PH5170

QUANTUM MECHANICS II

February–May 2021

Lecture schedule and meeting hours

- The course will consist of about 42 lectures, including about 8–10 tutorial sessions. However, note that there may not be separate tutorial sessions, and I will try to integrate them with the lectures.
- The duration of each lecture will be 50 minutes. We will be meeting on Google Meet. I will share the link over smail.
- The first lecture will be on Tuesday, February 2, and the last lecture will be on Thursday, May 6.
- We will meet thrice a week. The lectures are scheduled for 11:00–11:50 AM on Tuesdays, 10:00–10:50 AM on Wednesdays, and 8:00–8:50 AM on Thursdays.
- We may also meet during 5:00–5:50 PM on Fridays for either the quizzes or to make up for lectures that I may have to miss due to other unavoidable commitments. Changes in schedule, if any, will be notified sufficiently in advance.
- If you would like to discuss with me about the course outside the lecture hours, please send me an e-mail at `sriram@physics.iitm.ac.in`. We can converge on a mutually convenient time to meet and discuss online. I would request you to write to me from your smail addresses with the subject line containing the name of the course, i.e. PH5170: Quantum Mechanics II.

Information about the course

- All the information regarding the course such as the schedule of the lectures, the structure and the syllabus of the course, suitable textbooks and additional references will be available on the course's page on Moodle at the following URL:

  `https://courses.iitm.ac.in/`

- The exercise sheets and other additional material will also be made available on Moodle.
- A PDF file containing these information as well as completed quizzes will also be available at the link on this course at the following URL:

  `http://physics.iitm.ac.in/~sriram/professional/teaching/teaching.html`

  I will keep updating this file and the course's page on Moodle as we make progress.

Quizzes, end-of-semester exam and grading

- The grading will be based on three scheduled quizzes and an end-of-semester exam.
- I will consider the best two quizzes for grading, and the two will carry 25% weight each.
- The three quizzes will be held on February 26, March 26 and April 23. All the three dates are Fridays, and the quizzes will be held during 5:00–6:30 PM on these days.
- The end-of-semester exam will be held on Thursday, May 20, and the exam will carry 50% weight. We will converge on the time of the exam in due course.

Last updated on May 8, 2021

Syllabus and structure

Quantum Mechanics II

1. **Warming up with basic quantum mechanics**

   (a) The Schrodinger equation
   (b) Uncertainty principle – Commutation relations – Simultaneous observables
   (c) Essential mathematical formalism
   (d) Simple problems in one dimension – Particle in a box – Harmonic oscillator
   (e) The Schrodinger equation in three dimensions – The angular and the radial equations

Exercise sheet 1

2. **Rotation and angular momentum** [$\sim$ 14 lectures]

   (a) Rotation – Orbital and spin angular momentum
   (b) Angular momentum algebra – Eigenstates and eigenvalues of angular momentum
   (c) Addition of angular momenta – Clebsch-Gordan coefficients
   (d) Measurements of spin correlations – Bell's inequality
   (e) Irreducible tensor operators – Wigner-Eckart theorem

Exercise sheets 2, 3, 4 and 5

Quiz I

Additional exercises I

3. **Systems of identical particles** [$\sim$ 10 lectures]

   (a) Symmetric and antisymmetric wave functions
   (b) Bosons and Fermions – Pauli's exclusion principle – The Slater determinant
   (c) Ground state of Helium
   (d) The free electron gas – Degeneracy pressure
   (e) Bloch's theorem – Band structure in solids
   (f) Quantum statistical mechanics – Black body radiation

Exercise sheets 6 and 7

Quiz II

4. **Theory of scattering** [$\sim$ 6 lectures]

   (a) Scattering in non-relativistic quantum mechanics
   (b) Scattering amplitude and cross-section
   (c) Partial wave analysis – Phase shifts
   (d) The integral equation form of the Schrodinger equation – Born approximation
   (e) The optical theorem

Exercise sheets 8 and 9

Additional exercises II

Quiz III

5. **Relativistic quantum mechanics** [$\sim$ 6 lectures]

   (a) Elements of relativistic quantum mechanics
   (b) The Klein-Gordon equation
   (c) The Dirac equation – Dirac matrices – Spinors
   (d) Positive and negative energy solutions – Physical interpretation
   (e) Non-relativistic limit of the Dirac equation
   (f) Covariant form of the Dirac equation – Bilinear covariants
   (g) The hydrogen atom

Exercise sheets 10 and 11

End-of-semester exam

Advanced problems

Note: The topics in red could not be covered for want of time.

Basic textbooks

1. J. Bjorken and S. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1965).
2. L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Course of Theoretical Physics, Volume 3), Third Edition (Pergamon Press, New York, 1977).
3. D. J. Griffiths, *Introduction to Elementary Particles* (John Wiley, New York, 1987).
4. J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, Singapore, 1994).
5. D. J. Griffiths, *Introduction to Quantum Mechanics*, Second Edition (Pearson Education, Delhi, 2005).

Additional references

1. P. A. M. Dirac, *The Principles of Quantum Mechanics*, Fourth Edition (Oxford University Press, Oxford, 1958).
2. A. Messiah, *Quantum Mechanics*, Volumes 1 and 2 (North Holland, Amsterdam, 1961).
3. J. J. Sakurai, Advanced Quantum Mechanics (Addison-Wesley, Singapore, 1967).
4. F. Halzen and A. D. Martin, *Quarks and Leptons: An Introductory Course in Modern Particle Physics* (John Wiley, New York, 1984).
5. R. W. Robinett, *Quantum Mechanics*, Second Edition (Oxford University Press, Oxford, 2006).
6. F. Dyson, *Advanced Quantum Mechanics* (World Scientific, Singapore, 2007).
7. R. Shankar, *Principles of Quantum Mechanics*, Second Edition (Springer, Delhi, 2008).

Exercise sheet 1

Warming up with basic quantum mechanics

Some classical mechanics

1. *Non-relativistic particle in an electromagnetic field:* A non-relativistic particle that is moving in an electromagnetic field described by the scalar potential ϕ and the vector potential $\mathbf{A}$ is governed by the Lagrangian
$$L = \frac{m\,\mathbf{v}^2}{2} + q\,\left(\frac{\mathbf{v}}{c} \cdot \mathbf{A}\right) - q\,\phi,$$
where m and q are the mass and the charge of the particle, while c denotes the velocity of light. Show that the equation of motion of the particle is given by
$$m\,\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q\,\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right),$$
where $\mathbf{E}$ and the $\mathbf{B}$ are the electric and the magnetic fields given by
$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\,\frac{\partial \mathbf{A}}{\partial t} \qquad \text{and} \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$
Note: The scalar and the vector potentials, viz. ϕ and $\mathbf{A}$, are dependent on time *as well as* space. Further, given two vectors, say, $\mathbf{C}$ and $\mathbf{D}$, one can write,
$$\nabla\,(\mathbf{C} \cdot \mathbf{D}) = (\mathbf{D} \cdot \nabla)\;\mathbf{C} + (\mathbf{C} \cdot \nabla)\;\mathbf{D} + \mathbf{D} \times (\nabla \times \mathbf{C}) + \mathbf{C} \times (\nabla \times \mathbf{D})\,.$$
Also, since $\mathbf{A}$ depends on time as well as space, we have,
$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla)\;\mathbf{A}.$$

2. *Period associated with bounded, one-dimensional motion:* Determine the period of oscillation as a function of the energy, say, E, when a particle of mass m moves in a field governed by the potential $V(x) = V_0\,|x|^n$, where V_0 is a constant and n is a positive integer.

3. *Poisson brackets:* Recall that the Poisson bracket $\{A, B\}$ between two observables $A(q_i, p_i)$ and $B(q_i, p_i)$ is defined as
$$\{A, B\} = \sum_{i=1}^{N} \left(\frac{\partial A}{\partial q_i}\,\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\,\frac{\partial B}{\partial q_i}\right),$$
where q_i and p_i denote the generalized coordinates and the corresponding conjugate momenta, respectively, while N denotes the number of degrees of freedom of the system.
   (a) *Poisson brackets for position and momenta:* Establish the following relations: $\{q_i, q_j\} = 0$, $\{p_i, p_j\} = 0$, $\{q_i, p_j\} = \delta_{ij}$, $\dot{q}_i = \{q_i, H\}$ and $\dot{p}_i = \{p_i, H\}$, where H represents the Hamiltonian of the system, while δ_{ij} represents the standard Kronecker symbol.
   (b) *Poisson brackets for angular momenta:* Let (L_x, L_y, L_z) be the Cartesian components of the angular momentum vector $\boldsymbol{L}$, and let $L^2 = L_x^2 + L_y^2 + L_z^2$. Evaluate the following Poisson brackets: $\{L_x, L_y\}$, $\{L_y, L_z\}$ and $\{L_z, L_x\}$, $\{L_x, L^2\}$, $\{L_y, L^2\}$ and $\{L_z, L^2\}$.
   Hint: For the final set of Poisson brackets, it may be useful to establish and use the result $\{A, B\,C\} = \{A, B\}\,C + B\,\{A, C\}$.

4. *Phase portraits:* Draw the phase portraits of a particle moving in the following one dimensional potentials: (a) $U(x) = \alpha\,|x|^n$, (b) $U(x) = \alpha\,x^2 - \beta\,x^3$, (c) $U(x) = \alpha\,(x^2 - \beta^2)^2$, and (d) $U(\theta) = -\alpha\cos\theta$, where $(\alpha, \beta) > 0$ and $n > 2$.

Some basics of quantum mechanics

5. *Operators, expectation values and properties:* Recall that the expectation value of an operator $\hat{A}$ is defined as
$$\langle \hat{A} \rangle = \int \mathrm{d}x\ \Psi^* \hat{A}\, \Psi.$$
   (a) *Hermitian operators:* An operator $\hat{A}$ is said to be hermitian if
$$\langle \hat{A} \rangle = \langle \hat{A} \rangle^*.$$
   Show that the position, the momentum and the Hamiltonian operators are hermitian.

   (b) *Motivating the momentum operator:* Using the time-dependent Schrodinger equation, show that
$$\frac{\mathrm{d}\langle x \rangle}{\mathrm{d}t} = -\frac{i\,\hbar}{m} \int \mathrm{d}x\ \Psi^* \frac{\partial \Psi}{\partial x} = \langle \hat{p}_x \rangle,$$
   a relation which can be said to motivate the expression for the momentum operator, viz. that $\hat{p}_x = -i\,\hbar\, \partial/\partial x$.

   (c) *Ehrenfest's theorem:* Show that
$$\frac{\mathrm{d}\langle \hat{p}_x \rangle}{\mathrm{d}t} = -\left\langle \frac{\partial V}{\partial x} \right\rangle,$$
   a relation that is often referred to as the Ehrenfest's theorem.

   (d) *Virial theorem:* Let x and p_x denote the position and momentum of a particle moving in a given time-independent potential $V(x)$. Show that the system satisfies the equation
$$\frac{\langle \hat{p}_x^2 \rangle}{2\,m} = \frac{1}{2} \left\langle \hat{x}\, \widehat{\frac{\partial V}{\partial x}} \right\rangle,$$
   where the expectation values are evaluated in a *stationary* state described by a *real* wavefunction.

   Note: The above relation is often referred to as the virial theorem.

   Hint: Establish the virial theorem in the following two steps. To begin with, show that
$$\int\limits_{-\infty}^{\infty} \mathrm{d}x\, \psi\, x\, \frac{\mathrm{d}V}{\mathrm{d}x}\, \psi = -\langle \hat{V} \rangle - 2 \int\limits_{-\infty}^{\infty} \mathrm{d}x\, \frac{\mathrm{d}\psi}{\mathrm{d}x}\, x\, V\, \psi$$
   and then, using the time-independent Schrodinger equation, illustrate that
$$-2 \int\limits_{-\infty}^{\infty} \mathrm{d}x\, \frac{\mathrm{d}\psi}{\mathrm{d}x}\, x\, V\, \psi = E + \frac{\hbar^2}{2\,m} \int\limits_{-\infty}^{\infty} \mathrm{d}x\, \left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)^2.$$

6. *Spreading of wave packets:* A free particle has the initial wave function
$$\Psi(x,0) = A\, \mathrm{e}^{-a\,x^2},$$
where A and a are constants, with a being real and positive.

   (a) Normalize $\Psi(x,0)$.

   (b) Find $\Psi(x,t)$.

   (c) Plot $\Psi(x,t)$ at $t=0$ and for large t. Determine qualitatively what happens as time goes on?

   (d) Find $\langle \hat{x} \rangle$, $\langle \hat{x}^2 \rangle$, $\langle \hat{p} \rangle$, $\langle \hat{p}^2 \rangle$, Δx and Δp.

(e) Does the uncertainty principle hold? At what time does the system have the minimum uncertainty?

Particle in a box

7. *Particle in a box:* Consider a particle of mass m that confined is to a box with its walls at $x = 0$ and $x = a$.

   (a) *Particle in a box I:* At time $t = 0$, the particle is equally likely to be found over the domain $0 < x < a/2$.

      i. What is the initial wave function of the system, i.e. $\Psi(x, 0)$?
      Hint: Assume that the wave function is real, and make sure you normalize the wave function.

      ii. What is the probability that a measurement of the energy of the particle would yield the value $E_1 = \pi^2\,\hbar^2/(2\,m\,a^2)$?

   (b) *Particle in a box II:* At time $t = 0$, the wave function of the particle is given by

   $$\Psi(x, 0) = A\,\left[\psi_1(x) + \psi_2(x)\right],$$

   where ψ_1 and ψ_2 are the ground state and the first excited state of the system.

      i. Determine the constant A assuming that $\Psi(x, 0)$ is normalized.

      ii. What is $\Psi(x, t)$ for $t > 0$?

      iii. Determine the expectation values $\langle\hat{x}\rangle$ and $\langle\hat{p}_x\rangle$ in the state $\Psi(x, t)$. Express the results in terms of the angular frequency $\omega = \pi^2\,\hbar/(2\,m\,a^2)$.

      iv. What are probabilities of the system to be found in the ground and the first excited states? What is the expectation value of the Hamiltonian, i.e. $\langle\hat{H}\rangle$, in the state $\Psi(x, t)$?

   (c) *Particle in a box III:* The particle is described by the following wave function:

   $$\psi(x) = \begin{cases} A\;(x/a) & \text{for } 0 < x < a/2, \\ A\;[1 - (x/a)] & \text{for } a/2 < x < a, \end{cases}$$

   where A is a real constant. If the energy of the system is measured, what is the probability for finding the energy eigen value to be $E_n = n^2\,\pi^2\,\hbar^2/(2\,m\,a^2)$?

8. *Quantum revival:* Consider an arbitrary wavefunction describing a particle in the infinite square well.

   (a) Show that the wave function will return to its original form after a time $T_{\mathrm{Q}} = 4\,m\,a^2/(\pi\,\hbar)$.
   Note: The time T_{Q} is known as the *quantum revival time.*

   (b) Determine the classical revival time T_{C} for a particle of energy E bouncing back and forth between the walls.

   (c) What is the energy for which $T_{\mathrm{Q}} = T_{\mathrm{C}}$?

The simple harmonic oscillator

9. *Oscillating charge in an electric field:* Let a particle of mass m and charge q be oscillating in the simple harmonic potential $V(x) = m\,\omega^2\,x^2/2$. A constant electric field of strength $\mathcal{E}$ is turned on along the positive x-direction.

   (a) What is the complete potential influencing the charge in the presence of the electric field?

   (b) Draw the classical trajectory of the charge in phase space when the electric field has been turned on. How does it compare with the original trajectory?

   (c) What are the energy eigen values of the system when it is quantized?

(d) Plot the ground state wave function of the system with and without the electric field.

10. *Half-an-oscillator:* Determine the energy levels and the corresponding eigen functions of an oscillator which is subjected to the additional condition that the potential is infinite for $x \le 0$.

    Note: You do not have to separately solve the Schrodinger equation. You can easily identify the allowed eigen functions and eigen values from the solutions of the original, complete, oscillator!

11. *Wagging the dog:* Recall that the time-independent Schrodinger equation satisfied by a simple harmonic oscillator of mass m and frequency ω is given by

$$-\frac{\hbar^2}{2\,m}\,\frac{\mathrm{d}^2\psi_{_E}}{\mathrm{d}x^2} + \frac{1}{2}\,m\,\omega^2\,x^2\,\psi_{_E} = E\,\psi_{_E}.$$

    In terms of the dimensionless variable

$$\xi = \sqrt{\frac{m\,\omega}{\hbar}}\;x,$$

    the above time-independent Schrodinger equation reduces to

$$\frac{\mathrm{d}^2\psi_{_E}}{\mathrm{d}\xi^2} + \left(\mathcal{E} - \xi^2\right)\,\psi_{_E} = 0,$$

    where $\mathcal{E}$ is the energy expressed in units of $(\hbar\,\omega/2)$, and is given by

$$\mathcal{E} = \frac{2\,E}{\hbar\,\omega}.$$

    According to the 'wag-the-dog' method, one solves the above differential equation numerically, say, using Mathematica, varying $\mathcal{E}$ until a wave function that goes to zero at large ξ is obtained.

    Find the ground state energy and the energies of the first two excited states of the harmonic oscillator to five significant digits by the 'wag-the-dog' method.

12. *Constructing the energy eigen states of the harmonic oscillator:* If $\psi_n(x)$ denotes the n-th energy eigen state of the harmonic oscillator and $\hat{a}$ represents the lowering operator, then, recall that

$$\hat{a}\,\psi_n(x) = \sqrt{n}\,\psi_{n-1}(x).$$

    (a) Using the fact that $\hat{a}\,\psi_0(x) = 0$, construct the ground state wave function $\psi_0(x)$, ensuring that it is normalized.

    (b) Also, recall that

$$\hat{a}^{\dagger}\,\psi_n(x) = \sqrt{n+1}\,\psi_{n+1}(x),$$

    where $\hat{a}^{\dagger}$ is the raising operator. Operate the raising operator on $\psi_0(x)$ successively to construct the wavefunctions describing the first and the second excited states, i.e. $\psi_1(x)$ and $\psi_2(x)$.

13. *More about oscillators:* Let $|0\rangle$ represent the ground state of a one dimensional quantum oscillator. Show that

$$\langle 0|\mathrm{e}^{i\,k\,\hat{x}}|0\rangle = \exp - \left(k^2\,\langle 0|\hat{x}^2|0\rangle/2\right),$$

    where $\hat{x}$ is the position operator.

14. *Coherent states of the harmonic oscillator:* Consider states, say, $|\alpha\rangle$, which are eigen states of the annihilation (or, more precisely, the lowering) operator, i.e.

$$\hat{a}|\alpha\rangle = \alpha\,|\alpha\rangle,$$

    where α is a complex number.

    Note: The state $|\alpha\rangle$ is called the coherent state.

(a) Calculate the quantities $\langle\hat{x}\rangle$, $\langle\hat{x}^2\rangle$, $\langle\hat{p}_x\rangle$ and $\langle\hat{p}_x^2\rangle$ in the coherent state.

(b) Also, evaluate the quantities Δx and Δp_x in the state, and show that $\Delta x\ \Delta p_x = \hbar/2$.

(c) Like any other general state, the coherent state can be expanded in terms of the energy eigen states $|n\rangle$ of the harmonic oscillator as follows:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n\,|n\rangle.$$

Show that the quantities c_n are given by

$$c_n = \frac{\alpha^n}{\sqrt{n!}}\,c_0.$$

(d) Determine c_0 by normalizing $|\alpha\rangle$.

(e) Upon including the time dependence, show that the coherent state continues to be an eigen state of the lowering operator $\hat{a}$ with the eigen value evolving in time as

$$\alpha(t) = \alpha\,\mathrm{e}^{-i\,\omega\,t}.$$

Note: Therefore, a coherent state *remains* coherent, and continues to minimize the uncertainty.

(f) Is the ground state $|0\rangle$ itself a coherent state? If so, what is the eigen value α?

Some mathematical and conceptual aspects

15. *Probabilities in momentum space:* A particle of mass m is bound in the delta function well $V(x) = -a\,\delta(x)$, where $a > 0$. What is the probability that a measurement of the particle's momentum would yield a value greater than $p_0 = m\,a/\hbar$?

16. *The energy-time uncertainty principle:* Consider a system that is described by the Hamiltonian operator $\hat{H}$.

(a) Given an operator, say, $\hat{Q}$, establish the following relation:

$$\frac{\mathrm{d}\langle\hat{Q}\rangle}{\mathrm{d}t} = \frac{i}{\hbar}\,\langle[\hat{H},\hat{Q}]\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle,$$

where the expectation values are evaluated in a specific state.

(b) When $\hat{Q}$ does not explicitly depend on time, using the generalized uncertainty principle, show that

$$\Delta H\,\Delta Q \ge \frac{\hbar}{2}\left|\frac{\mathrm{d}\langle\hat{Q}\rangle}{\mathrm{d}t}\right|.$$

(c) Defining

$$\Delta t \equiv \frac{\Delta Q}{|\mathrm{d}\langle\hat{Q}\rangle/\mathrm{d}t|},$$

establish that

$$\Delta E\,\Delta t \ge \frac{\hbar}{2},$$

and interpret this result.

17. *Two-dimensional Hilbert space:* Imagine a system in which there are only two linearly independent states, viz.

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The most general state would then be a normalized linear combination, i.e.

$$|\psi\rangle = \alpha\,|1\rangle + \beta\,|2\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

with $|\alpha|^2+|\beta|^2 = 1$. The Hamiltonian of the system can, evidently, be expressed as a 2×2 hermitian matrix. Suppose it has the following form:

$$\mathsf{H} = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

where a and b are *real* constants. If the system starts in the state $|1\rangle$ at an initial time, say, $t = 0$, determine the state of the system at a later time t.

18. *A two level system:* The Hamiltonian operator of a certain two level system is given by

$$\hat{H} = E\,\Big(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|\Big),$$

where $|1\rangle$ and $|2\rangle$ form an orthonormal basis, while E is a number with the dimensions of energy.

(a) Find the eigen values and the normalized eigen vectors, i.e. as a linear combination of the basis vectors $|1\rangle$ and $|2\rangle$, of the above Hamiltonian operator.

(b) What is the matrix that represents the operator $\hat{H}$ in this basis?

19. *A three level system:* The Hamiltonian for a three level system is represented by the matrix

$$\mathsf{H} = \hbar\,\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Two other observables, say, A and B, are represented by the matrices

$$\mathsf{A} = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathsf{B} = \mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where ω, λ and μ are positive real numbers.

(a) Find the eigen values and normalized eigen vectors of H, A, and B.

(b) Suppose the system starts in the generic state

$$|\psi(t=0)\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

with $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$. Find the expectation values of H, A and B in the state at $t = 0$.

(c) What is $|\psi(t)\rangle$ for $t > 0$? If you measure the energy of the state at a time t, what are the values of energies that you will get and what would be the probability for obtaining each of the values?

(d) Also, arrive at the corresponding answers for the quantities A and B.

20. *Expectation values in momentum space:* Given a wave function, say, $\Psi(x, t)$, in the position space, the corresponding wave function in momentum space is given by

$$\Phi(p, t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{\sqrt{2\,\pi\,\hbar}}\,\Psi(x, t)\,\mathrm{e}^{-i\,p\,x/\hbar}.$$

(a) Show that, if the wavefunction $\Psi(x,t)$ is normalized in position space, it is normalized in momentum space as well, i.e.

$$\int_{-\infty}^{\infty} \mathrm{d}x\, |\Psi(x,t)|^2 = \int_{-\infty}^{\infty} \mathrm{d}p\, |\Phi(p,t)|^2 = 1.$$

(b) Recall that, in the position representation, the expectation value of the momentum operator can be expressed as follows:

$$\langle \hat{p} \rangle = -i\,\hbar \int_{-\infty}^{\infty} \mathrm{d}x\, \Psi^*(x,t)\, \frac{\partial \Psi(x,t)}{\partial x}.$$

Show that, in momentum space, it can be expressed as

$$\langle \hat{p} \rangle = \int_{-\infty}^{\infty} \mathrm{d}p\, p\, |\Phi(p,t)|^2.$$

(c) Similarly, show that the following expectation value of the position operator in position space

$$\langle \hat{x} \rangle = \int_{-\infty}^{\infty} \mathrm{d}x\, x\, |\Psi(x,t)|^2,$$

can be written as

$$\langle \hat{x} \rangle = i\,\hbar \int_{-\infty}^{\infty} \mathrm{d}p\, \Phi^*(p,t)\, \frac{\partial \Phi(p,t)}{\partial p}.$$

21. _The Schrodinger equation in momentum space:_ Assuming that the potential $V(x)$ can be expanded in a Taylor series, show that the time-dependent Schrodinger equation in momentum space can be written as

$$\frac{p^2}{2\,m}\, \Phi(p,t) + V\left(i\,\hbar\, \frac{\partial}{\partial p}\right)\, \Phi(p,t) = i\,\hbar\, \frac{\partial \Phi(p,t)}{\partial t}.$$

22. _Sequential measurements:_ An operator $\hat{A}$, representing the observable A, has two normalized eigen states ψ_1 and ψ_2, with eigen values a_1 and a_2. Operator $\hat{B}$, representing another observable B, has two normalized eigen states ϕ_1 and ϕ_2, with eigen values b_1 and b_2. These eigen states are related as follows:

$$\psi_1 = \frac{3}{5}\, \phi_1 + \frac{4}{5}\, \phi_2 \qquad \text{and} \qquad \psi_2 = \frac{4}{5}\, \phi_1 - \frac{3}{5}\, \phi_2.$$

(a) Observable A is measured and the value a_1 is obtained. What is the state of the system immediately after this measurement?

(b) If B is now measured, what are the possible results, and what are their probabilities?

(c) Immediately after the measurement of B, A is measured again. What is the probability of getting a_1?

The Schrodinger equation in three dimensions

23. _Particle in a three dimensional box:_ Consider a particle that is confined to a three dimensional box of side, say, a. In other words, the particle is free inside the box, but the potential energy is infinite on the walls of the box, thereby confining the particle to the box.

(a) Determine the energy eigen functions and the corresponding energy eigen values.

(b) Does there exist degenerate energy eigen states? Identify a few of them.

24. _Particle in a spherical well:_ Consider a particle that is confined to the following spherical well:

$$V(r) = \begin{cases} 0 & \text{for } r < a, \\ \infty & \text{for } r \geq a. \end{cases}$$

Find the energy eigen functions and the corresponding energy eigen values of the particle.

25. *Expectation values in the energy eigen states of the hydrogen atom:* Recall that, the normalized wavefunctions that describe the energy eigen states of the electron in the hydrogen atom are given by

$$\psi_{nlm}(r,\theta,\phi) = \left[\left(\frac{2}{n\,a_0}\right)^3 \frac{(n-l-1)!}{2\,n\,[(n+l)!]^3}\right]^{1/2} \mathrm{e}^{-r/(n\,a_0)} \left(\frac{2\,r}{n\,a_0}\right)^l L_{n-l-1}^{2\,l+1}(2\,r/n\,a_0)\;Y_l^m(\theta,\phi),$$

where $L_p^q(x)$ and Y_l^m represent the associated Laguerre polynomials and the spherical harmonics, respectively, while a_0 denotes the Bohr radius.

(a) Evaluate $\langle \hat{r}\rangle$ and $\langle \hat{r}^2\rangle$ for the electron in the ground state of the hydrogen atom, and express it in terms of the Bohr radius.

(b) Find $\langle \hat{x}\rangle$ and $\langle \hat{x}^2\rangle$ for the electron in the ground state of hydrogen.
Hint: Express r^2 as $x^2+y^2+z^2$ and exploit the symmetry of the ground state.

(c) Calculate $\langle \hat{x}^2\rangle$ in the state $n=2$, $l=1$ and $m=1$.
Note: This state is not symmetrical in x, y and z. Use $x = r\sin\theta\cos\phi$.

Exercise sheet 2

Spin

1. *Velocity on the surface of a spinning electron:* Consider the electron to be a classical solid sphere. Assume that the radius of the electron is given by the classical electron radius, viz.

$$r_{\rm c} = \frac{e^2}{4\,\pi\,\epsilon_0\,m_{\rm e}\,c^2}$$

where e and $m_{\rm e}$ denote the charge and the mass of the electron, while c represents the speed of light. Also, assume that the angular momentum of the electron is $\hbar/2$. Evaluate the speed on the surface of the electron under these conditions.

2. *Orienting spin along an arbitrary direction:* Let $\hat{\boldsymbol{n}}$ be a three-dimensional unit vector whose polar angle (i.e. the angle with respect to the z-axis) is θ and the azimuthal angle (i.e. the angle with respect to the x-axis, when the unit vector $\hat{\boldsymbol{n}}$ has been projected on to the x-y plane) is ϕ, i.e.

$$\hat{\boldsymbol{n}} = \sin\theta\,\cos\phi\,\hat{\imath} + \sin\theta\,\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k}.$$

   (a) Obtain the eigen values of the operator $\hat{\boldsymbol{S}}\cdot\hat{\boldsymbol{n}}$ describing a spin-$\frac{1}{2}$ particle.
   (b) Construct the corresponding eigen vectors.
   (c) Show that the eigen vectors are orthonormal.

3. *Larmor precession:* Consider a charged, spin-$\frac{1}{2}$ particle which is at rest in an external and uniform magnetic field, say, $\mathbf{B}$, that is oriented along the z-direction, i.e. $\mathbf{B} = B\,\hat{k}$, where B is a constant. The Hamiltonian of the particle is then given by

$$\hat{H} = -\gamma\,B\,\hat{S}_z,$$

where γ is known as the gyromagnetic ratio of the particle.

   (a) Determine the most general, time dependent, solution that describes the state of the particle.
   (b) Evaluate the expectation values of the operators $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ in the state.
   (c) Show that the expectation value of the operator $\hat{\mathbf{S}} = \hat{S}_x\,\hat{\imath} + \hat{S}_y\,\hat{\jmath} + \hat{S}_z\,\hat{k}$ is tilted at a constant angle with respect to the direction of the magnetic field and precesses about the field at the so-called Larmor frequency $\omega = \gamma\,B$.

4. *Mean values and uncertainties associated with spin operators:* An electron is in the spin state

$$\chi = A\begin{pmatrix} 3\,i \\ 4 \end{pmatrix}.$$

   (a) Determine the normalization constant A.
   (b) Find the expectation values of the operators $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ in the above state.
   (c) Evaluate the corresponding uncertainties, i.e. ΔS_x, ΔS_y and ΔS_z.
   (d) Examine if the products of any two of these quantities are consistent with the corresponding uncertainty principles.

5. *Spin-$\frac{3}{2}$ particle:* Consider a particle with spin $\frac{3}{2}$.

   (a) What are the eigen values and eigen states of the $\hat{S}_z$ operator for the system?
   (b) Determine the effects of the operators $\hat{S}_+$ and $\hat{S}_-$ on these eigen states.
   (c) Construct the matrices describing the operators $\hat{S}_+$ and $\hat{S}_-$.
   (d) Obtain the matrix describing the operator $\hat{S}_x$.

Exercise sheet 3

Orbital angular momentum

1. <u>*Operators describing orbital angular momentum:*</u> Show that

   (a) $\hat{L}_x = -i\,\hbar\,\left(-\sin\phi\,\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\,\frac{\partial}{\partial\phi}\right)$,

   (b) $\hat{L}_y = -i\,\hbar\,\left(\cos\phi\,\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\,\frac{\partial}{\partial\phi}\right)$,

   (c) $\hat{L}_\pm = \pm\hbar\,\mathrm{e}^{\pm i\,\phi}\,\left(\frac{\partial}{\partial\,\theta} \pm i\cot\theta\,\frac{\partial}{\partial\phi}\right)$,

   (d) $\hat{L}_z = -i\,\hbar\,\frac{\partial}{\partial\phi}$,

   (e) $\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\phi^2}\right]$.

2. <u>*Ehrenfest's theorem for angular momentum:*</u> Consider a particle moving in a potential $V(\boldsymbol{r})$.

   (a) If $\hat{\boldsymbol{L}}$ and $\hat{\boldsymbol{N}}$ are the operators representing angular momentum and torque, show that

   $$\frac{\mathrm{d}\langle\hat{\boldsymbol{L}}\rangle}{\mathrm{d}t} = \langle\hat{\boldsymbol{N}}\rangle,$$

   where $\boldsymbol{N} = \boldsymbol{r}\times(-\boldsymbol{\nabla}V)$.

   (b) Show that $\mathrm{d}\langle\hat{\boldsymbol{L}}\rangle/\mathrm{d}t = 0$ for any spherically symmetric potential, i.e. when $V(\boldsymbol{r}) = V(r)$.

3. <u>*Matrices representing the angular momentum operators:*</u> Earlier, we had discussed the matrices describing the operators $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ of a spin-$\frac{1}{2}$ system. Consider the matrices describing the angular momentum operators $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ corresponding to $l=1$.

   (a) List the allowed states $|l,m\rangle$ for $l=1$. What is the rank of the matrices in such a Hilbert space?

   (b) Construct the matrices representing $\hat{L}_z$ and $\hat{\boldsymbol{L}}^2$.

   (c) Construct the matrices describing $\hat{L}_+$ and $\hat{L}_-$, and thereby $\hat{L}_x$ and $\hat{L}_y$.

   Hint: Recall the effects of the operators $\hat{L}_z$, $\hat{\boldsymbol{L}}^2$, $\hat{L}_+$ and $\hat{L}_-$ on the states $|l,m\rangle$.

4. <u>*Spherical harmonics:*</u> Recall that, $\langle\theta,\phi|l,m\rangle = Y_l^m(\theta,\phi)$. Using the relations $\hat{L}_+|l,l\rangle = 0$ and $\hat{L}_z|l,l\rangle = l\,\hbar|l,l\rangle$, determine the functional form of $Y_l^l(\theta,\phi)$.

   Hint: Using the representations of $\hat{L}_+$ and $\hat{L}_z$ as differential operators, write the two relations above as differential equations and solve them.

   Note: You need not have to normalize the wavefunction.

5. <u>*Particle in a central potential:*</u> The wavefunction of a particle in a central potential $V(r)$ is given by

   $$\psi(\boldsymbol{x}) = (x+y+3\,z)\,f(r).$$

   (a) Is this wavefunction an eigenfunction of $\hat{\boldsymbol{L}}^2$? If so, what is the value of l? If not, what are the possible values of l we may obtain if $\boldsymbol{L}^2$ is measured?

   (b) What are the probabilities for the particle to be found in the various m_l states?

   (c) Suppose it is known somehow that the above wavefunction is an energy eigenfunction with eigenvalue E. How does one find the potential $V(r)$?

Exercise sheet 4

Addition of angular momentum

1. *Commutation relations involving angular momentum:* Given $\hat{\boldsymbol{J}} = \hat{\boldsymbol{J}}_1 + \hat{\boldsymbol{J}}_2$, show that

   (a) $[\hat{\boldsymbol{J}}^2, \hat{\boldsymbol{J}}_1^2] = [\hat{\boldsymbol{J}}^2, \hat{\boldsymbol{J}}_2^2] = 0$,

   (b) $[\hat{\boldsymbol{J}}^2, \hat{\boldsymbol{J}}_1] = 2\,i\,\hbar\,(\hat{\boldsymbol{J}}_1 \times \hat{\boldsymbol{J}}_2)$ and $[\hat{\boldsymbol{J}}^2, \hat{\boldsymbol{J}}_2] = 2\,i\,\hbar\,(\hat{\boldsymbol{J}}_2 \times \hat{\boldsymbol{J}}_1)$, so that $[\hat{\boldsymbol{J}}^2, \hat{\boldsymbol{J}}] = 0$,

   (c) $[\hat{\boldsymbol{J}}^2, \hat{J}_{1z}] \neq 0$, $[\hat{\boldsymbol{J}}^2, \hat{J}_{2z}] \neq 0$, but $[\hat{\boldsymbol{J}}^2, \hat{J}_z] = 0$,

   (d) $[\hat{\boldsymbol{J}}_1 \cdot \hat{\boldsymbol{J}}_2, \hat{\boldsymbol{J}}_1] = i\,\hbar\,(\hat{\boldsymbol{J}}_1 \times \hat{\boldsymbol{J}}_2)$ and $[\hat{\boldsymbol{J}}_1 \cdot \hat{\boldsymbol{J}}_2, \hat{\boldsymbol{J}}_2] = i\,\hbar\,(\hat{\boldsymbol{J}}_2 \times \hat{\boldsymbol{J}}_1)$, so that $[\hat{\boldsymbol{J}}_1 \cdot \hat{\boldsymbol{J}}_2, \hat{\boldsymbol{J}}] = 0$.

2. *Limits on j:* Starting from $j_{\rm max} = j_1 + j_2$ and

$$\sum_{j=j_{\rm min}}^{j_{\rm max}} (2\,j+1) = (2\,j_1+1)\,(2\,j_2+1),$$

   prove that $j_{\rm min} = |j_1 - j_2|$.

3. *Spins of mesons and baryons:* As you may know, mesons (such as pions or kaons) and baryons (such as protons or neutrons) consist of two and three quarks, respectively. The quarks are known to carry spin-$\frac{1}{2}$. Assume that the quarks are in their ground state so that their orbital angular momentum is zero.

   (a) What are the possible spins for the mesons?

   (b) What are the possible spins for the baryons?

4. *Determining the Clebsch-Gordan coefficients:* Given that $(j_1 = 1, j_2 = 1, j = 1, m = -1)$, express the state $|j_1\, j_2; j\, m\rangle$ in terms of the states $|j_1\, j_2; m_1\, m_2\rangle = |j_1\, m_1\rangle \otimes |j_2\, m_2\rangle$.

5. *Using the Clebsch-Gordan coefficients:* Consider two spin-1 particles that occupy the state

$$|s_1\, s_2; m_1\, m_2\rangle = |1\,1; 1\,0\rangle.$$

   (a) What is the probability of finding the system in an eigenstate of the total spin $\hat{\boldsymbol{S}}^2$ with quantum number $s = 1$?

   (b) What is the probability for $s = 2$?

Additional exercises I

Angular momentum

1. *Hamiltonian and angular momentum of a free particle:* Consider a free particle in three dimensions whose position is described in terms of the spherical polar coordinates (r, θ, ϕ). Let the conjugate momenta corresponding to these coordinates be p_r, p_θ and p_ϕ. Show that the Hamiltonian of the free particle can be expressed as
$$H = \frac{p_r^2}{2\,m} + \frac{L^2}{2\,m\,r^2},$$
where $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ is the angular momentum of the particle. Also, show that
$$L^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}.$$

2. *An operator relation involving angular momentum:* Show that
$$\hat{\boldsymbol{L}}^2 = \hat{\boldsymbol{x}}^2\,\hat{\boldsymbol{p}}^2 - (\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{p}})^2 + i\,\hbar\,\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{p}}.$$

3. *A model of angular momentum:* Let
$$\hat{L}_\pm = \left(\hat{a}_\pm^\dagger\,\hat{a}_\mp\right)\hbar, \qquad \hat{L}_z = \left(\hat{a}_+^\dagger\,\hat{a}_+ - \hat{a}_-^\dagger\,\hat{a}_-\right)\hbar/2 \qquad \text{and} \qquad \hat{N} = \hat{a}_+^\dagger\,\hat{a}_+ + \hat{a}_-^\dagger\,\hat{a}_-,$$
where $\hat{a}_\pm$ and $\hat{a}_\pm^\dagger$ are the annihilation and the creation operators of two *independent* simple harmonic oscillators satisfying the usual commutation relations. Show that
   (a) $[\hat{L}_z, \hat{L}_\pm] = \pm\,\hbar\,\hat{L}_\pm$,
   (b) $\hat{\mathbf{L}}^2 = \hat{N}\,\left[\left(\hat{N}/2\right) + 1\right]\,(\hbar^2/2)$,
   (c) $[\hat{\mathbf{L}}^2, \hat{L}_z] = 0$.

   Note: This representation of the angular momentum operators in terms of creation and the annihilation operators of oscillators is known as the Schwinger model.

4. *Particle in a central potential:* A particle in a spherically symmetric potential is known to be in an eigenstate of $\hat{\boldsymbol{L}}^2$ and $\hat{L}_z$ with eigenvalues $l\,(l+1)\,\hbar^2$ and $m\,\hbar$, respectively. Prove that the expectation values in the $|l, m\rangle$ states satisfy
$$\langle\hat{L}_x\rangle = \langle\hat{L}_y\rangle = 0, \quad \langle\hat{L}_x^2\rangle = \langle\hat{L}_y^2\rangle = \frac{\hbar^2}{2}\,\left[l\,(l+1) - m^2\right].$$

5. *Hamiltonian involving angular momentum:* The Hamiltonian of a system is described in terms of the angular momentum operators as follows:
$$\hat{H} = \frac{\hat{L}_x^2}{2\,I_1} + \frac{\hat{L}_y^2}{2\,I_2} + \frac{\hat{L}_z^2}{2\,I_3}.$$
   (a) What are the eigen values of the Hamiltonian when $I_1 = I_2$?
   (b) What are the eigen values of the Hamiltonian if the angular momentum of the system is unity and $I_1 \neq I_2$?

6. *Rotation of a spin-$\frac{1}{2}$ system:* Consider a sequence of Euler rotations represented by
$$\mathcal{D}^{(1/2)}(\alpha, \beta, \gamma) = \exp - \left(\frac{i\,\sigma_3\,\alpha}{2}\right)\,\exp - \left(\frac{i\,\sigma_2\,\beta}{2}\right)\,\exp - \left(\frac{i\,\sigma_3\,\gamma}{2}\right).$$
Since the rotations form a group, we expect the above sequence of rotations to be equivalent to a single rotation by an angle, say, θ, about some axis. Find the angle θ.

7. *Rotating an eigen state of orbital angular momentum:* Consider the orbital angular momentum eigenstate $|l = 2, m = 0\rangle$. If this state is rotated by an angle β about the y-axis, determine the probability for the new state to be found in $m = (0, \pm 1, \pm 2)$.

8. *Measuring components of angular momentum:* A beam of particles is subject to the simultaneous measurement of angular momentum variables $\boldsymbol{L}^2$ and L_z. The measurement yields the pairs of values $(l = 0, m = 0)$ and $(l = 1, m = -1)$ with probabilities $3/4$ and $1/4$, respectively.

   (a) Reconstruct the state of the beam immediately before the measurement.

   (b) The particles in the beam with $(l = 1, m = -1)$ are separated out and subjected to the measurement of L_x. What are the possible outcomes and their probabilities?

   (c) Construct the spatial wavefunctions of the states that could arise from second measurement.

9. (a) *Adding spins:* A particle of spin-1 and a particle of spin-2 are at rest in a configuration such that the total spin is 3, and its z-component is 1 (i.e. the eigenvalue of $\hat{S}_z$ is $\hbar$). If you measured the z-component of the angular momentum of the spin-2 particle, what values might you get, and what is the probability of each one?

   (b) *Adding orbital and spin angular momenta:* An electron with spin down is in the state ψ_{510} of the hydrogen atom. If you could measure the total angular momentum squared of the electron alone (not including the proton spin), what values might you get, and what is the probability of each?

10. *Interacting spins:* A system of two particles with spins $s_1 = \frac{3}{2}$ and $s_2 = \frac{1}{2}$ is described by the Hamiltonian

$$\hat{H} = \alpha\, \hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2,$$

with α being a given constant. The system is initially (say, at $t = 0$) in the following eigenstate of $\hat{\boldsymbol{S}}_1^2$, $\hat{\boldsymbol{S}}_2^2$, $\hat{\boldsymbol{S}}_{1z}$ and $\hat{\boldsymbol{S}}_{2z}$:

$$|s_1\, s_2; m_1\, m_2\rangle = \left|\tfrac{3}{2}\, \tfrac{1}{2}; \tfrac{1}{2}\, \tfrac{1}{2}\right\rangle.$$

   (a) Find the state of the system at times $t > 0$.

   (b) What is the probability of finding the system in the state $\left|\tfrac{3}{2}\, \tfrac{3}{2}; \tfrac{1}{2}\, -\tfrac{1}{2}\right\rangle$?

A request

As you know, I will be conducting the first online quiz of my course tomorrow. In this regard, I have a request to all of you who are attending the course and will be taking the quiz. My request is simple: *kindly do not copy.*

Let me elaborate. In all my years of teaching, I have always adopted the following two methods for the quizzes and exams I have conducted: (1) I have allowed the students to keep their own handwritten notes with them, and (2) I have never been strict with the time limit. These methods have been motivated by my belief that the quizzes and exams should free of stress, allowing the students to think clearly and perform at their best. I recall an instance when a set of Ph.D. students took nine hours to complete an exam!

I would like to adopt the same methods for my online quiz as well. During the quiz, you are welcome to look up your own handwritten notes or materials such as the recorded lectures and the solutions to exercises that are available on Moodle. I would request you not to look up books or the internet. Needless to add, I would request you not to consult your classmates or anyone else for that matter. Evidently, it will not be possible for me to monitor all of you remotely. I do not intend to get you to sign an honor code or warn you of dire consequences. I would prefer to trust all of you. I am hoping that you will not break my trust.

Let us not lose sight of the fact that we are here to learn. I should mention that I too learn as I teach. You stand to benefit in your understanding of the topic by working through the problems in the quiz. I will be available online on our regular online platform during the period of the quiz. If you are having any difficulty, you are welcome to speak to me. I will be happy to assist you to the extent that it does not unduly benefit you when compared to the other students.

I am also hoping that you will be able to finish the quiz within the stipulated time. If many of you are having difficulty in completing the quiz in time, I will be glad to permit a little more time to complete it.

To repeat my request, *kindly do not copy* either from books, the internet or your classmates during the quiz.

Good luck.

Quiz I

Rotation, angular momentum and spin

1. *Trace of angular momentum operators:* Calculate the trace of the following combinations of the angular momentum operators: (i) $\hat{J}_i$, (ii) $\hat{J}_i\,\hat{J}_j$ and (iii) $\hat{J}_i\,\hat{J}_j\,\hat{J}_k$. $\boxed{\text{3+3+4 marks}}$

2. *Behavior of the angular momentum operator under rotations:*

   (a) Show that $\boxed{\text{4 marks}}$

   $$\mathrm{e}^{\hat{A}}\,\hat{B}\,\mathrm{e}^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}\,[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}\,[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \cdots.$$

   Note: Such a relation is known as the Baker-Campbell-Hausdorff formula.

   (b) Using the above formula, evaluate the following effect of rotation, say, about the z-axis by an angle ϕ, on the operator $\hat{J}_x$: $\boxed{\text{4 marks}}$

   $$\mathrm{e}^{i\,\hat{J}_z\,\phi/\hbar}\,\hat{J}_x\,\mathrm{e}^{-i\,\hat{J}_z\,\phi/\hbar}.$$

   (c) Do you recognize the form of the transformation? Without calculating, guess the corresponding effects of rotation by an angle ϕ about the z-axis on the operators $\hat{J}_y$ and $\hat{J}_z$. $\boxed{\text{2 marks}}$

3. Rotation operator for $j=1$: Consider a system with angular momentum $j=1$.

   (a) Construct the matrix representing the operator $\hat{J}_y$. $\boxed{\text{3 marks}}$

   (b) Calculate $\hat{J}_y^2$ and $\hat{J}_y^3$. Do you see a pattern? $\boxed{\text{3 marks}}$

   (c) Construct the matrix describing the rotation operator $\mathrm{e}^{-i\,\hat{J}_y\,\phi/\hbar}$. $\boxed{\text{4 marks}}$

4. *Hamiltonian of a spin-$\frac{1}{2}$ system:* Consider a particle with spin-$\frac{1}{2}$ that is described by the following Hamiltonian: $\hat{H} = A\hbar\,\hat{S}_z + B\,\hat{S}_x^2$, where A and B are constants.

   (a) Construct the matrix describing the Hamiltonian of the system. $\boxed{\text{4 marks}}$

   (b) Determine the energy levels of the system. $\boxed{\text{3 marks}}$

   (c) What are energy eigen states of system? $\boxed{\text{3 marks}}$

5. *Addition of spin:* Consider a composite system of two particles with spins $s_1 = 1$ and $s_2 = \frac{1}{2}$.

   (a) What are the possible values of the total spin s of the system? $\boxed{\text{1 mark}}$

   (b) List out all the states $|s,\,m\rangle$ of the composite system, where $s\,(s+1)\,\hbar^2$ and $m\,\hbar$ are the eigen values of the operators $\hat{\boldsymbol{S}}^2 = (\hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2)^2$ and $\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$. $\boxed{\text{2 marks}}$

   (c) Express the states $|s,\,m\rangle$ in terms of the states $|s_1,\,m_1\rangle$ and $|s_2,\,m_2\rangle$. $\boxed{\text{7 marks}}$

Exercise sheet 5

Spherical tensors and the Wigner-Eckart theorem

1. *Decomposing a dyadic:* Consider the dyadic $T_{ij} = u_i\, v_j$, which is a product of the two vectors u_i and v_j.

   (a) Show that the dyadic can be expressed as

   $$T_{ij} = T_{ij}^{(0)} + T_{ij}^{(1)} + T_{ij}^{(2)},$$

   where

   $$T_{ij}^{(0)} = \frac{1}{3}\begin{pmatrix} \boldsymbol{u}\cdot\boldsymbol{v} & 0 & 0 \\ 0 & \boldsymbol{u}\cdot\boldsymbol{v} & 0 \\ 0 & & \boldsymbol{u}\cdot\boldsymbol{v} \end{pmatrix},$$

   $$T_{ij}^{(1)} = \frac{1}{2}\begin{pmatrix} 0 & (u_x\, v_y - u_y\, v_x) & (u_x\, v_z - u_z\, v_x) \\ -(u_x\, v_y - u_y\, v_x) & 0 & (u_y\, v_z - u_z\, v_y) \\ -(u_x\, v_z - u_z\, v_x) & -(u_y\, v_z - u_z\, v_y) & 0 \end{pmatrix} = \frac{1}{2}\,\epsilon_{ijk}\,(\boldsymbol{u}\times\boldsymbol{v})_k\,,$$

   $$T_{ij}^{(2)} = \begin{pmatrix} u_x\, v_x - \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{3} & \frac{(u_x\, v_y + u_y\, v_x)}{2} & \frac{(u_x\, v_z + u_z\, v_x)}{2} \\ \frac{(u_x\, v_y + u_y\, v_x)}{2} & u_y\, v_y - \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{3} & \frac{(u_y\, v_z + u_z\, v_y)}{2} \\ \frac{(u_x\, v_z + u_z\, v_x)}{2} & \frac{(u_y\, v_z + u_z\, v_y)}{2} & u_z\, v_z - \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{3} \end{pmatrix}.$$

   (b) Count the degrees of freedom associated with the dyadic T_{ij} and show that the total degrees of freedom associated with $T_{ij}^{(0)}$, $T_{ij}^{(1)}$, $T_{ij}^{(2)}$ amount to the same number of degrees of freedom.

2. *Constructing spherical tensors:* Recall that the spherical harmonics $Y_l^m(\theta,\phi)$ for $l=0$, $l=1$ and $l=2$ are given by

   $$Y_0^0(\theta,\phi) = \sqrt{\frac{1}{4\,\pi}},$$
   $$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\,\pi}}\cos\theta, \quad Y_1^{\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\,\pi}}\sin\theta\, \mathrm{e}^{\pm i\,\phi},$$
   $$Y_2^0(\theta,\phi) = \sqrt{\frac{5}{16\,\pi}}\left(3\cos^2\theta - 1\right), \quad Y_2^{\pm 1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\,\pi}}\sin\theta\cos\theta\, \mathrm{e}^{\pm i\,\phi},$$
   $$Y_2^{\pm 2}(\theta,\phi) = \mp\sqrt{\frac{15}{32\,\pi}}\sin^2\theta\, \mathrm{e}^{\pm 2\,i\,\phi},$$

   (a) Show that we can write

   $$Y_1^0(x,y,z) = \sqrt{\frac{3}{4\,\pi}}\,\frac{z}{r}, \quad Y_1^{\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\,\pi}}\,\frac{x\pm iy}{r},$$
   $$Y_2^0(x,y,z) = \sqrt{\frac{5}{16\,\pi}}\,\frac{2\,z^2 - x^2 - y^2}{r^2}, \quad Y_2^{\pm 1}(x,y,z) = \mp\sqrt{\frac{15}{8\,\pi}}\,\frac{(x + i\,y)\,z}{r^2},$$
   $$Y_2^{\pm 2}(x,y,z) = \sqrt{\frac{15}{32\,\pi}}\,\frac{(x\pm i\,y)^2}{r^2},$$

   where, as usual, $r = \sqrt{x^2+y^2+z^2}$.

   (b) The above properties motivates us to define the following set of orthonormal unit vectors:

   $$\boldsymbol{e}_0 = \hat{\boldsymbol{z}}, \quad \boldsymbol{e}_\pm = \mp\frac{1}{\sqrt{2}}\,(\hat{\boldsymbol{x}} \mp i\,\hat{\boldsymbol{y}})\,.$$

   Decompose a vector, say, $\boldsymbol{u}$, in the new basis. Also, show that the scalar product between two vectors, say, $\boldsymbol{u}$ and $\boldsymbol{v}$, can be expressed in terms of the components in the new basis as follows:

   $$\boldsymbol{u}\cdot\boldsymbol{v} = -u_+\,v_- + u_0\,v_0 - u_-\,v_+ = \sum_q (-1)^q\, u_q\, v_{-q}.$$

(c) Using the above structure of spherical harmonics and the components $T_{ij}^{(0)}$, $T_{ij}^{(1)}$, $T_{ij}^{(2)}$ of the dyadic T_{ij} in the previous problem, show that we can write the spherical tensors $T_0^{(0)}$, $T_q^{(1)}$ and $T_q^{(2)}$ as follows:

$$\begin{aligned}
T_0^{(0)} &= -\frac{1}{3}\,\boldsymbol{u}\cdot\boldsymbol{v} = \frac{1}{3}\,(u_+\,v_- - u_0\,v_0 + u_-\,v_+)\,,\\
T_q^{(1)} &= \frac{1}{\sqrt{2}\,i}\,(\boldsymbol{u}\times\boldsymbol{v})_q\,,\\
T_0^{(2)} &= \frac{1}{\sqrt{6}}\,(u_+\,v_- + 2\,u_0\,v_0 + u_-\,v+)\,,\\
T_{\pm 1}^{(2)} &= \frac{1}{\sqrt{2}}\,(u_\pm\,v_0 + u_0\,v_\pm)\,,\\
T_{\pm 2}^{(2)} &= u_\pm\,v_\pm.
\end{aligned}$$

(d) Using the above expressions, show that the components $T_{ij}^{(0)}$, $T_{ij}^{(1)}$, $T_{ij}^{(2)}$ of the dyadic can be expressed in terms of the spherical tensors as follows:

$$\begin{aligned}
T_{ij}^{(0)} &= \frac{1}{3}\begin{pmatrix} -T_0^{(0)} & 0 & 0\\ 0 & -T_0^{(0)} & 0\\ 0 & & -T_0^{(0)}\end{pmatrix},\\
T_{ij}^{(1)} &= \begin{pmatrix} 0 & i\,\sqrt{2}\,T_0^{(1)} & T_1^{(1)} + T_{-1}^{(1)}\\ -i\,\sqrt{2}\,T_0^{(1)} & 0 & -i\left(T_1^{(1)} - T_{-1}^{(1)}\right)\\ -\left(T_1^{(1)} + T_{-1}^{(1)}\right) & i\left(T_1^{(1)} - T_{-1}^{(1)}\right) & 0\end{pmatrix},\\
T_{ij}^{(2)} &= \frac{1}{2}\begin{pmatrix} T_2^{(2)} + T_{-2}^{(2)} - \sqrt{\frac{2}{3}}\,T_0^{(2)} & -i\left(T_2^{(2)} - T_{-2}^{(2)}\right) & -T_1^{(2)} + T_{-1}^{(2)}\\ -i\left(T_2^{(2)} - T_{-2}^{(2)}\right) & -T_2^{(2)} - T_{-2}^{(2)} - \sqrt{\frac{2}{3}}\,T_0^{(2)} & i\left(T_1^{(2)} + T_{-1}^{(2)}\right)\\ -T_1^{(2)} + T_{-1}^{(2)} & i\left(T_1^{(2)} + T_{-1}^{(2)}\right) & \sqrt{\frac{8}{3}}\,T_0^{(2)}\end{pmatrix}.
\end{aligned}$$

(e) Consider the case wherein $\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{r}$. In such a case, establish the connection between the components of the second rank tensor $T_{ij} = x_i\,x_j$, and the corresponding spherical tensor $T_q^{(k=2)}$.

3. *Recursion relations:* The complete set of eigenkets $|j_1\,j_2;j\,m\rangle$ allows us to write the state $|j_1\,j_2;j\,m\rangle$ as follows:

$$|j_1\,j_2;j\,m\rangle = \sum_{m_1'}\sum_{m_2'}|j_1\,j_2;m_1'\,m_2'\rangle\langle j_1\,j_2;m_1'\,m_2'|j_1\,j_2;j\,m\rangle.$$

(a) Determine the effect of operating $\hat{J}_\pm = \hat{J}_{1+} \pm \hat{J}_{2+}$ on the above relation.

(b) Take the scalar product of the resulting relation with $\langle j_1\,j_2;m_1\,m_2|$ to arrive at the following recursion relation describing the Clebsch-Gordan coefficients:

$$\begin{aligned}
\sqrt{(j \mp m)\ (j \pm m + 1)}\ &\langle j_1\,j_2;m_1\,m_2|j_1\,j_2;j\,m\pm 1\rangle\\
&= \sqrt{(j_1 \pm m_1)\ (j_1 \mp m_1 + 1)}\,\langle j_1\,j_2;m_1 \mp 1\,m_2|j_1\,j_2;j\,m\rangle\\
&\quad + \sqrt{(j_2 \pm m_2)\ (j_2 \mp m_2 + 1)}\,\langle j_1\,j_2;m_1\,m_2 \mp 1|j_1\,j_2;j\,m\rangle.
\end{aligned}$$

(c) Recall that the spherical tensor $\hat{T}_q^{(k)}$ satisfies the following commutation relations:

$$[\hat{J}_z,\hat{T}_q^{(k)}] = q\,\hbar\,\hat{T}_q^{(k)},\quad [\hat{J}_\pm,\hat{T}_q^{(k)}] = \sqrt{(k \mp q)\ (k \pm q + 1)}\,\hbar\,\hat{T}_{q\pm 1}^{(k)}.$$

Utilize these commutation relations to establish that

$$\begin{aligned}\sqrt{(j'\pm m')\ (j'\mp m'+1)}\,\langle n',j',m'\mp 1|\hat{T}_q^{(k)}|n,j,m\rangle \\ = \sqrt{(j\mp m)\ (j\pm m+1)}\,\langle n',j',m'|\hat{T}_q^{(k)}|n,j,m\pm 1\rangle \\ + \sqrt{(k\mp q)\ (k\pm q+1)}\,\langle n',j',m'|\hat{T}_{q\pm 1}^{(k)}|n,j,m\rangle.\end{aligned}$$

4. *Wigner-Eckart theorem:* Compare the above recursion relations involving the Clebsch-Gordan coefficients and the transition elements involving the spherical tensor $\hat{T}_q^{(k)}$ to show that

$$\langle n',j',m'|\hat{T}_q^{(k)}|n,j,m\rangle = \frac{1}{\sqrt{2\,j+1}}\,\langle j,k;m,q|j,k;j',m'\rangle\,\langle n',j'||\hat{T}^{(k)}||n,j\rangle,$$

where the double vertical bar indicates a matrix element that is independent of m and m'.

Note: We should point out that the first factor on the right hand side is the Clebsch-Gordan coefficient. Also, note that, we have not included the index q in the spherical tensor on the right hand side. It is because of the reason that the matrix element indicated by the double vertical bars is actually independent of q as well (apart from being independent of m and m'). This implies that, if we can evaluate the transition element involving the spherical tensor $\hat{T}_q^{(k)}$ for a particular q, we can arrive at all the remaining transition elements using the Clebsch-Gordan coefficients. This is essentially the statement of the Wigner-Eckart theorem.

5. *Utilizing the Wigner-Eckart theorem:* Let $|n,l,m\rangle$ denote the states of the hydrogen atom and let us define

$$\mathcal{A} = \langle n'=3,l'=2,m'=2|\,\hat{x}\,\hat{y}\,|n=3,l=0,m=0\rangle.$$

For $T_{ij} = x_i\,x_j$, using the Wigner-Eckart theorem, express the matrix elements $\langle n'=3,l'=2,m'|\hat{T}_{ij}\,|n=3,l=0,m=0\rangle$ for all i and j in terms of the quantity $\mathcal{A}$.

Exercise sheet 6

Systems of identical particles – Concepts

1. *Exchange forces:* Consider a pair of free identical particles of mass m. For simplicity, let us assume that they are moving in one dimension. Also, let us ignore their spin. The particles are localized around the points $+a$ and $-a$ and are described by the following real wave functions:

$$\psi_{\pm}(x) = \left(\frac{\beta}{\pi}\right)^{1/4} \exp - \left[\beta\,(x \mp a)^2\right].$$

Evidently, a well-localized state corresponds to $\beta \gg a^{-2}$.

(a) Write down the wave function of the system for the cases wherein the identical particles are bosons or fermions.

(b) Calculate the expectation value of the energy and show that, if the two particles are fermions, then there is an effective repulsion between them.

(c) Compare the result with the case of two identical bosons.

2. *Wavefunctions of three bosons or fermions:* Let $\psi_{f_i}(\xi)$ be the wavefunctions of single particle states normalized to unity, where f_i are the quantum numbers of some complete set.

(a) Write down the normalized wavefunctions for states of a system consisting of three identical weakly interacting bosons which can occupy single particle states with given quantum numbers f_1, f_2 and f_3.

(b) Also, write down the corresponding wavefunctions if the particles are fermions.

3. *Slater determinant:* Consider a solution of the Schrodinger equation of the following form:

$$\hat{H}\,\psi_E(x_1, x_2, \ldots, x_N) = E\,\psi_E(x_1, x_2, \ldots, x_N),$$

which describes N particles. If the total Hamiltonian is given by

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \cdots + \hat{H}_N,$$

where $\hat{H}_i$ denotes the Hamiltonian of the i-th particle, then we can write

$$\psi_E(x_1, x_2, \ldots, x_N) = \psi_{E_1}(x_1)\,\psi_{E_2}(x_2)\,\ldots\psi_{E_N}(x_N),$$

with $E = E_1+E_2+\cdots+E_N$. Prove, say, by induction, that a completely antisymmetric wavefunction describing the system of N particles can be written as a determinant as follows:

$$\psi(x_1, x_2, \cdots, x_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_{E_1}(x_1) & \psi_{E_1}(x_2) & \psi_{E_1}(x_3) & \cdots & \psi_{E_1}(x_N) \\ \psi_{E_2}(x_1) & \psi_{E_2}(x_2) & \psi_{E_2}(x_3) & \cdots & \psi_{E_2}(x_N) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \psi_{E_N}(x_1) & \psi_{E_N}(x_2) & \psi_{E_N}(x_3) & \cdots & \psi_{E_N}(x_N) \end{vmatrix}.$$

Note: The above determinant is often referred to as the Slater determinant.

4. *Ground state energy of a pair of interacting electrons:* Consider a pair of electrons constrained to move in one dimension in the total spin $s = 1$ state. The electrons interact through the following attractive potential:

$$V(x) = \begin{cases} 0 & \text{for } |x_1 - x_2| > a, \\ -V_0 & \text{for } |x_1 - x_2| \le a, \end{cases}$$

where $V_0 > 0$. Find the lowest energy eigenvalue in the case where the total momentum of the two electrons vanishes.

5. *Energy of the ground state of Helium:* Earlier, when we had discussed the energy of the ground state of the Helium atom, we had ignored the interaction between the two electrons.

   (a) Calculate the energy of the ground state of the system when the interaction between the two electrons is accounted for. How does the result compared with the experimentally observed value?

   (b) Can you account for the difference?

Exercise sheet 7

Systems of identical particles – Applications

1. *Fermi energy of copper:* The density of copper is $8.96\,\mathrm{gm/cm^3}$, and its atomic weight is $63.5\,\mathrm{gm/mole}$.

   (a) Assume $q = 1$, calculate the Fermi energy for copper and express it in electron volts.

   (b) What is the corresponding velocity of the electrons if one assumes that the Fermi energy can be written as $E_{\mathrm{F}} = m\,v^2/2$? Is it safe to assume that the electrons in copper are nonrelativistic?

   (c) At what temperature would the characteristic thermal energy (i.e. $k_{\mathrm{B}}\,T$, where k_{B} is the Boltzmann constant and T is the temperature in Kelvin) be equal to the Fermi energy, for copper?

   Note: This is called the Fermi temperature. As long as the actual temperature is substantially below the Fermi temperature, the material can be regarded as 'cold', with most of the electrons in the ground-state configuration. Since the melting point of copper is $1356\,\mathrm{K}$, solid copper is always cold.

   (d) Calculate the degeneracy pressure of copper in the electron gas model.

2. *White dwarfs:* Certain cold stars (called white dwarfs) are stabilized against gravitational collapse by the degeneracy pressure of their electrons. Assuming constant density, the radius R of such an object can be calculated as follows:

   (a) Write the total electron energy in terms of the radius R, the number of nucleons (protons and neutrons) N, the number of electrons per nucleon q, and the mass of the electron m_{e}.

   (b) Calculate the gravitational energy of a uniformly dense sphere. Express the result in terms of G (the constant of gravitation), R, N and the mass of a nucleon, say, m_{n}.

   Note: The gravitational energy will be negative.

   (c) Find the radius for which the sum of the electron and the gravitational energies is a minimum.

   (d) Express the radius R in terms of the number of particles N.

   (e) Determine the radius, in kilometers, of a white dwarf with the mass of the sun.

   (f) Determine the Fermi energy, in electron volts, for the white dwarf and compare it with the rest energy of an electron. What do you notice?

3. *Chandrasekhar limit:* We can extend the theory of a free electron gas to the relativistic domain by replacing the classical kinetic energy $E = p^2/(2\,m)$, with the relativistic formula $E^2 = (p^2\,c^2 + m^2\,c^4) - m^2\,c^4$. Note that the momentum is related to the wave vector in the usual way, viz. $\boldsymbol{p} = h\,\boldsymbol{k}$. Specifically, in the extreme relativistic limit, we have $E \simeq p\,c = \hbar\,c\,k$.

   (a) Replace $\hbar^2\,k^2/(2\,m)$ in the energy associated with the electrons by the ultra-relativistic expression $\hbar\,c\,k$ and calculate the total energy of the relativistic electrons.

   (b) Repeat the arguments of the previous exercise concerning the electron and gravitational energies for the case of the ultra-relativistic electron gas.

   (c) In this case, one finds that, there is no stable minimum, regardless of the radius R. Find the critical number of nucleons N_{c} such that gravitational collapse occurs for $N > N_{\mathrm{c}}$.

   Note: This is called the Chandrasekhar limit.

   (d) Express the corresponding stellar mass in terms of the mass of the Sun.

4. *Bose-Einstein condensation:* Consider a collection of bosons at a finite temperature T. Let $\mu(T)$ be the chemical potential for the system.

(a) Show that, for bosons, the chemical potential must always be less than the minimum allowed energy.

(b) In particular, for the ideal bose gas (identical bosons in the three-dimensional infinite square well), $\mu(T) < 0$ for all T. Show that, in this case, $\mu(T)$ monotonically increases as T decreases, assuming that the total number N and the volume V are held constant.

(c) Bose condensation corresponds to a situation wherein all the particles crowd into the ground state. This occurs as we lower T and $\mu(T)$ hits zero. Evaluate the integral describing the total number of particles N for $\mu = 0$, and determine the critical temperature T_{c} at which this occurs.

(d) Find the critical temperature for ^{4}He.

Note: The density of ^{4}He at this temperature is $0.15\,\mathrm{gm/cm^3}$. Also, the experimental value of the critical temperature in ^{4}He is $2.17\,\mathrm{K}$.

5. *Wien displacement and Stefan-Boltzmann laws:* Consider black body radiation at a temperature T.

(a) Determine the energy density in the wavelength range $\mathrm{d}\lambda$.

(b) Use this expression to derive the following Wien displacement law for the wavelength λ_{max} at which the energy density of the black body radiation is a maximum:

$$\lambda_{\mathrm{max}} = \frac{b}{T},$$

where the quantity b is known as the Wien's constant.

Note: You will encounter a transcendental equation, which you will need to solve numerically.

(c) Determine the value of Wien's constant.

(d) Derive the following Stefan-Boltzmann formula describing the total energy density of black-body radiation:

$$\rho = \frac{4\,\sigma}{c}\,T^4,$$

where σ is a constant known as the Stefan's constant.

(e) Determine the value of Stefan's constant.

Quiz II

Systems of identical particles

1. *Energy of a collection of electrons or bosons:* Consider a system of electrons or bosons that are confined to a box.

   (a) If there are 18 electrons are inside a cubical box of width L, what is the Fermi energy of the system of electrons? $\boxed{\text{2 marks}}$

   (b) What is the total energy of the electrons? $\boxed{\text{3 marks}}$

   (c) What will be the total energy if instead of electrons, there are 18 bosons inside the box? $\boxed{\text{2 marks}}$

   (d) If N electrons are inside a *one-dimensional box of width* L, what will be the total energy of the electrons? $\boxed{\text{3 marks}}$

   Note: In the case of electrons, recall that, there can be two electrons with opposite spins in a given state with a set of quantum numbers describing the spatial wave function.

2. *Two simple harmonic oscillators:* Consider a system of two, non-interacting, simple harmonic oscillators in one-dimension, both with mass m and frequency ω.

   (a) Let x_1 and x_2 denote the coordinates of the two oscillators. What is the Hamiltonian of the system? What are the energy eigen values of the system and what are the corresponding energy eigen functions? $\boxed{\text{3 marks}}$

   (b) Instead of the coordinates x_1 and x_2, let us describe the system in terms of the coordinates $X = (x_1 + x_2)/2$ and $x = x_1 - x_2$. What is the Hamiltonian of the system in terms of the coordinates X and x? Determine the corresponding energy eigen values and the energy eigen functions of the system. $\boxed{\text{7 marks}}$

3. *Energy of interacting bosons:* Two identical bosons, each of mass m, move in the one-dimensional harmonic oscillator potential $V(x) = (m/2)\,\omega^2\,x^2$. The bosons also interact with each other through the potential

$$V_{\rm int}(x_1, x_2) = \alpha\,{\rm e}^{-\beta\,(x_1 - x_2)^2},$$

   where β is a positive quantity.

   (a) What are the ground state energy and the ground state wave function of the system in the absence of the interaction? $\boxed{\text{2 marks}}$

   (b) Write down the expression describing the correction to the ground state energy due to the interaction at the first order in perturbation theory. $\boxed{\text{2 marks}}$

   (c) Evaluate the correction to the ground state energy due to the interaction at the first order in perturbation theory. $\boxed{\text{6 marks}}$

4. *Energy of different types of interacting particles:* Two particles of mass m are placed in a rectangular box of sides $a > b > c$ in the lowest energy state of the system that are compatible with the properties of the particles listed below. The particles interact with each other according to the potential $V(\boldsymbol{r}_1 - \boldsymbol{r}_2) = A\,\delta^{(3)}(\boldsymbol{r}_1 - \boldsymbol{r}_2)$, where, evidently, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ denote the positions of the two particles. Using first order perturbation theory, calculate the energy of the system under the following conditions:

   (a) The particles distinguishable, $\boxed{\text{2 marks}}$

   (b) The particles are indistinguishable and have spin zero, $\boxed{\text{5 marks}}$

   (c) The particles are indistinguishable, have spin one-half, and the spins are parallel. $\boxed{\text{3 marks}}$

5. *Energy of particles in a box:* Consider a system of three bosons with spin zero that are confined to a one-dimensional box. It is known that the total energy of the system, when it is in the lowest energy state, is $3\,\mathrm{eV}$. Given this information, determine the total energy of the following systems of particles, when the system is in the ground (i.e. the lowest energy) state: $\boxed{\text{1+2+2+3+2 marks}}$

   (a) Five bosons,

   (b) Five electrons,

   (c) Five electrons, with all their spins being parallel,

   (d) Five particles with spin-$\frac{3}{2}$,

   (e) Five particles with spin-$\frac{3}{2}$, with all their spins being the same.

Exercise sheet 8

Theory of scattering – Essentials

1. *Classical scattering by a hard sphere:* Consider a small ball which is incident on a large sphere of radius, say, R, and bouncing off elastically.
   (a) Express the impact parameter b of the incident ball in terms of the scattering angle θ.
   (b) Evaluate the differential cross-section $D(\theta)$.
   (c) What is the corresponding total cross-section?

2. *Rutherford scattering in classical mechanics:* An incident particle of charge q and kinetic energy E scatters off a heavy, stationary particle with charge Q.
   (a) Derive the expression relating the impact parameter b to the scattering angle θ.
   (b) Determine the differential scattering cross-section $D(\theta)$.
   (c) Show that the corresponding total cross-section is infinite.

3. *Rayleigh's formula:* Establish the following Rayleigh's formula:
$$\mathrm{e}^{i\,k\,z} = \sum_{l=0}^{\infty} i^l\,(2\,l+1)\,j_l(k\,r)\,P_l(\cos\theta),$$
   where $z = r\cos\theta$, while $j_l(x)$ and $P_l(x)$ denote the spherical Bessel function and the Legendre polynomial, respectively.

4. *Green's function for the Helmholtz equation:* Derive the Green's function $G(\boldsymbol{r})$ associated with the following Helmholtz equation:
$$(\boldsymbol{\nabla}^2 + k^2)\,G(\boldsymbol{r}) = \delta^{(3)}(\boldsymbol{r}),$$
   where $\delta^{(3)}(\boldsymbol{r})$ denotes the three-dimensional Dirac delta function.

5. *The Lippmann-Schwinger equation and the integral form of the Schrodinger equation:* Consider the Hamiltonian of a system which can be expressed as $H = H_0 + V$, where H_0 is the Hamiltonian of the free particle. Let $|\psi_0\rangle$ denote the energy eigen ket of the free particle with energy E_0, i.e.
$$\hat{H}_0\,|\psi_0\rangle = E_0\,|\psi_0\rangle.$$
   Also, let $|\psi\rangle$ denote an energy eigen ket of the complete system with eigen value E satisfying the equation
$$(\hat{H}_0 + \hat{V})\,|\psi\rangle = E\,|\psi\rangle.$$
   (a) Argue that we can rewrite the above equation as
$$|\psi_\pm\rangle = |\psi_0\rangle + \frac{1}{E - H_0 \pm i\,\epsilon}\,\hat{V}\,|\psi_\pm\rangle,$$
   where ϵ is an infinitesimal, positive definite quantity.
   Note: This equation is known as the Lippmann-Schwinger equation.
   (b) Show that, in the position representation, the above equation corresponds to the following integral form of the Schrodinger equation:
$$\psi(\boldsymbol{r}) = \psi_0(\boldsymbol{r}) - \frac{m}{2\,\pi\,\hbar^2}\int \mathrm{d}^3\boldsymbol{r}'\,\frac{\mathrm{e}^{-i\,k\,|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\,V(\boldsymbol{r}')\,\psi(\boldsymbol{r}'),$$
   where $\psi_0(\boldsymbol{r}) = \langle\boldsymbol{r}|\psi_0\rangle$ and $\psi(\boldsymbol{r}) = \langle\boldsymbol{r}|\psi\rangle$.
   (c) For appropriate $V(\boldsymbol{r})$ and E, explicitly check that the ground state of the hydrogen atom satisfies above the integral form of the Schrodinger equation.

Exercise sheet 9

Theory of scattering – Examples

1. *Quantum scattering by a 'hard' sphere:* Consider a 'hard' repulsive spherical potential of the following form:
$$V(r) = \begin{cases} \infty & \text{for } r \le a, \\ 0 & \text{for } r > a. \end{cases}$$
Using the method of partial waves, determine the corresponding scattering amplitude and the total cross-section.

   Note: This can be considered to be the quantum mechanical version of classical scattering by a hard sphere.

2. *Scattering by a 'soft' sphere:* Now, consider the case wherein the repulsive spherical potential is 'softened' to a finite value in the following manner:
$$V(r) = \begin{cases} V_0 & \text{for } r \le a, \\ 0 & \text{for } r > a, \end{cases}$$
where $V_0 > 0$.

   (a) Working in the Born approximation, determine the scattering amplitude, the differential cross-section and the total cross-section for the above potential at low energies.

   (b) What is the corresponding result for arbitrary energies? Does it reduce to the result obtained at low energies?

3. *Yukawa and Rutherford scattering:* Consider the following spherically symmetric Yukawa potential:
$$V(r) = \beta\, \frac{\mathrm{e}^{-\mu\, r}}{r},$$
where β and μ are constants.

   (a) Working in the Born approximation, evaluate the corresponding scattering amplitude and differential cross-section.

   (b) Determine the scattering amplitude and differential cross-section for the case wherein $\beta = Q\, q/(4\,\pi\,\epsilon_0)$ and $\mu = 0$, which corresponds to Rutherford scattering. How does the differential cross-section compare with the classical result?

4. *Scattering by a spherical shell:* Consider scattering by a spherical shell described by the following potential:
$$V(r) = \alpha\, \delta^{(1)}(r - r_0),$$
where α and r_0 are constants.

   (a) Using the method of partial waves, calculate the scattering amplitude at low energies.

   (b) Evaluate the corresponding differential and total cross-sections.

   (c) Using the Born approximation, calculate the scattering amplitude and the differential cross-section at low energies. Does the result match the one obtained using the method of partial waves?

   (d) Working in the Born approximation, calculate the scattering amplitude for arbitrary energies.

5. *Beyond the first order Born approximation:* Earlier, using the first order Born approximation, we had calculated the scattering amplitude at low energies in the following potential:

$$V(r) = \begin{cases} V_0 & \text{for } r \leq a, \\ 0 & \text{for } r > a, \end{cases}$$

where $V_0 > 0$. Evaluate the correction to the scattering amplitude due to the second order Born approximation.

Additional exercises II

From systems of identical particles to the theory of scattering

1. *Possible configurations of identical bosons:* Three identical bosons with spin $s = 1$ are in the same orbital states described by the wavefunction $\psi(\boldsymbol{r})$.

   (a) Write down the normalized spin functions for the total system.

   (b) How many independent states are possible?

   (c) What are the possible values of the total spin of the system?

2. *Ortho and para states:* Two identical spin-$\frac{1}{2}$ fermions move in one dimension inside an infinite square well described by the potential

$$V(x) = \begin{cases} 0 & \text{for } 0 < x < L, \\ \infty & \text{otherwise.} \end{cases}$$

   (a) Write down the ground state wave function and the ground state energy when the two particles are in the triplet spin state, referred to as the ortho state.

   (b) What is the ground state wave function and the corresponding energy when they are in the singlet spin state, called the para state?

   (c) Let us now suppose that the two particles are interacting mutually through the short range attractive potential $V(x) = \lambda\, \delta(x_1 - x_2)$, with $\lambda > 0$. Use first order perturbation theory to calculate the corrections to the above energies due to this mutual interaction.

3. *Ground state and Fermi energies of fermions:* Consider N identical spin-$\frac{1}{2}$ particles which are subjected to the one-dimensional simple harmonic oscillator potential.

   (a) What is the Fermi energy of the system?

   (b) What is the energy of the ground state of the system?

   (c) What are the ground state and Fermi energies if we ignore the mutual interactions and assume N to be very large?

4. *Neutron stars:* Stars that are heavier than the Chandrasekhar limit will not form white dwarfs, but they will collapse further, finally becoming neutron stars under certain conditions. This is due to the fact that, at very high density, inverse beta decay, i.e. $e + p \to n + \nu$, converts virtually all of the protons and electrons into neutrons, liberating neutrinos in the process, which carry off energy. Eventually, neutron degeneracy pressure stabilizes the collapse, just as electron degeneracy does for the white dwarfs.

   (a) Calculate the radius of a neutron star with the mass of the sun.

   (b) Also calculate the neutron Fermi energy, and compare it with the rest energy of a neutron. Is it reasonable to treat such a star non-relativistically?

5. *Distinguishable particles in a thermal bath:* Consider a collection of N distinguishable, non-interacting, quantum particles which are in thermal equilibrium at a finite temperature T. Assume that each of these particles experiences the simple harmonic potential.

   (a) Working in one spatial dimension, evaluate the chemical potential and the total energy of the collection of particles.

   (b) What are the energies in the quantum and classical limits, viz. when $k_{\mathrm{B}}\, T \ll \hbar\, \omega$ and $k_{\mathrm{B}}\, T \gg \hbar\, \omega$? Can you interpret the results?

   (c) What are the corresponding results in three spatial dimensions?

6. *Phase shift in one-dimensional scattering:* A particle of mass m and energy E is incident from the left on the potential

$$V(x) = \begin{cases} 0 & \text{for } x < -a, \\ -V_0 & \text{for } -a \leq x \leq 0, \\ \infty & \text{for } x > 0, \end{cases}$$

where $V_0 > 0$. Consider an incoming wave $A\,\mathrm{e}^{i\,k\,x}$, where $k = \sqrt{2\,m\,E}/\hbar$ with $E > 0$, which is reflected by the above potential.

   (a) Find the reflected wave.

   (b) What is the amplitude of the reflected wave? Does the reflection alter the amplitude of incident wave?

   (c) Assuming that the well is deep, i.e. $-V_0 \ll E$, determine the shift in the phase of the reflected wave.

7. *Partial waves and phase shifts:* Recall that, using the method of partial waves, we had obtained the scattering amplitude to be

$$f(\theta) = \sum_{l=0}^{\infty} (2\,l+1)\,a_l\,P_l(\cos\theta),$$

where $P_l(x)$ denotes the Legendre polynomials and a_l was called the l-th partial wave amplitude.

   (a) Using the above result, arrive at the corresponding expression for the differential cross-section and show that total cross-section can be written as

$$\sigma = \sum_{l=0}^{\infty} (2\,l+1)\,|a_l|^2.$$

   (b) Focusing on a particular l, show that the partial wave amplitude a_l can be expressed in terms of the phase shift δ_l as follows:

$$a_l = \frac{1}{k}\,\mathrm{e}^{i\,\delta_l}\,\sin(\delta_l).$$

   (c) Use this form of a_l to arrive an expression for the total cross-section in terms of the phase shifts δ_l.

8. *The optical theorem:* According to the so-called optical theorem, the imaginary part of the scattering amplitude $f(\theta)$ along the forward direction is related to the total cross-section σ through the following relation:

$$\mathrm{Im.}\,f(\theta=0) = \frac{k\,\sigma}{4\,\pi}.$$

Use the results from the previous exercise to establish the theorem.

9. *Scattering in one dimension:* Consider scattering by a potential $V(x)$ in one spatial dimension. In such a case, we can write the time-independent Schrodinger equation as

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + k^2\,\psi = Q,$$

where $k = \sqrt{2\,m\,E}/\hbar$ and $Q = (2\,m/\hbar^2)\,V\,\psi$.

   (a) Obtain the Green's function for the above Schrodinger equation.

   (b) Use it to construct the integral form of the equation.

   (c) Develop the Born approximation for one-dimensional scattering and obtain an expression for the reflection coefficient in terms of the potential $V(x)$.

(d) Use the Born approximation to determine the reflection and transmission coefficients for the cases of the delta function potential

$$V(x) = -\alpha\, \delta^{(1)}(x),$$

with $\alpha > 0$, and the finite square well potential of the following form:

$$V(x) = \begin{cases} -V_0 & \text{for } -a < x < a, \\ 0 & \text{for } -|x| > a, \end{cases}$$

with $V_0 > 0$.

(e) Compare the results from the Born approximation with the exact ones.

10. *Scattering by an electric dipole:* Consider an electric dipole consisting of two electric charges e and $-e$ located at, say, $x = -a$ and $x = a$. A particle of charge e and mass m described by the wave vector $\boldsymbol{k} = k\,\hat{\boldsymbol{z}}$ is incident on the dipole. Calculate the scattering amplitude in the Born approximation. Find the directions at which the differential cross section is maximal.

Quiz III

Theory of scattering

1. *Scattering by a repulsive potential in classical mechanics:* Consider scattering by a central potential of the following form in classical mechanics:

$$V(r) = \frac{\alpha}{r^2},$$

where $\alpha > 0$. Let $E = m\,v^2/2$ be the energy of the incoming particle, with m being its mass and v being its velocity when it is far away from the scattering centre. Also, let b be the impact parameter of the particle.

   (a) Determine the scattering angle θ as a function of the impact parameter b and the asymptotic incoming velocity v. 7 marks

   (b) Obtain the corresponding differential cross-section. 3 marks

2. *Classical scattering by a spherical potential well:* Consider a particle with impact parameter b that is being scattered by a spherical potential well of depth $-V_0$ and radius a.

   (a) In fact, to begin with, consider two regions with constant potentials, say, V_1 and V_2, separated by a planar surface. A particle with initial velocity $\boldsymbol{v}_1$ travels across the surface from the region described by the potential V_1 to the region characterized by the potential V_2. Let θ_1 and θ_2 be the angles formed by the velocity vector with respect to the normal at the interface on either side. Using conservation of momentum and energy, establish that 2 marks

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{1 + \frac{2\,(V_1 - V_2)}{m\,v_1^2}}.$$

   (b) Using this result, show that, in the case of scattering by the spherical potential well of depth $-V_0$, the impact parameter b is related to the scattering angle through the relation 4 marks

$$b^2 = a^2\,\frac{n^2\sin^2(\theta/2)}{n^2 + 1 - 2\,n\cos(\theta/2)},$$

   where the quantity n is given by

$$n = \sqrt{1 + \frac{2\,V_0}{m\,v^2}}$$

   with v being the initial velocity of the incoming particle.

   (c) Determine the associated differential cross-section. 2 marks

   (d) What is the total cross-section? 2 marks

   Note:

$$\int_1^n \mathrm{d}x\,\frac{(x-1)\,(n^2-x)}{(n^2+1-2\,x)^2} = \frac{1}{2}.$$

3. *Scattering in one dimension:* Consider scattering by a potential $V(x)$ in one spatial dimension. In such a case, we can write the time-independent Schrodinger equation as

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + k^2\,\psi = Q,$$

where $k = \sqrt{2\,m\,E}/\hbar$ and $Q = (2\,m/\hbar^2)\,V\,\psi$.

(a) Obtain the Green's function for the above Schrodinger equation. $\boxed{\text{4 marks}}$

(b) Use it to construct the integral form of the equation. $\boxed{\text{1 mark}}$

(c) Develop the Born approximation for one-dimensional scattering and obtain an expression for the reflection coefficient in terms of the potential $V(x)$. $\boxed{\text{3 marks}}$

(d) Use the Born approximation to determine the reflection and transmission coefficients for the cases of the delta function potential

$$V(x) = -\alpha\, \delta^{(1)}(x),$$

with $\alpha > 0$. $\boxed{\text{2 marks}}$

4. *Phase shift in one-dimensional scattering:* A particle of mass m and energy E is incident from the left on the potential

$$V(x) = \begin{cases} 0 & \text{for } x < -a, \\ -V_0 & \text{for } -a \le x \le 0, \\ \infty & \text{for } x > 0, \end{cases}$$

where $V_0 > 0$.

(a) Consider an incoming wave $A\, e^{i\,k\,x}$, where $k = \sqrt{2\,m\,E}/\hbar$ with $E > 0$, which is reflected by the above potential. Find the reflected wave. $\boxed{\text{5 marks}}$

(b) What is the amplitude of the reflected wave? Does the reflection alter the amplitude of incident wave? $\boxed{\text{3 marks}}$

(c) Express the wave function in the region $x < -a$ as

$$\psi(x) = A\,\left[e^{i\,k\,x} - e^{i\,(2\,\delta - k\,x)}\right].$$

In other words, after refection, the wave can be considered to have suffered a phase shift characterized by the quantity δ. Assuming that the well is deep, i.e. $-V_0 \ll E$, determine the shift in the phase δ of the reflected wave. $\boxed{\text{2 marks}}$

5. *Scattering cross-section in the first Born approximation:* Using the first Born approximation, calculate the scattering amplitude, the differential cross-section and the total cross-section for a particle moving the central potential $V(r) = V_0 \exp - (r^2/a^2)$. $\boxed{\text{5+1+4 marks}}$

Exercise sheet 10

The Dirac equation and solutions describing the free particle

1. *Properties of the Dirac matrices:* Recall that the four Dirac matrices α_i with $i = (1, 2, 3)$ and β are defined as

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathcal{I} & 0 \\ 0 & -\mathcal{I} \end{pmatrix},$$

where σ_i and $\mathcal{I}$ are the 2×2 Pauli matrices and the unit matrix given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(a) Show that the Dirac matrices satisfy the following relations:

$$\alpha_i\, \alpha_j + \alpha_j\, \alpha_i = 2\, \delta_{ij}, \quad \alpha_i\, \beta + \beta\, \alpha_i = 0, \quad \alpha_i^2 = \beta^2 = 1.$$

(b) Show that the sum of the eigen values is zero for all the four matrices.

(c) Determine the eigen values of all the Dirac matrices.

2. *Pauli's Hamiltonian and the gyromagnetic ratio of the electron:* Consider an electron in an external magnetic field $\boldsymbol{B}$. In non-relativistic quantum mechanics, the interaction of the magnetic moment associated with the spin of the electron $\boldsymbol{S}$ and the magnetic field is described by the following Hamiltonian originally due to Pauli:

$$\hat{H} = -\frac{e\, g}{2\, m\, c}\, \hat{\boldsymbol{S}} \cdot \boldsymbol{B} = -\frac{e\, g\, \hbar}{4\, m\, c}\, \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{B},$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ denotes the Pauli matrices, and the quantity g is known as the gyromagnetic ratio. Evidently, e and m denote the charge and the mass of the electron, while c represents the velocity of light.

Note: The gyromagnetic ratio g of a system is defined as the ratio of its magnetic moment and angular momentum.

(a) Show that, in the absence of the magnetic field, the Hamiltonian of the free, non-relativistic electron can be expressed as

$$\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2\, m} = \frac{1}{2\, m}\, (\sigma \cdot \hat{\boldsymbol{p}})^2\,.$$

(b) In the presence of an external magnetic field, we can replace the momentum operator $\hat{\boldsymbol{p}}$ by $\hat{\boldsymbol{p}} - (e/c)\, \boldsymbol{A}$, where $\boldsymbol{A}$ is the vector potential describing the magnetic field. In such a case, the Hamiltonian of the electron can be written as

$$\hat{H} = \frac{1}{2\, m}\, \left[\sigma \cdot \left(\hat{\boldsymbol{p}} - \frac{e}{c}\, \boldsymbol{A}\right)\right]^2.$$

Using the properties of the Pauli matrices, show that the interaction of the electron's spin with the magnetic field is described by Pauli's Hamiltonian above with $g = 2$.

Note: For this reason, the gyromagnetic ratio of the electron is said to be 2.

3. *Properties of the gamma matrices:* Recall that the gamma matrices $\gamma^\mu = (\gamma^0, \gamma^i)$ are defined as

$$\gamma^0 = \beta, \quad \gamma^i = \beta\, \alpha_i.$$

Show that the relations for the matrices α_i and β (listed in the previous exercise) can be expressed as

$$\gamma^\mu\, \gamma^\nu + \gamma^\nu\, \gamma^\mu = 2\, \eta^{\mu\nu}\, \mathcal{I},$$

where $\eta^{\mu\nu}$ is the metric tensor describing the Minkowski spacetime and $\mathcal{I}$ represents the 4×4 unit matrix.

4. <u>*Spinors describing particles and anti-particles:*</u> Recall that the solutions to the Dirac equation which governs relativistic free particles can be expressed as

$$\psi(\boldsymbol{r},t) = A\,\mathrm{e}^{-(i/\hbar)\,(E\,t-\boldsymbol{p}\cdot\boldsymbol{r})}\,u(E,\boldsymbol{p}),$$

where E and $\boldsymbol{p} = (p_x, p_y, p_z)$ represent the energy and momentum of the particle, while $u(E,\boldsymbol{p})$ denotes a bispinor and A is a constant. The spinors $u(E,\boldsymbol{p})$ that describe particles are given by

$$u^{(1)}(E,\boldsymbol{p}) = N \begin{pmatrix} 1 \\ 0 \\ \frac{c\,p_z}{E+m\,c^2} \\ \frac{c\,(p_x+i\,p_y)}{E+m\,c^2} \end{pmatrix}, \quad u^{(2)}(E,\boldsymbol{p}) = N \begin{pmatrix} 0 \\ 1 \\ \frac{c\,(p_x-i\,p_y)}{E+m\,c^2} \\ \frac{-c\,p_z}{E+m\,c^2} \end{pmatrix},$$

where $E = \sqrt{p^2\,c^2 + m^2\,c^4}$ and N is the normalization constant. The spinors s $v(E,\boldsymbol{p})$ that describe anti-particles are given by

$$v^{(1)}(E,\boldsymbol{p}) = N \begin{pmatrix} \frac{c\,(p_x-i\,p_y)}{E+m\,c^2} \\ \frac{-c\,p_z}{E+m\,c^2} \\ 0 \\ 1 \end{pmatrix}, \quad v^{(2)}(E,\boldsymbol{p}) = -N \begin{pmatrix} \frac{c\,p_z}{E+m\,c^2} \\ \frac{c\,(p_x+i\,p_y)}{E+m\,c^2} \\ 1 \\ 0 \end{pmatrix}.$$

(a) Show that, in $u^{(1)}$ and $u^{(2)}$, the bottom two components (i.e. u_B) are smaller than the top two components (i.e. u_A) by the factor of v/c in the non-relativistic limit.

(b) The scalar product between two spinors, say, u and v, is defined as $u^\dagger\,v$. Determine the normalization constant N for all the above spinors by setting $u^\dagger\,u = 2\,E/c$.

(c) Show that $u^{(1)}$ and $u^{(2)}$ are orthogonal. Similarly, show that $v^{(1)}$ and $v^{(2)}$ are also orthogonal.

(d) Are $u^{(1)}$ and $u^{(2)}$ orthogonal to $v^{(1)}$ and $v^{(2)}$?

5. <u>*From the Dirac equation to the Klein-Gordon equation:*</u> Recall that, in its covariant form, the Dirac equation can be expressed as

$$(i\,\hbar\,\gamma^\mu\,\partial_\mu - m\,c)\;\psi = 0,$$

where $\partial_\mu = \partial/\partial\,x^\mu$ and ψ represents a spinor. By applying the operator $\gamma^\nu\,\partial_\nu$ on the above equation, show that all the components of the Dirac spinor satisfy the following Klein-Gordon equation:

$$\left(\partial^\mu\,\partial_\mu + \frac{m^2\,c^2}{\hbar^2}\right)\,\psi = \left(\Box + \frac{m^2\,c^2}{\hbar^2}\right)\,\psi = 0.$$

Exercise sheet 11

Spin and helicity of particles described by the Dirac equation

1. *Eigen spinors of $\hat{S}_z$:* Recall that the spin operator describing the Dirac particles is given by

$$\hat{\boldsymbol{S}} = \frac{\hbar}{2}\,\boldsymbol{\Sigma},$$

where

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}.$$

We had seen earlier that the spinors $\left\{u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}\right\}$ (listed in previous exercise sheet) are *not* eigen spinors of $\hat{\boldsymbol{S}}_z$ operator. Show that, when $p_x = p_y = 0$ so that $p_z = p$, these spinors prove to be eigen spinors of $\hat{\boldsymbol{S}}_z$. What are the corresponding eigen values?

2. *Spin of particles described by the Dirac field:* Show that every bispinor is an eigen state of the operator $\hat{\boldsymbol{S}}^2$. Determine the corresponding eigen value and identify the spin of the particle described by the Dirac equation.

3. *Conservation of total angular momentum:* Consider a relativistic particle described by the Dirac equation.

   (a) What is the Hamiltonian $\hat{H}$ of the relativistic particle?

   (b) Evaluate the commutator of $\hat{H}$ with the orbital angular momentum $\hat{\boldsymbol{L}} = \hat{\boldsymbol{r}} \times \hat{\boldsymbol{p}}$.

   (c) Also, calculate the commutator of $\hat{H}$ with the spin angular momentum $\hat{\boldsymbol{S}} = (\hbar/2)\,\hat{\boldsymbol{\Sigma}}$.

   (d) Show that, while the orbital and spin angular momenta $\hat{\boldsymbol{L}}$ and $\hat{\boldsymbol{S}}$ are not conserved individually, the total angular momentum $\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}$ is conserved.

4. *Spinors with definite helicity:* Construct normalized spinors, say, $u^{(+)}$ and $u^{(-)}$, representing an electron of momentum $\boldsymbol{p}$ with helicity ± 1. In other words, obtain solutions to $(\gamma^\mu\, p_\mu - m\,c)\; u = 0$, with positive energy E, which are eigen spinors of the helicity operator $\boldsymbol{p}\cdot\boldsymbol{\Sigma}$ with eigenvalues ± 1.

5. *Massless Dirac particles and helicity:* Consider the case of massless particles described by the Dirac equation.

   (a) Writing the Dirac bispinor ψ in terms of the spinors describing the particles and anti-particles, say, ϕ and χ, arrive at the equations governing these spinors in Fourier space (or, equivalently, in the energy-momentum space).

   (b) Let us define two new spinors (which are linear combinations of the original spinors ϕ and χ) as follows:

$$\psi_\pm = \frac{1}{\sqrt{2}}\,(\chi \pm \phi).$$

   Arrive at the equations describing the spinors $\psi_\pm$.

   (c) Show that the $\psi_\pm$ spinors describe particles and anti-particles with definite helicity.

Note: Unlike electrons and positrons which can be described by states with different helicities (as we saw in the previous exercise), neutrinos and anti-neutrinos have definite helicities. While neutrinos are left handed (i.e. they have helicity -1), anti-neutrinos are right handed (i.e. they have helicity $+1$). Therefore, they can be described by spinors such as ψ_- and ψ_+, respectively.