Lecture 40-41: Failure analysis (static failure theories)

Joshua Pribe

Fall 2019

Lecture Book: Ch. 15

PURDUE
ENGINEERING

Motivation

- We have spent the last few classes finding the **state of stress** at various points in a body due to **combined loading**
  - We have seen various combinations of normal stresses and shear stresses

- **Mohr's circle** gives us a way to **compare** different states of stress
  - For **any** state of stress, we can identify three important parameters: the **two in-plane principal stresses** and the **absolute maximum shear stress**

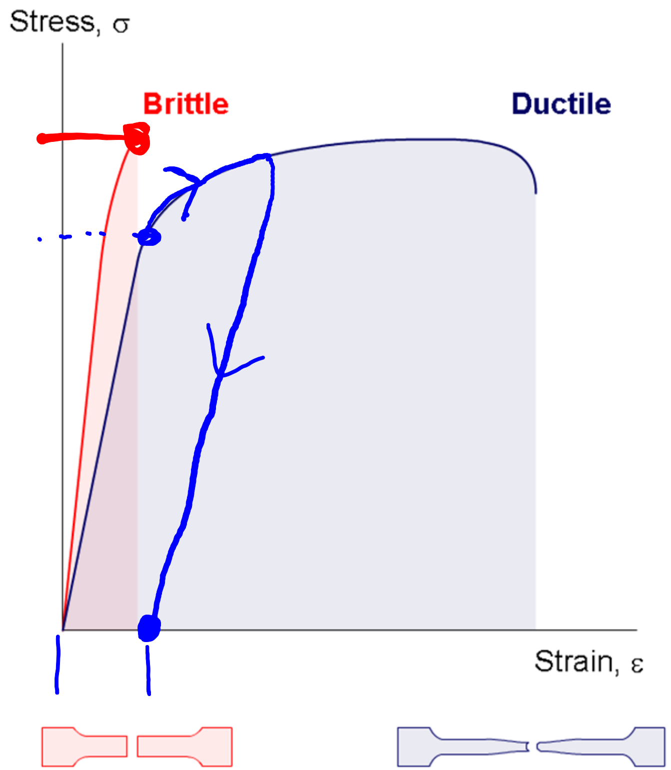
- Now: how can we **use** this information to predict whether a point in a body will **fail**?
  - First, we need to define what "failure" means…this depends on the type of material!

Failure theories overview

- Use the results of “simple” tests to formulate hypotheses
  - Usual hypothesis: the mechanism that causes failure in a tensile test is *the same mechanism* that causes failure in more complex stress states
- We have different failure theories for brittle and ductile materials
- “All models are wrong, but some are useful”

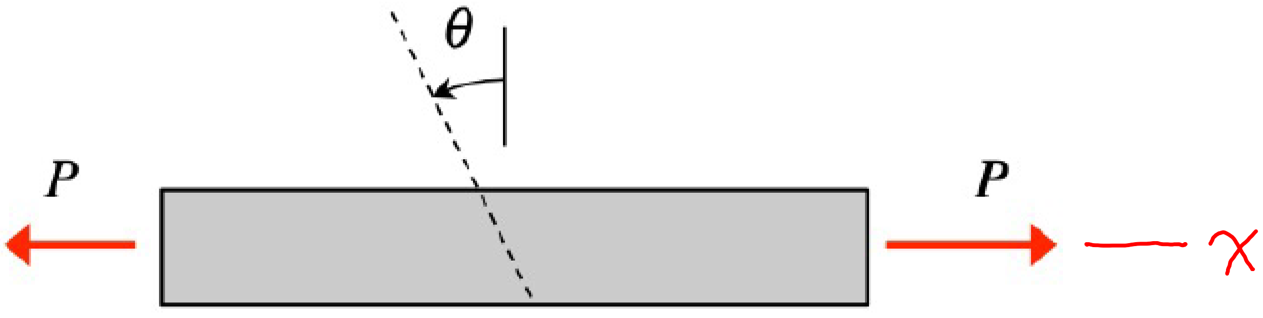
fracture ← σ_u

permanent ← σ_y
deformation

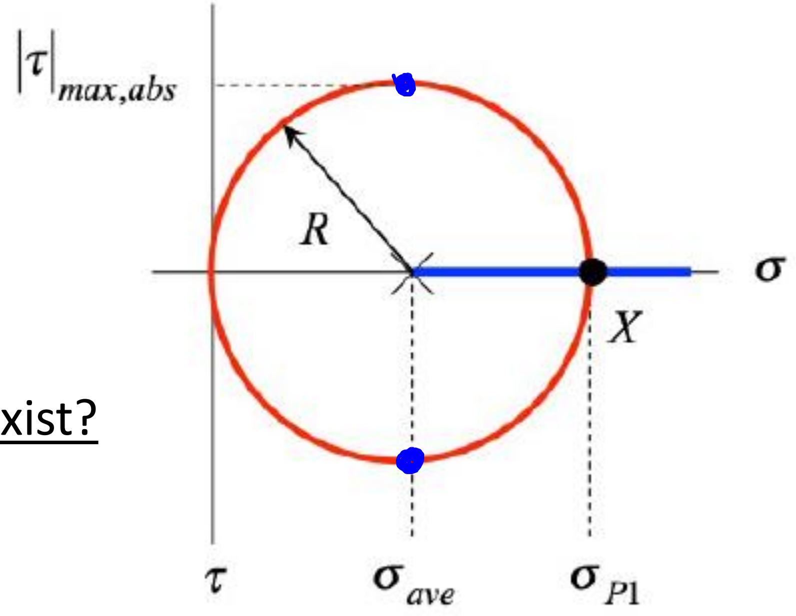


The tensile test **Lecture Book: Ch. 15, pg. 2**

Free-body diagram



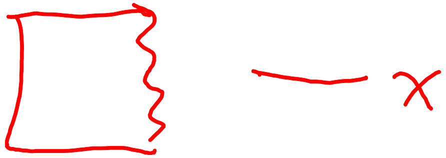
Mohr's circle at any point on any cross section



What is the maximum normal stress? On which plane does this stress exist?

$\sigma_{P1} = \sigma = P/A = \sigma_x$

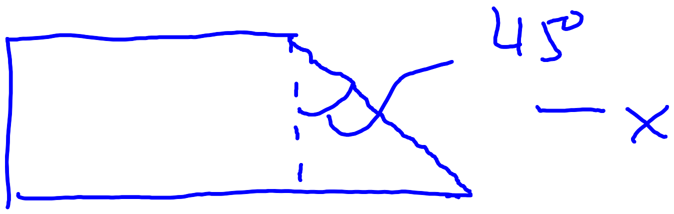
$\theta = 0°$



What is the maximum shear stress? On which planes does this stress exist?

$\tau_{max}^{abs} = R = \sigma/2$

$\theta = \pm 45°$

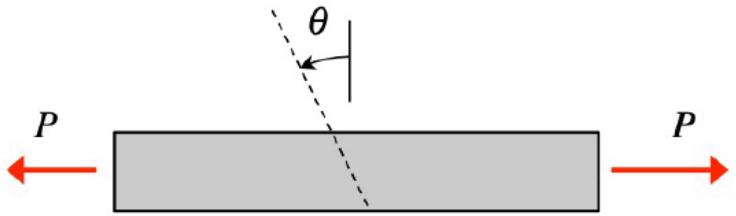


Brittle failure: Maximum normal stress theory

<u>Hypothesis</u>: A brittle material fractures when the maximum principal stress equals or exceeds the ultimate normal stress when fracture occurs in a tensile test

Define this "ultimate normal stress" as the **ultimate strength**

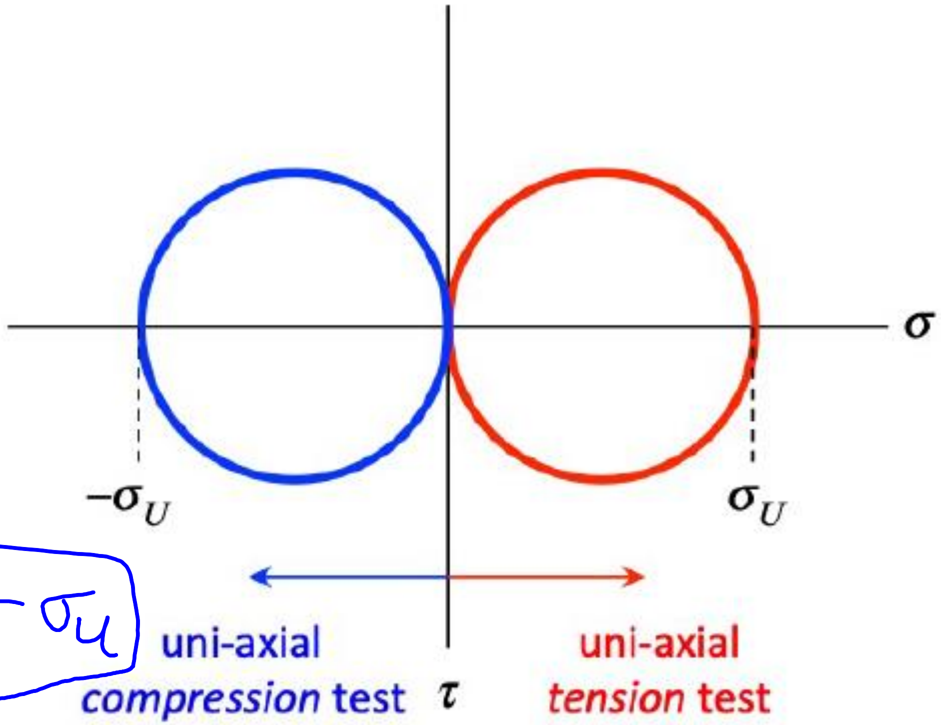
Assumption: the ultimate strength in tension and compression is the same

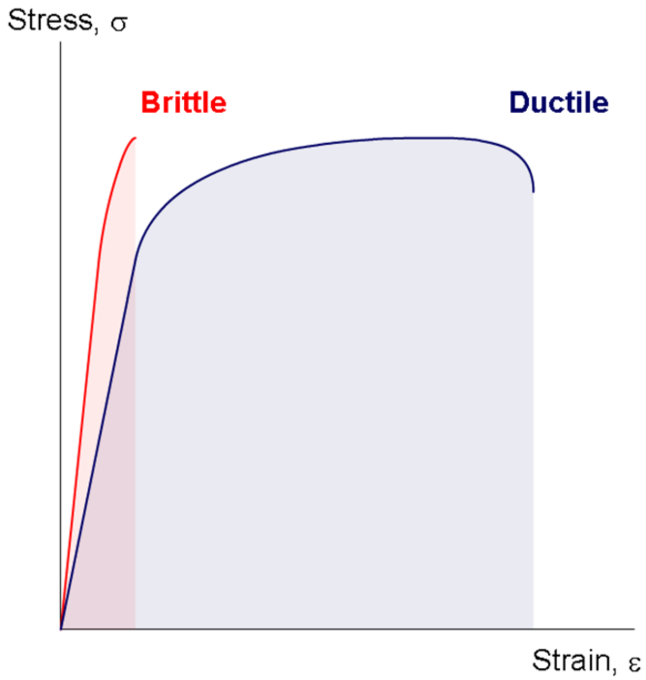


tensile: $\frac{P}{A} = \sigma_u$

$\boxed{\sigma_{p1} \geq \sigma_u}$

Comp: $\frac{P}{A} = -\sigma_u \rightarrow \boxed{\sigma_{p2} \leq -\sigma_u}$



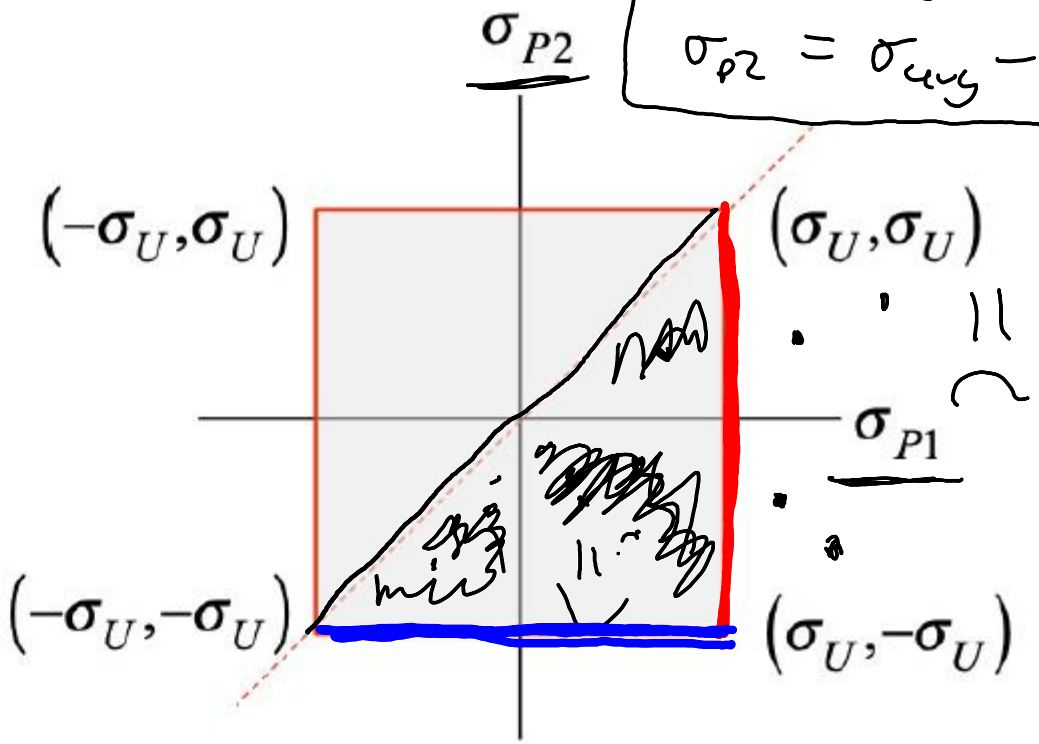


Lecture Book: Ch. 15, pg. 11

Brittle failure: Maximum normal stress theory

We can visualize the failure boundary in principal stress space

$$\boxed{\begin{aligned} \sigma_{p1} &= \sigma_{avg} + R \\ \sigma_{p2} &= \sigma_{avg} - R \end{aligned}}$$



Failure criteria

$$\sigma_{p1} \geq \sigma_U \longrightarrow \boxed{FS = \frac{\sigma_u}{\sigma_{p1}}}$$

$$\sigma_{p2} \leq -\sigma_U \longrightarrow \boxed{FS = \left| \frac{\sigma_u}{\sigma_{p2}} \right|}$$

Factor of safety

The "real" FS is whichever of these is smaller

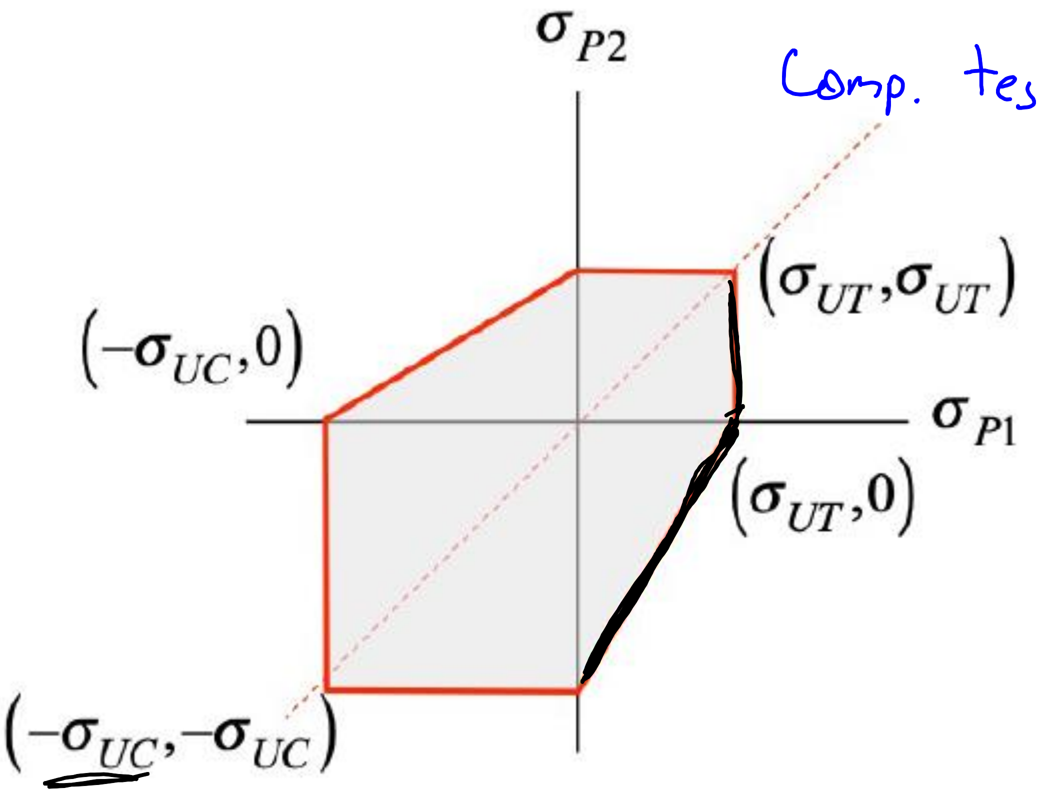
Brittle failure: Mohr's theory

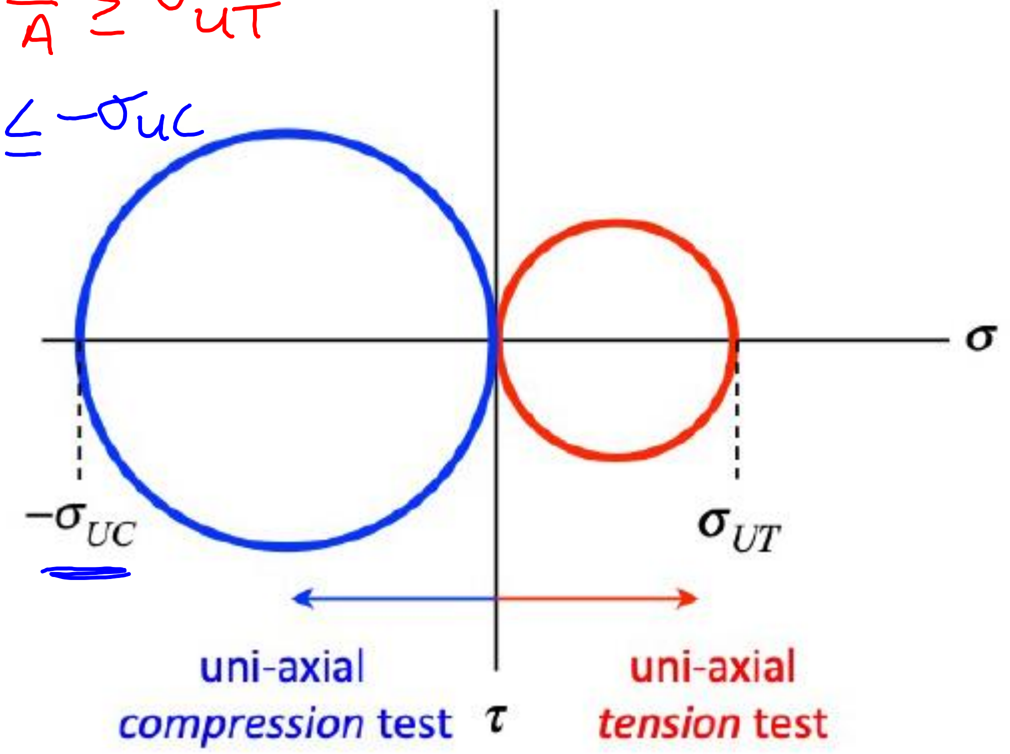
$\sigma_{UC} > \sigma_{UT}$

Modification to maximum normal stress theory based on the observation that many materials are stronger in compression than they are in tension, i.e. $\sigma_{UT} < \sigma_{UC}$, and the maximum normal stress theory is non-conservative when the principal stresses have different signs

tensile test → $\frac{P}{A} \geq \sigma_{UT}$

Comp. test → $\frac{P}{A} \leq -\sigma_{UC}$





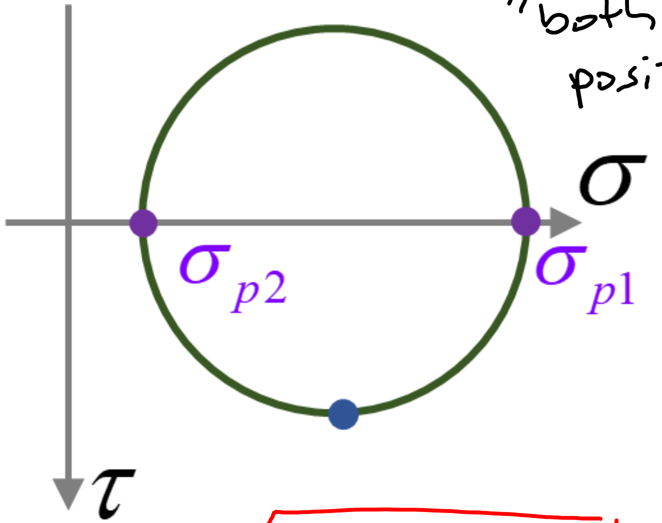
Lecture Book: Ch. 15, pg. 12

Brittle failure: Mohr's theory

For a general state of plane stress, there are three possible situations **Lecture Book: Ch. 15, pg. 6**

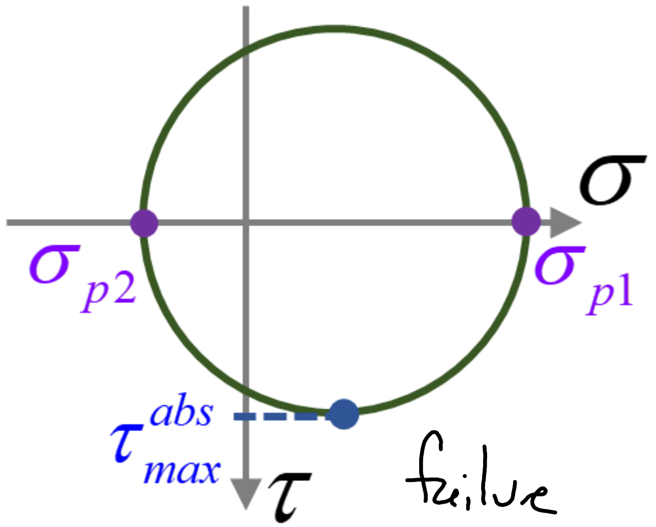
<u>Case 1:</u> $\underline{\sigma_{p1} > \sigma_{p2} > 0}$

"both positive"



failure: $\boxed{\sigma_{p1} \geq \sigma_{UT}}$

<u>Case 2:</u> $\sigma_{p1} > 0 > \sigma_{p2}$

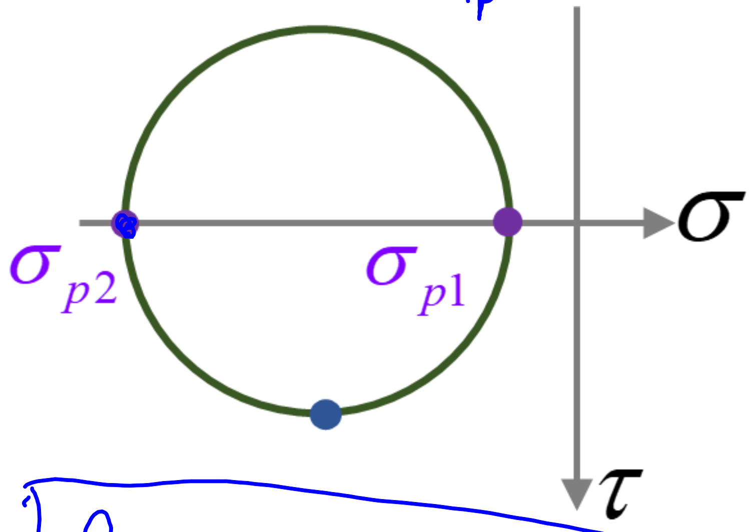


failure

$$\boxed{\frac{\sigma_{p1}}{\sigma_{UT}} - \frac{\sigma_{p2}}{\sigma_{UC}} \geq 1}$$

<u>Case 3:</u> $0 > \sigma_{p1} > \sigma_{p2}$

"both comp."

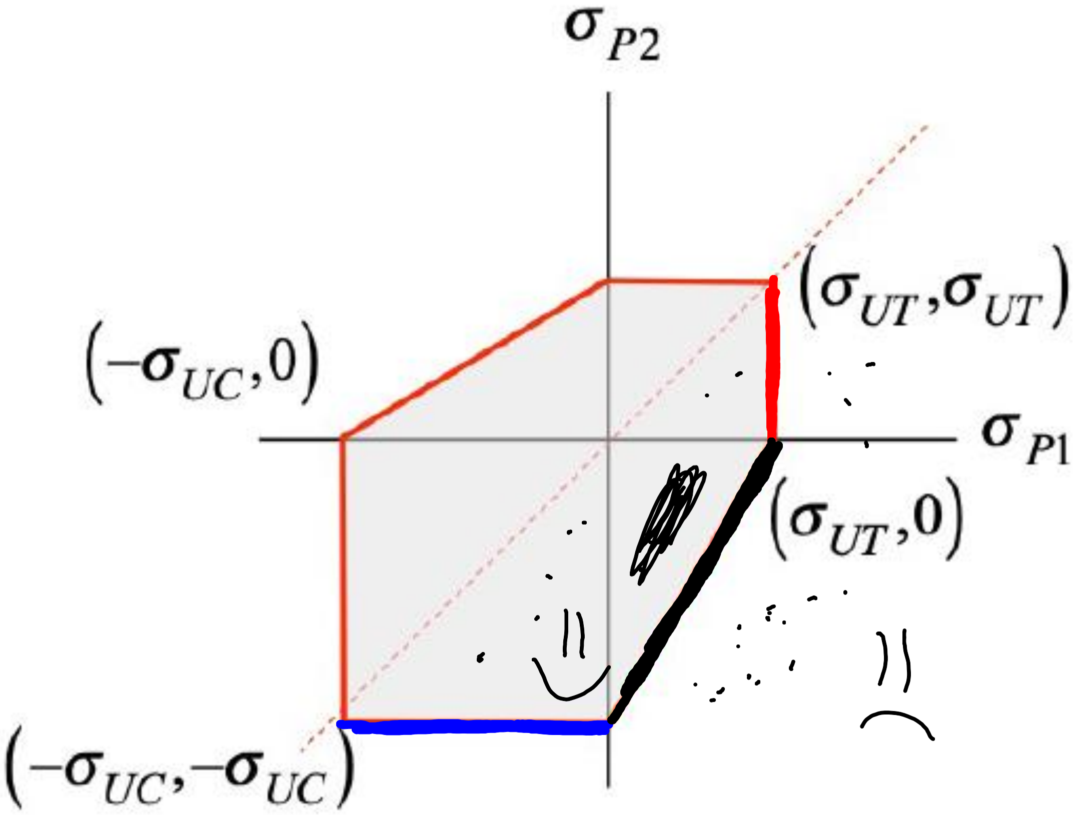


$\boxed{\text{failure: } \sigma_{p2} \leq -\sigma_{UC}}$

Lecture Book: Ch. 15, pg. 12

Brittle failure: Mohr's theory

We can visualize the failure boundary in principal stress space



Failure criteria

Factor of safety

Case 1: $\sigma_{p1} > \sigma_{p2} > 0$

$\sigma_1 = \sigma_{p1}$

$\sigma_2 = \sigma_{p2}$

$\sigma_3 = 0$

$$\sigma_{p1} \geq \sigma_{UT}$$

$\rightarrow FS = \frac{\sigma_{UT}}{\sigma_{p1}}$

Case 2: $\sigma_{p1} > 0 > \sigma_{p2}$

$\sigma_1 = \sigma_{p1}$

$\sigma_2 = 0$

$\sigma_3 = \sigma_{p2}$

$$\frac{\sigma_{p1}}{\sigma_{UT}} - \frac{\sigma_{p2}}{\sigma_{UC}} \geq 1$$

$\rightarrow FS = \frac{1}{\frac{\sigma_{p1}}{\sigma_{UT}} - \frac{\sigma_{p2}}{\sigma_{UC}}}$

Case 3: $0 > \sigma_{p1} > \sigma_{p2}$

$\sigma_1 = 0$

$\sigma_2 = \sigma_{p1}$

$\sigma_3 = \sigma_{p2}$

$$\sigma_{p2} \leq -\sigma_{UC}$$

$\rightarrow FS = \left|\frac{\sigma_{UC}}{\sigma_{p2}}\right|$

Lecture Book: Ch. 15, pg. 12

Re-order the principal stresses → $\sigma_1 > \sigma_2 > \sigma_3$

$$\frac{\sigma_1}{\sigma_{UT}} - \frac{\sigma_3}{\sigma_{UC}} \geq 1$$

Brittle failure: Summary

Maximum normal stress theory

Failure criterion

$$\sigma_{p1} \geq \sigma_U \quad \text{or} \quad \sigma_{p2} \leq -\sigma_U$$

Factor of safety

$$FS = \left|\frac{\sigma_U}{\sigma_{p1}}\right| \quad \text{or} \quad FS = \left|\frac{\sigma_U}{\sigma_{p2}}\right|$$

(whichever is *smaller* is the real factor of safety)

Mohr's failure theory

Failure criteria (3 possible cases based on the signs of the principal stresses

$$\sigma_{p1} > \sigma_{p2} > 0: \ \sigma_{p1} \geq \sigma_{UT}$$

$$0 > \sigma_{p1} > \sigma_{p2}: \ \sigma_{p2} \leq -\sigma_{UC}$$

$$\sigma_{p1} > 0 > \sigma_{p2}: \ \frac{\sigma_{p1}}{\sigma_{UT}} - \frac{\sigma_{p2}}{\sigma_{UC}} \geq 1$$

$$\boxed{\frac{\sigma_1}{\sigma_{UT}} - \frac{\sigma_3}{\sigma_{UC}} \geq 1}$$

Factor of safety

$$\sigma_{p1} > \sigma_{p2} > 0: \ FS = \frac{\sigma_{UT}}{\sigma_{p1}}$$

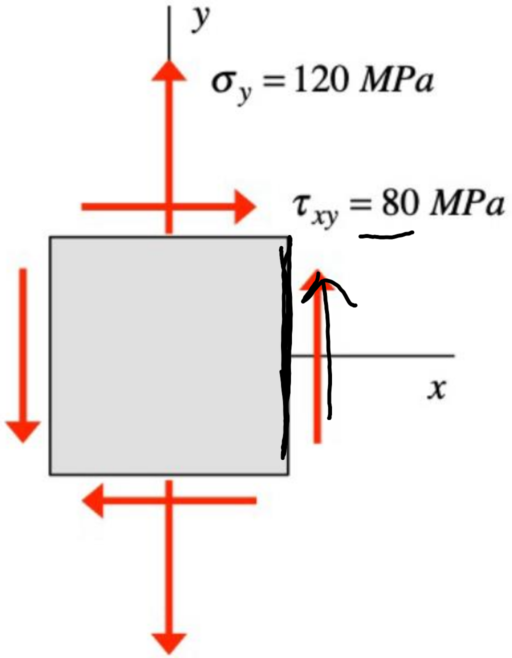
$$0 > \sigma_{p1} > \sigma_{p2}: \ FS = \left|\frac{\sigma_{UC}}{\sigma_{p2}}\right|$$

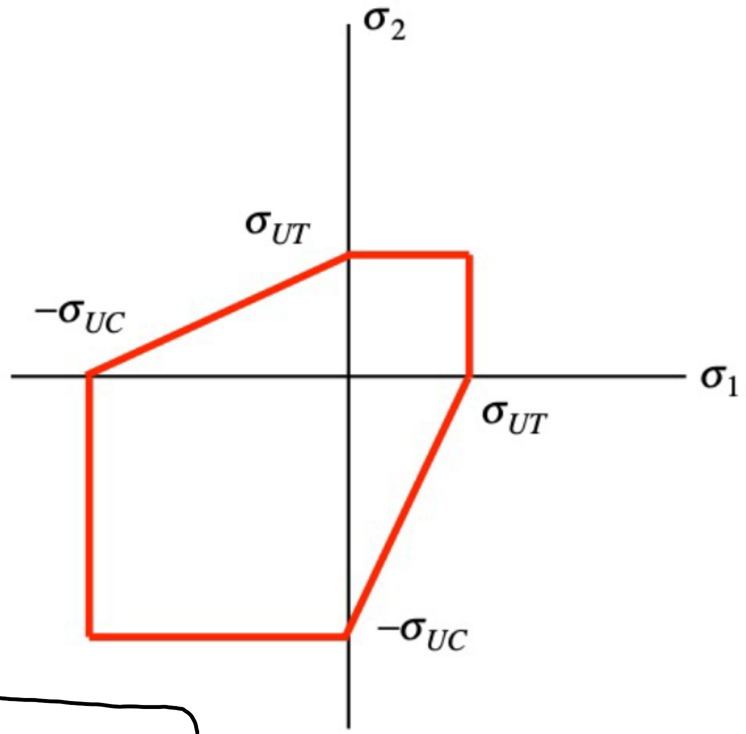
$$\sigma_{p1} > 0 > \sigma_{p2}: \ FS = \frac{1}{\dfrac{\sigma_{p1}}{\sigma_{UT}} - \dfrac{\sigma_{p2}}{\sigma_{UC}}} = \frac{\sigma_{UT}\sigma_{UC}}{\sigma_{p1}\sigma_{UC} - \sigma_{p2}\sigma_{UT}}$$

Re-order $\sigma_1 > \sigma_2 > \sigma_3 \rightarrow FS = \dfrac{1}{\dfrac{\sigma_1}{\sigma_{UT}} - \dfrac{\sigma_3}{\sigma_{UC}}}$

Example 15.7

The state of stress shown exists at a location in a component made of a brittle material with $\sigma_{UC} = 850$ MPa and $\sigma_{UT} = 170$ MPa. According to Mohr's theory, has the material failed?





$\sigma_x = 0$

$\sigma_y = +120$ MPa

$\tau_{xy} = +80$ MPa

$\sigma_{avg} = \dfrac{120+0}{2} = 60$ MPa

$R = \sqrt{\left(\dfrac{0-120}{2}\right)^2 + (80)^2} = \sqrt{60^2 + 80^2} = 100$ MPa

$\Rightarrow \boxed{\begin{aligned} \sigma_{p1} &= \sigma_{avg} + R = 160 \text{ MPa} \\ \sigma_{p2} &= \sigma_{avg} - R = -40 \text{ MPa} \end{aligned}}$

$\sigma_1 = \sigma_{p1} = 160$ MPa $\quad \sigma_2 = 0 \quad \sigma_3 = \sigma_{p2} = -40$ MPa

$\cancel{\sigma_{p1} \geq \sigma_{uT}}$ ~~OR~~ $\cancel{\sigma_{p2} \leq \sigma_{uC}}$ $\sigma_{p3} = 0$

$$* \boxed{\frac{\sigma_{p1}}{\sigma_{uT}} - \frac{\sigma_{p2}}{\sigma_{uC}} \geq 1} * \rightarrow \frac{160\text{ MPa}}{170\text{ MPa}} - \frac{(-40\text{ MPa})}{850\text{ MPa}}$$

or

$$* \frac{\sigma_1}{\sigma_{uT}} - \frac{\sigma_3}{\sigma_{uC}} \geq 1 * \quad \text{where } \sigma_1 > \sigma_2 > \sigma_3$$

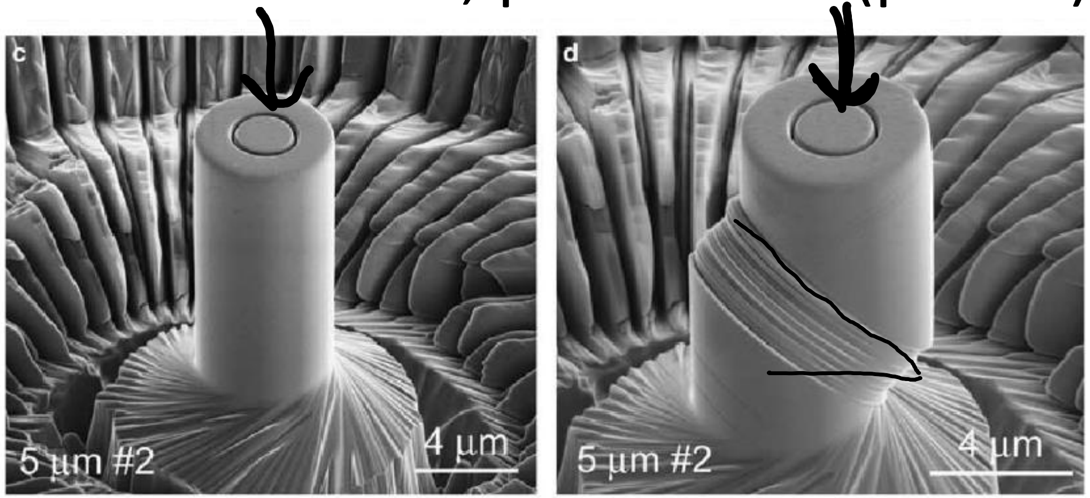
$$= 0.99 \leq 1 \Rightarrow \boxed{\text{Safe}}$$

What value of σ_{p2} would cause failure? $\rightarrow \sigma_{p2}^{fail}$

$$\rightarrow \frac{\sigma_{p1}}{\sigma_{uT}} - \frac{\sigma_{p2}^{fail}}{\sigma_{uC}} = 1 \rightarrow \sigma_{p2}^{fail} = -50\text{ MPa}$$

Ductile failure: Maximum shear stress theory

- On the microscale, permanent (plastic) deformation occurs by “slip”



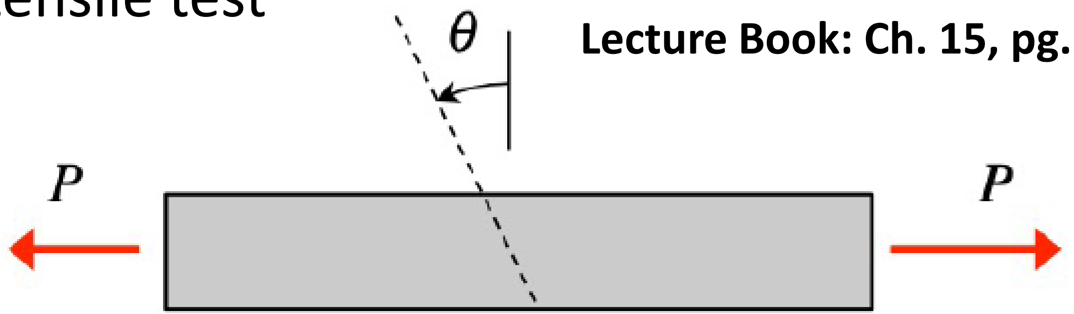
"micropillar compression"

- Failure in a tensile test of a ductile material often looks very similar

Aluminum – failure due to normal or shear stress?

Ductile failure: Maximum shear stress theory

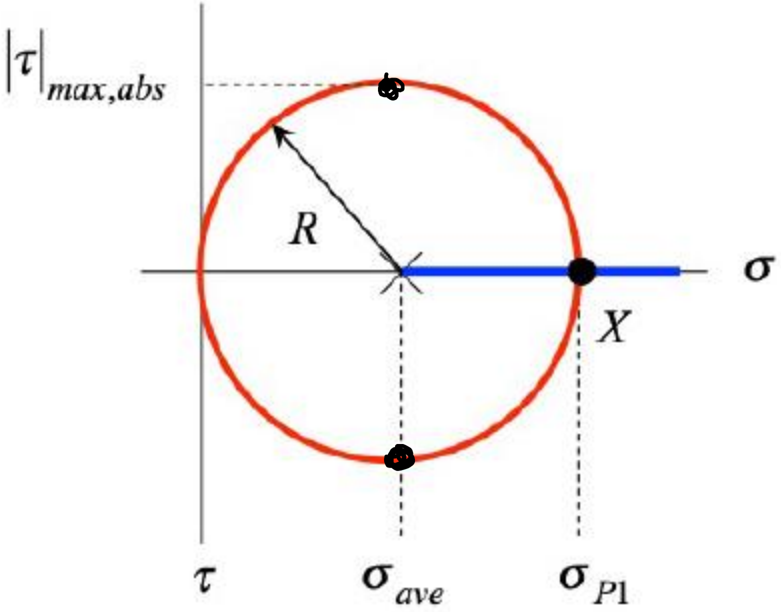
Hypothesis: for *any* stress state, yielding of a ductile material occurs when the *absolute maximum shear stress* equals or exceeds the maximum shear stress when yielding occurs in a tensile test

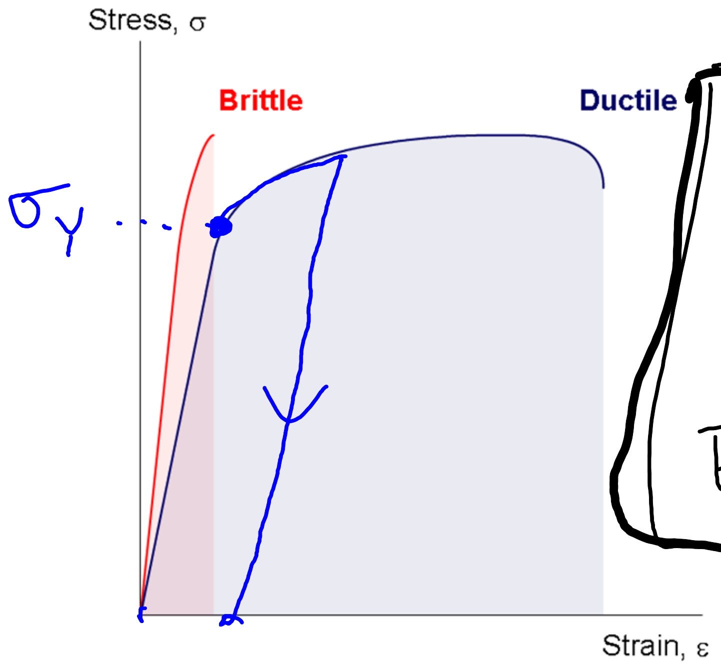


Lecture Book: Ch. 15, pg. 5

Uniaxial tension → $\tau_{max}^{abs} = R = \frac{\sigma}{2}$

failure → $\sigma_{P1} = \sigma = \sigma_Y$





Failure criterion:

$$\tau_{max}^{abs} \geq \frac{\sigma_Y}{2}$$

Factor of safety: $FS = \frac{\sigma_Y}{2\tau_{max}^{abs}}$

Lecture Book: Ch. 15, pg. 5

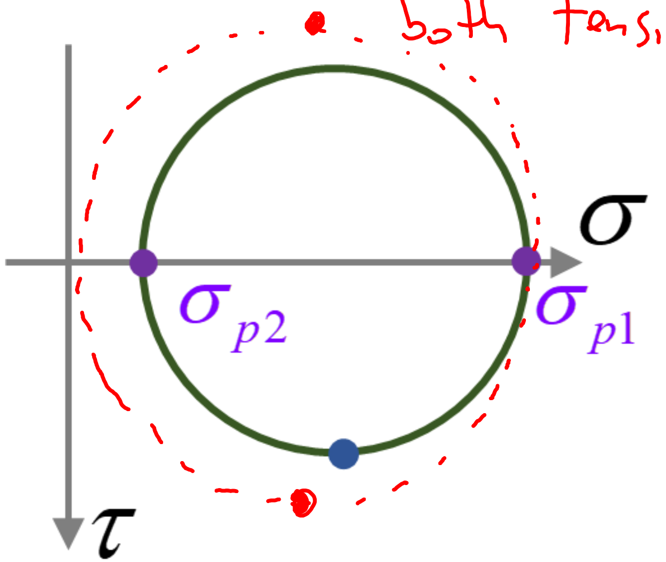
Ductile failure: Maximum shear stress theory

For a general state of plane stress, there are three possible situations

Lecture Book: Ch. 15, pg. 6

<u>Case 1:</u> $\sigma_{p1} > \sigma_{p2} > 0$

"both tensile"

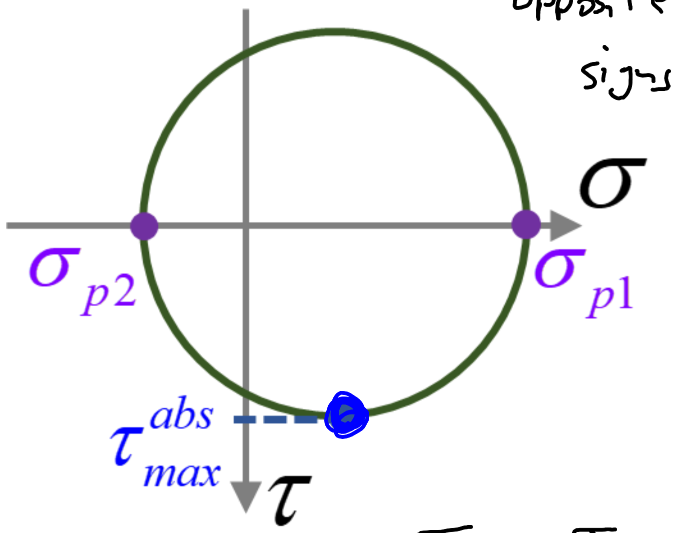


$\tau_{max}^{abs} = \sigma_{p1}/2$

Failure: $\tau_{max}^{abs} = \frac{\sigma_{p1}}{2} \geq \frac{\sigma_Y}{2}$

<u>Case 2:</u> $\sigma_{p1} > 0 > \sigma_{p2}$

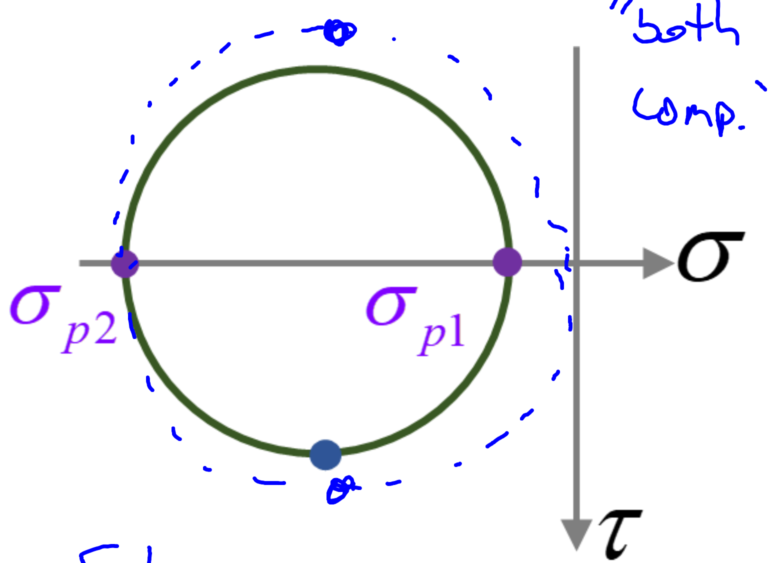
"opposite signs"



$\tau_{max}^{abs} = R = \frac{\sigma_{p1} - \sigma_{p2}}{2}$

<u>Case 3:</u> $0 > \sigma_{p1} > \sigma_{p2}$

"both comp."



Failure:

$\tau_{max}^{abs} = \left| \frac{\sigma_{p2}}{2} \right| \geq \frac{\sigma_Y}{2}$

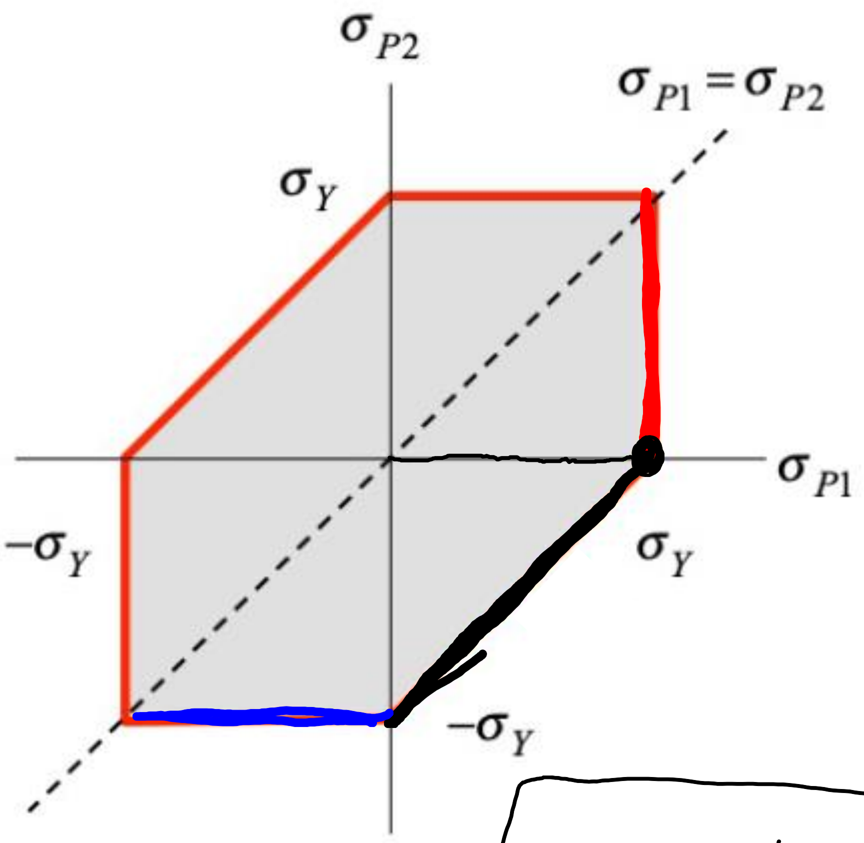
Reorder the principal stresses so that $\sigma_1 > \sigma_2 > \sigma_3$

$$\boxed{\tau_{max}^{abs} = \left| \frac{\sigma_1 - \sigma_3}{2} \right|} \longrightarrow \boxed{\tau_{max}^{abs} \geq \frac{\sigma_Y}{2}}$$ is the failure criterion

Ductile failure: Maximum shear stress theory

We can visualize the failure boundary in principal stress space

Lecture Book: Ch. 15, pg. 7



Failure criteria

Case 1: $\sigma_{p1} > \sigma_{p2} > 0$

$\sigma_{p1} \geq \sigma_Y$

$\sigma_1 = \sigma_{p1}$, $\sigma_2 = \sigma_{p2}$, $\sigma_3 = 0$

Case 2: $\sigma_{p1} > 0 > \sigma_{p2}$

$\sigma_{p1} - \sigma_{p2} \geq \sigma_Y$

$\sigma_1 = \sigma_{p1}$, $\sigma_2 = 0$, $\sigma_3 = \sigma_{p2}$

Case 3: $0 > \sigma_{p1} > \sigma_{p2}$

$|\sigma_{p2}| \geq \sigma_Y$

$\sigma_1 = 0$, $\sigma_2 = \sigma_{p1}$, $\sigma_3 = \sigma_{p2}$

$$\boxed{\tau_{max}^{abs} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| \geq \sigma_Y}$$

Factor of safety

$$FS = \frac{\sigma_Y}{2\tau_{max}^{abs}} = \frac{\sigma_Y}{|\sigma_1 - \sigma_3|}$$

Ductile failure: Maximum distortional energy theory

von Mises proposed a <u>different hypothesis</u>: yielding occurs when the *distortion energy density* equals or exceeds the distortion energy density when yielding occurs in a tensile test

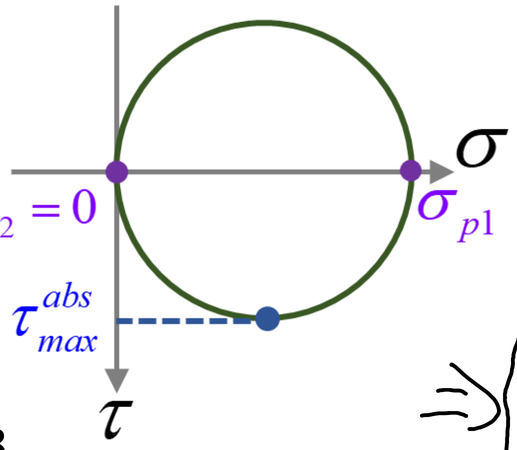
Evidence: a material subjected to purely hydrostatic stress $(\sigma_{p1} = \sigma_{p2} = \sigma_{p3})$ *never* yields

Total elastic strain energy density = change of volume + distortion (change of shape)

$$\bar{u} = \frac{1}{2E}\left[\sigma_{p1}^2 + \sigma_{p2}^2 - 2\nu\sigma_{p1}\sigma_{p2}\right] \qquad \bar{u}_v = \frac{1}{2G}\left(\sigma_{p1} + \sigma_{p2}\right) \qquad \bar{u}_d = \frac{1}{6G}\left(\sigma_{p1}^2 - \sigma_{p1}\sigma_{p2} + \sigma_{p2}^2\right)$$

<u>For yielding in the tensile test</u>

$$\bar{u}_{d,yield} = \frac{1}{6G}\sigma_Y^{\ 2}$$



So, our failure criterion for *<u>any plane stress state</u>* is

$$\frac{1}{6G}\boxed{\left(\sigma_{p1}^2 - \sigma_{p1}\sigma_{p2} + \sigma_{p2}^2\right)} = \frac{1}{6G}\boxed{\sigma_Y}^{2}$$

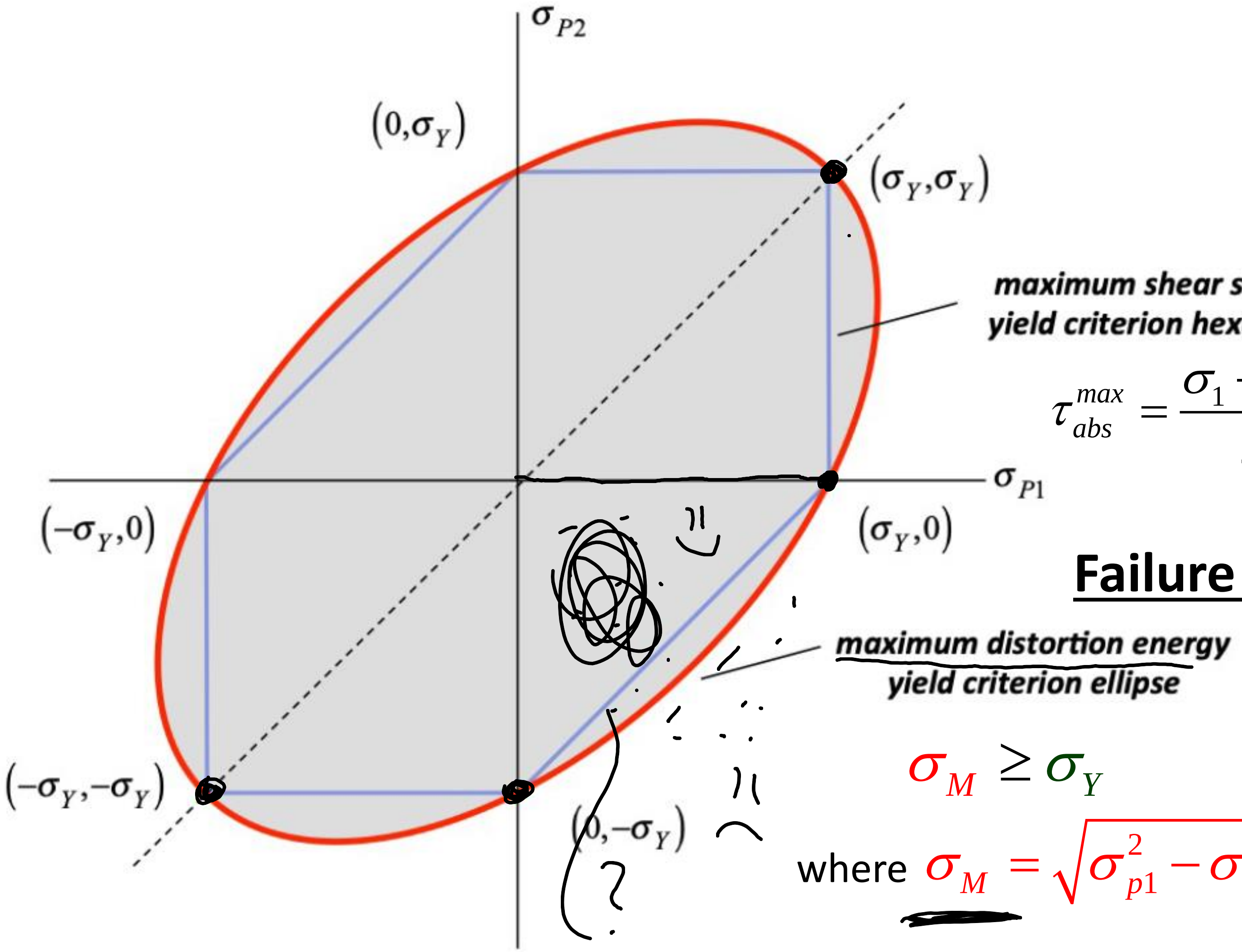
Von Mises stress: $\sigma_M = \sqrt{\sigma_{p1}^2 - \sigma_{p1}\sigma_{p2} + \sigma_{p2}^2}$

$\Rightarrow$ failure criterion: $\sigma_M \geq \sigma_Y$

Lecture Book: Ch. 15, pg. 8

Ductile failure: Maximum distortional energy theory

In principal stress space, the maximum distortional energy failure boundary is an ellipse



$$\tau_{abs}^{max} = \frac{\sigma_1 - \sigma_3}{2} \geq \frac{\sigma_Y}{2}$$

Failure criterion

$$\sigma_M \geq \sigma_Y$$

where $\sigma_M = \sqrt{\sigma_{p1}^2 - \sigma_{p1}\sigma_{p2} + \sigma_{p2}^2}$

Factor of safety

$$FS = \frac{\sigma_Y}{\sigma_M}$$

Ductile failure: Summary

Maximum shear stress theory

Failure criterion: $\tau_{max}^{abs} \geq \dfrac{\sigma_Y}{2}$

3 possible cases for τ_{max}^{abs} based on signs of principal stresses

$$\sigma_{p1} > \sigma_{p2} > 0: \ \sigma_{p1} \geq \sigma_Y \qquad 0 > \sigma_{p1} > \sigma_{p2}: \ |\sigma_{p2}| \geq \sigma_Y$$

$$\sigma_{p1} > 0 > \sigma_{p2}: \ \sigma_{p1} - \sigma_{p2} \geq \sigma_Y$$

Or, if you re-order the principal stresses so $\sigma_1 > \sigma_2 > \sigma_3$,

$\boxed{\tau_{abs}^{max} = \dfrac{\sigma_1 - \sigma_3}{2} \geq \dfrac{\sigma_Y}{2}}$ is the failure criterion for all cases

Factor of safety: $\boxed{FS = \dfrac{\sigma_Y}{2\tau_{max}^{abs}} = \dfrac{\sigma_Y}{\sigma_1 - \sigma_3}}$

Maximum distortional energy (von Mises) theory

Failure criterion (based on the von Mises stress):

$$\boxed{\sigma_M = \sqrt{\sigma_{p1}^2 - \sigma_{p1}\sigma_{p2} + \sigma_{p2}^2} \geq \sigma_Y}$$

Factor of safety: $FS = \dfrac{\sigma_Y}{\sigma_M}$

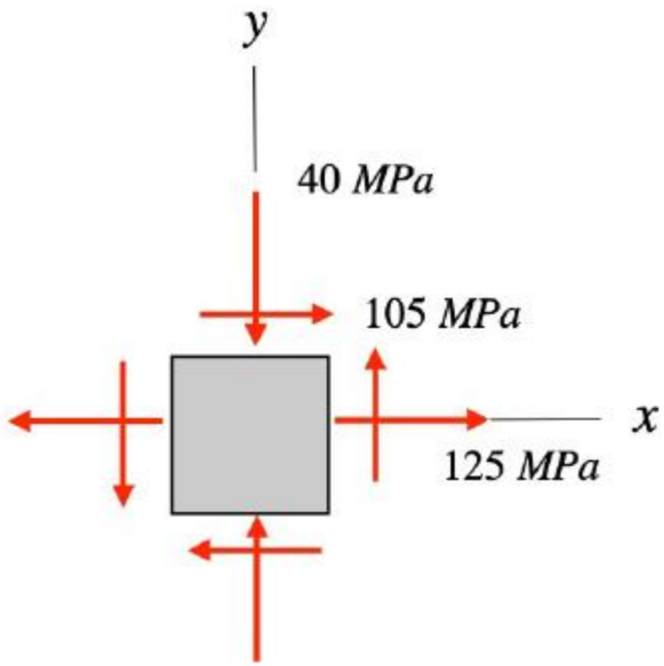
Or, equivalently,

$\sigma_1 > \sigma_2 > \sigma_3$

$$\Rightarrow \boxed{\sigma_M = \sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_1-\sigma_3)^2\right]}}$$

Example 15.1

$85/2 = 42.5$



The state of stress shown is in a component made of a ductile material with a **yield strength** of $\sigma_Y = 250\ \mathrm{MPa}$. Does the maximum shear stress theory predict failure for the material? Does the maximum distortion energy theory predict failure for the material?

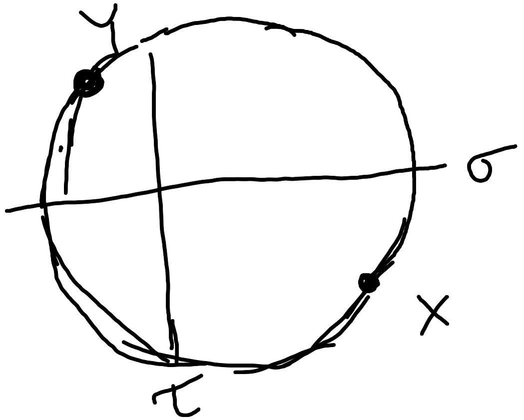
$\sigma_x = +125$ MPa

$\sigma_y = -40$ MPa

$\tau_{xy} = +105$ MPa

$$\sigma_{avg} = \frac{125 + (-40)}{2} = 42.5 \text{ MPa}$$

$$R = \sqrt{\left(\frac{125-(-40)}{2}\right)^2 + (105)^2} = 133.5 \text{ MPa}$$



$\sigma_{p1} = 176$ MPa

$\sigma_{p2} = -91$ MPa

$\sigma_{p3} = 0$

MDE / von Mises

$\sigma_Y = 250$ MPa

$$\sigma_M = \sqrt{\sigma_{p1}^2 - \sigma_{p1}\sigma_{p2} + \sigma_{p2}^2} = 235.2 \text{ MPa}$$

$$= \sqrt{176^2 - (176)(-91) + (-91)^2}$$

$$\boxed{\sigma_M < \sigma_Y \Rightarrow \text{safe by MDE!}}$$

MSS

$\sigma_1 = 176$ MPa

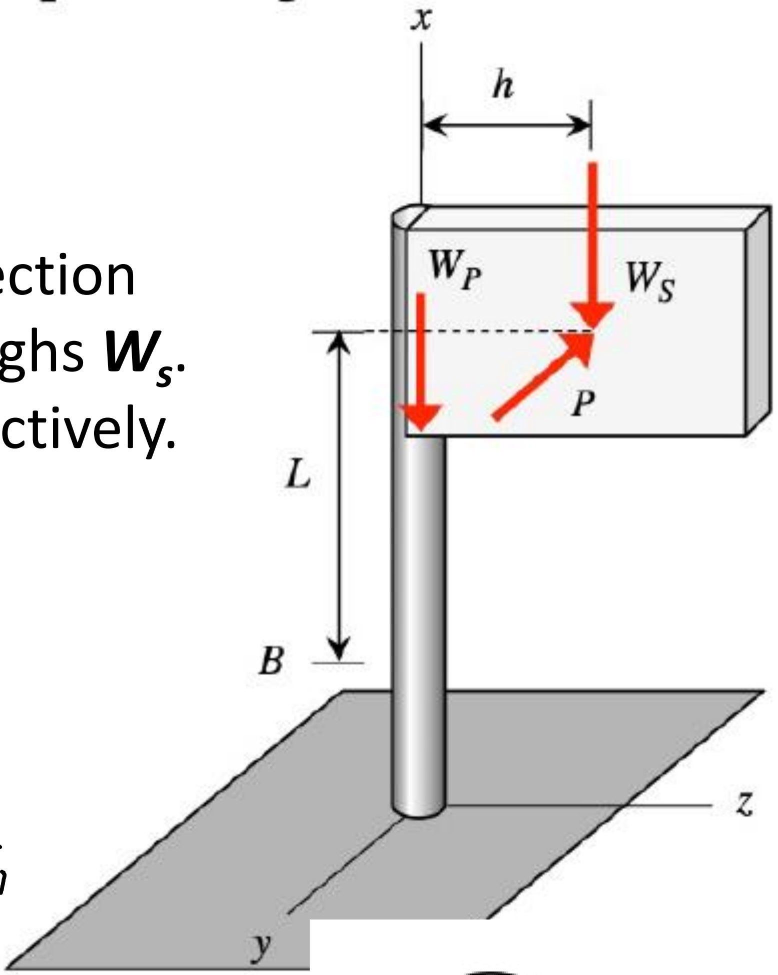
$\sigma_2 = 0$

$\sigma_3 = -91$ MPa

$$\Rightarrow \boxed{\begin{aligned} \tau_{max}^{abs} &= \left|\frac{\sigma_1 - \sigma_3}{2}\right| = 133.5 \text{ MPa} > \frac{\sigma_Y}{2} = 125 \text{ MPa} \\ FS &= \frac{\sigma_Y}{2\tau_{max}^{abs}} = \frac{\sigma_Y}{\sigma_1 - \sigma_3} = 0.94 < 1 \Rightarrow \text{fail by MSS} \end{aligned}}$$

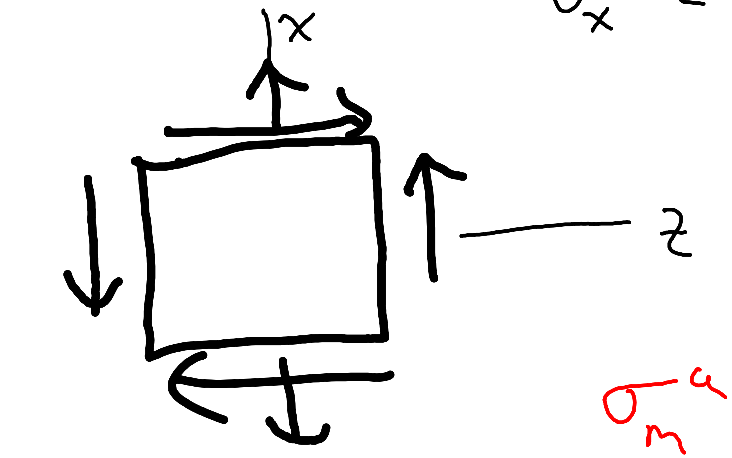
Revisit Example 14.12

Wind blowing on a sign produces a resultant force ***P*** in the –y direction at the point shown. The support pole weighs $\boldsymbol{W_P}$ and the sign weighs $\boldsymbol{W_s}$. The pole is a pipe with outer and inner diameters $\boldsymbol{d_o}$ and $\boldsymbol{d_i}$, respectively.



What are the factors of safety for points *a* and *b* according to the maximum distortion energy theory if the pole is made from an aluminum alloy with a yield strength of 20 ksi? $\longrightarrow \sigma_Y = 20$ ksi

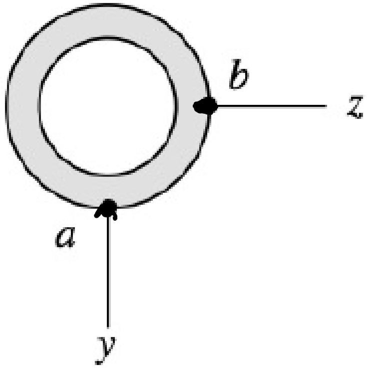
Point a



$$\sigma_x^a = \frac{-W_P - W_S}{A} + \frac{P L d_o}{2 I_{zz}} = 9433\,\text{psi}$$

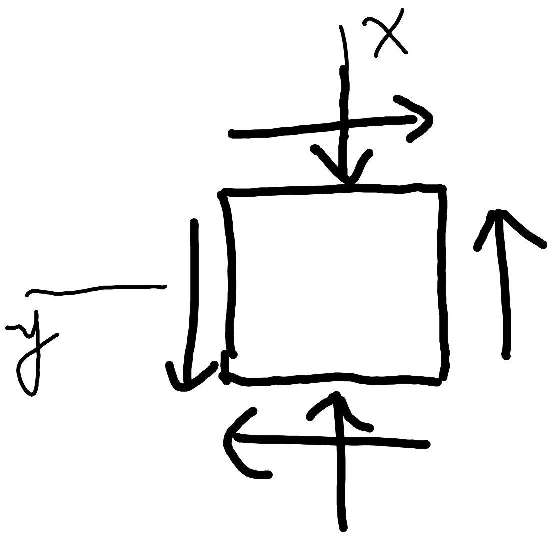
$$\tau_{xz}^a = \frac{(Ph)d}{2 I_p} = 871\,\text{psi}$$

$$\sigma_m^a = 9.6\,\text{ksi} \Rightarrow \boxed{FS^a = \frac{20}{9.1} = 2.1}$$



pipe cross section at B

Point b



$$\sigma_x^b = \frac{-W_p - W_s}{A} - \frac{W_s h d_o}{2 I_{yy}} = {}^{-}3025 \text{ psi}$$

$$\tau_{xy}^b = \frac{-(Ph)d_o}{2 I_p} - \frac{2P}{A} = {}^{-}938 \text{ psi}$$

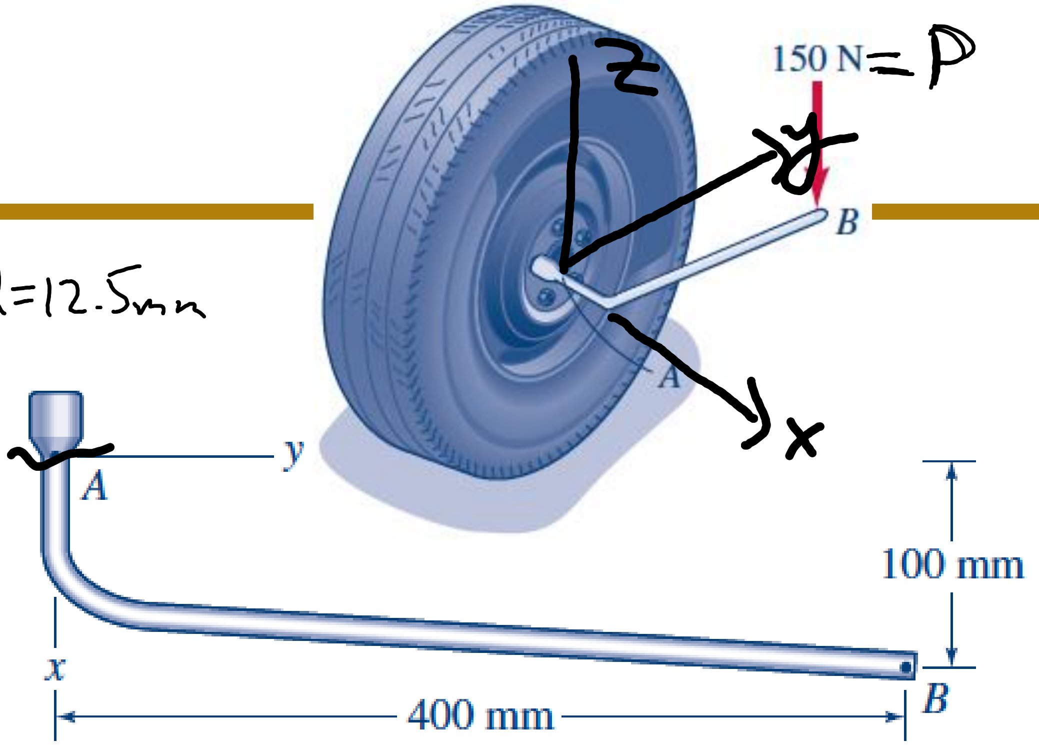
$$\sigma_m^b = 3.4 \text{ ksi} \Rightarrow \boxed{FS^b = \frac{20}{3.4} = 5.9}$$

the "real" factor of safety is the smaller one (point a is closer to failure)

Bonus example

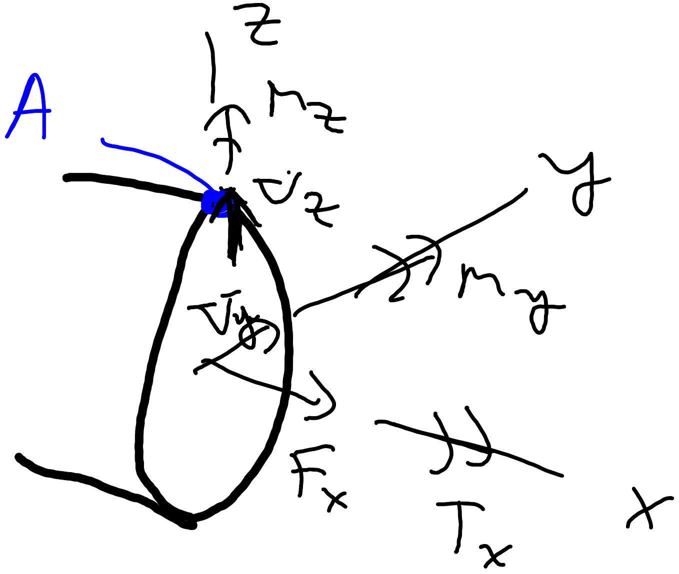
Determine the principal stresses and the maximum shear stress at point A (i.e., the point on top of the wrench handle). The diameter of the circular cross section is 12.5 mm. $\rightarrow d = 12.5\,mm$

all lengths in mm, stresses in MPa, forces in N units work out!



If the wrench is made of a ductile material with a yield strength of 300 MPa, what value of the force will cause yielding at point A according to the maximum shear stress theory? How about the maximum distortion energy theory?

$\sigma_Y = 300\ MPa$



$$\vec{F} = -P\hat{k}$$

$$\vec{r} = (100\,\hat{i} + 400\,\hat{j})\ mm$$

$$\cancel{F_x}^{0}\,\hat{i} + \cancel{V_y}^{0}\,\hat{j} + V_z\,\hat{k} = \vec{F} = -P\,\hat{k} \qquad \boxed{V_z = -P}$$

$$T_x\,\hat{i} + M_y\,\hat{j} + \cancel{M_z}^{0}\,\hat{k} = \vec{r} \times \vec{F} = -400P\,\hat{i} + 100P\,\hat{j} + 0\,\hat{k}$$

$$\boxed{T_x = -400P \qquad M_y = 100P}$$

Load	Stress dist	Stress at a
$V_z = -P$		0
$T_x = -400P$		$\tau_{xy} = \dfrac{+(400P)(d/2)}{I_p}$
$M_y = 100P$		$\sigma_x = \dfrac{-(100P)(d/2)}{I}$

Combine stresses

$$\sigma_x^A = \frac{-3200P}{\pi d^3} \qquad \tau_{xy}^A = \frac{6400P}{\pi d^3}$$

Note: the units work out in all of these equations since there is a "mm" dimension embedded in the numerator

Principal stresses + failure

$$\sigma_{avg} = \frac{-1600P}{\pi d^3}$$

$$R = \frac{6597P}{\pi d^3}$$

$$\sigma_{p1} = \sigma_{avg} + R = \frac{4997P}{\pi d^3}$$

$$\sigma_{p2} = \sigma_{avg} - R = \frac{-8197P}{\pi d^3}$$

$$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \begin{aligned} \sigma_1 &= \sigma_{p1} \\ \sigma_2 &= 0 \\ \sigma_3 &= \sigma_{p2} \end{aligned}$$

MSS

$$\tau_{max}^{abs} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = \frac{6547P}{\pi d^3} = \frac{\sigma_Y}{2} \Rightarrow \boxed{P_{MSS} = 139.5\,N}$$

MDE

$$\sigma_m = \sigma_Y \Rightarrow \boxed{P_{MDE} = 159.5\,N}$$

$$\sigma_m = \frac{11538P}{\pi d^3}$$