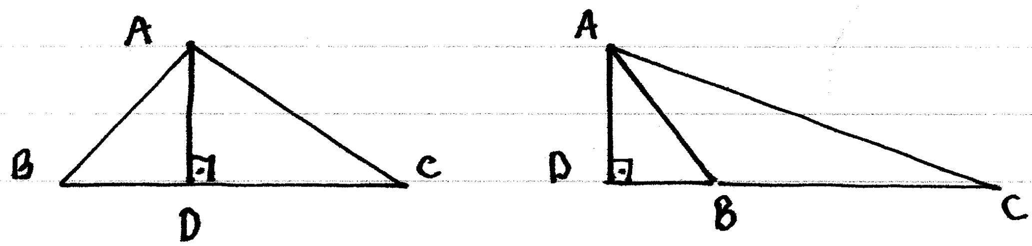
Consider an arbitrary triangle $\overset{\Delta}{ABC}$.



$a = BC$, $b = CA$, $c = AB$.

① Law of sines $\longrightarrow$ $\boxed{\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}}$

Proof



- Bring the height AD with $AD \perp BC$.

From $\overset{\Delta}{ADC}$: $AD = AC \cdot \sin C = b \sin C$ (1)

From $\overset{\Delta}{ADB}$: $AD = AB \cdot \sin B = c \sin B$ (2)

From (1) and (2):

$$b \sin C = c \sin B \Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C}$$

Similarly we get $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$. □

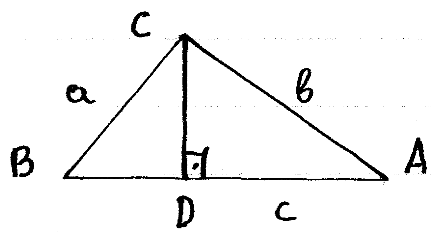
- We now use the projection laws to prove the law of cosines.

Projection laws

$$\boxed{\begin{aligned} c &= a\cos B + b\cos A \\ a &= b\cos C + c\cos B \\ b &= c\cos A + a\cos C \end{aligned}}$$

Proof

Case 1: $B \leqslant \pi/2$ (acute angle)



▶ Bring the height $CD \perp AB$ with $D \in AB$.

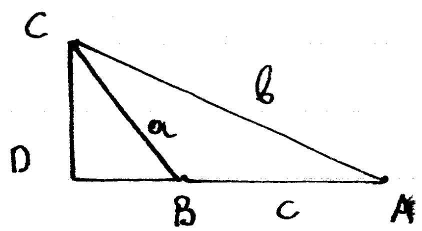
From $B\hat{D}C$: $BD = BC\cos B = a\cos B$. (1)

From $C\hat{D}A$: $AD = AC\cos A = b\cos A$ (2)

From (1) and (2):

$c = AB = BD + AD = a\cos B + b\cos A$.

Case 2: $B > \pi/2$



▶ Bring the height $CD \perp AB$ with $B \in AD$.

From $B\hat{D}C$: $BD = BC\cos(C\hat{B}D) = a\cos(\pi - B) =$

$= -a\cos B$. (3)

From $C\hat{D}A$: $AD = AC\cos A = b\cos A$ (4)

From (3) and (4):

$c = AB = AD - BD = b\cos A - (-a\cos B) =$

$= a\cos B + b\cos A$.

Repeat argument for the other two equations.

② Law of cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = c^2 + a^2 - 2ca\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Proof

$$\begin{aligned}
b^2 + c^2 &= b(c\cos A + a\cos C) + c(a\cos B + b\cos A) \\
&= bc\cos A + ab\cos C + ac\cos B + bc\cos A \\
&= 2bc\cos A + (ab\cos C + ac\cos B) = \\
&= 2bc\cos A + a(b\cos C + c\cos B) \\
&= 2bc\cos A + a^2 \Rightarrow
\end{aligned}$$

$$\Rightarrow a^2 = b^2 + c^2 - 2bc\cdot\cos A$$

Repeat argument for the other two equations.

Solving general triangles

- We use the law of sines when given:
  a) 1 side + 2 angles
  b) 2 sides + angle not between them
- We use the law of cosines when given
  c) 3 sides
  d) 2 sides + angle between them.
- We also note that for any triangle angle, A, we have $0 < A < \pi$, and therefore:
  $\sin A = x \Leftrightarrow A = \text{Arcsin}(x) \vee A = \pi - \text{Arcsin}(x)$
  $\cos A = x \Leftrightarrow A = \text{Arccos}(x)$.
- When solving $\sin A = x$ we use the following triangle property to accept or reject solutions.
  $a < b \Leftrightarrow A < B$
  $b < c \Leftrightarrow B < C$
  $c < a \Leftrightarrow C < A$ etc.

EXAMPLES

a) 2 sides + angle not between them. (one solution)

Given: $a = 5$, $b = 6$, $B = 60°$.

Since: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} \Rightarrow$

$$\Rightarrow \sin A = \frac{a \sin B}{b} = \frac{5 \sin 60°}{6} = \frac{5 \cdot (\sqrt{3}/2)}{6} = \frac{5\sqrt{3}}{12} \Rightarrow$$

$$\Rightarrow A = \text{Arcsin}\left(\frac{5\sqrt{3}}{12}\right) \vee A = \pi - \text{Arcsin}\left(\frac{5\sqrt{3}}{12}\right) \Rightarrow$$

$$\Rightarrow A \cong 46^\circ \vee A \approx 180 - 46 = 134. \quad (1)$$

Since $\left.\begin{array}{l} a < b \Rightarrow A < B \\ B = 60^\circ \end{array}\right\} \Rightarrow \underline{A \cong 46^\circ}$ (one solution)

$$\underline{C} = 180 - A - B \cong 180 - 46 - 60 = \underline{74^\circ}$$

Finally:

$$\frac{c}{\sin C} = \frac{b}{\sin B} \Rightarrow$$

$$\Rightarrow c = \frac{b \sin C}{\sin B} = \frac{6 \sin 74^\circ}{\sin 60^\circ} \approx 6.66$$

Thus: $a = 5 \quad A \cong 46^\circ$

$b = 6 \quad B = 60^\circ$

$c \cong 6.66 \quad C \cong 74^\circ$.

b) <u>2 sides + angle not between them</u> (2 solutions)

Given: $A = 30^\circ$, $a = 2$, $b = 3$

$$\frac{b}{\sin B} = \frac{a}{\sin A} \Rightarrow$$

$$\Rightarrow \sin B = \frac{b \sin A}{a} = \frac{3 \sin 30^\circ}{2} = \frac{3 \cdot (1/2)}{2} = \frac{3}{4} \Rightarrow$$

$$\Rightarrow B = \text{Arcsin}(3/4) \cong 48^\circ \vee B \cong 180^\circ - 48^\circ = 132^\circ$$

Since $a < b \Rightarrow A < B \Rightarrow 30 < B$

Both solution for B are valid, thus there are two possible triangles.

c) 2 sides + angle not between them (no solution)

Given: $A = 45^0$, $a = 2$, $b = 8$

$$\frac{b}{\sin B} = \frac{a}{\sin A} \Rightarrow$$

$$\Rightarrow \sin B = \frac{b \sin A}{a} = \frac{8 \sin 45^0}{2} = 4 \cdot \frac{\sqrt{2}}{2} = 2\sqrt{2} > 1$$

thus no triangle is possible.

d) 3 sides

Given $a = \sqrt{3}/2$, $b = \sqrt{2}/2$, $c = (\sqrt{6} + \sqrt{2})/4$

Note that:

$a^2 = 3/4$ and $b^2 = 2/4 = 1/2$ and

$$c^2 = \frac{(\sqrt{6} + \sqrt{2})^2}{16} = \frac{6 + 2\sqrt{12} + 2}{16} = \frac{8 + 4\sqrt{3}}{16} =$$

$$= \frac{2 + \sqrt{3}}{4}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{\frac{1}{2} + \frac{2+\sqrt{3}}{4} - \frac{3}{4}}{\cancel{2} \cdot \frac{\sqrt{2}}{\cancel{2}} \cdot \frac{\sqrt{6} + \sqrt{2}}{4}} =$$

$$= \frac{2 + 2 + \sqrt{3} - 3}{\sqrt{2}(\sqrt{6} + \sqrt{2})} = \frac{1 + \sqrt{3}}{2 + \sqrt{12}} = \frac{1 + \sqrt{3}}{2 + 2\sqrt{3}} =$$

$$= \frac{1 + \sqrt{3}}{2(1 + \sqrt{3})} = \frac{1}{2} = \cos 60^0 \Rightarrow \underline{A = 60^0}.$$

$$\cos B = \frac{c^2+a^2-b^2}{2ac} = \frac{\frac{2+\sqrt{3}}{4}+\frac{3}{4}-\frac{2}{4}}{\cancel{2}\cdot\frac{\sqrt{3}}{\cancel{2}}\cdot\frac{\sqrt{6}+\sqrt{2}}{4}} =$$

$$= \frac{2+\sqrt{3}+3-2}{\sqrt{3}\,(\sqrt{6}+\sqrt{2})} = \frac{3+\sqrt{3}}{\sqrt{3}\,(\sqrt{6}+\sqrt{2})} = \frac{\sqrt{3}\,(\sqrt{3}+1)}{\sqrt{3}\sqrt{2}\,(\sqrt{3}+1)} =$$

$$= \frac{1}{\sqrt{2}} = \cos 45^\circ \Rightarrow B = 45^\circ.$$

$$C = 180^\circ - A - B = 180^\circ - 60^\circ - 45^\circ = 75^\circ.$$

e) <u>2 sides + angle in between</u>

Given: $a=2$, $b=3$, $C=30^\circ$

$$c^2 = a^2+b^2-2ab\cos C = 2^2+3^2-2\cdot 2\cdot 3\cdot \cos 30^\circ =$$

$$= 4+9-12\cdot(\sqrt{3}/2) = 13-6\sqrt{3} \Rightarrow$$

$$\Rightarrow c = \sqrt{13-6\sqrt{3}}$$

$$\cos B = \frac{c^2+a^2-b^2}{2ac} = \frac{(13-6\sqrt{3})+2^2-3^2}{2\cdot 2\cdot\sqrt{13-6\sqrt{3}}} =$$

$$= \frac{13-6\sqrt{3}+4-9}{4\sqrt{13-6\sqrt{3}}} = \frac{8-6\sqrt{3}}{4\sqrt{13-6\sqrt{3}}} =$$

$$= \frac{4-3\sqrt{3}}{2\sqrt{13-6\sqrt{3}}} \Rightarrow B = \operatorname{Arccos}\left(\frac{4-3\sqrt{3}}{2\sqrt{13-6\sqrt{3}}}\right)$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{3^2+(13-6\sqrt{3})-2^2}{2\cdot 3\cdot\sqrt{13-6\sqrt{3}}} =$$

$$= \frac{9+13-6\sqrt{3}-4}{6\sqrt{13-6\sqrt{3}}} = \frac{18-6\sqrt{3}}{6\sqrt{13-6\sqrt{3}}} =$$

$$= \frac{3-\sqrt{3}}{\sqrt{13-6\sqrt{3}}} \Rightarrow A = \mathrm{Arccos}\left(\frac{3-\sqrt{3}}{\sqrt{13-6\sqrt{3}}}\right)$$

EXERCISES

(2) Solve the following general triangles $A\hat{B}C$:

a) $a=1$, $b=3$, $B=30°$
b) $a=2$, $b=1$, $B=75°$
c) $a=3$, $b=4$, $B=45°$
d) $a=3$, $b=4$, $c=5$
e) $a=2$, $b=\sqrt{6}$, $c=1+\sqrt{3}$
f) $A=60°$, $B=45°$, $a=5$
g) $a=3$, $b=\sqrt{2}$, $C=45°$
h) $a=1$, $b=\sqrt{3}$, $C=60°$

→ To confirm your answer use a calculator to verify that it satisfies the Mollweide identity:

$$\boxed{\frac{b-c}{a}\cos\left(\frac{A}{2}\right)=\sin\left(\frac{B-C}{2}\right)}$$

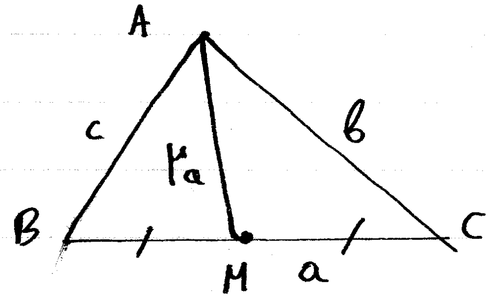
(3) Consider a triangle $A\hat{B}C$. Let AD be the bisector of the angle A with D a point on BC. Show that

$$\frac{DB}{DC}=\frac{AB}{AC}$$

(Hint: Use the law of sines to calculate DB, DC

(4) Let $\overset{\triangle}{ABC}$ be a triangle and let AM be a median with M on BC such that $BM = CM$. If $\mu_a = AM$, show that

$$b^2 + c^2 = 2\mu_a^2 + \frac{a^2}{2}.$$



(Hint: Use law of cosines to calculate μ_a).

(5) Show the Mollweide identities; for any triangle

a) $$\frac{b-c}{a}\cos\left(\frac{A}{2}\right) = \sin\left(\frac{B-C}{2}\right)$$

b) $$\frac{b+c}{a}\sin\left(\frac{A}{2}\right) = \cos\left(\frac{B-C}{2}\right)$$

c) $$\frac{b-c}{b+c} = \tan\left(\frac{B-C}{2}\right)\tan\left(\frac{A}{2}\right)$$

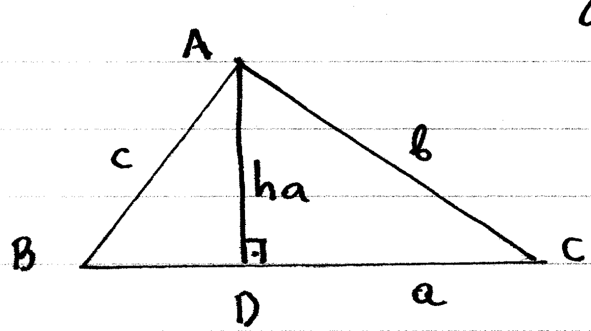
(Hint: Use law of sines to write a, b, c in terms of $\sin A, \sin B, \sin C$. Then use the sum to product identities)

$\hookrightarrow$ The identity in 5c is the lesser-known <u>law of the tangents.</u>

d) $$\frac{b^2-c^2}{a^2}\sin A = \sin(B-C)$$

▼ Area of triangles

Let $\overset{\triangle}{ABC}$ be a triangle with heights h_a, h_b, h_c.



It is well-known that the area of $\overset{\triangle}{ABC}$ is given by

$$\boxed{A = \frac{a h_a}{2} = \frac{b h_b}{2} = \frac{c h_c}{2}}$$

We note that: $h_a = c \sin B$
$h_b = a \sin C$
$h_c = b \sin A$

and therefore

$$\boxed{A = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ac \sin B}$$

We will now show that

$$\boxed{\begin{array}{l} A = \sqrt{s(s-a)(s-b)(s-c)} \\ 2s = a+b+c \end{array}}$$ (Heron's formula)

Proof

$$A = \frac{1}{2} ac \sin B \Rightarrow$$

$$\Rightarrow A^2 = \frac{1}{4} a^2 c^2 \sin^2 B = \frac{1}{4} a^2 c^2 (1 - \cos^2 B) =$$

$$= \frac{1}{4} a^2 c^2 (1 - \cos B)(1 + \cos B) =$$

$$= \frac{1}{4} a^2 c^2 \left[1 - \frac{a^2 + c^2 - b^2}{2ac}\right]\left[1 + \frac{a^2 + c^2 - b^2}{2ac}\right]$$

$$= \frac{1}{4} \frac{a^2 c^2}{(2ac)^2} (2ac - a^2 - c^2 + b^2)(2ac + a^2 + c^2 - b^2)$$

$$= \frac{1}{16} [b^2 - (a-c)^2][(a+c)^2 - b^2] =$$

$$= \frac{1}{16} [b - a + c][b + a - c][a + c - b][a + c + b]$$

$$= \frac{(a+b+c)}{2} \frac{(-a+b+c)}{2} \frac{(a-b+c)}{2} \frac{(a+b-c)}{2}$$

Note that

$$s - a = \frac{a+b+c}{2} - a = \frac{a+b+c-2a}{2} = \frac{-a+b+c}{2}$$

$$s - b = \frac{a-b+c}{2} \quad \text{and} \quad s - c = \frac{a+b-c}{2}$$

thus $A^2 = s(s-a)(s-b)(s-c) \Rightarrow$

$$\Rightarrow A = \sqrt{s(s-a)(s-b)(s-c)} \qquad \square$$

EXERCISES

⑥ Find the area of triangles with

a) $a=1$, $b=2$, $c=2$

b) $a=2$, $b=4$, $c=3$

c) $a=1$, $b=2$, $C=60°$

d) $a=2$, $b=1$, $C=45°$

⑦ Show that for any triangle $\hat{ABC}$:

a) $\sin\left(\frac{B}{2}\right)=\sqrt{\frac{(s-c)(s-a)}{ca}}$ (Hint: Use cos2a identities and the law of cosines)

b) $\cos\left(\frac{B}{2}\right)=\sqrt{\frac{s(s-b)}{ca}}$

⑧ Consider a triangle $\hat{ABC}$ and let AD be the bisector of the angle A with D on BC. Use the result of exercise 3 to show that

a) $DB=\frac{ac}{b+c}$ and $DC=\frac{bc}{b+c}$

b) $\delta_a=AD=\frac{ac}{b+c}\frac{2\sin(B/2)\cos(B/2)}{\sin(A/2)}$

(Hint: Use law of sines on $\hat{ABD}$)

c) $\delta_a=\frac{2\sqrt{bc}}{b+c}\cdot\sqrt{s(s-a)}$

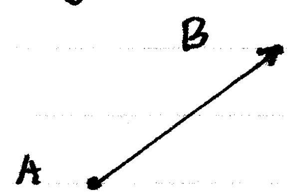
(Hint: Use exercise 7)

PRE6: Vectors

VECTORS

Definitions

- A vector is a line segment with an established direction. If A, B are two points, then $\vec{AB}$ represents the vector defined by the line segment AB with direction from A to B.

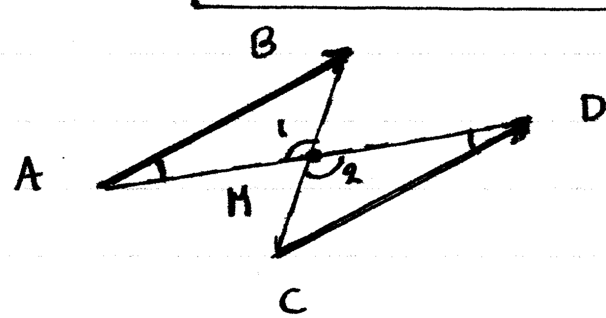


A = initial point
B = terminal point

Vector Equality

Def : Let $\vec{AB}$, $\vec{CD}$ be two vectors. Let $M = AD \cap BC$. We then define vector equality as follows:

$$\boxed{\vec{AB} = \vec{CD} \Leftrightarrow \begin{cases} AM = MD \\ BM = MC \end{cases}}$$



- interpretation: If $\vec{AB} = \vec{CD}$ then $AB = CD$ and $AB \parallel CD$ and $\vec{AB}$ and $\vec{CD}$ have "the same direction".

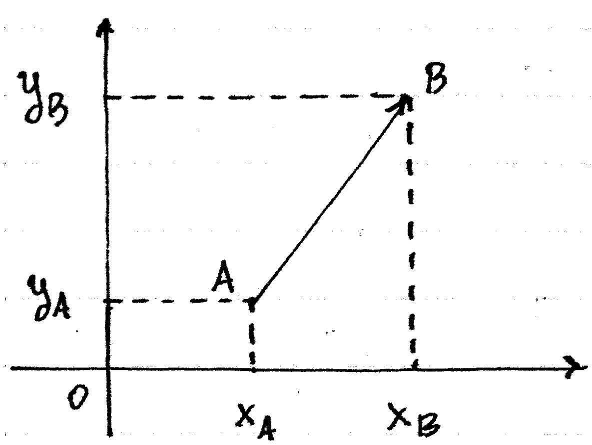
<u>Prop</u> : $\boxed{\vec{AB} = \vec{CD} \Rightarrow AB = CD \wedge AB \parallel CD}$

<u>Proof</u>

Let $\hat{M}_1 = A\hat{M}B$ and $\hat{M}_2 = C\hat{M}D$.
Let $\hat{C} = B\hat{C}D$ and $\hat{D} = A\hat{D}C$.
By definition : $\vec{AB} = \vec{CD} \Rightarrow AM = MD \wedge BM = MC$ (1)
We also note that : $\hat{M}_1 = \hat{M}_2$ (vertical angles) (2)
From (1) and (2) : $A\overset{\Delta}{M}B = C\overset{\Delta}{M}D$ (3)
From (3) : <u>$AB = CD$.</u>
From (3) : $\hat{A} = \hat{D} \Rightarrow$ <u>$AB \parallel CD$</u> (equal interior alternating angles) □

- <u>Vector representation</u>

Consider a cartesian coordinate system with axis $x'Ox$ and $y'Oy$. Let $\vec{AB}$ be a vector with



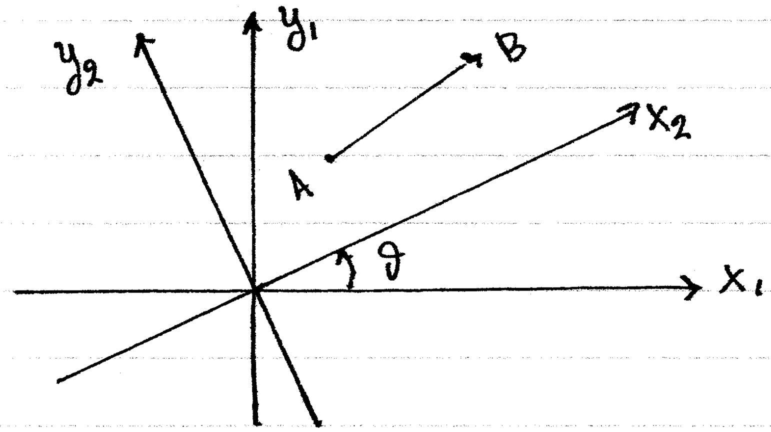
$A(x_A, y_A)$ and $B(x_B, y_B)$.
We represent :
$\vec{AB} = (x_B - x_A, y_B - y_A)$
Note that the same vector may have different representations in different coordinate systems.

• Zero vector

We define the zero vector as $\vec{0} = (0,0)$ for any coordinate system. Note that for any point A:

$$\vec{AA} = (x_A - x_A, y_A - y_A) = (0,0) = \vec{0}.$$

• Rotation of coordinate system



Consider a coordinate system consisting of an x_1-axis and y_1-axis. We define a new coordinate system, by rotating counterclockwise by angle ϑ; consisting of an x_2-axis and y_2-axis.

Let $\vec{AB}$ be a vector. If

$\vec{AB} = (x_1, y_1)$ in the x_1y_1 coordinate system

$\vec{AB} = (x_2, y_2)$ in the x_2y_2 coordinate system

then

$$\boxed{\begin{aligned} x_2 &= x_1 \cos\vartheta + y_1 \sin\vartheta \\ y_2 &= -x_1 \sin\vartheta + y_1 \cos\vartheta \end{aligned}}$$

We write equivalently : $\boxed{(x_2,y_2) = R(\vartheta)(x_1,y_1)}$

It can be shown that

$$\boxed{R(\vartheta_1)R(\vartheta_2)(x_1,y_1) = R(\vartheta_1+\vartheta_2)(x_1,y_1)}$$

- <u>Magnitude of vector</u>

- Let $\vec{a} = (a_1, a_2)$ be a vector. We define:

$$\boxed{|\vec{a}| = \sqrt{a_1^2 + a_2^2}}$$

- $|\vec{a}|$ represents the <u>length</u> of the vector $\vec{a}$. It follows that for two points A, B:

$$|\vec{AB}| = |\vec{BA}| = AB.$$

- We note that $|\vec{a}|$ is invariant under rotation:

<u>Thm</u> : $\boxed{|R(\vartheta)\vec{a}| = |\vec{a}|}$

<u>Proof</u>

Let $\vec{a} = (a_1, a_2)$ and $R(\vartheta)\vec{a} = (b_1, b_2)$. It follows that

$b_1 = a_1\cos\vartheta + a_2\sin\vartheta$

$b_2 = -a_1\sin\vartheta + a_2\cos\vartheta$

and therefore:

$$b_1^2 + b_2^2 = (a_1\cos\vartheta + a_2\sin\vartheta)^2 + (-a_1\sin\vartheta + a_2\cos\vartheta)^2 =$$

$$= \underline{a_1^2\cos^2\vartheta} + 2a_1a_2\cos\vartheta\sin\vartheta + \underline{\underline{a_2^2\sin^2\vartheta}} + \underline{a_1^2\sin^2\vartheta}$$

$$- 2a_1a_2\cos\vartheta\sin\vartheta + \underline{\underline{a_2^2\cos^2\vartheta}} =$$

$$= a_1^2(\cos^2\vartheta + \sin^2\vartheta) + a_2^2(\cos^2\vartheta + \sin^2\vartheta) =$$

$$= a_1^2 + a_2^2 \Rightarrow$$

$$\Rightarrow |R(\theta)\vec{a}| = \sqrt{b_1^2 + b_2^2} = \sqrt{a_1^2 + a_2^2} = |\vec{a}|. \quad \square$$

EXAMPLE

a) For $\vec{a} = (\sqrt{2}-1, \sqrt{2}+1)$, evaluate $|\vec{a}|$.

Solution

$$|\vec{a}| = \sqrt{(\sqrt{2}-1)^2 + (\sqrt{2}+1)^2} =$$

$$= \sqrt{2 - 2\sqrt{2} + 1 + 2 + 2\sqrt{2} + 1} = \sqrt{6} \quad \square$$

b) Rotate the vector $\vec{a} = (2,1)$ by $-30°$

Solution

Rotate the axis in the opposite direction: $+30°$!!

Let $(x,y) = R(30°)\vec{a} = R(30°)(2,1)$. Then

$$x = 2\cos 30° + 1\sin 30° = 2(\sqrt{3}/2) + 1\cdot(1/2) =$$

$$= \sqrt{3} + 1/2 = \frac{2\sqrt{3}+1}{2}$$

$$y = -2\sin 30° + 1\cos 30° = -2\cdot(1/2) + 1\cdot(\sqrt{3}/2) =$$

$$= -1 + \sqrt{3}/2 = \frac{\sqrt{3}-2}{2}. \text{ Thus } R(30°)\vec{a} = \left(\frac{2\sqrt{3}+1}{2}, \frac{\sqrt{3}-2}{2}\right).$$

EXERCISES

① Let A, B, C be three points with $A(2,1)$, $B(3,3)$, $C(1,5)$.

a) Evaluate $|\vec{AB}|$.

b) Rotate $\vec{BC}$ by 45°.

c) Rotate $\vec{AC}$ by 15°.

② Evaluate $|\vec{a}|$ with

a) $\vec{a} = \left(\sqrt{2+\sqrt{2}}, \sqrt{2-\sqrt{2}}\right)$

b) $\vec{a} = (3+\sqrt{2}, 3-\sqrt{2})$

c) $\vec{a} = (2, 1-\sqrt{2})$

d) $\vec{a} = (2+3\sqrt{2}, 1-\sqrt{2})$

③ Let $A(1+\sqrt{2}, 1-\sqrt{2})$ and $B(1-\sqrt{2}, 1+\sqrt{2})$
Rotate $\vec{AB}$ by 30°.

④ Let $A(2,-1)$ and $B(-1,-1)$.
Rotate $\vec{AB}$ by 15°.

↳ To rotate a vector by angle ϑ we must rotate the axises by angle $-\vartheta$. Thus to rotate $\vec{a}$ by angle ϑ we calculate
$\vec{b} = R(-\vartheta)\,\vec{a}$.

▼ Vector operations

We define 3 vector operations:

a) Vector sum
b) Scalar product
c) Inner product (dot product).

● Vector sum

Let $\vec{a}, \vec{b}$ be vectors with $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$. Then we define

$$\boxed{\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2)}$$

▶ Properties

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$	commutative
$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$	associative
$\vec{a} + \vec{0} = \vec{a}$	neutral element

We also define $\underline{-\vec{a} = (-a_1, -a_2)}$ and therefore

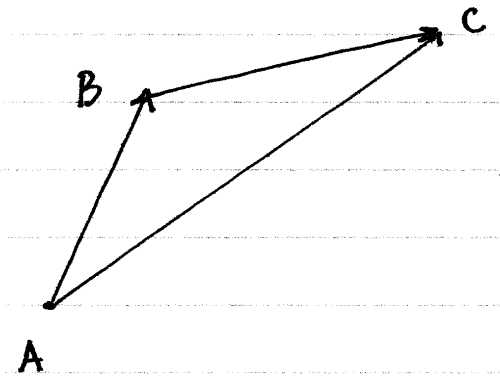
$\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$	inverse element

▶ Geometric interpretation

Thm : For any three points A, B, C:

$$\boxed{\vec{AB} + \vec{BC} = \vec{AC}}$$

Proof



Let $A(x_A, y_A)$,
$B(x_B, y_B)$,
$C(x_C, y_C)$.
Then:

$$\vec{AB} + \vec{BC} = (x_B - x_A, y_B - y_A) + (x_C - x_B, y_C - y_B) =$$
$$= (x_B - x_A + x_C - x_B, y_B - y_A + y_C - y_B) =$$
$$= (x_C - x_A, y_C - y_A) = \vec{AC}. \qquad \square$$

● Scalar product

Let $\vec{a} = (a_1, a_2)$. Then we define:

$$\boxed{\lambda \vec{a} = (\lambda a_1, \lambda a_2), \ \forall \lambda \in \mathbb{R}}$$

▶ Properties

$\lambda(\vec{a}+\vec{b}) = \lambda\vec{a} + \lambda\vec{b}$	distributive
$(\lambda+\mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$	distributive
$(\lambda\mu)\vec{a} = \lambda(\mu\vec{a}) = \mu(\lambda\vec{a})$	associative
$1\vec{a} = \vec{a}$	neutral element
$0\vec{a} = \mathbf{0}$	
$\lambda\vec{\mathbf{0}} = \mathbf{0}$	

▶ We also define:

$$\vec{a} - \vec{b} = \vec{a} + (-1)\vec{b} = (a_1 - b_1, a_2 - b_2).$$

▶ Define the unit vectors: $\vec{i} = (1,0)$ and $\vec{j} = (0,1)$.

Then:

$$\vec{a} = (a_1, a_2) = a_1\vec{i} + a_2\vec{j}.$$

EXAMPLES

a) If $\vec{a}=(2,1)$ and $\vec{b}=(3,2)$, evaluate $\vec{c}=2\vec{a}+3\vec{b}$.

Solution

$$\vec{c}=2\vec{a}+3\vec{b}=2(2,1)+3(3,2)=$$
$$=(4,2)+(9,6)=(4+9,2+6)=(13,8).$$

b) If $\vec{a}=(x+1,y)$ and $\vec{b}=(x-1,x+y)$, find all x,y such that $\vec{a}-2\vec{b}=\mathbf{0}$.

Solution

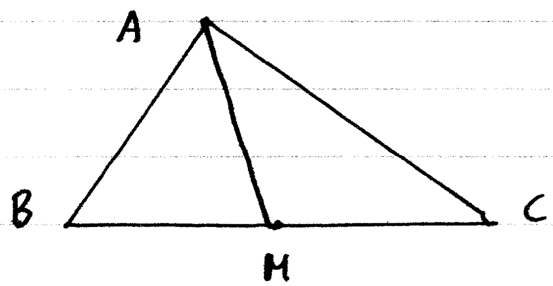
$$\vec{a}-2\vec{b}=(x+1,y)-2(x-1,x+y)=$$
$$=(x+1,y)+(-2x+2,-2x-2y)$$
$$=(x+1-2x+2,\,y-2x-2y)=$$
$$=(-x+3,\,-2x-y)$$

It follows that

$$\vec{a}-2\vec{b}=\mathbf{0}\Leftrightarrow\begin{cases}-x+3=0\\-2x-y=0\end{cases}\Leftrightarrow\begin{cases}x=3\\-2\cdot3-y=0\end{cases}\Leftrightarrow$$
$$\Leftrightarrow\begin{cases}x=3\\-6-y=0\end{cases}\Leftrightarrow\begin{cases}x=3\\y=-6\end{cases}$$
$$\Leftrightarrow(x,y)=(3,-6).$$

c) Let $\overset{\triangle}{ABC}$ be a triangle and let M be the midpoint of BC. Show that
$$\vec{AM} = (1/2)(\vec{AB} + \vec{AC}).$$

Solution



M midpoint of $BC \Rightarrow BM = (1/2)\,BC \Rightarrow$
$$\Rightarrow \vec{BM} = (1/2)\,\vec{BC}.$$

It follows that

$$\vec{AM} = \vec{AB} + \vec{BM} = \vec{AB} + (1/2)\vec{BC} = \vec{AB} + (1/2)(\vec{BA} + \vec{AC})$$
$$= \vec{AB} + (1/2)(-\vec{AB} + \vec{AC}) =$$
$$= (1 - 1/2)\vec{AB} + (1/2)\vec{AC} = (1/2)\vec{AB} + (1/2)\vec{AC} =$$
$$= (1/2)(\vec{AB} + \vec{AC}).$$

EXERCISES

(5) Given the vectors

$$\vec{a} = (\sqrt{3} - 2, \sqrt{3} + 2)$$
$$\vec{b} = (\sqrt{3} + 1, \sqrt{3} - 1)$$

evaluate:

$$\vec{c} = (\sqrt{3} - 1)(\vec{a} + \vec{b}).$$

(6) Given the vectors

$$\vec{a} = (x + y\sqrt{2}, x - y\sqrt{2})$$
$$\vec{b} = (x - y\sqrt{3}, x + y\sqrt{3})$$
$$\vec{c} = (1, 2)$$

find all values of $x, y \in \mathbb{R}$ such that

$$2\vec{a} - \vec{b} = \vec{c}.$$

(7) Let $\vec{a}, \vec{b}$ be two vectors. Let O, A, B, C be points such that

$$\vec{OA} = \vec{a} + \vec{b}, \quad \vec{OB} = 2\vec{a} + 3\vec{b}, \quad \vec{OC} = 5\vec{a} + 9\vec{b}.$$

Show that $\vec{AB} = 4\vec{AC}$.

(8) Let $\hat{ABC}$ be a triangle with $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$. Find the coordinates of the point G that satisfies

$$\vec{GA} + \vec{GB} + \vec{GC} = \vec{0}.$$

(9) Let $\overset{\Delta}{ABC}$ be a triangle. If D is the midpoint of AB and E the midpoint of AC, show that

$$\vec{DE} = (1/2)\vec{AB}$$

(10) Let $\overset{\Delta}{ABC}$ be a triangle. If D is the midpoint of BC, E the midpoint of CA, F the midpoint of AB, then show that

$$\vec{AD} + \vec{BE} + \vec{CF} = \vec{0}$$

(Hint: First show that $\vec{AD} = (1/2)(\vec{AB} + \vec{AC})$, etc.)

Inner Product

Let $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ be two vectors.
We define

$$\boxed{\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2}$$

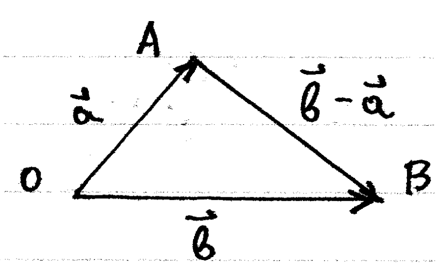
Properties

$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$	commutative
$(\lambda \vec{a}) \cdot \vec{b} = \vec{a} \cdot (\lambda \vec{b}) = \lambda (\vec{a} \cdot \vec{b})$	associative
$(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$	distributive
$\lvert a \rvert^2 = \vec{a} \cdot \vec{a}$	norm

inner product theorem

- Let $\vec{a}, \vec{b}$ be two vectors and let ϑ be the angle between $\vec{a}$ and $\vec{b}$. Then:

$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos\vartheta}$$



Proof

Let $\vec{a} = \vec{OA}$ and $\vec{b} = \vec{OB}$. From the law of cosines on $O\hat{A}B$:

$$|\vec{b}-\vec{a}|^2 = AB^2 = OA^2 + OB^2 - 2OA \cdot OB \cdot \cos\vartheta =$$
$$= |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\vartheta \qquad (1)$$

Also note that:

$$|\vec{b}-\vec{a}|^2 = (\vec{b}-\vec{a})\cdot(\vec{b}-\vec{a}) =$$
$$= \vec{b}\cdot\vec{b} - \vec{b}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{a} =$$
$$= |b|^2 + |a|^2 - 2(\vec{a}\cdot\vec{b}) \qquad (2)$$

From (1) and (2):

$$|a|^2 + |b|^2 - 2|\vec{a}||\vec{b}|\cos\vartheta = |\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a}\cdot\vec{b}) \Rightarrow$$
$$\Rightarrow -2(\vec{a}\cdot\vec{b}) = -2|\vec{a}||\vec{b}|\cos\vartheta \Rightarrow$$
$$\Rightarrow \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\vartheta \qquad \square$$

It follows that the angle ϑ between two vectors $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ satisfies:

$$\boxed{\cos\vartheta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{a_1 b_1 + a_2 b_2}{\sqrt{a_1^2 + a_2^2}\,\sqrt{b_1^2 + b_2^2}}}$$

- Orthogonal vectors

Thm: $\boxed{\vec{a} \perp \vec{b} \Leftrightarrow \vec{a}\cdot\vec{b} = 0}$

Proof

$$\vec{a} \perp \vec{b} \Leftrightarrow \vartheta = \pi/2 \vee \vartheta = 3\pi/2 \Leftrightarrow \cos\vartheta = 0 \Leftrightarrow$$
$$\Leftrightarrow \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = 0 \Leftrightarrow \vec{a}\cdot\vec{b} = 0 \qquad \square$$

EXAMPLES

a) If $\vec{a} = (3,1)$ and $\vec{b} = (2,4)$, then evaluate $\lambda = (\vec{a} - \vec{b}) \cdot \vec{b}$.

Solution

$$\lambda = (\vec{a} - \vec{b}) \cdot \vec{b} = [(3,1) - (2,4)] \cdot (2,4) =$$
$$= (3-2, 1-4) \cdot (2,4) = (1,-3) \cdot (2,4) =$$
$$= 1 \cdot 2 + (-3) \cdot 4 = 2 - 12 = -10.$$

b) If $\vec{a} = (1,2)$ and $\vec{b} = (2,3)$, then find $\cos\theta$ of the angle θ between $\vec{a}$ and $\vec{b}$.

Solution

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{(1,2)\cdot(2,3)}{|(1,2)| \cdot |(2,3)|} =$$
$$= \frac{1 \cdot 2 + 2 \cdot 3}{\sqrt{1^2+2^2}\ \sqrt{2^2+3^2}} = \frac{2+6}{\sqrt{1+4}\ \sqrt{4+9}} =$$
$$= \frac{8}{\sqrt{5}\ \sqrt{13}} = \frac{8}{\sqrt{65}} = \frac{8\sqrt{65}}{65}$$

c) Find all x such that $\vec{a} = (x, x+1)$ and $\vec{b} = (x+1, 3)$ are orthogonal.

Solution

We note that:

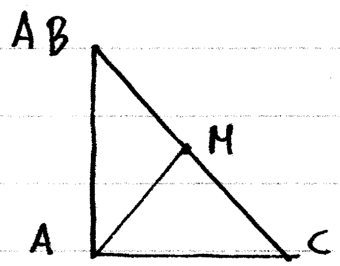
$$\vec{a}\cdot\vec{b} = (x, x+1)\cdot(x+1, 3) =$$
$$= x(x+1) + (x+1)3 = (x+1)(x+3).$$

It follows that

$$\vec{a}\perp\vec{b} \Leftrightarrow \vec{a}\cdot\vec{b} = 0 \Leftrightarrow (x+1)(x+3) = 0 \Leftrightarrow$$
$$\Leftrightarrow x+1 = 0 \vee x+3 = 0 \Leftrightarrow$$
$$\Leftrightarrow x = -1 \vee x = -3.$$

d) Consider a triangle $A\hat{B}C$ with $\hat{A} = 90°$. Let M be the midpoint of BC. Show that $AM = BC/2$.

<u>Solution</u>



Since $\hat{A} = 90° \Rightarrow AB \perp AC \Rightarrow$
$$\Rightarrow \overrightarrow{AB}\cdot\overrightarrow{AC} = 0 \quad (1)$$

We also recall that
$$\overrightarrow{AM} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$$

Note that

$$|\overrightarrow{AB} + \overrightarrow{AC}|^2 = (\overrightarrow{AB} + \overrightarrow{AC})\cdot(\overrightarrow{AB} + \overrightarrow{AC}) =$$
$$= \overrightarrow{AB}\cdot\overrightarrow{AB} + 2(\overrightarrow{AB}\cdot\overrightarrow{AC}) + \overrightarrow{AC}\cdot\overrightarrow{AC} =$$
$$= |\overrightarrow{AB}|^2 + 2\cdot 0 + |\overrightarrow{AC}|^2 = AB^2 + AC^2 =$$
$$= BC^2 \Rightarrow |\overrightarrow{AB} + \overrightarrow{AC}| = BC \Rightarrow$$
$$\Rightarrow AM = |\overrightarrow{AM}| = \left|\frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})\right| = \frac{1}{2}|\overrightarrow{AB} + \overrightarrow{AC}| =$$
$$= \frac{BC}{2}.$$

EXERCISES

(11) Evaluate $\vec{a}\cdot\vec{b}$ given

a) $\vec{a} = (1+\sqrt{2}, 1-\sqrt{3})$
$\vec{b} = (1-\sqrt{2}, 1+\sqrt{3})$

b) $\vec{a} = (x+y, 2x)$, $\vec{b} = (x+y, -y)$

c) $\vec{a} = (x+y, 3xy)$, $\vec{b} = (x+y)(x+y, -1)$

(12) Show that $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2$.

(13) Let $\vec{a}, \vec{b}$ such that $|\vec{a}| = 2$ and $|\vec{b}| = 3$ and let $\theta = \pi/3$ be the angle from $\vec{a}$ to $\vec{b}$. Show that:

$$(\vec{a}-2\vec{b})\cdot(3\vec{a}+2\vec{b}) = -21.$$

(14) If $|\vec{a}| = 1$ and $|\vec{b}| = \sqrt{2}$ and the angle from $\vec{a}$ to $\vec{b}$ is $\theta = 3\pi/4$, then evaluate $|\vec{c}|$ with $\vec{c} = 3\vec{a} - 2\vec{b}$.

(15) Let $\vec{a} = (2,1)$ and $\vec{b} = (2+\sqrt{3}, 1-2\sqrt{3})$. Show that the angle θ between $\vec{a}$ and $\vec{b}$ satisfies $\cos\theta = 1/2$.

(16) If $|\vec{a}| = |\vec{b}| = 1$ and θ is the angle from $\vec{a}$ to $\vec{b}$ is $\theta = 2\pi/3$, show that the angle from $\vec{c} = 2\vec{a}+\vec{b}$ to $\vec{d} = \vec{a} - 2\vec{b}$ satisfies $\cos\varphi = \sqrt{21}/14$.

(17) Let $\vec{a} = ((x-1)\sqrt{3}, 2x)$ and $\vec{b} = (-\sqrt{3}, 1)$.
If ϑ is the angle from $\vec{a}$ to $\vec{b}$ show that
$\cos\vartheta = 1/2 \Leftrightarrow x = \pm 1$.

(18) Given the points $A(-2,2)$ and $B(1,1)$, find a point C on the y-axis such that $AC \perp BC$.

(19) Let (c) be a circle with center O. Let AB be a diameter and let C be another point on the circle. Show that $AC \perp CB$.

(20) Let $\vec{a}, \vec{b}, \vec{c}$ be vectors. Show that:

a) $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}| \Rightarrow \vec{a} \perp \vec{b}$.

b) $\vec{a} \perp [(\vec{a}\cdot\vec{b})\vec{c} - (\vec{a}\cdot\vec{c})\vec{b}]$

c) $\vec{a} \perp (\vec{b}-\vec{c})$ and $\vec{b} \perp (\vec{c}-\vec{a}) \Rightarrow \vec{c} \perp (\vec{a}-\vec{b})$

PRE7: Sequences and series

INTRODUCTION TO SERIES

Sequences and series

Recall that:

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$
$$\mathbb{N}^* = \{1, 2, 3, \dots\}$$

Definition : Any function $a : \mathbb{N} \to \mathbb{R}$ or $a : \mathbb{N}^* \to \mathbb{R}$ is called a real sequence (or just sequence) and we write:

$$a_n = a(n) \ , \forall n \in \mathbb{N}$$

Defining a sequence

There are two methods for defining a sequence (a_n) :

1) Directly → We provide a formula for directly calculating a_n.

e.g. $a_n = \dfrac{(-1)^n}{2n}$, $\forall n \in \mathbb{N}$.

2) Recursively → We define the first few terms of the sequence and a recursive formula give the next term in terms of previous terms.

e.g.: $(a_n):\begin{cases} a_1=2 \\ a_{n+1}=3a_n-1 \end{cases}$

e.g.: $(a_n):\begin{cases} a_1=1 \wedge a_2=1 \\ a_n=a_{n-1}+a_{n-2} \end{cases}$ ← Fibonacci sequence.

Series

A series is a sequence s_n defined via a partial sum of the terms of a sequence a_n. For example:

$$s_n = a_1+a_2+\cdots+a_n.$$

▸ Notation: $\boxed{\sum_{n=p}^{q} a_n = a_p+a_{p+1}+\cdots+a_q}$

We note that:

$$\sum_{n=p}^{q}(a_n+b_n)=\sum_{n=p}^{q}a_n+\sum_{n=p}^{q}b_n$$

$$\sum_{n=p}^{q}(a_n-b_n)=\sum_{n=p}^{q}a_n-\sum_{n=p}^{q}b_n$$

$$\sum_{n=p}^{q}ca_n=c\sum_{n=p}^{q}a_n$$

Basic Sums

$S_1(n) = \sum_{k=1}^{n} k = 1+2+\cdots+n = \dfrac{n(n+1)}{2}$
$S_2(n) = \sum_{k=1}^{n} k^2 = 1^2+2^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$
$S_3(n) = \sum_{k=1}^{n} k^3 = 1^3+2^3+\cdots+n^3 = \dfrac{n^2(n+1)^2}{4} = [S_1(n)]^2$

Proof

▶ For $S_1(n)$

We note that $(x+1)^2 = x^2+2x+1$.

For $x=1: \; 2^2 = 1^2+2\cdot 1+1$

$x=2: \; 3^2 = 2^2+2\cdot 2+1$

$\vdots$

$x=n: \; (n+1)^2 = n^2+2n+1$

Add the equations above:

$$[2^2+3^2+\cdots+(n+1)^2] = [1^2+2^2+\cdots+n^2]+2S_1(n)+n \Leftrightarrow$$

$$\Leftrightarrow (n+1)^2 = 1+2S_1(n)+n \Leftrightarrow$$

$$\Leftrightarrow 2S_1(n) = (n+1)^2-1-n = n^2+2n+1-1-n =$$

$$= n^2+n = n(n+1) \Leftrightarrow$$

$$\Leftrightarrow S_1(n) = \frac{n(n+1)}{2}.$$

- **For $S_2(n)$**

We note that $(x+1)^3 = x^3+3x^2+3x+1$

For $x=1$: $2^3 = 1^3+3\cdot 1^2+3\cdot 1+1$

$x=2$: $3^3 = 2^3+3\cdot 2^2+3\cdot 2+1$

$\vdots$

$x=n$: $(n+1)^3 = n^3+3n^2+3n+1$

Add the equations above:

$[2^3+\cdots+(n+1)^3] = [1^3+\cdots+n^3]+3S_2(n)+3S_1(n)+n \Leftrightarrow$

$\Leftrightarrow (n+1)^3 = 1+3S_2(n)+3S_1(n)+n \Leftrightarrow$

$$\Leftrightarrow 3S_2(n) = (n+1)^3-1-3S_1(n)-n$$

$$= (n+1)^3 - 3\frac{n(n+1)}{2} - (n+1) =$$

$$= (n+1)\left[(n+1)^2 - \frac{3n}{2} - 1\right] =$$

$$= (n+1)\left[n^2+2n+1-\frac{3n}{2}-1\right] =$$

$$= (n+1)\left(n^2+\frac{n}{2}\right) = n(n+1)\left(n+\frac{1}{2}\right) =$$

$$= \frac{1}{2}n(n+1)(2n+1) \Leftrightarrow$$

$$\Leftrightarrow S_2(n) = \frac{n(n+1)(2n+1)}{6}.$$

▸ For $S_3(n)$

We note that $(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$

$x=1: \; 2^4 = 1^4 + 4\cdot 1^3 + 6\cdot 1^2 + 4\cdot 1 + 1$

$x=2: \; 3^4 = 2^4 + 4\cdot 2^3 + 6\cdot 2^2 + 4\cdot 2 + 1$

$\vdots$

$x=n: \; (n+1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1$

Adding the above equations:

$$2^4 + \cdots + (n+1)^4 = [1^4 + \cdots + n^4] + 4S_3(n) + 6S_2(n) + 4S_1(n) + n$$

$$\Leftrightarrow (n+1)^4 = 1 + 4S_3(n) + 6S_2(n) + 4S_1(n) + n$$

$$\Leftrightarrow 4S_3(n) = (n+1)^4 - (n+1) - 6S_2(n) - 4S_1(n) =$$

$$= (n+1)^4 - (n+1) - 6\frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} =$$

$$= (n+1)^4 - (n+1) - n(n+1)(2n+1) - 2n(n+1)$$

$$= (n+1)[(n+1)^3 - 1 - n(2n+1) - 2n] =$$

$$= (n+1)[(n+1)^3 - n(2n+1) - (2n+1)] =$$

$$= (n+1)[(n+1)^3 - (n+1)(2n+1)] =$$

$$= (n+1)(n+1)[(n+1)^2 - (2n+1)]$$

$$= (n+1)^2[n^2 + 2n + 1 - 2n - 1] = n^2(n+1)^2 \Leftrightarrow$$

$$\Leftrightarrow S_3(n) = \frac{n^2(n+1)^2}{4} = [S_1(n)]^2. \qquad \square$$

EXAMPLES

a) $S_n = 1\cdot 3 + 2\cdot 5 + 3\cdot 7 + \cdots + n(2n+1)$

Solution

$$S_n = 1\cdot 3 + 2\cdot 5 + 3\cdot 7 + \cdots + n(2n+1) = \sum_{K=1}^{n} K(2K+1) =$$

$$= \sum_{K=1}^{n} (2K^2 + K) = 2\sum_{K=1}^{n} K^2 + \sum_{K=1}^{n} K =$$

$$= 2S_2(n) + S_1(n) = 2\,\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} =$$

$$= n(n+1)\left[\frac{2n+1}{3} + \frac{1}{2}\right] = \frac{1}{6}\,n(n+1)[2(2n+1)+3]$$

$$= \frac{n(n+1)(4n+2+3)}{6} = \frac{n(n+1)(4n+5)}{6}.$$

b) $S_n = 1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3$

Solution

$$S_n = 1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = \sum_{K=1}^{n} (2K-1)^3 =$$

$$= \sum_{K=1}^{n} (8K^3 - 3(2K)^2 + 3(2K) - 1) =$$

$$= \sum_{K=1}^{n} (8K^3 - 12K^2 + 6K - 1) =$$

$$= 8S_3(n) - 12S_2(n) + 6S_1(n) - n =$$

$$= 8\frac{n^2(n+1)^2}{4} - 12\frac{n(n+1)(2n+1)}{6} + 6\frac{n(n+1)}{2} - n =$$

$$= 2n^2(n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n =$$

$$= n(n+1)[2n(n+1) - 2(2n+1) + 3] - n =$$

$$= n(n+1)[2n^2 + 2n - 4n - 2 + 3] - n$$

$$= n(n+1)(2n^2 - 2n + 1) - n$$

Application to arithmetic series

Def: $\boxed{\left.\begin{array}{l}(a_n) \text{ arithmetic} \\ \text{sequence}\end{array}\right\} \Leftrightarrow \forall n \in \mathbb{N} : a_{n+1} = a_n + c}$

- It is easy to see that if (a_n) is an arithmetic sequence, then

$$\boxed{a_n = a_1 + (n-1)c \ , \forall n \in \mathbb{N}}$$

Thm: $\boxed{\left.\begin{array}{l}(a_n) \text{ arithmetic} \\ \text{sequence}\end{array}\right\} \Rightarrow \sum_{k=1}^{n} a_k = \frac{n(a_1 + a_n)}{2}}$

Proof

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} [a_1 + (k-1)c] = a_1 n + c \sum_{k=1}^{n} (k-1)$$

$$= a_1 n + c \sum_{k=0}^{n-1} k = a_1 n + c S_1(n-1) =$$

$$= a_1 n + \frac{(n-1)[(n-1)+1]}{2} \cdot c = a_1 n + \frac{cn(n-1)}{2} =$$

$$= \frac{a_1 n}{2} + \frac{a_1 n}{2} + \frac{cn(n-1)}{2} =$$

$$= \frac{a_1 n}{2} + \frac{n}{2}[a_1 + c(n-1)] = \frac{a_1 n}{2} + \frac{n}{2} \cdot a_n =$$

$$= \frac{n(a_1 + a_n)}{2}. \qquad \square$$

EXERCISES

① Show that:

a) $1\cdot 2+2\cdot 3+\cdots+n(n+1)=(1/3)\,n(n+1)(n+2)$
b) $1\cdot 2+2\cdot 5+\cdots+n(3n-1)=n^2(n+1)$
c) $1^2+3^2+\cdots+(2n-1)^2=(1/3)n(2n-1)(2n+1)$
d) $1^3+3^3+\cdots+(2n-1)^3=n^2(2n^2-1)$
e) $1\cdot 2^2+2\cdot 3^2+\cdots+n(n+1)^2=(1/12)\,n(n+1)(n+2)(3n+5)$
f) $1^2\cdot 2+2^2\cdot 3+\cdots+n^2(n+1)=(1/12)n(n+1)(n+2)(3n+1)$
g) $1^2\cdot 3+2^2\cdot 5+\cdots+n^2(2n+1)=(1/6)\,n(n+1)(3n^2+5n+1)$
h) $1\cdot 3^2+2\cdot 5^2+\cdots+n(2n+1)^2=(1/6)n(n+1)(6n^2+14n+7)$

Geometric sums

$$\boxed{G_a(n) = 1 + a + a^2 + \dots + a^n = \frac{1-a^{n+1}}{1-a}}$$

Proofs

We note that

$$G_a(n) = 1 + a + a^2 + \dots + a^n \qquad (1)$$

$$aG_a(n) = a + a^2 + a^3 + \dots + a^{n+1} \qquad (2)$$

Subtract (2) from (1):

$$G_a(n) - aG_a(n) = (1 + a + \dots + a^n) - (a + a^2 + \dots + a^{n+1})$$

$$= 1 - a^{n+1} \Rightarrow$$

$$\Rightarrow (1-a)\, G_a(n) = 1 - a^{n+1} \Rightarrow$$

$$\Rightarrow G_a(n) = \frac{1-a^{n+1}}{1-a} \qquad \square$$

Application to geometric sequences

Def: (a_n) geometric sequence $\Leftrightarrow \forall n \in \mathbb{N}^*: a_{n+1} = \lambda a_n$

It follows that

$$\boxed{a_n = a_1 \lambda^{n-1}, \ \forall n \in \mathbb{N}^*}$$

Thm: $\boxed{\begin{array}{l}(a_n) \text{ geometric} \Rightarrow S_n = a_1 + \cdots + a_n = \\ \text{sequence} \qquad\qquad\quad = \dfrac{a_1(1-\lambda^n)}{1-\lambda}\end{array}}$

Proof

$$S_n = a_1 + \cdots + a_n = \sum_{k=1}^{n} a_1 \lambda^{k-1} = a_1 \sum_{k=1}^{n} \lambda^{k-1} =$$

$$= a_1 \sum_{k=0}^{n-1} \lambda^k = a_1 G_\lambda(n-1) = a_1 \frac{1-\lambda^n}{1-\lambda} =$$

$$= \frac{a_1(1-\lambda^n)}{1-\lambda} \qquad \square$$

EXAMPLES

a) $\displaystyle\sum_{k=0}^{n} \left(\frac{2}{3}\right)^k$.

Solution

$$S_n = \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k = \frac{1-(2/3)^{n+1}}{1-(2/3)} = \frac{1-(2/3)^{n+1}}{1/3} =$$

$$= 3\left[1-(2/3)^{n+1}\right] = \frac{3\left[3^{n+1}-2^{n+1}\right]}{3^{n+1}} =$$

$$= \frac{3^{n+1}-2^{n+1}}{3^n}.$$

b) $\sum_{k=0}^{n} (-1)^k \left(\frac{1}{3}\right)^{2k}$.

Solution

$$S_n = \sum_{k=0}^{n} (-1)^k \left(\frac{1}{3}\right)^{2k} = \sum_{k=0}^{n} \left[-\left(\frac{1}{3}\right)^2\right]^k =$$

$$= \sum_{k=0}^{n} \left(-\frac{1}{9}\right)^k = \frac{1-(-1/9)^{n+1}}{1-(-1/9)} = \frac{1-(-1)^{n+1}(1/9)^{n+1}}{1+1/9}$$

$$= \frac{1+(-1)^n(1/9)^{n+1}}{10/9} = \frac{9}{10}\,\frac{1}{9^{n+1}}\left[9^{n+1}+(-1)^n\right]$$

$$= \frac{9^{n+1}+(-1)^n}{9^n \cdot 10}.$$

c) $\sum_{k=n}^{2n} \left(\frac{\sqrt{2}}{2}\right)^{k+2}$

Solution

$$S_n = \sum_{k=n}^{2n} \left(\frac{\sqrt{2}}{2}\right)^{k+2} = \left(\frac{\sqrt{2}}{2}\right)^2 \sum_{k=n}^{2n} \left(\frac{\sqrt{2}}{2}\right)^k =$$

$$= \frac{1}{2} \sum_{k=0}^{n} \left(\frac{\sqrt{2}}{2}\right)^{k+n} = \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)^n \sum_{k=0}^{n} \left(\frac{\sqrt{2}}{2}\right)^k =$$

$$= \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)^n \frac{1-(\sqrt{2}/2)^{n+1}}{1-(\sqrt{2}/2)} = \left(\frac{\sqrt{2}}{2}\right)^n \frac{1-(\sqrt{2}/2)^{n+1}}{2-\sqrt{2}} =$$

$$= \left(\frac{\sqrt{2}}{2}\right)^n \frac{\left[1-(\sqrt{2}/2)^{n+1}\right](2+\sqrt{2})}{2^2-(\sqrt{2})^2} =$$

$$= \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)^n (2+\sqrt{2})\left[1-(\sqrt{2}/2)^{n+1}\right] =$$

$$= \frac{2+\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\right)^n\left[1-\left(\frac{\sqrt{2}}{2}\right)^{n+1}\right]$$

Infinite geometric series

$$\boxed{-1 < a < 1 \Rightarrow \sum_{k=0}^{+\infty} a^k = \frac{1}{1-a}}$$

EXAMPLES

a) $\sum\limits_{k=0}^{+\infty} (\sqrt{3}-1)^k$

Solution

Since $1 < \sqrt{3} < 2 \Rightarrow 0 < \sqrt{3}-1 < 1 \Rightarrow$

$$\Rightarrow S = \sum_{k=0}^{+\infty} (\sqrt{3}-1)^k = \frac{1}{1-(\sqrt{3}-1)} = \frac{1}{1-\sqrt{3}+1} =$$

$$= \frac{1}{2-\sqrt{3}} = \frac{2+\sqrt{3}}{(2-\sqrt{3})(2+\sqrt{3})} = \frac{2+\sqrt{3}}{2^2-(\sqrt{3})^2} =$$

$$= \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}.$$

b) $\sum\limits_{k=2}^{+\infty} (\sqrt{2}-1)^k$

Solution

$$s = \sum_{k=2}^{+\infty} (\sqrt{2}-1)^k = \sum_{k=0}^{+\infty} (\sqrt{2}-1)^k - (\sqrt{2}-1)^0 - (\sqrt{2}-1)^1 =$$

$$= \frac{1}{1-(\sqrt{2}-1)} - 1 - (\sqrt{2}-1) = \frac{1}{1-\sqrt{2}+1} - 1 - \sqrt{2} + 1 =$$

$$= \frac{1}{2-\sqrt{2}} - \sqrt{2} = \frac{2+\sqrt{2}}{(2-\sqrt{2})(2+\sqrt{2})} - \sqrt{2} =$$

$$= \frac{2+\sqrt{2}}{2^2-(\sqrt{2})^2} - \sqrt{2} = \frac{2+\sqrt{2}}{2} - \sqrt{2} = \frac{2+\sqrt{2}-2\sqrt{2}}{2} =$$

$$= \frac{2-\sqrt{2}}{2}.$$

c) $\displaystyle\sum_{k=n}^{+\infty} \left(\frac{1}{3}\right)^{k-1}$

Solution

$$S = \sum_{k=n}^{+\infty} \left(\frac{1}{3}\right)^{k-1} = \left(\frac{1}{3}\right)^{-1} \sum_{k=n}^{+\infty} \left(\frac{1}{3}\right)^k =$$

$$= 3\left[\sum_{k=0}^{+\infty} \left(\frac{1}{3}\right)^k - \sum_{k=0}^{n-1} \left(\frac{1}{3}\right)^k\right] =$$

$$= 3\left[\frac{1}{1-1/3} - \frac{1-(1/3)^n}{1-1/3}\right] =$$

$$= 3.\left[\frac{1-(1-(1/3)^n)}{2/3}\right] = 3.\frac{3}{2}.\left[1-1+(1/3)^n\right] =$$

$$= \frac{9}{2}\left(\frac{1}{3}\right)^n.$$

EXERCISES

② Evaluate the following sums:

a) $\sum_{k=0}^{n} \left(\frac{1}{3}\right)^{k}$

b) $\sum_{k=0}^{n} (-1)^{k} \cdot \left(\frac{1}{2}\right)^{2k+1}$

c) $\sum_{k=0}^{n} \left(\frac{1}{2}\right)^{2k-1}$

d) $\sum_{k=0}^{n} 2^{k/2}$

e) $\sum_{k=0}^{n} (\sqrt{2})^{2k-1}$

f) $\sum_{k=0}^{n} (-1)^{k+1} (\sqrt{3})^{k-1}$

$\mapsto$ Try $n=1$ or $n=2$ to check your answer.

③ Similarly, evaluate the following sums:

a) $\sum_{k=3}^{n+3} 2^{k}$

b) $\sum_{k=3}^{2n+1} (\sqrt{3})^{k}$

c) $\sum_{k=n}^{2n-1} (-1)^{k} \left(\frac{1}{3}\right)^{k+1}$

d) $\sum_{k=n+1}^{2n} (\sqrt{2})^k$

e) $\sum_{k=2n}^{3n+1} \left(\frac{2}{3}\right)^k$

f) $\sum_{k=n}^{2n} (-1)^k (1+\sqrt{2})^{2k}$

④ Similarly, evaluate the following infinite sums:

a) $\sum_{k=0}^{+\infty} \left(\frac{1}{3}\right)^k$

b) $\sum_{k=0}^{+\infty} (-1)^k \left(\frac{1}{\sqrt{2}}\right)^{k+1}$

c) $\sum_{k=2}^{+\infty} (-1)^k \left(\frac{1}{\sqrt{3}}\right)^k$

d) $\sum_{k=n}^{+\infty} \left(\frac{2}{5}\right)^k$

e) $\sum_{k=n+1}^{+\infty} \left(\frac{2}{\sqrt{3}}\right)^k$

f) $\sum_{k=2n+1}^{+\infty} (\sqrt{2}-1)^k$

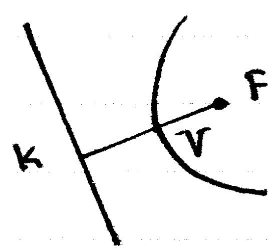
PRE8: Conic sections

INTRODUCTION TO ANALYTICAL GEOMETRY

▼ Parabola

- Let (ℓ) be a line and let a point $F \notin (\ell)$. Then (c) is a parabola with
  a) Focus F
  b) Directrix (ℓ)
  if and only if $\boxed{M \in (c) \Leftrightarrow MF = d(M, (\ell))}$

- Let $FK \perp (\ell)$ with $K \in (\ell)$. Let V be the midpoint of FK. We claim that $V \in (c)$.

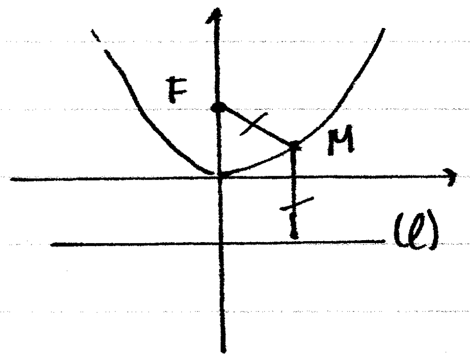


Proof : $VF = VK = d(V, (\ell)) \Rightarrow V \in (c)$. □

Thm :

$$\boxed{\left.\begin{array}{l}(c) \text{ parabola with} \\ \text{focus } F(0, p) \\ \text{directrix } (\ell): y = -p\end{array}\right\} \Rightarrow (c): x^2 = 4py}$$

Proof



Let $M \in (c)$. Then

$MF = \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + (y-p)^2}$

$d(M,(\ell)) = |y-(-p)| = |y+p|$.

It follows that

$$M \in (c) \Leftrightarrow MF = d(M,(\ell)) \Leftrightarrow \sqrt{x^2+(y-p)^2} = |y+p|^2$$
$$\Leftrightarrow x^2 + (y-p)^2 = (y+p)^2 \Leftrightarrow$$
$$\Leftrightarrow x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2 \Leftrightarrow$$
$$\Leftrightarrow x^2 - 2py = 2py \Leftrightarrow \underline{x^2 = 4py}. \qquad \square$$

In general:

$F(x_0+p, y_0)$ $(\ell): x = x_0 - p$ $\Downarrow$ $(c): (y-y_0)^2 = 4p(x-x_0)$	$F(x_0, y_0+p)$ $(\ell): y = y_0 - p$ $\Downarrow$ $(c): (x-x_0)^2 = 4p(y-y_0)$

EXAMPLES

a) Find the parabola (c) with focus $F(1,3)$ and directrix $(\ell): y=1$

Solution

In general, for focus $F(x_0, y_0+p)$ and directrix $(\ell): y = y_0 - p$ the corresponding parabola is $(c): (x-x_0)^2 = 4p(y-y_0)$.

Then,

$$\begin{cases} \text{Focus } F(1,3) \\ \text{Directrix } (\ell): y=1 \end{cases} \Leftrightarrow \begin{cases} x_0 = 1 \\ y_0 + p = 3 \\ y_0 - p = 1 \end{cases} + \Leftrightarrow \begin{cases} x_0 = 1 \\ 2y_0 = 4 \\ y_0 + p = 3 \end{cases}$$

$$2y_0 = 4$$

$$\Leftrightarrow \begin{cases} x_0 = 1 \\ y_0 = 2 \\ 2+p = 3 \end{cases} \Leftrightarrow \begin{cases} x_0 = 1 \\ y_0 = 2 \\ p = 1 \end{cases}$$

and it follows that

$$(c): (x-1)^2 = 4 \cdot 1 \cdot (y-2) \Leftrightarrow x^2 - 2x + 1 = 4y - 8 \Leftrightarrow$$

$$\Leftrightarrow x^2 - 2x + 1 - 4y + 8 = 0 \Leftrightarrow$$

$$\Leftrightarrow x^2 - 2x - 4y + (1+8) = 0$$

$$\Leftrightarrow x^2 - 2x - 4y + 9 = 0.$$

Thus:

$$(c): x^2 - 2x - 4y + 9 = 0$$

B) Find the focus and directrix of the parabola

$(c): y^2-2x-6y+7=0$

<u>Solution</u>

Since,

$$(c): y^2-2x-6y+7=0 \Leftrightarrow (y^2-6y+9)-2x+7-9=0 \Leftrightarrow$$
$$\Leftrightarrow (y-3)^2-2x-2=0 \Leftrightarrow (y-3)^2=2x+2 \Leftrightarrow$$
$$\Leftrightarrow (y-3)^2=2(x+1) \Leftrightarrow (y-3)^2=4\cdot(1/2)(x-(-1))$$

In general, $(c): (y-y_0)^2 = 4p(x-x_0)$ has focus $F(x_0+p, y_0)$ and $(\ell): x=x_0-p$. It follows that

$$x_0=-1 \wedge y_0=3 \wedge p=1/2 \Rightarrow \begin{cases} \text{Focus } F(-1+1/2, 3) \\ \text{Directrix } (\ell): x=(-1)-1/2 \end{cases}$$

$$\Rightarrow \begin{cases} \text{Focus } F(-1/2, 3) \\ \text{Directrix } (\ell): x=-3/2 \end{cases}$$

↳ Curves with equations of the form

$(c): x^2+Ax+By+C=0$ or

$(c): y^2+Ax+By+C=0$

are sometimes parabolas. To rewrite in standard form

$(c): (y-y_0)^2 = 4p(x-x_0)$ or

$(c): (x-x_0)^2 = 4p(y-y_0)$

we complete the square as shown in the example above.

EXERCISES

(1) Find the equation for the parabola with focus F and directrix (ℓ) with

a) $F(1,2)$, $(\ell): x=-1$
b) $F(-1,3)$, $(\ell): x=2$
c) $F(0,0)$, $(\ell): x=-2$
d) $F(2,5)$, $(\ell): y=1$
e) $F(-2,3)$, $(\ell): y=3$
f) $F(-1,-3)$, $(\ell): y=-2$

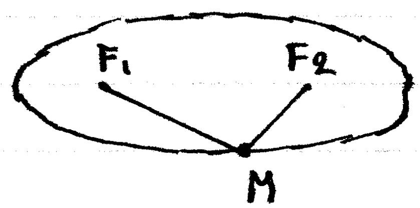
(2) Find the focus and directrix of the following parabolas:

a) $x^2+4x+2y+1=0$
b) $x^2+6x+3y-1=0$
c) $y^2+2x+8y+3=0$
d) $y^2+3x-4y+2=0$
e) $x^2-x+y-1=0$
f) $x^2+3x-2y+5=0$

Ellipse

Let F_1, F_2 be two points. An ellipse (c) with foci F_1 and F_2 is any curve such that

$$\boxed{M \in (c) \Leftrightarrow MF_1 + MF_2 = 2a}$$



Here $a \in (0, +\infty)$ is a constant.
We also define:
(a) Focal distance: $F_1F_2 = 2c$
(b) Eccentricity: $e = c/a$

Prop: $\boxed{0 < c < a}$

Proof

We apply the triangle inequality to $\overset{\triangle}{MF_1F_2}$:

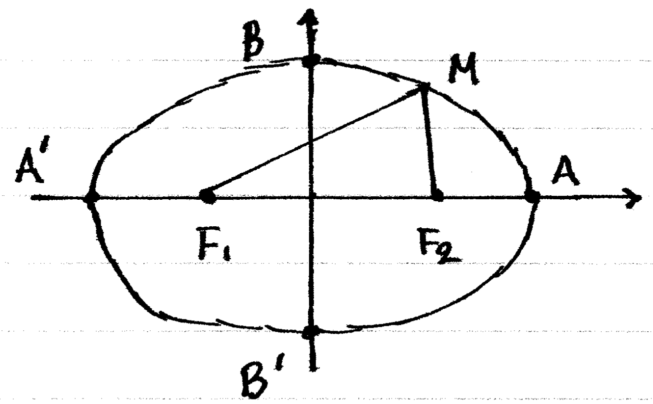
$$\begin{aligned} 2c &= F_1F_2 && [\text{def}] \\ &< MF_1 + MF_2 && [\text{triangle ineq.}] \\ &= 2a && [M \in (c)] \end{aligned}$$

$\Rightarrow c < a \Rightarrow 0 < c < a$ $\square$

Equation of the ellipse

Consider an ellipse (c) with foci $F_1(-c,0)$ and $F_2(c,0)$. Then, for $M(x,y)$:

$$\boxed{M \in (c) \Leftrightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \quad \text{with} \quad \boxed{a^2 = b^2 + c^2}$$



Note that
$A(a,0)$
$A'(-a,0)$
$B(0,b)$
$B'(0,-b)$

Terminology:

a) Vertices: A, A', B, B'

b) Major axis: AA'

c) Minor axis: BB'

d) Focal radii:
$r_1 = F_1M$
$r_2 = F_2M$

It can also be shown that for $M(x,y)$:

$r_1 = MF_1 = a + \dfrac{cx}{a}$	$r_2 = MF_2 = a - \dfrac{cx}{a}$

We now prove the above statements:

Thm: $\boxed{M(x,y) \in (c) \Leftrightarrow \begin{cases} r_1 = a + cx/a \\ r_2 = a - cx/a \end{cases}}$

Proof

$(\Rightarrow)$

Assume $M(x,y) \in (c)$.

It follows that

$$r_1 + r_2 = MF_1 + MF_2 = 2a \quad (1)$$

Also note that:

$$\left.\begin{aligned} r_1^2 &= MF_1^2 = (x+c)^2 + y^2 \\ r_2^2 &= MF_2^2 = (x-c)^2 + y^2 \end{aligned}\right\} \Rightarrow$$

$$\begin{aligned} \Rightarrow r_1^2 - r_2^2 &= [(x+c)^2 + y^2] - [(x-c)^2 + y^2] = \\ &= (x+c)^2 - (x-c)^2 = \\ &= x^2 + 2cx + c^2 - (x^2 - 2cx + c^2) = \\ &= 2cx + 2cx = 4cx \Rightarrow \end{aligned}$$

$$\Rightarrow (r_1 - r_2)(r_1 + r_2) = 4cx \Rightarrow (r_1 - r_2)2a = 4cx \Rightarrow$$

$$\Rightarrow r_1 - r_2 = \frac{2cx}{a} \quad (2).$$

From (1) and (2):

$$\begin{cases} r_1 + r_2 = 2a \\ r_1 - r_2 = 2cx/a \end{cases} \Leftrightarrow \begin{cases} r_1 = a + cx/a \\ r_1 + r_2 = 2a \end{cases} \Leftrightarrow$$

$$2r_1 = 2a + 2cx/a$$

$$\Leftrightarrow \begin{cases} r_1 = a + cx/a \\ r_2 = 2a - r_1 = 2a - (a + cx/a) = a - cx/a \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} r_1 = a + cx/a \\ r_2 = a - cx/a \end{cases}$$

$(\Leftarrow)$: Assume that $\begin{cases} r_1 = a + cx/a \\ r_2 = a - cx/a \end{cases}$

Then:

$$MF_1 + MF_2 = r_1 + r_2 = (a + cx/a) + (a - cx/a)$$
$$= 2a \Rightarrow M \in (c). \quad \square$$

Thm: $\boxed{M(x,y) \in (c) \Leftrightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - c^2} = 1}$

Proof

$(\Rightarrow)$: Assume that $M(x,y) \in (c) \Rightarrow \underline{r_1 = a + cx/a}$ (1)

Recall that $\underline{r_1^2 = (x+c)^2 + y^2}$. (2)

From (1) and (2):

$$(a + cx/a)^2 = (x+c)^2 + y^2 \Leftrightarrow$$
$$\Leftrightarrow a^2 + 2 \cdot cx + (cx/a)^2 = x^2 + 2cx + c^2 + y^2 \Leftrightarrow$$
$$\Leftrightarrow x^2 + 2cx + c^2 + y^2 - a^2 - 2\ cx - (c^2/a^2)x^2 = 0 \Leftrightarrow$$
$$\Leftrightarrow (1 - c^2/a^2)x^2 + y^2 = a^2 - c^2 \Leftrightarrow$$
$$\Leftrightarrow \frac{a^2 - c^2}{a^2} x^2 + y^2 = a^2 - c^2 \Leftrightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \quad (3)$$

$(\Leftarrow)$: Assume that $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - c^2} \underset{(3)}{=} 1 \Rightarrow$

$$\left.\begin{aligned} &\Rightarrow (a + cx/a)^2 = (x+c)^2 + y^2 \quad (4) \\ &\quad r_1^2 = (x+c)^2 + y^2 \end{aligned}\right\} \Rightarrow r_1^2 = (a + cx/a)^2$$

$\Rightarrow \underline{r_1 = a + cx/a.}$ (5).

From (4), replace c with $-c$:

$$\left.\begin{array}{l}(a - cx/a)^2 = (x-c)^2 + y^2 \\ r_2^2 = (x-c)^2 + y^2\end{array}\right\} \Rightarrow r_2^2 = (a - cx/a)^2 \Rightarrow$$

$$\Rightarrow \underline{r_2 = a - cx/a} \quad (6)$$

From (5) and (6): $M(x,y) \in (C)$. $\square$

- <u>General equation of the ellipse</u>

$$\boxed{M(x,y) \in (C) \Leftrightarrow \frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1}$$

Vertices: $A(x_0+a, y_0)$, $A'(x_0-a, y_0)$
$B(x_0, y_0+b)$, $B'(x_0, y_0-b)$

$a>b$	$a<b$
Foci: $F_1(x_0-c, y_0)$ $F_2(x_0+c, y_0)$ with $c^2 = a^2 - b^2$	Foci: $F_1(x_0, y_0-c)$ $F_2(x_0, y_0+c)$ with $c^2 = b^2 - a^2$

EXAMPLES

a) Find the foci and eccentricity of the ellipse
$(c): x^2+3y^2+4x+6y+3=0$

Solutions

$$(c): x^2+3y^2+4x+6y+3=0 \Leftrightarrow$$
$$\Leftrightarrow (x^2+4x+4)+(3y^2+6y+3)-4=0$$
$$\Leftrightarrow (x+2)^2+3(y^2+2y+1)=4$$
$$\Leftrightarrow (x+2)^2+3(y+1)^2=4 \Leftrightarrow$$
$$\Leftrightarrow \frac{(x+2)^2}{4}+\frac{3(y+1)^2}{4}=1 \Leftrightarrow$$
$$\Leftrightarrow \frac{(x-(-2))^2}{2^2}+\frac{(y-(-1))}{(2/\sqrt{3})^2}=1.$$

It follows that:

$x_0=-2,\ y_0=-1,\ a=2,\ b=2/\sqrt{3}.$

Since $a>b \Rightarrow$

$\Rightarrow$ Foci: $F_1(x_0-c, y_0)$ and $F_2(x_0+c, y_0)$

with

$$c^2=a^2-b^2=2^2-(2/\sqrt{3})^2=4-4/3=4\cdot 2/3 \Rightarrow$$
$$\Rightarrow c=\frac{2\sqrt{2}}{\sqrt{3}}=\frac{2\sqrt{6}}{3}$$

It follows that $F_1(-2-2\sqrt{6}/3, -1)$ and $F_2(-2+2\sqrt{6}/3, -1)$.

Eccentricity: $e=\frac{c}{a}=\frac{2\sqrt{6}/3}{2}=\frac{\sqrt{6}}{3}.$

b) Find the equation of the ellipse with foci $F_1(2,1)$ and $F_2(2,5)$ and major axis $AA' = 12$.

Solution

Let $(c): \dfrac{(x-x_0)^2}{a^2} + \dfrac{(y-y_0)^2}{b^2} = 1$.

$2a = AA' = 12 \Rightarrow a = 6$.

$2c = F_1F_2 = |y_{F_1} - y_{F_2}| = |1-5| = 4 \Rightarrow c = 2$.

Since $F_1F_2 \parallel y\text{-axis} \Rightarrow a < b \Rightarrow$

$\Rightarrow b^2 = a^2 + c^2 = 6^2 + 2^2 = 36 + 4 = 40 \Rightarrow b = 2\sqrt{10}$.

Since O midpoint of F_1F_2:

$x_0 = x_{F_1} = 2$

$y_0 = \dfrac{1}{2}(y_{F_1} + y_{F_2}) = \dfrac{1}{2}(1+5) = \dfrac{6}{2} = 3$

Thus:

$(c): \dfrac{(x-2)^2}{36} + \dfrac{(y-3)^2}{40} = 1$

c) Find the equation of the ellipse with foci $F_1(2,2)$ and $F_2(5,2)$ and eccentricity $e = 1/2$.

Solution

Let $(c): \dfrac{(x-x_0)^2}{a^2} + \dfrac{(y-y_0)^2}{b^2} = 1$

$2c = F_1F_2 = |x_{F_1} - x_{F_2}| = |2-5| = 3 \Rightarrow c = 3/2$

$e = c/a \Rightarrow a = \dfrac{c}{e} = \dfrac{3/2}{1/2} = 3$

Since $F_1F_2 // x\text{-axis} \Rightarrow a > b \Rightarrow a^2 = b^2 + c^2 \Rightarrow$

$\Rightarrow b^2 = a^2 - c^2 = 3^2 - (3/2)^2 = 9[1 - 1/4] = 9 \cdot 3/4 = 27/4$

$\Rightarrow b = \dfrac{3\sqrt{3}}{2}$.

O midpoint of F_1F_2, thus:

$x_0 = \dfrac{1}{2}(x_{F_1} + x_{F_2}) = \dfrac{1}{2}(2+5) = \dfrac{7}{2}$

$y_0 = y_{F_1} = 2.$

It follows that:

$(c): \dfrac{(x-7/2)^2}{3^2} + \dfrac{(y-2)^2}{27/4} = 1 \Leftrightarrow$

$\Leftrightarrow \dfrac{(2x-7)^2}{4 \cdot 3^2} + \dfrac{4(y-2)^2}{27} = 1.$

$\Leftrightarrow \dfrac{(2x-7)^2}{36} + \dfrac{4(y-2)^2}{27} = 1.$

EXERCISES

③ Find the foci and eccentricity of the following ellipses:

a) $x^2+2y^2+6x+8y+1=0$
b) $2x^2+3y^2-6x+6y-2=0$
c) $5x^2+2y^2-10x+12y+3=0$
d) $3x^2+y^2+30x+12y+2=0$
e) $3x^2+4y^2-6x-16y+3=0$
f) $2x^2+y^2+12x+4y-1=0$.

④ Find an equation for the ellipse with focus F_1, F_2 and eccentricity e; with

a) $F_1(-1,0)$, $F_2(2,0)$, $e=1/2$
b) $F_1(2,2)$, $F_2(2,6)$, $e=2/3$
c) $F_1(1,-\sqrt{2})$, $F_2(1,+\sqrt{2})$, $e=1/\sqrt{2}$
d) $F_1(1-\sqrt{2},3)$, $F_2(1+\sqrt{2},3)$, $e=1/3$
e) $F_1(-1,\sqrt{2})$, $F_2(-1,2\sqrt{2})$, $e=\sqrt{2}-1$.

▼ Hyperbola

Let F_1, F_2 be two points. A hyperbola (c) with focus F_1 and F_2 is a set of points such that

$$\boxed{M \in (c) \Leftrightarrow |MF_1 - MF_2| = 2a}$$

with $a \in (0, +\infty)$ a constant. We also define:

(a) Focal distance: $F_1F_2 = 2c$

(b) Eccentricity: $e = c/a$

Prop: $\boxed{e > 1}$

Proof

Apply triangle inequality to $M\hat{F_1}F_2$:

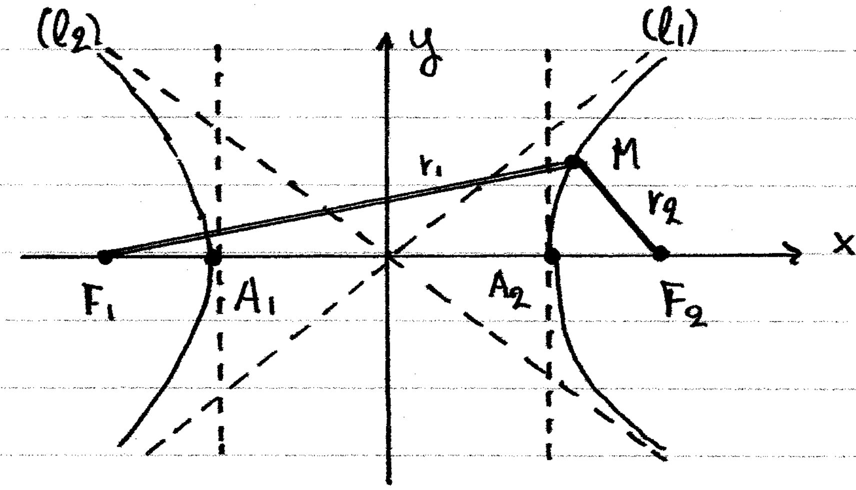
$$\begin{aligned} 2a &= |MF_1 - MF_2| < && [\text{def}] \\ &< F_1F_2 && [\text{triangle inequality}] \\ &= 2c \Rightarrow && [\text{def}] \end{aligned}$$

$$\Rightarrow a < c \Rightarrow e = \frac{c}{a} > 1 \qquad \square$$

❶ Equation of hyperbola

Consider a hyperbola (c) with $F_1(-c,0)$ and $F_2(c,0)$. Then:

$$\boxed{M(x,y) \in (c) \Leftrightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1} \text{ with } \boxed{c^2 = b^2 + a^2}$$



Terminology:

a) Vertices ⟶ $A_1(-a,0)$ and $A_2(a,0)$

b) Asymptotes

Focus-Asymptote distance

$(\ell_1):\ y = \frac{b}{a}x$	$(\ell_2):\ y = -\frac{b}{a}x$

(!) $\boxed{d(F_1,(\ell_1)) = d(F_1,(\ell_2)) = d(F_2,(\ell_1)) = d(F_2,(\ell_2)) = b}$

c) Focal radii

$r_1 = MF_1 = \left\lvert \frac{cx}{a} + a \right\rvert$	$r_2 = MF_2 = \left\lvert \frac{cx}{a} - a \right\rvert$

Proof of equation of the ellipse

Let (c) be a hyperbola with focii $F_1(-c,0)$ and $F_2(c,0)$ with $c>0$ such that

$$M(x,y) \in (c) \Leftrightarrow |MF_1 - MF_2| = 2a \quad , \text{ with } a>0$$

First we show that:

Prop : $$\boxed{M(x,y) \in (C) \Leftrightarrow \begin{cases} r_1 = MF_1 = |cx/a + a| \\ r_2 = MF_2 = |cx/a - a| \end{cases}}$$

Proof

$(\Rightarrow)$: Assume that $M(x,y) \in (c)$. Then:

$$r_1^2 = MF_1^2 = (x_M - x_{F_1})^2 + (y_M - y_{F_1})^2 =$$
$$= (x-(-c))^2 + (y-0)^2 = (x+c)^2 + y^2$$
$$r_2^2 = MF_2^2 = (x_M - x_{F_2})^2 + (y_M - y_{F_2})^2 =$$
$$= (x-c)^2 + (y-0)^2 = (x-c)^2 + y^2.$$

It follows that

$$(r_1 - r_2)(r_1 + r_2) = r_1^2 - r_2^2 = [(x+c)^2 + y^2] - [(x-c)^2 + y^2]$$
$$= (x+c)^2 - (x-c)^2 =$$
$$= x^2 + 2cx + c^2 - x^2 + 2cx - c^2 =$$
$$= 2cx + 2cx = 4cx$$

thus:

$$\begin{cases} (r_1 - r_2)(r_1 + r_2) = 4cx & (1) \\ |r_1 - r_2| = 2a \end{cases}$$

since $M(x,y) \in (c) \Rightarrow |MF_1 - MF_2| = 2a \Rightarrow |r_1 - r_2| = 2a$.

Case 1: Assume $x=0 \Rightarrow 4cx=0 \Rightarrow r_1^2-r_2^2=0 \Rightarrow$
$\Rightarrow r_1=r_2 \Rightarrow MF_1=MF_2 \Rightarrow$
$\Rightarrow |MF_1-MF_2|=0 \neq 2a \leftarrow$ contradiction.

Case 2: Assume $x>0 \Rightarrow 4cx>0 \Rightarrow (r_1-r_2)(r_1+r_2)>0$
$\Rightarrow r_1-r_2>0$, therefore

$$(1) \Leftrightarrow \begin{cases} (r_1-r_2)(r_1+r_2)=4cx \\ r_1-r_2=2a \end{cases} \Leftrightarrow \begin{cases} 2a(r_1+r_2)=4cx \\ r_1-r_2=2a \end{cases} \Leftrightarrow$$
$$\Leftrightarrow \begin{cases} r_1+r_2=4cx/2a=2cx/a \\ r_1-r_2=2a \end{cases} \Leftrightarrow$$
$$\Leftrightarrow \begin{cases} r_1=cx/a+a \\ r_2=cx/a-a. \end{cases}$$

Case 3: Assume $x<0 \Rightarrow 4cx<0 \Rightarrow (r_1-r_2)(r_1+r_2)<0$
$\Rightarrow r_1-r_2<0$, therefore

$$(1) \Leftrightarrow \begin{cases} (r_1-r_2)(r_1+r_2)=4cx \\ r_1-r_2=-2a \end{cases} \Leftrightarrow \begin{cases} -2a(r_1+r_2)=4cx \\ r_1-r_2=-2a \end{cases}$$
$$\Leftrightarrow \begin{cases} r_1+r_2=-2cx/a \\ r_1-r_2=-2a \end{cases} \Leftrightarrow \begin{cases} r_1=-cx/a-a \\ r_2=-cx/a+a \end{cases}$$

From cases 1, 2, 3 above:
$$(1) \Leftrightarrow \begin{cases} r_1=|cx/a+a| \\ r_2=|cx/a-a| \end{cases}$$

($\Leftarrow$): Assume that

$$r_1 = MF_1 = |\,cx/a + a| \quad (1)$$
$$r_2 = MF_2 = |\,cx/a - a|$$

Without making any assumptions, we show again that

$$(r_1 - r_2)(r_1 + r_2) = 4cx \quad (2)$$

Case 1: Assume that $x = 0 \overset{(1)}{\Longrightarrow} r_1 = r_2 = |a| = a$

Also:

$$\left.\begin{array}{l} r_1^2 = (x+c)^2 + y^2 = c^2 + y^2 \\ r_1^2 = a^2 \end{array}\right\} \Rightarrow c^2 + y^2 = a^2 \Rightarrow$$

$\Rightarrow y^2 = a^2 - c^2 < 0$ (since $a < c$) $\Rightarrow$

$\Rightarrow y^2 < 0 \leftarrow$ Contradiction.

Case 2: Assume that $x \neq 0$. Under this assumption, from cases 2,3 of the ($\Rightarrow$) argument above, we can show, without making any further assumptions, that:

$$\begin{cases} (r_1 - r_2)(r_1 + r_2) = 4cx \\ |r_1 - r_2| = 2a \end{cases} \Leftrightarrow \begin{cases} r_1 = |\,cx/a + a| \\ r_2 = |\,cx/a - a| \end{cases}$$

It follows that

$(1) \Rightarrow |r_1 - r_2| = 2a \Rightarrow |MF_1 - MF_2| = 2a \Rightarrow$

$\Rightarrow M(x, y) \in (C)$. $\square$

We will now show that

Thm: $\boxed{M(x,y) \in (c) \Leftrightarrow \dfrac{x^2}{a^2} - \dfrac{y^2}{c^2-a^2} = 1}$

Proof

$(\Rightarrow)$: Assume that $M(x,y) \in (c)$.
We show again that $r_1^2 = MF_1^2 = (x+c)^2 + y^2$. Then
$M(x,y) \in (c) \Rightarrow r_1 = \left| \dfrac{cx}{a} + a \right| \Rightarrow r_1^2 = \left(\dfrac{cx}{a} + a \right)^2$

$$\Rightarrow (x+c)^2 + y^2 = \left(\frac{cx}{a} + a \right)^2. \quad (1).$$

Note that:

$$(1) \Leftrightarrow x^2 + 2cx + c^2 + y^2 = \frac{c^2x^2}{a^2} + 2cx + a^2 \Leftrightarrow$$

$$\Leftrightarrow x^2 + c^2 + y^2 = \frac{c^2x^2}{a^2} + a^2 \Leftrightarrow$$

$$\Leftrightarrow \left[1 - \frac{c^2}{a^2} \right] x^2 + y^2 = a^2 - c^2 \Leftrightarrow$$

$$\Leftrightarrow \frac{a^2-c^2}{a^2} x^2 + y^2 = a^2 - c^2 \Leftrightarrow$$

$$\Leftrightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1 \Leftrightarrow \frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = 1. \quad (2)$$

Thus $(1) \Rightarrow \dfrac{x^2}{a^2} - \dfrac{y^2}{c^2-a^2} = 1.$

$(\Leftarrow)$: Assume that $\dfrac{x^2}{a^2} - \dfrac{y^2}{c^2-a^2} = 1 \quad (3)$

Using (2) we have

$$(3) \Rightarrow (x+c)^2 + y^2 = \left(\frac{cx}{a}+a\right)^2 \Rightarrow r_1^2 = \left(\frac{cx}{a}+a\right)^2$$

$$\Rightarrow r_1 = \left|\frac{cx}{a}+a\right| \quad (4)$$

Furthermore:

$$r_2^2 = (x-c)^2 + y^2 = (x-c)^2 + \left[\left(\frac{cx}{a}+a\right)^2 - (x+c)^2\right] =$$

$$= \left(\frac{cx}{a}+a\right)^2 + (x^2-2cx+c^2) - (x^2+2cx+c^2) =$$

$$= \left(\frac{cx}{a}+a\right)^2 - 4cx = \left(\frac{cx}{a}-a\right)^2 \Rightarrow$$

$$\Rightarrow r_2 = \left|\frac{cx}{a}-a\right| \quad (5).$$

From (4) and (5) it follows that $M(x,y) \in (c)$. $\square$

We now show that:

- $\boxed{\text{At } x \to \pm\infty,\ (c) \text{ approaches the lines } (\ell_{1,2}): y = \pm \dfrac{b}{a}x}$

Proof

$$M(x,y) \in (c) \Leftrightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Leftrightarrow b^2x^2 - a^2y = a^2b^2$$

$$\Leftrightarrow a^2y^2 = b^2x^2 - a^2b^2 \Leftrightarrow$$

$$\Leftrightarrow y^2 = \frac{b^2x^2 - a^2b^2}{a^2} = \frac{b^2(x^2-a^2)}{a^2} =$$

$$= \frac{b^2x^2}{a^2}\left[1 - \frac{a^2}{x^2}\right] \Leftrightarrow$$

$$\Leftrightarrow y = \pm \frac{bx}{a}\sqrt{1 - \frac{a^2}{x^2}}$$

For $x \to \pm\infty$: $\sqrt{1-a^2/x^2} \longrightarrow 1$

thus $y/x \sim \pm b/a$.

By symmetry the two lines have to intersect at the origin, thus:

$$(l_{1,2}): y = \pm \frac{b}{a}x . \qquad \square$$

- $\boxed{d(F_1,(l_1)) = d(F_1,(l_2)) = d(F_2,(l_1)) = d(F_2,(l_2)) = b}$

Proof

Recall that in general, the distance of the point $M(x_0,y_0)$ from the line $(l): Ax+By+C=0$ is given by:

$$d(M,(l)) = \frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}$$

For $F_1(-c,0)$ and $(\ell_1): y = \frac{b}{a}x \Leftrightarrow bx - ay = 0$
we have:

$$d(F_1,(\ell_1)) = \frac{|bx_{F_1} - ay_{F_1}|}{\sqrt{b^2+(-a)^2}} = \frac{|b\cdot(-c) - a\cdot 0|}{\sqrt{a^2+b^2}} =$$

$$= \frac{|-bc|}{\sqrt{c^2}} = \frac{|b||c|}{|c|} = |b| = b$$

Similar argument gives:

$$d(F_2,(\ell_1)) = d(F_1,(\ell_2)) = d(F_2,(\ell_2)) = b \quad \square$$

General Equation of the hyperbola

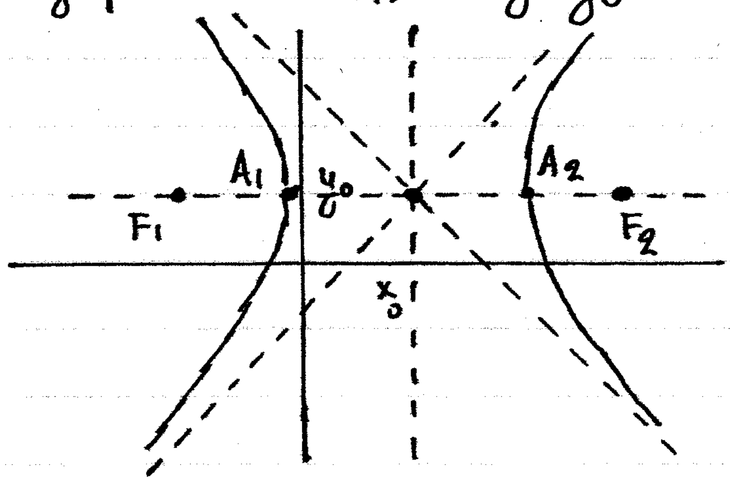
1) $$\boxed{M(x,y) \in (C) \Leftrightarrow \frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1}$$

$c^2 = a^2 + b^2$

Focus: $F_1(x_0 - c, y_0)$, $F_2(x_0 + c, y_0)$

Vertices: $A_1(x_0 - a, y_0)$, $A_2(x_0 + a, y_0)$

Asymptotes: $(\ell_{1,2})$: $y - y_0 = \pm \frac{b}{a}(x - x_0)$.



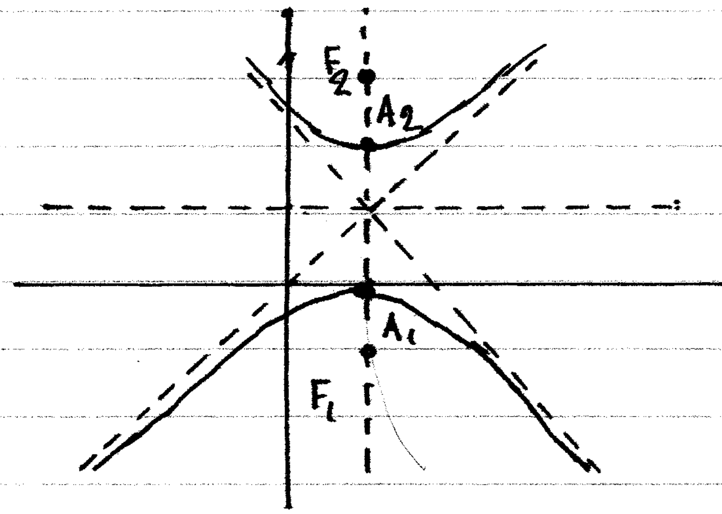
2) $$\boxed{M(x,y) \in (C) \Leftrightarrow \frac{(y-y_0)^2}{a^2} - \frac{(x-x_0)^2}{b^2} = 1}$$

$c^2 = a^2 + b^2$

Focus: $F_1(x_0, y_0 - c)$, $F_2(x_0, y_0 + c)$

Vertices: $A_1(x_0, y_0 - a)$, $A_2(x_0, y_0 + a)$

Asymptotes: $(\ell_{1,2})$: $x - x_0 = \pm \frac{b}{a}(y - y_0)$.


F_2
A_2
A_1
F_1

EXAMPLES

a) Find the foci, vertices, and asymptotes of the hyperbola

$$(c): x^2-2y^2+4x-12y-20=0.$$

Solution

$(c): x^2-2y^2+4x-12y-20=0 \Leftrightarrow$

$\Leftrightarrow (x^2+4x+4)-2(y^2+6y+9)-20-4+18=0 \Leftrightarrow$

$\Leftrightarrow (x+2)^2-2(y+3)^2-6=0 \Leftrightarrow$

$\Leftrightarrow (x+2)^2-2(y+3)^2=6 \Leftrightarrow$

$$\Leftrightarrow \frac{(x+2)^2}{6}-\frac{(y+3)^2}{3}=1$$

$$\Leftrightarrow \frac{(x-(-2))^2}{(\sqrt{6})^2}-\frac{(y-(-3))^2}{(\sqrt{3})^2}=1$$

For $a=\sqrt{6}$ and $b=\sqrt{3}$:

$c^2=a^2+b^2=(\sqrt{6})^2+(\sqrt{3})^2=6+3=9 \Rightarrow$

$\Rightarrow c=3$

It follows that

a) Focus: $F_1(-2-3, \quad -3)=F_1(-5,-3)$

$F_2(-2+3,-3)=F_2(1,-3)$

b) Vertices: $A_1(-2+\sqrt{6},-3)$

$A_2(-2-\sqrt{6},-3)$

c) Asymptotes:

$$(\ell_{1,2}): \ y-(-3) = \pm \frac{\sqrt{3}}{\sqrt{6}} (x-(-2)) \Leftrightarrow$$

$$\Leftrightarrow y+3 = \pm \frac{1}{\sqrt{2}} (x+2) \Leftrightarrow \sqrt{2}(y+3) = \pm (x+2) \Leftrightarrow$$

$$\Leftrightarrow \mp (x+2) + \sqrt{2}(y+3) = 0 \ .$$

thus:

$$(\ell_1): \ (x+2) + \sqrt{2}(y+3) = 0 \Leftrightarrow$$

$$\Leftrightarrow \underline{x + \sqrt{2}y + (2+3\sqrt{2}) = 0}$$

and

$$(\ell_2): \ -(x+2) + \sqrt{2}(y+3) = 0 \Leftrightarrow$$

$$\Leftrightarrow \underline{-x + \sqrt{2}y + (3\sqrt{2} - 2) = 0}.$$

b) Find the hyperbola with $F_1(1,2)$, $F_2(6,2)$ focis and vertices $A_1(2,2)$ and $A_2(5,2)$.

Solution

$2a = A_1A_2 = |x_{A_2} - x_{A_1}| = |5-2| = 3 \Rightarrow \underline{a = 3/2}$

$2c = F_1F_2 = |x_{F_2} - x_{F_1}| = |6-1| = 5 \Rightarrow \underline{c = 5/2}$

$c^2 = a^2 + b^2 \Rightarrow$

$\Rightarrow b^2 = c^2 - a^2 = (5/2)^2 - (3/2)^2 = \frac{25-9}{4} =$

$= \frac{16}{4} = 4 \Rightarrow \underline{b = 2}.$

Origin O midpoint of F_1F_2 thus:

$$x_0 = \frac{x_{F_1} + x_{F_2}}{2} = \frac{1+6}{2} = \frac{7}{2}$$

$$y_0 = \frac{y_{F_1} + y_{F_2}}{2} = \frac{2+2}{2} = 2$$

It follows that:

$$(c): \frac{(x-7/2)^2}{(3/2)^2} - \frac{(y-2)^2}{2^2} = 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{4(x-7/2)^2}{9} - \frac{(y-2)^2}{4} = 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{(2x-7)^2}{9} - \frac{(y-2)^2}{4} = 1 \Leftrightarrow$$

$$\Leftrightarrow 4(2x-7)^2 - 9(y-2)^2 = 36 \Leftrightarrow$$

$$\Leftrightarrow 4(4x^2 - 28x + 49) - 9(y^2 - 4y + 4) = 36$$

$$\Leftrightarrow 16x^2 - 112x + 196 - 9y^2 + 36y - 36 = 36$$
$$\Leftrightarrow 16x^2 - 9y^2 - 112x + 36y + (196 - 36 - 36) = 0$$
$$\Leftrightarrow 16x^2 - 9y^2 - 112x + 36y + 124 = 0$$

thus:

$(C): 16x^2 - 9y^2 - 112x + 36y + 124 = 0.$

c) Find the hyperbola with focus $F_1(1-\sqrt{2},1)$ and $F_2(1+\sqrt{2},1)$ and asymptotes $(\ell_{1,2}): y-1=\pm 2(x-1)$.

Solution

$$2c = F_1F_2 = |x_{F_2}-x_{F_1}| = |(1+\sqrt{2})-(1-\sqrt{2})| =$$
$$= |1+\sqrt{2}-1+\sqrt{2}| = |2\sqrt{2}| = 2\sqrt{2} \Rightarrow c=\sqrt{2}$$

$(\ell_{1,2}): y-1=\pm 2(x-1)$ asymptotes $\Rightarrow$

$$\Rightarrow \frac{b}{a}=2 \Rightarrow \underline{b=2a}.$$

It follows that

$$\begin{cases} b=2a \\ c^2=a^2+b^2 \end{cases} \Leftrightarrow \begin{cases} b=2a \\ a^2+(2a)^2=(\sqrt{2})^2 \end{cases} \Leftrightarrow \begin{cases} b=2a \\ a^2+4a^2=2 \end{cases}$$

$$\Leftrightarrow \begin{cases} b=2a \\ 5a^2=2 \end{cases} \Leftrightarrow \begin{cases} b=2a \\ a^2=2/5 \end{cases} \Leftrightarrow \begin{cases} b=2\sqrt{10}/5 \\ a=\dfrac{\sqrt{2}}{\sqrt{5}}=\dfrac{\sqrt{10}}{5} \end{cases}$$

We also note that

$$x_0 = \frac{x_{F_1}+x_{F_2}}{2} = \frac{(1-\sqrt{2})+(1+\sqrt{2})}{2} =$$

$$= \frac{2}{2} = 1 \text{ and}$$

$$y_0 = 1.$$

and therefore:

$$(c): \frac{(x-1)^2}{2/5} - \frac{(y-1)^2}{4/5} = 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{5(x-1)^2}{2} - \frac{5(y-1)^2}{4} = 1 \Leftrightarrow$$

$$\Leftrightarrow 10(x-1)^2 - 5(y-1)^2 = 4 \Leftrightarrow$$

$$\Leftrightarrow 10(x^2-2x+1) - 5(y^2-2y+1) = 4 \Leftrightarrow$$

$$\Leftrightarrow 10x^2 - 20x + 10 - 5y^2 + 10y - 5 = 4 \Leftrightarrow$$

$$\Leftrightarrow 10x^2 - 5y^2 - 20x + 10y + (10-5-4) = 0$$

$$\Leftrightarrow 10x^2 - 5y^2 - 20x + 10y + 1 = 0$$

Thus:

$$(c): \underline{10x^2 - 5y^2 - 20x + 10y + 1 = 0}$$

EXERCISES

⑤ Find the focii, vertices, and asymptotes of the following hyperbolas

a) $2x^2 - y^2 + 4x - 2y + 3 = 0$
b) $x^2 - 3y^2 + 6x + 6y + 1 = 0$
c) $x^2 - 5y^2 + 10x - 20y + 10 = 0$
d) $3x^2 - 4y^2 + 12x + 40y - 3 = 0$
e) $2x^2 - 2y^2 + 24x + 28y - 7 = 0$

⑥ Find an equation of the hyperbola with
a) Focus $F_1(3,3)$, $F_2(7,3)$;
Vertices $A_1(4,3)$, $A_2(6,3)$
b) Focus $F_1(2, 1-\sqrt{3})$, $F_2(2, 1+\sqrt{3})$;
Vertices $A_1(2,0)$, $A_2(2,2)$
c) Focus $F_1(1,-1)$, $F_2(3,-1)$;
asymptote (ℓ): $y+1 = 3(x-2)$
d) Focus $F_1(2,1)$, $F_2(2,7)$;
asymptote (ℓ): $y-4 = 2(x-2)$
(careful with this one!)

References

The following references were consulted during the preparation of these lecture notes.

(1) Pistofides (1988): "Algebra. I.", unpublished lecture notes.
(2) Pistofides (1989): "Algebra. II.", unpublished lecture notes.
(3) Xenou (1994): "Algebra and Analytic Geometry. 1", , Ekdoseis ZHTH.
(4) Xenou (1995): "Algebra B", Ekdoseis ZHTH.

Lecture notes by Pistofides are available for download at

http://www.math.utpa.edu/lf/OGS/pistofides.html