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Lectures on Functional Calculus

– 21st International Internet Seminar –

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Preface

The present text is the slightly reworked version of the lecture notes for the course phase of the “21st International Internet Seminar”, which was organized in the academic year 2017/2018 by an international team from the universities of Bordeaux, Kiel, Wuppertal and Salerno.

The topic was “Functional Calculus” and it was my responsability to provide the necessary text. For about 14 weeks from mid October 2017 until the beginning of February 2018, I had to provide a “virtual lecture” each weak, comprising study material and exercises. As a rule, it was one chapter per week, with only one exception (Chapter 6 was distributed over two weeks). The “supplementary material” was included for more experienced participants and could be left out on first reading without harm.

The participants, under the supervision of “local coordinators”, read and discussed the material and worked on exercises. We provided an online forum, where the participants could communicate with me or among each other. The present reworking of the original text is to a large extent my reaction on the comments, remarks and questions posted on that forum.

This brings me straight to the *acknowledgments* I owe all the people who in one or the other way helped me to shape the text. In first place, I want to thank my doctoral students Florian Pannasch and Marco Peruzzetto for spending so much time in reading the manuscript and checking the exercises, often under time pressure. You have done a tremendous job!

Next, my warmest thanks go to my colleagues and friends Hendrik Vogt (Bremen) and Jürgen Voigt (Dresden), who contributed numerous astute comments based on their meticulous reading of the manuscript. I learn so much from the discussions with you, and I feel honored that you find it all worth while!

Then I want to express my gratitude to all the participants who took the effort to post a question or a comment on the forum, even if it was sometimes only about a typo or a wrong reference: Zafrar Abderrahim, Pappu Anuragi, Sahiba Arora, Sebastian Bechtel (who gets special thanks for having been by far the most devoted commentator), Johannes Becker, Clemens Bombach,

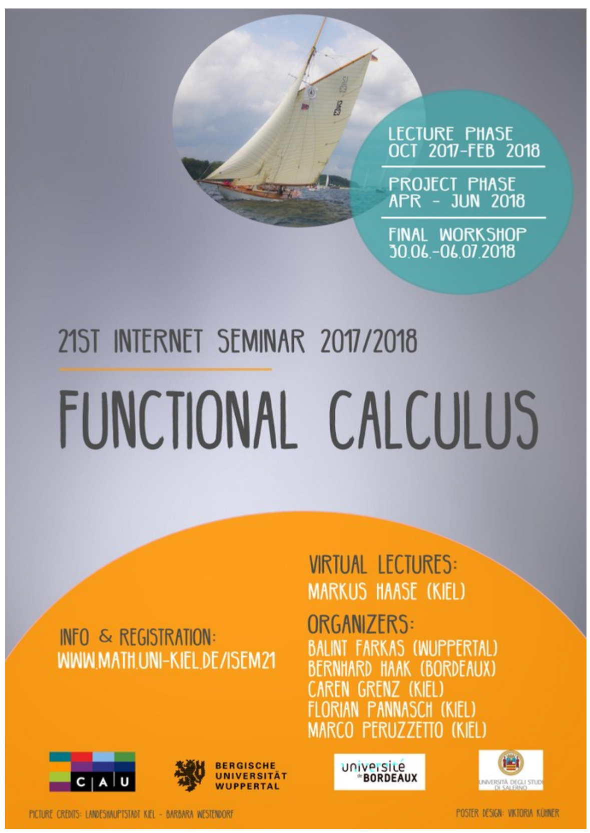
Arjan Cornelissen, Sonja Cox, Benedikt Diederichs, Moritz Egert, Thomas Eiter, David Erzmann, Alexander Friedrich, Fabian Gabel, Max Griehl, Satbir Malhi, Gabriel McCracken, Jan Meichsner, Ike Mulder, Ahmed Sani, Lukas Schröter, Jonas Tibke, Benjamin Waßermann.

Not least, I am indebted to our students from Kiel who contributed much through their questions and comments and engaged me in stimulating discussions during our weekly meetings: Christopher Babin, Lukas Betz, Alexander Dobrick, Lukas Hagedorn and Goran Nakerst.

I also would like to acknowledge the support of my institution, the Department of Mathematics at Kiel University, in particular my colleagues in the Analysis group, who enabled me a semester free from other teaching duties.

Finally, I would like to thank Jana, Fenja and Julius for being so loving and supportive all the time.

Kiel, March 2018 *Markus Haase*



Contents

Chapter 1

Holomorphic Functions of Bounded Operators

1.1 Polynomial Functional Calculus

Let X be a Banach space (over $\mathbb{C}$) and $A \in \mathcal{L}(X)$ a bounded operator on it. Then for every polynomial

$$p = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \in \mathbb{C}[z] \tag{1.1}$$

we can form the (likewise bounded) operator

$$p(A) := a_0 + a_1 A + a_2 A^2 + \cdots + a_n A^n \in \mathcal{L}(X).$$

The space of polynomials $\mathbb{C}[z]$ is a unital algebra and the mapping

$$\Psi : \mathbb{C}[z] \to \mathcal{L}(X), \qquad p \mapsto \Psi(p) := p(A)$$

is a (unital) algebra homomorphism. In other words, Ψ is a **representation** of the unital algebra $\mathbb{C}[z]$ on the vector space X by bounded operators.

Obviously, any representation Ψ of $\mathbb{C}[z]$ on X by bounded operators is of the form above: simply *define* $A := \Psi(z)$ and find $\Psi(p) = p(A)$ for all $p \in \mathbb{C}[z]$.

One may call Ψ a **calculus** since it reduces calculations with operators to calculations with other (here: formal) objects. Strictly speaking, however, this calculus is not (yet) a *functional* calculus, since a polynomial is not a function. It can become a function when it is interpreted as one, but this needs the specification of a *domain*, say $D \subseteq \mathbb{C}$.

Let us denote, for a polynomial $p \in \mathbb{C}[z]$, its interpretation as a function on D by $p|_D$. Then the indeterminate, which we have called "z" here, is mapped to the function

$$z|_D : D \to \mathbb{C}, \qquad z \mapsto z.$$

Here and in all what follows, we shall denote this function by $\mathbf{z}$. If p has the form given in (1.1) then

$$p|_D = a_0 + a_1\mathbf{z} + a_2\mathbf{z}^2 + \cdots + a_n\mathbf{z}^n.$$

Let us take $D = \mathbb{C}$, which is somehow the most natural choice. Then the mapping $p \mapsto p|_\mathbb{C}$ is injective. Hence we can take its inverse and compose it with Ψ from above to obtain a unital algebra homomorphism

$$\Phi : \{\text{polynomial functions on } \mathbb{C}\} \to \mathcal{L}(X), \qquad \Phi(p|_\mathbb{C}) = \Psi(p).$$

Now this is indeed a *functional* calculus, called the **polynomial functional calculus**. Note that

$$A = \Phi(\mathbf{z})$$

and Φ is uniquely determined by this value.

1.2 Power Series Functional Calculi

So far, the property that A is bounded was not used at all. Everything was pure algebra. Taking boundedness into account we can extend the polynomial functional calculus to a functional calculus of entire functions. Namely, each entire function $f \in \mathrm{Hol}(\mathbb{C})$ has a unique power series representation

$$f = a_0 + a_1\mathbf{z} + a_2\mathbf{z}^2 + \cdots + a_n\mathbf{z}^n \cdots = \sum_{n=0}^{\infty} a_n\mathbf{z}^n.$$

The convergence is absolute and locally uniform, so that in particular

$$\sum_{n=0}^{\infty} |a_n|\, r^n < \infty \quad \text{for each } r \in \mathbb{R}_+.$$

Hence, we can define

$$f(A) := \Phi(f) := \sum_{n=0}^{\infty} a_n A^n \in \mathcal{L}(X),$$

the series being absolutely convergent in the Banach space $\mathcal{L}(X)$. Clearly, if f is a polynomial (function), this definition of $f(A)$ agrees with the previous. The so obtained mapping

$$\Phi : \mathrm{Hol}(\mathbb{C}) \to \mathcal{L}(X), \qquad f \mapsto f(A)$$

is a homomorphism of unital algebras, because the usual rules of manipulation with absolutely convergent series are the same in $\mathbb{C}$ and in $\mathcal{L}(X)$ (actually in all Banach algebras). Notably, the multiplicativity of Φ follows from the fact that the *Cauchy product formula*

$$\Big(\sum_{n=0}^{\infty} a_n\Big)\Big(\sum_{n=0}^{\infty} b_n\Big) = \sum_{n=0}^{\infty}\Big(\sum_{j=0}^{n} a_j b_{n-j}\Big)$$

for absolutely convergent series holds in $\mathbb{C}$ as well as in $\mathcal{L}(X)$.

The functional calculus for entire functions works for *every* bounded operator A. If we take $\|A\|$ or even all the norms $\|A^n\|$, $n \in \mathbb{N}$, into account, we can extend the calculus further, for instance as follows:

Fix $r > 0$ such that $M(A,r) := \sup_{n\geq 0} \|A^n\| r^{-n} < \infty$. (One can take $r = \|A\|$, but in general $r < \|A\|$ is possible.) Abbreviate $\mathbb{D}_r := \mathrm{B}(0,r)$ (and $\mathbb{D} = \mathbb{D}_1$), and denote by $\mathrm{A}^1_+(\mathbb{D}_r)$ the set of functions $f : \overline{\mathbb{D}_r} \to \mathbb{C}$ that can be represented as a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (|z| \leq r) \tag{1.2}$$

such that

$$\|f\|_{\mathrm{A}^1_+} := \|f\|_{\mathrm{A}^1_+(\mathbb{D}_r)} := \sum_{n=0}^{\infty} |a_n|\, r^n < \infty. \tag{1.3}$$

Note that each $f \in \mathrm{A}^1_+(\mathbb{D}_r)$ is holomorphic on $\mathbb{D}_r$ and continuous on $\overline{\mathbb{D}_r} = \mathrm{B}[0,r]$. In particular, the coefficients a_n are uniquely determined by the function f, so $\|f\|_{\mathrm{A}^1_+}$ is well defined. Using the Cauchy product formula, it is easy to see that $\mathrm{A}^1_+(\mathbb{D}_r)$ is a unital Banach algebra. (Exercise 1.2).

For $f \in \mathrm{A}^1_+(\mathbb{D}_r)$ with representation (1.2) we can now define

$$f(A) := \Phi(f) := \sum_{n=0}^{\infty} a_n A^n.$$

The condition (1.3) ensures that this power series is absolutely convergent and computations as before show that

$$\Phi : \mathrm{A}^1_+(\mathbb{D}_r) \to \mathcal{L}(X), \qquad f \mapsto f(A)$$

is a homomorphism of unital algebras. Moreover, one has the norm estimate

$$\|f(A)\| \leq M(A,r)\, \|f\|_{\mathrm{A}^1_+} \qquad (f \in \mathrm{A}^1_+(\mathbb{D}_r))$$

as is easily seen.

Remark 1.1. The case $r = 1$ is particularly important. An operator with $M(A,1) < \infty$ is called **power bounded**. So, power bounded operators have a (bounded) $\mathrm{A}^1_+(\mathbb{D})$-calculus, a fact that will become important later in this course.

Note that in the above construction one has $A = \Phi(\mathbf{z})$, and by this condition Φ is uniquely determined as a continuous unital algebra homomorphism $\mathrm{A}^1_+(\mathbb{D}_r) \to \mathcal{L}(X)$.

1.3 The Dunford–Riesz Calculus

The above power series functional calculi for an operator A use only information about the asymptotic behaviour of the norms $\|A^n\|$. In this section we shall construct a holomorphic functional calculus for A that is based purely on the *location of its spectrum*.

Review of Elementary Spectral Theory

Recall from an elementary functional analysis course that the **spectrum** of $A \in \mathcal{L}(X)$ is the set

$$\sigma(A) := \{\lambda \in \mathbb{C} \mid \lambda - A \,\text{is not invertible}\}.$$

(We write $\lambda - A$ instead of $\lambda\mathrm{I} - A$.) The complement $\rho(A) := \mathbb{C} \setminus \sigma(A)$, an open subset of $\mathbb{C}$, is called the **resolvent set** of A. The mapping

$$R(\cdot, A) : \rho(A) \to \mathcal{L}(X), \qquad R(\lambda, A) := (\lambda - A)^{-1}$$

is the **resolvent**. It satisfies the **resolvent identity**

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A) \qquad (\lambda, \mu \in \rho(A)) \tag{1.4}$$

and is *holomorphic*. (See Appendix A.3 for an introduction to vector-valued holomorphic mappings.)

A number $\lambda \neq 0$ belongs to $\rho(A)$ whenever the series $\sum_{n=0}^{\infty} \lambda^{-(n+1)} A^n$ converges, in which case this series equals $R(\lambda, A)$. (This follows from the Neumann series theorem.) In particular, $\lambda \in \rho(A)$ if $|\lambda| > \|A\|$ and then

$$\|R(\lambda, A)\| \le \frac{1}{|\lambda| - \|A\|}.$$

Hence, the spectrum is compact and not empty (by Liouville's theorem), except in the case $X = \{0\}$. The **spectral radius**

$$r(A) := \sup\{|\lambda| \mid \lambda \in \sigma(A)\}$$

satisfies $r(A) \le \|A\|$.

Dunford–Riesz Calculus, “Baby” Version

We start with a “baby” version of the Dunford–Riesz calculus and take $s > r(A)$, i.e., $\sigma(A) \subseteq \mathbb{D}_s$. For a function $f \in \mathrm{Hol}(\mathbb{D}_s)$ define

$$\Phi(f) := \frac{1}{2\pi\mathrm{i}} \int_{\gamma_r} f(z) R(z, A)\,\mathrm{d}z \tag{1.5}$$

where $0 < r < s$ is such that still $\sigma(A) \subseteq \mathbb{D}_r$ and γ_r is the contour which traverses the boundary of $\mathbb{D}_r$ once in counterclockwise direction. E.g., $\gamma_r(t) := r\mathrm{e}^{2\pi\mathrm{i}t}$, $t \in [0, 1]$. (The integral here is elementary and covered by results collected in Appendix A.2.) Note that the integral in (1.5) does not depend on the choice of $r \in (r(A), s)$, by Cauchy's theorem (note Theorem A.6).

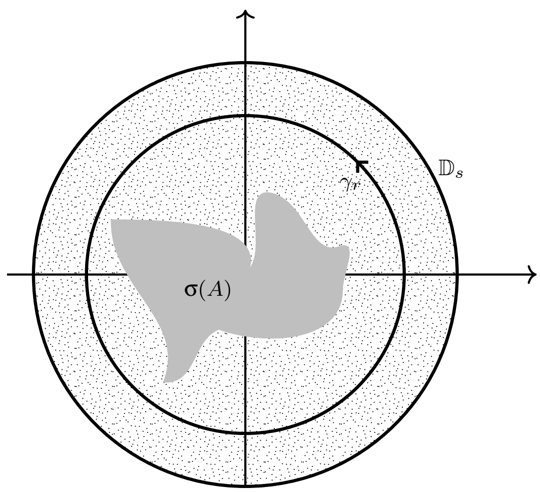


Fig. 1.1 The function $f(\cdot)R(\cdot, .A)$ is holomorphic on $\mathbb{D}_s \setminus \sigma(A)$.

Remark 1.2. The “intuition” behind formula (1.5) is as follows: suppose that $X = \mathbb{C}$ is the one-dimensional space, and A is multiplication with $a \in \mathbb{C}$. Then $\sigma(A) = \{a\}$ and hence $|a| < r$. If f is a polynomial, then $f(A)$ is multiplication with $f(a)$, so one expects the same behavior for other functions f. Now, Cauchy's theorem in an elementary form tells that if $f \in \mathrm{Hol}(\mathbb{D}_s)$ and $|a| < r < s$, then

$$f(a) = \frac{1}{2\pi\mathrm{i}} \int_{\gamma_r} f(z) \frac{1}{z - a}\,\mathrm{d}z. \tag{1.6}$$

Thinking of $\Phi(f)$ as $f(A)$ and $R(z, A)$ as $\frac{1}{z-A}$ and replacing formally a by A in (1.6) yields (1.5).

Theorem 1.3. *Let A be a bounded operator on a Banach space X and let $s > 0$ such that $\sigma(A) \subseteq \mathbb{D}_s$. Then the mapping*

$$\Phi : \mathrm{Hol}(\mathbb{D}_s) \to \mathcal{L}(X)$$

given by (1.5) *is a homomorphism of unital algebras such that $\Phi(\mathbf{z}) = A$. Moreover, it is continuous with respect to local uniform convergence, i.e.: if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathrm{Hol}(\mathbb{D}_s)$ converging to f locally uniformly on $\mathbb{D}_s$, then $\Phi(f_n) \to \Phi(f)$ in operator norm.*

Proof. It is clear from the definition (1.5) that Φ is linear and has the claimed continuity property. The multiplicativity of Φ is a consequence of the resolvent identity, Fubini's theorem (for continuous functions on rectangles) and Cauchy's theorem. To wit, let $f,\ g \in \mathrm{Hol}(\mathbb{D}_s)$ and choose $r(A) < r < r' < s$. Then

$$\begin{aligned}
\Phi(f)\Phi(g) &= \frac{1}{(2\pi \mathrm{i})^2} \int_{\gamma_r} \int_{\gamma_{r'}} f(z) g(w) R(z,A) R(w,A) \,\mathrm{d}w \,\mathrm{d}z \\
&= \frac{1}{(2\pi \mathrm{i})^2} \int_{\gamma_r} \int_{\gamma_{r'}} f(z) g(w) \frac{1}{w-z} \big(R(z,A) - R(w,A) \big) \,\mathrm{d}w \,\mathrm{d}z \\
&= \frac{1}{(2\pi \mathrm{i})^2} \int_{\gamma_r} f(z) \int_{\gamma_{r'}} \frac{g(w)}{w-z} \,\mathrm{d}w \, R(z,A) \,\mathrm{d}z \\
&\qquad - \frac{1}{(2\pi \mathrm{i})^2} \int_{\gamma_{r'}} \int_{\gamma_r} \frac{f(z)}{w-z} \,\mathrm{d}z \, g(w) R(w,A) \,\mathrm{d}w.
\end{aligned}$$

By Cauchy's theorem,

$$\frac{1}{2\pi i} \int_{\gamma_{r'}} \frac{g(w)}{w-z} \,\mathrm{d}w = g(z) \qquad \text{for each } z \in \gamma_r([0,1]),$$

since $r' > r$. For the same reason

$$\int_{\gamma_r} \frac{f(z)}{w-z} \,\mathrm{d}z = 0 \qquad \text{for each } w \in \gamma_{r'}([0,1]).$$

Hence, we obtain

$$\Phi(f)\Phi(g) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_r} f(z) g(z) R(z,A) \,\mathrm{d}z = \Phi(fg),$$

as desired.

It remains to show that $\Phi(\mathbf{z}) = A$ and $\Phi(\mathbf{1}) = \mathrm{I}$. For the former, note that $zR(z,A) = \mathrm{I} + AR(z,A)$ for each $z \in \rho(A)$. Hence, by Cauchy's theorem

$$\Phi(\mathbf{z}) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_r} zR(z,A) \,\mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_r} AR(z,A) \,\mathrm{d}z = A\Phi(\mathbf{1}).$$

So it remains to establish the latter. To this end, note that

$$\begin{aligned}\Phi(\mathbf{1}) &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma_r} R(z,A)\,\mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_r} \frac{\mathrm{I}}{z} + \frac{AR(z,A)}{z}\,\mathrm{d}z \\ &= \mathrm{I} + \frac{1}{2\pi \mathrm{i}} \int_{\gamma_r} \frac{AR(z,A)}{z}\,\mathrm{d}z,\end{aligned}$$

and this holds for *each* $r > r(A)$. Letting $r \to \infty$ yields

$$\int_{\gamma_r} \frac{AR(z,A)}{z}\,\mathrm{d}z \to 0 \qquad (r \to \infty)$$

by an elementary estimate employing that $\|AR(z,A)\| = O(|z|^{-1})$ as $|z| \to \infty$. Hence, $\Phi(\mathbf{1}) = \mathrm{I}$ as desired. □

The Spectral Radius Formula

After a closer look on Theorem 1.3 we realize that the constructed functional calculus is still of power series type. Indeed, any holomorphic function $f \in \mathrm{Hol}(\mathbb{D}_s)$ has a (locally uniformly convergent) power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (|z| < s).$$

Hence, by Theorem 1.3

$$\Phi(f) = \sum_{n=0}^{\infty} a_n \Phi(\mathbf{z})^n = \sum_{n=0}^{\infty} a_n A^n.$$

However, it was impossible to *define* Φ by the formula $\Phi(f) := \sum_{n=0}^{\infty} a_n A^n$, because the convergence of the series could not be guaranteed. The information we lacked here was the following famous formula, which now comes as a corollary.

Corollary 1.4 (Spectral Radius Formula). *Let $A \in \mathcal{L}(X)$ be a bounded operator on a Banach space X. Then*

$$r(A) = \inf_{n \in \mathbb{N}} \|A^n\|^{\frac{1}{n}} = \lim_{n \to \infty} \|A^n\|^{\frac{1}{n}}.$$

Proof. The second identity is trivial if $\|A^n\| = 0$ for some $n \in \mathbb{N}$. In the other case, it follows from Fekete's Lemma 1.5 below with $a_n := \log(\|A^n\|)$.

Suppose that $|\lambda| > \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$. Then the power series

$$\sum_{n=0}^{\infty} \lambda^{-(n+1)} A^n$$

converges. Hence, as mentioned above, $\lambda \in \rho(A)$. This yields the inequality

$$r(A) \leq \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}.$$

For the converse, take $r(A) < r$. Then

$$A^n = \Phi(\mathbf{z}^n) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_r} z^n R(z, A)\, \mathrm{d}z.$$

Taking norms and estimating crudely yields a constant C_r (independent of n) such that

$$\|A^n\| \leq C_r r^{n+1}.$$

Taking n-th roots and letting $n \to \infty$ we obtain

$$\lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} \leq r.$$

As $r > r(A)$ was arbitrary, we are done. □

The following analytical lemma was used in the proof above.

Lemma 1.5 (Fekete). *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that*

$$a_{n+m} \leq a_n + a_m.$$

Then $\lim\limits_{n\to\infty} \dfrac{a_n}{n} = \inf\limits_{n\in\mathbb{N}} \dfrac{a_n}{n}$ *in* $\mathbb{R} \cup \{-\infty\}$.

Proof. Let $c := \inf_{n\in\mathbb{N}} \frac{a_n}{n}$ and take $c < d$. Then there is $m \in \mathbb{N}$ such that $a_m < md$. Let $C := \max_{1\leq l\leq m} a_l$. Each $n \in \mathbb{N}$ can be written as $n = km + l$, with $k \in \mathbb{N}_0$ and $1 \leq l \leq m$. With this choice,

$$a_n = a_{km+l} \leq k a_m + a_l \leq (n-l)\frac{a_m}{m} + C \leq n\frac{a_m}{m} + C + |a_m|\,.$$

Hence, $\frac{a_n}{n} \leq \frac{a_m}{m} + \frac{C+|a_m|}{n} < d$ for all sufficiently large $n \in \mathbb{N}$, and this proves the claim. □

Dunford–Riesz Calculus, Full Version

The baby version of the Dunford–Riesz calculus has the disadvantage that the domain of the considered functions is just a disc. A partition of the spectrum in disjoint parts, for instance, cannot be "seen" by these functions, and hence neither by the functional calculus.

More advanced versions of the Dunford–Riesz calculus remedy this. They differ from the baby version precisely in the domain of the functions and the corresponding contours used in the Cauchy integrals. We are going to state the "full" version, which is the most general. It simply assumes that U is an open subset of $\mathbb{C}$ containing $\sigma(A)$ and the functions f are holomorphic on U.

To handle such generality, one needs a sophisticated version of the Cauchy integral theorem, the so-called "global" version. However, in many concrete cases simpler domains with simpler contours will do. So if the following is too far away from what you know from your complex variables courses, just think of the set U as being a union of finitely many disjoint discs and the cycles Γ as a disjoint union of circular arcs within these discs.

We recall some notions of complex analysis, see for example [3, Chapter 10] about the "global Cauchy theorem". A **cycle** is a formal $\mathbb{Z}$-combination

$$\Gamma = n_1\gamma_1 \oplus n_2\gamma_2 \oplus \cdots \oplus n_k\gamma_k, \tag{1.7}$$

where the γ_k are closed paths (parametrized over $[0,1]$, say) in the plane. An integral over Γ is defined as

$$\int_\Gamma f(z)\,\mathrm{d}z = \sum_{j=1}^{k} n_j \int_{\gamma_j} f(z)\,\mathrm{d}z.$$

The **trace** of a cycle Γ as in (1.7) is

$$\Gamma^* := \bigcup \{\gamma_j^* \mid n_j \neq 0\},$$

where γ_j^* is just the image of γ_j as a mapping from $[0,1]$ to $\mathbb{C}$. We say that Γ is a cycle **in** a subset $O \subseteq \mathbb{C}$ if $\Gamma^* \subseteq O$. The **index** of $a \in \mathbb{C} \setminus \Gamma^*$ is

$$\chi_\Gamma(a) := \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{1}{z-a}\,\mathrm{d}z.$$

It is an integer number (since all γ_j are closed). The **interior** of a cycle Γ is

$$\mathrm{Int}(\Gamma) = \{a \in \mathbb{C} \setminus \Gamma^* \mid \chi_\Gamma(a) \neq 0\}.$$

A cycle Γ is **positively oriented**[1] if

$$\chi_\Gamma(a) \in \{0,1\} \quad \text{for all } a \in \mathbb{C} \setminus \Gamma^*.$$

We need the following "global" version of Cauchy's theorem.

[1] This is Conway's terminology from [1].

Theorem 1.6. *Let X be a Banach space, $O \subseteq \mathbb{C}$ open, $f \in \mathrm{Hol}(O;X)$ and Γ a cycle in O such that*[2] $\mathrm{Int}(\Gamma) \subseteq O$. *Then*

$$\chi_\Gamma(a) f(a) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(z)}{z-a}\,\mathrm{d}z \quad \textit{for all } a \in O \setminus \Gamma^*$$

and

$$\int_\Gamma f(z)\,\mathrm{d}z = 0.$$

For the proof of the case $X = \mathbb{C}$ see [3, Thm. 10.35]. The Banach space version follows from the scalar version by applying linear functionals, see Appendix A.3.

In addition, we need the following result about the existence of positively oriented cycles.

Theorem 1.7. *Let $O \subseteq \mathbb{C}$ be an open subset of the plane and $K \subseteq O$ a compact subset. Then there is a positively oriented cycle Γ in $O \setminus K$ such that $K \subseteq \mathrm{Int}(\Gamma) \subseteq O$.*

The proof can be found in [3, Thm. 13.5]. Note that for the cycle Γ guaranteed by Theorem 1.7 one has Cauchy's formula

$$f(a) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(z)}{z-a}\,\mathrm{d}z \quad \text{for all } a \in K$$

by Theorem 1.6.

Now back to functional calculus. Let $A \in \mathcal{L}(X)$ be a bounded operator on a Banach space X, and let $U \subseteq \mathbb{C}$ be open such that $\sigma(A) \subseteq U$. Choose (by Theorem 1.7) a positively oriented cycle Γ in $U \setminus \sigma(A)$ such that $\sigma(A) \subseteq \mathrm{Int}(\Gamma) \subseteq U$.

Then define, for $f \in \mathrm{Hol}(U)$,

$$\Phi(f) := \frac{1}{2\pi \mathrm{i}} \int_\Gamma f(z) R(z,A)\,\mathrm{d}z. \tag{1.8}$$

This definition does not depend on the choice of Γ. Indeed, if Λ is another cycle meeting the requirements, then the cycle $\Gamma \oplus -\Lambda$ satisfies

$$\mathrm{Int}(\Gamma \oplus -\Lambda) \subseteq U \setminus \sigma(A).$$

Theorem 1.6 with $O := U \setminus \sigma(A)$ hence yields

$$0 = \int_{\Gamma \oplus -\Lambda} f(z) R(z,A)\,\mathrm{d}z = \int_\Gamma f(z) R(z,A)\,\mathrm{d}z - \int_\Lambda f(z) R(z,A)\,\mathrm{d}z.$$

[2] The condition $\mathrm{Int}(\Gamma) \subseteq O$ is sometimes rephrased by saying that "Γ is nullhomologous in O".

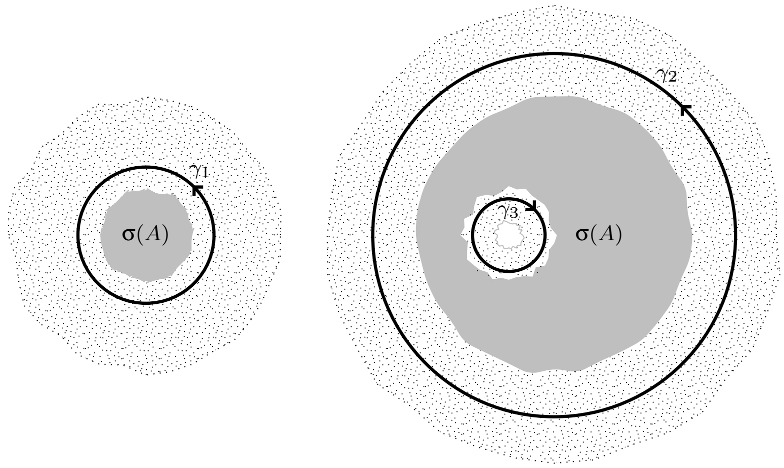


Fig. 1.2 The region U is dotted. The cycle $\Gamma = \gamma_1 \oplus \gamma_2 \oplus \gamma_3$ is positively oriented, lies entirely in $U \setminus \sigma(A)$ and has $\sigma(A)$ in its interior.

As a consequence (how precisely?) we obtain that Φ, called the **(full) Dunford–Riesz calculus** for A on U, is an extension of the "baby" Dunford–Riesz calculus considered above.

Now we obtain the analogue of Theorem 1.3, with an analogous proof.

Theorem 1.8. *Let A be a bounded operator on a Banach space X and let $U \subseteq \mathbb{C}$ be open such that $\sigma(A) \subseteq U$. Then the mapping*

$$\Phi : \mathrm{Hol}(U) \to \mathcal{L}(X)$$

given by (1.8) *is a homomorphism of unital algebras such that $\Phi(\mathbf{z}) = A$. Moreover, it is continuous with respect to locally uniform convergence.*

Proof. Linearity and the continuity property are clear, again. Since Φ extends the "baby" Dunford–Riesz calculus, $\Phi(\mathbf{1}) = \mathrm{I}$ and $\Phi(\mathbf{z}) = A$. The multiplicativity $\Phi(fg) = \Phi(f)\Phi(g)$ is proved almost literally as before. One only needs to choose the cycle for computing $\Phi(g)$, say Λ, in $O \setminus (\sigma(A) \cup \Gamma^*)$ such that

$$\sigma(A) \cup \Gamma^* \subseteq \mathrm{Int}(\Lambda) \subseteq U.$$

Since Γ^* is compact, this is possible. □

Remark 1.9. The Dunford–Riesz calculus as a homomorphism of unital algebras $\mathrm{Hol}(U) \to \mathcal{L}(X)$, continuous with respect to locally uniform convergence, is uniquely determined by the requirement that $\Phi(\mathbf{z}) = A$. This can be seen as follows: If $U = \mathbb{C}$ then one is in the power series case, and the claim is clear. If $U \neq \mathbb{C}$, then for $\lambda \in \mathbb{C} \setminus U$ one has $\Phi((\lambda - \mathbf{z})^{-1}) = R(\lambda, A)$ (why?). Hence, Φ is determined on rational functions with poles outside of

U. By a consequence of Runge's theorem, each $f \in \mathrm{Hol}(U)$ can be approximated locally uniformly by a sequence of such rational functions, see [1, Cor. VIII.1.14]. Hence, Φ is determined on f, too.

1.4 From Bounded to Unbounded Operators

So far, we have only considered functional calculi of bounded operators. Also, the range of the functional calculus consisted also of bounded operators only. However, restricting oneself entirely to bounded operators is quite unnatural, for two reasons. Firstly, many interesting and highly relevant operators for which functional calculi can be defined are not bounded. Secondly, quite some natural operations of "functional calculus type" leave the class of bounded operators.

Suppose, for instance, that A is a bounded operator (still) on a Banach space X. If A is injective one can form the operator A^{-1}, defined on the range $\mathrm{ran}(A)$ of A. The operator A^{-1} can (and should) be regarded as the result of inserting A into the scalar function $z \mapsto \frac{1}{z}$. However, only in special cases will A^{-1} be again bounded.

Let us look at an example. The **Volterra operator** V on $\mathrm{L}^p(0,1)$ is given by

$$Vf(t) := \int_0^t f(s)\,\mathrm{d}s \qquad (f \in \mathrm{L}^p(0,1)).$$

It is bounded and injective. Its inverse, defined on $\mathrm{ran}(V)$ is the first derivative. It follows from elementary theory of Sobolev spaces in one dimension[3] that

$$\mathrm{W}^{1,\mathrm{p}}(0,1) = \mathrm{ran}(V) \oplus \mathbb{C}\mathbf{1}$$

where $\mathbf{1}$ denotes the function which is equal to 1 everywhere. And $V^{-1} = \frac{\mathrm{d}}{\mathrm{d}s}$ (the weak derivative), defined on

$$\mathrm{ran}(V) = \{u \in \mathrm{W}^{1,\mathrm{p}}(0,1) \mid u(0) = 0\}.$$

Note that V^{-1} is not bounded with respect to the L^p-norm.

Now, from a functional calculus point of view, the bounded operator V and the unbounded operator V^{-1} are pretty much the same thing, because each definition of $f(V)$ for some complex function f yields a definition of $g(V^{-1})$ for $g = f(\mathbf{z}^{-1})$. Of course the same works in the converse direction, so the functional calculus theories of V and of V^{-1} are equivalent.

These few remarks should convince you that it is advised to have a good basic knowledge about unbounded operators before delving deeper into functional calculus theory. And this is what we are going to do in the next chapter.

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[3] If you are not familiar with this theory, don't feel pushed to catch up now. Eventually, there will be an appendix on that. If you can't wait, look into [2, Chap.10] for the L^2-case.

1.5 Supplement: Functional Calculus for Matrices

In this supplement we present, for the case of a finite-dimensional space $X = \mathbb{C}^d$, an alternative, completely algebraic way of constructing the Dunford–Riesz calculus.

Let $A \in \mathbb{C}^{d\times d}$ be a complex square matrix of dimension $d \in \mathbb{N}$. The natural polynomial functional calculus

$$\Psi : \mathbb{C}[z] \to \mathbb{C}^{d\times d}, \qquad p \mapsto p(A)$$

has a non-trivial kernel (for reasons of dimension), which must be an ideal of the ring $\mathbb{C}[z]$. As every ideal in $\mathbb{C}[z]$ is a principal ideal (due to the presence of division with remainder and the Euclidean algorithm), there is a monic polynomial m_A, the so-called **minimal polynomial**, with the property that

$$p(A) = 0 \quad \iff \quad m_A \,|\, p \ \text{ in } \ \mathbb{C}[z]$$

for each $p \in \mathbb{C}[z]$. In symbols:

$$\ker(\Psi) = (m_A) := \{m_A p \mid p \in \mathbb{C}[z]\},$$

the principal ideal generated by m_A. By standard algebra, the homomorphism Ψ induces a homomorphism

$$\widehat{\Psi} : \mathbb{C}[z]/(m_A) \to \mathbb{C}^{d\times d}.$$

Let $U \subseteq \mathbb{C}$ be open with $\sigma(A) \subseteq U$. We shall construct a unital algebra homomorphism

$$\Phi : \mathrm{Hol}(U) \to \mathbb{C}^{d\times d}$$

such that $\Phi(\mathbf{z}) = A$.

We are done if we can show that the natural homomorphism

$$\eta : \mathbb{C}[z]/(m_A) \to \mathrm{Hol}(U)/m_A\cdot \mathrm{Hol}(U), \qquad \eta(p + (m_A)) := p|_U + m_A\cdot \mathrm{Hol}(U)$$

which arises from interpreting a polynomial p as a holomorphic function $p|_U$ on U, is an *isomorphism*. Indeed, we then can take its inverse η^{-1} and let Φ be defined as the composition of the maps

$$\Phi \ : \ \mathrm{Hol}(U) \xrightarrow{\ s\ } \mathrm{Hol}(U)/m_A\cdot \mathrm{Hol}(U) \xrightarrow{\eta^{-1}} \mathbb{C}[z]/(m_A) \xrightarrow{\widehat{\Psi}} \mathbb{C}^{d\times d},$$

where s is the canonical surjection. In order to achieve our goal, we need to delve a little deeper into the structure of the polynomial m_A.

By the fundamental theorem of algebra, m_A factorizes into linear factors

$$m_A = \prod_{j=1}^{k} (z - \lambda_j)^{e_j}$$

for certain pairwise different $\lambda_j \in \mathbb{C}$ and numbers $e_j \in \mathbb{N}$. Linear algebra shows that the λ_j are precisely the eigenvalues of A:

$$\sigma(A) = \sigma_{\mathrm{p}}(A) = \{\lambda_1, \dots, \lambda_k\}.$$

In particular, U contains all the zeroes of m_A, and that implies the injectivity of η, as follows. Suppose that p is a polynomial such that there is $f \in \mathrm{Hol}(U)$ with $p|_U = m_A|_U\, f$ in $\mathrm{Hol}(U)$. Then for each $j = 1, \dots, k$ the holomorphic (polynomial) function $p|_U$ must have a zero of order e_j at $z = \lambda_j$. By the uniqueness of power series representation, this implies that the polynomial $m_j := (z - \lambda_j)^{e_j}$ divides p even in $\mathbb{C}[z]$ (and not just in $\mathrm{Hol}(U)$). Since the polynomials m_j are mutually prime, m_A divides p in $\mathbb{C}[z]$, i.e. $p = 0$ in $\mathbb{C}[z]/(m_A)$.

The proof of the surjectivity of η requires a little more work. It is particularly interesting since it will yield an algorithm for the functional calculus Φ. Observe that the polynomials

$$r_l := \frac{m_A}{m_l} = \prod_{j \neq l} m_j \qquad (l = 1, \dots, k)$$

satisfy

$$\gcd\big(r_1, \dots, r_k\big) = 1.$$

Hence, again by a consequence of the Euclidean algorithm, there are polynomials $q_j \in \mathbb{C}[z]$ such that

$$1 = \sum_{j=1}^{k} q_j r_j.$$

Now suppose that $f \in \mathrm{Hol}(U)$ is given. We want to find a polynomial p such that $m_A \,|\, f - p$ in $\mathrm{Hol}(U)$. (Note that eventually $\Phi(f) := p(A)$.) As a holomorphic function on U, f has a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_{jn} (z - \lambda_j)^n$$

around λ_j, for each $j = 1, \dots, k$. (Recall that $a_{jn} = \frac{f^n(\lambda_j)}{n!}$.) Let

$$p_j := \sum_{n=0}^{e_j - 1} a_{jn} (z - \lambda_j)^n \qquad (j = 1, \dots, k)$$

and

$$p := \sum_{j=1}^{k} p_j \, q_j \, r_j .$$

By construction, $f - p_j$ has a zero of order e_j at λ_j and hence $m_j \mid f - p_j$ in $\mathrm{Hol}(U)$. Consequently,

$$f - p = \sum_{j=1}^{k} (f - p_j) \, q_j \, r_j = m_A \sum_{j=1}^{k} \frac{f - p_j}{m_j} \, q_j \;\in\; m_A \cdot \mathrm{Hol}(U)$$

as desired. Let us summarize our findings.

Theorem 1.10. *Let $A \in \mathbb{C}^{d \times d}$ and let U be an open subset of $\mathbb{C}$ containing $\sigma(A)$. Then there is a unique unital algebra homomorphism*

$$\Phi : \mathrm{Hol}(U) \to \mathbb{C}^{d \times d}$$

such that $\Phi(\mathbf{z}) = A$.

Proof. Existence has been shown above. For uniqueness, note that the requirements on Φ determine Φ on all polynomial functions. In particular, $\Phi(m_A) = 0$, where as above m_A is the minimal polynomial of A. Hence, if $f \in \mathrm{Hol}(U)$ and $p \in \mathbb{C}[z]$ is such that $m_A \mid f - p$ in $\mathrm{Hol}(U)$ then $0 = \Phi(f - p) = \Phi(f) - \Phi(p)$. Since we know already that such a polynomial p can always be found, Φ is indeed uniquely determined. □

The (unique) unital algebra homomorphism $\Phi : \mathrm{Hol}(U) \to \mathbb{C}^{d \times d}$ with $\Phi(\mathbf{z}) = A$ is called the **matrix functional calculus** for the matrix A. With the notation from above,

$$\Phi(f) = p(A) = \sum_{j=1}^{k} \sum_{n=0}^{e_j - 1} \frac{f^{(n)}(\lambda_j)}{n!} (A - \lambda_j)^n q_j(A) r_j(A). \tag{1.9}$$

The matrices $(A - \lambda_j)^n q_j(A) r_j(A)$ for $j = 1, \dots, k$ and $n = 0, \dots, e_j - 1$ do not depend on f, and this is advantageous from a computational point of view.

Corollary 1.11. *Let $A \in \mathbb{C}^{d \times d}$, let U be an open subset of $\mathbb{C}$ containing $\sigma(A)$ and let $\Phi : \mathrm{Hol}(U) \to \mathbb{C}^{d \times d}$ be the matrix functional calculus for A. Then Φ is continuous with respect to locally uniform convergence.*

Proof. Let $(f_m)_m$ be a sequence in $\mathrm{Hol}(U)$ such that $f_m \to 0$ locally uniformly. Then, by a standard result of complex function theory, $f_m^{(n)} \to 0$ locally uniformly for all $n \in \mathbb{N}_0$. Hence, the claim follows from the explicit formula (1.9). □

By uniqueness, the matrix functional calculus must coincide with the Dunford–Riesz calculus.

Exercises

1.1 (Spectral Mapping Theorem (Polynomials)). Let A be a bounded operator on a Banach space X and let $p \in \mathbb{C}[z]$. Show that

$$\sigma(p(A)) = p(\sigma(A)) := \{p(\lambda) \mid \lambda \in \sigma(A)\}.$$

1.2. Fix $r > 0$, define $\mathbb{D}_r := \{z \in \mathbb{C} \mid |z| < r\}$ and let $\mathrm{A}^1_+(\mathbb{D}_r)$ be the set of functions $f : \overline{\mathbb{D}_r} \to \mathbb{C}$ that can be represented as a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (z \in \mathbb{C},\ |z| \le r)$$

such that

$$\|f\|_{A^r_+} := \sum_{n=0}^{\infty} |a_n|\, r^n < \infty.$$

Show that $\mathrm{A}^1_+(\mathbb{D}_r)$ is a Banach algebra.

1.3. Consider, for $\operatorname{Re}\alpha > 0$ the function

$$f_\alpha(z) := (1 - z)^\alpha \qquad (z \in \mathbb{D}).$$

(We use the principal branch of the logarithm to define the fractional power.)

a) Determine the power series representation of f_α and show that $\|f_\alpha\|_{\mathrm{A}^1_+} = 2$ for $0 < \alpha < 1$. [Hint: Look at the signs of the Taylor coefficients.]
b) Show that $f_\alpha \in \mathrm{A}^1_+(\mathbb{D})$ for all $\operatorname{Re}\alpha > 0$. [Hint: Reduce first to the case $0 < \operatorname{Re}\alpha < 1$; then estimate by comparing with the case $0 < \alpha < 1$. May be a little tricky.]

1.4. Let A be a power-bounded operator on a Banach space X. Describe, in what respect the $\mathrm{A}^1_+(\mathbb{D})$-calculus differs from the Dunford–Riesz calculus for A.

1.5. Provide an alternative proof of the inequality $\lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} \le r(A)$ (in the proof of Corollary 1.4) by using the following consequence of Theorem 1.3: *If $\sum_{n=0}^{\infty} a_n z^n$ is a power series with a radius of convergence $r > r(A)$, then $\sum_{n=0}^{\infty} a_n A^n$ converges in $\mathcal{L}(X)$.*

1.6. Prove the following continuous version of Fekete's Lemma: *Let $f : (0,\infty) \to \mathbb{R}$ be a function, bounded on compact subintervals and satisfying*

$$f(x + y) \le f(x) + f(y) \qquad (x,\ y > 0).$$

Then $\lim_{x\to\infty} \frac{f(x)}{x} = \inf_{x>0} \frac{f(x)}{x}$.

1.7. Let X and Y be Banach spaces and A and B bounded operators on X and Y, respectively. Show that the following assertions are equivalent for an operator $T \in \mathcal{L}(X;Y)$:

(i) $TA = BT$.

(ii) $Tf(A) = f(B)T$ for each $f \in \mathrm{Hol}(U)$ and open set $U \supseteq \sigma(A) \cup \sigma(B)$.

1.8. Let A be a bounded operator on a Banach space X and suppose that the spectrum $\sigma(A)$ of A can be written as the union

$$\sigma(A) = K_1 \cup K_2, \qquad K_1 \cap K_2 = \emptyset$$

of disjoint compact sets K_1 and K_2. Show that there exists $P \in \mathcal{L}(X)$ with $P^2 = P$ and such that the following holds:

1) $X_1 := \mathrm{ran}(P)$ and $X_2 := \ker(P)$ are A-invariant.

2) $\sigma(A|_{X_1}) = K_1$ and $\sigma(A|_{X_2}) = K_2$.

1.9 (Spectral Mapping Theorem). Let A be a bounded operator on a Banach space X, let $U \supseteq \sigma(A)$ open, and let $f \in \mathrm{Hol}(U)$. Show that

$$f(\sigma(A)) = \sigma(f(A)).$$

References

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Chapter 2
Unbounded Operators and Abstract Functional Calculus

2.1 Unbounded Operators

We start with an introduction to unbounded operators and their spectral theory. This will be brief and cursory so that we can quickly pass on to more substantial material. For a more detailed account the reader is referred to Appendix A.4.

By an (unbounded) operator from a Banach space X to a Banach space Y we mean a linear mapping $A : \mathrm{dom}(A) \to X$ where $\mathrm{dom}(A)$ is a linear subspace of X. An operator is *closed* if one has the implication

$$\mathrm{dom}(A) \ni x_n \to x \in X,\ Ax_n \to y \in Y \quad \Rightarrow \quad x \in \mathrm{dom}(A),\ Ax = y.$$

Equivalently, A is closed if $\mathrm{dom}(A)$ endowed with the **graph norm**

$$\|x\|_A := \|x\|_X + \|Ax\|_Y \qquad (x \in \mathrm{dom}(A))$$

is a Banach space. The set of all closed operators is denoted by

$$\mathcal{C}(X;Y) \qquad (\text{and } \mathcal{C}(X) := \mathcal{C}(X;X)).$$

We reserve the term **bounded operator** for the elements of $\mathcal{L}(X;Y)$. By the closed graph theorem, an operator A is bounded if and only if it is closed and satisfies $\mathrm{dom}(A) = X$ (i.e., it is **fully defined**).

Operators A can be identified with their **graph**

$$\mathrm{graph}(A) = \{(x,y) \in X \oplus Y \mid x \in \mathrm{dom}(A),\ Ax = y\},$$

which is a linear subspace of $X \oplus Y$ (a so-called **linear relation**). The operator is closed if and only if its graph is closed in $X \oplus Y$. Set theory purists would claim that an operator actually *is the same* as its graph. So we are justified to identify these concepts, and will do so henceforth. We use the notation "$Ax = y$" synonymously with $(x, y) \in A$, so that "$x \in \mathrm{dom}(A)$" is tacit here.

Identifying operators with (certain) linear relations gives immediate meaning to the inclusion statement "$A \subseteq B$". Equivalently, this can be expressed through the implication

$$Ax = y \quad \Rightarrow \quad Bx = y \qquad (x \in X,\ y \in Y)$$

or through:

$$\mathrm{dom}(A) \subseteq \mathrm{dom}(B) \quad \text{and} \quad B|_{\mathrm{dom}(A)} = A.$$

An operator A is **closable** if there is a closed operator B such that $A \subseteq B$. In this case, there is a smallest closed operator containing A, named the **closure** of A and denoted by $\overline{A}$. (This is consistent with the interpretation as linear subspaces of $X \oplus Y$.) A subspace $D \subseteq \mathrm{dom}(A)$ is a **core** for a closed operator A, if A is the closure of $A|_D$.

Algebra for (unbounded) operators is more complicated than for bounded ones due to domain issues. For example, the **sum** $A + B$ of two operators is defined by

$$\mathrm{dom}(A + B) := \mathrm{dom}(A) \cap \mathrm{dom}(B), \qquad (A + B)x := Ax + Bx.$$

The composition AB of two operators (when meaningful) is defined by

$$ABx = z \quad \overset{\text{def.}}{\Longleftrightarrow} \quad \exists\, y : Bx = y \,\wedge\, Ay = z.$$

Sum and product are (trivially) associative, but distributivity fails in general. The most important fact to remember about products is: *If A is closed and B is bounded, then AB is closed.* (See Exercise 2.2.)

For an operator A, its inverse A^{-1} is the linear relation defined by

$$(y, x) \in A^{-1} \quad \overset{\text{def.}}{\Longleftrightarrow} \quad Ax = y.$$

The inverse A^{-1} is an operator if and only if A is injective. Obviously, A^{-1} is closed if and only if A is closed. An operator A is **invertible** if A^{-1} is a bounded operator.

Spectral Theory. The notions of spectrum and resolvent (set) for general operators A on a Banach space X are exactly the same as for bounded operators. The **resolvent set** is

$$\rho(A) := \{\lambda \in \mathbb{C} \mid \lambda - A \text{ is invertible}\},$$

where $\lambda - A := \lambda \mathrm{I} - A$. The **resolvent** is the mapping

$$R(\cdot, A) : \rho(A) \to \mathcal{L}(X), \qquad R(\lambda, A) := (\lambda - A)^{-1}.$$

The **spectrum** is $\sigma(A) := \mathbb{C} \setminus \rho(A)$, but it is sometimes helpful to think of ∞ as a spectral value of an operator which is *not* bounded. (Actually, for spectral theory it is much more convenient to work not just with operators but with general linear relations, see Appendix A.4.)

As for bounded operators, the resolvent set is open and the resolvent is holomorphic and satisfies the **resolvent identity**. Differentiating yields $\frac{\mathrm{d}}{\mathrm{d}z} R(z, A) = -R(z, A)^2$ and the usual power series representation obtains (see Corollary A.15).

Eigenvalues and **approximate eigenvalues** and the corresponding notions of **point spectrum** $\sigma_{\mathrm{p}}(A)$ and **approximate point spectrum** $\sigma_{\mathrm{a}}(A)$ are defined as for bounded operators, see Lemma A.17 and A.18.

2.2 Multiplication Operators

Let $\Omega = (\Omega, \Sigma, \mu)$ be any measure space. For an essentially measurable function $a : \Omega \to \mathbb{C}$ we define the corresponding **multiplication operator** M_a on $\mathrm{L}^p(\Omega)$, $1 \le p \le \infty$, by

$$M_a f := af \quad \text{for} \quad f \in \mathrm{dom}(M_a) := \{g \in \mathrm{L}^p(\Omega) \mid ag \in \mathrm{L}^p(\Omega)\}.$$

The next theorem collects the relevant properties. We restrict to *semi-finite* measure spaces for convenience (see Appendix A.1).

Theorem 2.1. *Let $\Omega = (\Omega, \Sigma, \mu)$ be a semi-finite measure space. Then for a measurable function $a : \Omega \to \mathbb{C}$ the multiplication operator M_a on $\mathrm{L}^p(\Omega)$, $1 \le p < \infty$, has the following properties:*

a) *M_a is a closed operator.*

b) *M_a is injective if and only if $\mu[\, a = 0\,] = 0$. In this case, $M_a^{-1} = M_{a^{-1}}$.*[1]

c) *M_a is densely defined.*

d) *M_a is bounded if and only if $a \in \mathrm{L}^\infty(\Omega)$. In this case $\|M_a\| = \|a\|_{\mathrm{L}^\infty}$.*

e) *If $b : \Omega \to \mathbb{C}$ is measurable, then $M_a M_b \subseteq M_{ab}$ and $M_a + M_b \subseteq M_{a+b}$. Moreover,*
$$\mathrm{dom}(M_a M_b) = \mathrm{dom}(M_b) \cap \mathrm{dom}(M_{ab}).$$

f) *If $b \in \mathrm{L}^\infty(\Omega)$ then*
$$M_a M_b = M_{ab} \quad \textit{and} \quad M_a + M_b = M_{a+b}.$$

g) *$\sigma(M_a) = \sigma_{\mathrm{a}}(M_a) = \mathrm{essran}(a)$, the essential range of a.*

h) $\lambda \in \rho(M_a) \quad \Longrightarrow \quad R(\lambda, M_a) = M_{\frac{1}{\lambda - a}}$.

[1] Here as in most cases, a^{-1} is synonymous with $\frac{1}{a}$.

i) *Up to equality μ-almost everywhere, a is uniquely determined by M_a.*

Proof. a) Suppose that $(f_n, af_n) \to (f, g)$ in $\mathrm{L}^p \oplus \mathrm{L}^p$. Passing to a subsequence if necessary we may suppose that $f_n \to f$ almost everywhere. But then $af = g$. Hence $f \in \mathrm{dom}(M_a)$ and $g = M_a f$.

b) M_a is not injective precisely when there is $0 \neq f \in \mathrm{L}^p$ such that $af = 0$, i.e. $[\, f \neq 0\,] \subseteq [\, a = 0\,]$. Since the measure is semi-finite, such a function f can be found precisely when $\mu[\, a = 0\,] > 0$.

If $\mu[\, a = 0\,] = 0$ then a^{-1} is an essentially measurable $\mathbb{C}$-valued function and $af = g$ μ-almost everywhere if and only if $f = a^{-1}g$ μ-almost everywhere. Hence $M_a^{-1} = M_{a^{-1}}$ as claimed.

c) Let $f \in \mathrm{L}^p(\Omega)$, define $A_n := [\, |a| \le n\,]$ and let $f_n = \mathbf{1}_{A_n} f$. Then $f_n \in \mathrm{L}^p$ and $f_n \to f$ in L^p. Moreover, $|af_n| \le n\,|f|$, hence $f_n \in \mathrm{dom}(M_a)$.

d) It is easy to see that $a \in \mathrm{L}^\infty$ implies that M_a is bounded and $\|M_a\| \le \|a\|_{\mathrm{L}^\infty}$. For the converse, suppose that M_a is bounded and fix $c > 0$ such that $\mu[\, |a| \ge c\,] > 0$. Then by the semi-finiteness of the measure space Ω there is a set $A \subseteq [\, |a| \ge c\,]$ of strictly positive but finite measure. Applying M_a to $\mathbf{1}_A$ yields

$$\mu(A)c^p \le \int_\Omega |a\mathbf{1}_A|^p = \|M_a \mathbf{1}_A\|_p^p \le \|M_a\|^p \mu(A).$$

It follows that $c \le \|M_a\|$, and the claim is proved.

e) and f) are straightforward.

g) Passing to $\lambda - a$ if necessary we only need to show that

$$0 \in \boldsymbol{\sigma}(M_a) \quad \Longrightarrow \quad 0 \in \mathrm{essran}(a) \quad \Longrightarrow \quad 0 \in \boldsymbol{\sigma}_{\mathrm{a}}(M_a).$$

If $0 \notin \mathrm{essran}(a)$ then there is $\varepsilon > 0$ such that $\mu[\, |a| \le \varepsilon\,] = 0$. Hence M_a is injective, and $M_a^{-1} = M_{1/a}$. But $\|a^{-1}\|_{\mathrm{L}^\infty} \le \frac{1}{\varepsilon} < \infty$ and hence M_a^{-1} is bounded. It follows that $0 \in \boldsymbol{\rho}(M_a)$.

Suppose that $0 \in \mathrm{essran}(a)$. Then for each $n \in \mathbb{N}$ one has $\mu\big[\, |a| \le \frac{1}{n}\,\big] > 0$. By the semi-finiteness we can find a set $A_n \subseteq \big[\, |a| \le \frac{1}{n}\,\big]$ of finite and strictly positive measure. Let $c_n = \|\mathbf{1}_{A_n}\|_p^{-1}$ and $f_n := c_n \mathbf{1}_{A_n}$. Then $\|f_n\|_p = 1$ and $f_n \in \mathrm{dom}(M_a)$ and $|M_a f_n| = |af_n| \le \frac{1}{n}\,|f_n| \to 0$ in $\mathrm{L}^p(\Omega)$. Hence, $(f_n)_n$ is an approximate eigenvector for 0, and $0 \in \boldsymbol{\sigma}_{\mathrm{a}}(M_a)$ as claimed.

h) is clear.

i) Suppose that a and b are measurable functions on Ω such that $M_a = M_b$ as operators on $\mathrm{L}^p(\Omega)$. For $n \in \mathbb{N}$ and $f \in \mathrm{L}^p$ define $f_n := f\mathbf{1}_{[\,|a|+|b| \le n\,]}$. Then $f_n \in \mathrm{dom}(M_a) \cap \mathrm{dom}(M_b)$ and hence

$$a\mathbf{1}_{[\,|a|+|b|\le n\,]} f = M_a f_n = M_b f_n = b\mathbf{1}_{[\,|a|+|b|\le n\,]} f.$$

It follows that $a\mathbf{1}_{[\,|a|+|b|\le n\,]} = b\mathbf{1}_{[\,|a|+|b|\le n\,]}$ almost everywhere for each $n \in \mathbb{N}$ and hence $a = b$ almost everywhere. □

Multiplication operators can be considered on other function spaces, for instance on the space $\mathrm{C_b}(\Omega)$, where Ω is a metric space (Exercise 2.5), or on the space $\mathrm{C}_0(\Omega)$, where Ω is a locally compact space. (Here, the multiplicators a are continuous functions.)

2.3 Functional Calculus for Multiplication Operators

From now on we fix a (semi-finite) measure space $\Omega = (\Omega, \Sigma, \mu)$, a μ-essentially measurable function $a : \Omega \to \mathbb{C}$ and a number $1 \le p < \infty$ and consider the operator $A := M_a$ on $X := \mathrm{L}^p(\Omega)$. Moreover, let us write

$$K := \operatorname{essran}(a) = \sigma(A) \subseteq \mathbb{C},$$

which is a closed subset of $\mathbb{C}$. It is easy to see (Exercise 2.6) that

$$a \in K \quad \mu\text{-almost everywhere.}$$

Equivalently, the Borel measure $\nu := a_*\mu$ is supported on K:

$$K = \operatorname{essran}(a) = \operatorname{supp}(a_*\mu) = \operatorname{supp}(\nu).$$

Hence, if $f : K \to \mathbb{C}$ is ν-almost everywhere measurable, then $f \circ a$ is μ-almost everywhere measurable, and we can form the closed(!) operator

$$\Phi(f) := M_{f \circ a}$$

on $X = \mathrm{L}^p(\Omega)$. The set $\mathrm{L}^0(K, \nu)$ of ν-almost everywhere measurable functions (modulo ν-a.e. null functions) on K is an algebra. The emerging mapping

$$\Phi : \mathrm{L}^0(K, \nu) \to \mathcal{C}(X)$$

is called the **functional calculus** for the operator $A = M_a$. Here is a listing of some essential properties.

Theorem 2.2. *Let $\Omega = (\Omega, \Sigma, \mu)$ be a semi-finite measure space, let $a \in \mathrm{L}^0(\Omega)$ and $1 \le p < \infty$, $K := \operatorname{essran}(a)$ and $\nu := a_*\mu$ as above. Let, furthermore,*

$$\Phi : \mathrm{L}^0(K, \nu) \to \{\textit{operators on } \mathrm{L}^p(\Omega)\}, \qquad \Phi(f) = M_{f \circ a}$$

be the associated functional calculus. Then for all $f, g \in \mathrm{L}^0(K, \nu)$ and $\lambda \in \mathbb{C}$ the following statements hold:

a) *The operator $\Phi(f)$ is closed.*

b) $\Phi(\mathbf{1}) = \mathrm{I}$.

c) $\lambda \Phi(f) \subseteq \Phi(\lambda f) \quad$ *and* $\quad \Phi(f) + \Phi(g) \subseteq \Phi(f + g)$.

d) $\Phi(f)\Phi(g) \subseteq \Phi(fg)$ *with*

$$\operatorname{dom}(\Phi(f)\Phi(g)) = \operatorname{dom}(\Phi(g)) \cap \operatorname{dom}(\Phi(fg)).$$

e) $\Phi(f)$ *is injective if and only if* $\nu[\, f = 0\,] = 0$, *and in this case* $\Phi(f)^{-1} = \Phi(\frac{1}{f})$.

f) $\Phi(f)$ *is a bounded operator if and only if* $f \in \mathrm{L}^\infty(K, \nu)$.

g) *If* $(f_n)_n$ *is a bounded sequence in* $\mathrm{L}^\infty(K, \nu)$ *with* $f_n \to f$ ν*-almost everywhere, then* $\Phi(f_n) \to \Phi(f)$ *strongly on* $\mathrm{L}^p(\Omega)$.

Proof. a) Since $\Phi(f) = M_{f\circ a}$, this follows from Theorem 2.1.a).

b) This is obvious.

c) and d) follow from Theorem 2.1.e).

e) follows from Theorem 2.1.b).

f) follows from Theorem 2.1.d).

g) It follows from the hypotheses that $f_n \circ a \to f \circ a$ μ-almost everyhere. If $x \in X = \mathrm{L}^p(\Omega)$ is arbitrary, then

$$\Phi(f_n)x = (f_n \circ a)x \to (f \circ a)x = \Phi(f)x$$

by Lebesgue's theorem. □

In the following we shall base an abstract definition of "functional calculus" on some of the just listed properties. Before, let us note an interesting fact: the functional calculus Φ for M_a is determined by its restriction to those $f \in \mathrm{L}^0(K, \nu)$ that yield *bounded* operators. In fact, for $f \in \mathrm{L}^0(K, \nu)$ let

$$h := \frac{1}{1 + |f|^2} \quad \text{and} \quad g := \frac{f}{1 + |f|^2}.$$

Then h, $g \in \mathrm{L}^\infty(K, \nu)$. Moreover, $f = \frac{g}{h}$ and $\Phi(h)$ is injective, hence

$$\Phi(h)^{-1}\Phi(g) = \Phi(\tfrac{1}{h})\Phi(g) = \Phi(\tfrac{g}{h}) = \Phi(f),$$

by Theorem 2.2.d) and since $\Phi(g)$ is a bounded operator.

2.4 Abstract Functional Calculus (I) — Definition

With the concrete model of multiplication operators at hand, we now turn to a more axiomatic treatment of functional calculi.

Let $\mathcal{F}$ be an algebra with a unit element $\mathbf{1}$ and let X be a Banach space. A mapping

$$\Phi : \mathcal{F} \to \mathcal{C}(X)$$

from $\mathcal{F}$ to the set of closed operators on X is called a **proto-calculus** (or: $\mathcal{F}$-proto-calculus) on X if the following axioms are satisfied ($f,\, g \in \mathcal{F},\, \lambda \in \mathbb{C}$):

(FC1) $\Phi(\mathbf{1}) = \mathrm{I}$.

(FC2) $\lambda\Phi(f) \subseteq \Phi(\lambda f)$ and $\Phi(f) + \Phi(g) \subseteq \Phi(f+g)$.

(FC3) $\Phi(f)\Phi(g) \subseteq \Phi(fg)$ and

$$\mathrm{dom}(\Phi(f)\Phi(g)) = \mathrm{dom}(\Phi(g)) \cap \mathrm{dom}(\Phi(fg)).$$

An element $f \in \mathcal{F}$ is called **Φ-bounded** if $\Phi(f) \in \mathcal{L}(X)$. The set of Φ-bounded elements is denoted by

$$\mathcal{F}_\Phi := \{f \in \mathcal{F} \mid \Phi(f) \in \mathcal{L}(X)\} = \Phi^{-1}(\mathcal{L}(X)).$$

For each $f \in \mathcal{F}$ the set of its **Φ-regularizers** is

$$\mathrm{Reg}_\Phi(f) := \{e \in \mathcal{F}_\Phi \mid ef \in \mathcal{F}_\Phi\}.$$

The definition of a *regularizer* here differs from and is more general than the one given in our earlier work [1]. It has been noticed in [2] (eventually published as [3]) that such a relaxation of terminology is useful.

The following theorem summarizes basic properties of a proto-calculus.

Theorem 2.3. *Let $\Phi : \mathcal{F} \to \mathcal{C}(X)$ be a proto-calculus on a Banach space X. Then the following assertions hold ($f,\, g \in \mathcal{F},\, \lambda \in \mathbb{C}$):*

a) *If $\lambda \neq 0$ or $\Phi(f) \in \mathcal{L}(X)$ then $\Phi(\lambda f) = \lambda\Phi(f)$.*

b) *If $\Phi(g) \in \mathcal{L}(X)$ then*

$$\Phi(f) + \Phi(g) = \Phi(f+g) \quad \textit{and} \quad \Phi(f)\Phi(g) = \Phi(fg).$$

c) *If $fg = \mathbf{1}$ then $\Phi(g)$ is injective and $\Phi(g)^{-1} \subseteq \Phi(f)$. If, in addition $fg = gf$, then $\Phi(g)^{-1} = \Phi(f)$.*

d) *The set $\mathcal{F}_\Phi$ of Φ-bounded elements is a unital subalgebra of $\mathcal{F}$ and*

$$\Phi : \mathcal{F}_\Phi \to \mathcal{L}(X)$$

is an algebra homomorphism.

e) *The set $\mathrm{Reg}_\Phi(f)$ of Φ-regularizers of f is a left ideal in $\mathcal{F}_\Phi$.*

Proof. a) One has

$$\Phi(f) = \Phi(\lambda^{-1}\lambda f) \supseteq \lambda^{-1}\Phi(\lambda f) \supseteq \lambda^{-1}\lambda\Phi(f) = \Phi(f).$$

Hence, all inclusions are equalities, and the assertion follows.

b) By Axiom (FC2) and a)

$$\begin{aligned} \Phi(f) = \Phi(f+g-g) &\supseteq \Phi(f+g) + \Phi(-g) = \Phi(f+g) - \Phi(g) \\ &\supseteq \Phi(f) + \Phi(g) - \Phi(g) \overset{!}{=} \Phi(f). \end{aligned}$$

Hence, all inclusions are equalities and the first assertion in b) follows. For the second, note that by Axiom (FC3) $\Phi(f)\Phi(g) \subseteq \Phi(fg)$ with

$$\operatorname{dom}(\Phi(f)\Phi(g)) = \operatorname{dom}(\Phi(g)) \cap \operatorname{dom}(\Phi(fg)) = \operatorname{dom}(\Phi(fg)),$$

hence we are done.

c) By (FC3), if $fg = \mathbf{1}$ then $\Phi(f)\Phi(g) \subseteq \Phi(fg) = \Phi(\mathbf{1}) = \mathrm{I}$. Hence, $\Phi(g)$ is injective and $\Phi(f) \supseteq \Phi(g)^{-1}$. If $fg = gf$ then, by symmetry, $\Phi(f)$ is injective too, and $\Phi(g) \supseteq \Phi(f)^{-1}$. This yields $\Phi(f) = \Phi(g)^{-1}$ as desired.

d) and e) follow directly from b). □

Determination. Let $\Phi : \mathcal{F} \to \mathcal{C}(X)$ be a proto-calculus on X and $e, f \in \mathcal{F}$. Then, by Axiom (FC3),

$$\Phi(e)\Phi(f) \subseteq \Phi(ef). \tag{2.1}$$

If in addition $\Phi(e), \Phi(ef) \in \mathcal{L}(X)$, i.e., if e is a Φ-regularizer of f, then (2.1) simply means that

$$\forall\, x,\, y \in X : \quad \Phi(f)x = y \quad \Rightarrow \quad \Phi(ef)x = \Phi(e)y.$$

In very special situations, e.g. if e has a left inverse in $\mathcal{F}$, the converse implication holds (Exercise 2.7). In that case, we say that e is **Φ-determining** for f.

More generally, a subset $\mathcal{M} \subseteq \operatorname{Reg}_\Phi(f)$ is said to be **Φ-determining** for $f \in \mathcal{F}$ if one has

$$\Phi(f)x = y \quad \Longleftrightarrow \quad \forall e \in \mathcal{M} :\ \Phi(ef)x = \Phi(e)y \tag{2.2}$$

for all $x, y \in X$. (As explained, only the implication "$\Leftarrow$" is interesting here.)

Lemma 2.4. *Let Φ be an $\mathcal{F}$-proto-calculus on a Banach space X, let $f \in \mathcal{F}$ and let $\mathcal{M} \subseteq \operatorname{Reg}_\Phi(f)$ be Φ-determining for f. Then the following assertions hold:*

a) $\bigcap_{e \in \mathcal{M}} \ker(\Phi(e)) = \{0\}$.

b) *Each set $\mathcal{M}'$ with $\mathcal{M} \subseteq \mathcal{M}' \subseteq \operatorname{Reg}_\Phi(f)$ is also determining for f.*

c) *If $T \in \mathcal{L}(X)$ commutes with all operators $\Phi(e)$ and $\Phi(ef)$ for $e \in \mathcal{M}$, then T commutes with $\Phi(f)$.*

Recall that the last statement simply means $T\Phi(f) \subseteq \Phi(f)T$.

Proof. a) follows since $\Phi(f)$ is an operator and not just a linear relation.
b) is clear.
c) Suppose that $\Phi(f)x = y$. Then, for each $e \in \mathcal{M}$,

$$\Phi(ef)Tx = T\Phi(ef)x = T\Phi(e)y = \Phi(e)Ty.$$

Since $\mathcal{M}$ is Φ-determining for f, $\Phi(f)Tx = Ty$. But this just means that $T\Phi(f) \subseteq \Phi(f)T$. □

A proto-calculus $\Phi : \mathcal{F} \to \mathcal{C}(X)$ is called a **calculus** (or: $\mathcal{F}$-calculus) if in addition to (FC1)–(FC3) the following fourth axiom is satisfied:

(FC4) For each $f \in \mathcal{F}$ the set $\mathrm{Reg}_\Phi(f)$ is Φ-determining for f.

The "Determination Axiom" (FC4) ensures that an $\mathcal{F}$-proto-calculus Φ is determined *in an algebraic* way by its restriction to the set of Φ-bounded elements. If $\Phi(\mathcal{F}) \subseteq \mathcal{L}(X)$, i.e., if every $f \in \mathcal{F}$ is Φ-bounded, then (FC4) is trivially satisfied (why?).

Remarks 2.5. 1) The restriction of a proto-calculus $\Phi : \mathcal{F} \to \mathcal{C}(X)$ to a unital subalgebra $\mathcal{F}'$ of $\mathcal{F}$ is again a proto-calculus, sloppily called a **subcalculus**. Even if the original proto-calculus satisfies (FC4), this need not be the case for the subcalculus. (It does if the subcalculus has the same set of Φ-bounded elements.) Similarly, the subcalculus may satisfy (FC4) while the original one does not.

2) Given a proto-calculus $\Phi : \mathcal{F} \to \mathcal{C}(X)$, the set $\mathcal{F}_\Phi$ of Φ-bounded elements is a unital subalgebra and the restriction of Φ to $\mathcal{F}_\Phi$ a homomorphism of unital algebras. Let us call this the **bounded subcalculus**. As already mentioned, it trivially satisfies (FC4).

3) A proto-calculus $\Phi : \mathcal{F} \to \mathcal{C}(X)$ such that $\Phi(\mathcal{F}) \subseteq \mathcal{L}(X)$ (equivalently: $\mathcal{F} = \mathcal{F}_\Phi$) is nothing else then a *representation* of the unital algebra $\mathcal{F}$ by bounded operators.
So with our definition of a (proto-)calculus we generalize representation theory towards unbounded operators. However, Axiom (FC4) (if satisfied) guarantees that the whole calculus is determined by the bounded subcalculus and we are confronted only with a "tame" kind of unboundedness.

4) If the algebra $\mathcal{F}$ is not an algebra of functions, an $\mathcal{F}$-calculus is—strictly speaking—not a *functional* calculus. However, although we shall try to be consistent with this distinction, a little sloppyness should be allowed here. After all, many commutative algebras are *isomorphic* to function algebras.

Example 2.6. Let $\Omega = (\Omega, \Sigma, \mu)$ be a semi-finite measure space and $a : \Omega \to \mathbb{C}$ a μ-essentially measurable function. Then, with $\nu = a_*\mu$, $K := \operatorname{essran}(a)$ and $X = \mathrm{L}^p(\Omega)$, the mapping

$$\Phi : \mathrm{L}^0(K, \nu) \to \mathcal{C}(X), \qquad \Phi(f) := M_{f \circ a}$$

is an $\mathrm{L}^0(K, \nu)$-calculus on X. The set of Φ-bounded elements is $\mathrm{L}^0(K, \nu)_\Phi = \mathrm{L}^\infty(K, \nu)$. For $f \in \mathrm{L}^0(K, \nu)$ each function

$$(1 + |f|^2)^{-\alpha}, \qquad \alpha \geq \tfrac{1}{2},$$

is a Φ-determining Φ-regularizer for f.

2.5 Abstract Functional Calculus (II) — Generators

The definition of a (proto-)calculus given so far is, admittedly, very general. In many cases, the algebra $\mathcal{F}$ is actually a subalgebra of functions (or of equivalence classes of functions) on some subset $D \subseteq \mathbb{C}$ of the complex plane and contains the special function $\mathbf{z} := (z \mapsto z)$. (E.g., in our multiplicator example, $D = K = \operatorname{essran}(a)$ and $\mathcal{F} = \mathrm{L}^0(K, \nu)$.) In that case, the operator $A := \Phi(\mathbf{z})$ is a distinguished closed operator on X, and the $\mathcal{F}$-calculus Φ can be called a functional calculus **for** A and A is called the **generator** of the functional calculus Φ. Moreover, the notation

$$f(A) := \Phi(f)$$

is employed.

Corollary 2.7. *Let A be the generator of an $\mathcal{F}$-proto-calculus on a Banach space X, where $\mathcal{F}$ is an algebra of functions on some subset $D \subseteq \mathbb{C}$. Let $\lambda \in \mathbb{C}$ such that $\frac{1}{\lambda - \mathbf{z}} \in \mathcal{F}$. Then $\lambda - A$ is injective and the following assertions are equivalent:*

(i) $\lambda \in \rho(A)$;

(ii) $(\frac{1}{\lambda - \mathbf{z}})(A) \in \mathcal{L}(X)$.

In this case, $R(\lambda, A) = (\frac{1}{\lambda - \mathbf{z}})(A)$.

Proof. Let the given $\mathcal{F}$-calculus be called Φ. Note that $\Phi(\lambda \mathbf{1}) = \lambda \Phi(\mathbf{1}) = \lambda \mathrm{I} \in \mathcal{L}(X)$ and hence

$$\Phi(\lambda - \mathbf{z}) = \lambda \mathrm{I} - A.$$

Since $\mathbf{1} = (\lambda - \mathbf{z}) \frac{1}{\lambda - \mathbf{z}} = \frac{1}{\lambda - \mathbf{z}} (\lambda - \mathbf{z})$, by Theorem 2.3 the operator $\lambda - A$ is injective and $(\lambda - A)^{-1} = \Phi(\frac{1}{\lambda - \mathbf{z}})$. So $\lambda \in \rho(A)$ if and only if this operator is bounded. □

Example 2.8. Let Φ be the $\mathrm{L}^0(K,\nu)$-calculus for some multiplication operator M_a as before. Then $A = \Phi(\mathbf{z}) = M_a$, so our terminology is consistent. For $f \in \mathrm{L}^0(K,\nu)$ one then has (by definition)

$$f(M_a) = M_{f\circ a}.$$

E.g., since A is multiplication by a, e^A is multiplication by e^a. Note that here $K = \operatorname{essran}(a) = \boldsymbol{\sigma}(A)$ and $\lambda - A$ is injective *precisely when* $(\lambda - \mathbf{z})^{-1} \in \mathrm{L}^0(K,\nu)$. (Why?)

Corollary 2.7 hints at situations when one can identify a generator of a functional calculus Φ even in the case when the coordinate function $\mathbf{z}$ itself is not contained in $\mathcal{F}$, but some functions of the form $\frac{1}{\lambda-\mathbf{z}}$ are. This is treated in the following supplement.

Supplement: Linear Relations as Generators

Let $\Phi : \mathcal{F} \to \mathcal{C}(X)$ be an $\mathcal{F}$-calculus, where $\mathcal{F}$ is an algebra of functions defined on some set $D \subseteq \mathbb{C}$. We suppose in addition that the set

$$\Lambda_\Phi := \{\lambda \in \mathbb{C} \mid (\lambda - \mathbf{z})^{-1} \in \mathcal{F}\}$$

is not empty. Then the following result yields a unique linear relation A on X such that

$$\Phi\big((\lambda - \mathbf{z})^{-1}\big) = (\lambda - A)^{-1} \qquad \text{for all } \lambda \in \Lambda_\Phi.$$

Theorem 2.9. *Let $\Phi : \mathcal{F} \to \mathcal{C}(X)$ be an $\mathcal{F}$-calculus, where $\mathcal{F}$ is an algebra of functions on some set $D \subseteq \mathbb{C}$. Then the linear relation*

$$\lambda - \Phi\Big(\frac{1}{\lambda - \mathbf{z}}\Big)^{-1}$$

is independent of $\lambda \in \Lambda_\Phi$.

Proof. Let $\lambda, \mu \in \Lambda_\Phi$ and define $r_\lambda(z) := \frac{1}{\lambda - z}$ and $r_\mu(z) := \frac{1}{\mu - z}$. Then the resolvent identity

$$(\mu - \lambda) r_\lambda r_\mu = r_\lambda - r_\mu$$

holds. Now fix $e_\mu \in \operatorname{Reg}_\Phi(r_\mu)$ and $e_\lambda \in \operatorname{Reg}_\Phi(r_\lambda)$. Then $e := e_\mu e_\lambda$ is a regularizer for both r_λ and r_μ. Let, furthermore, $x, y \in X$. Note that

$$(x,y) \in \lambda - \Phi(r_\lambda)^{-1} \quad\iff\quad \Phi(r_\lambda)(\lambda x - y) = x$$

and likewise for μ instead of λ. We shall show that

$$\Phi(r_\lambda)(\lambda x - y) = x \quad\Longrightarrow\quad \Phi(r_\mu)(\mu x - y) = x.$$

By symmetry, this will suffice for establishing the claim.

So abbreviate $u := \lambda x - y$ and suppose that $\Phi(r_\lambda)u = x$. Then $\Phi(e_\lambda r_\lambda)u = \Phi(e_\lambda)x$ and hence

$$\begin{aligned}\Phi(e_\lambda)\Phi(e_\mu r_\mu)(\mu x - y) &= \Phi(e r_\mu)((\mu - \lambda)x + u) \\ &= (\mu - \lambda)\Phi(e_\mu r_\mu)\Phi(e_\lambda)x + \Phi(e r_\mu)u \\ &= (\mu - \lambda)\Phi(e_\mu r_\mu)\Phi(e_\lambda r_\lambda)u + \Phi(e r_\mu)u \\ &= \Phi(e(\mu - \lambda)r_\mu r_\lambda + e r_\mu)u \\ &= \Phi(e r_\lambda)u = \Phi(e_\mu)\Phi(e_\lambda r_\lambda)u = \Phi(e_\mu)\Phi(e_\lambda)x \\ &= \Phi(e_\lambda)\Phi(e_\mu)x.\end{aligned}$$

Since e_λ ranges over all of $\mathrm{Reg}_\Phi(r_\lambda)$,

$$\Phi(e_\mu r_\mu)(\mu x - y) = \Phi(e_\mu)x.$$

But this holds for all $e_\mu \in \mathrm{Reg}_\Phi(r_\mu)$, hence by Axiom (FC4) it follows that

$$\Phi(r_\mu)(\mu x - y) = x. \qquad \square$$

Suppose that Φ is an $\mathcal{F}$-calculus as above, and such that $\Lambda_\Phi \neq \emptyset$. Then we call the linear relation

$$A_\Phi := \lambda - \Phi((\lambda - \mathbf{z})^{-1})^{-1} \qquad (\lambda \in \Lambda_\Phi)$$

the **generator** of the calculus Φ.

Exercises

2.1. Let X and Y be Banach spaces and let $A : X \supseteq \mathrm{dom}(A) \to Y$ be a linear operator.

a) Prove that the following assertions are equivalent:
   (i) A is closed.
   (ii) Whenever $\mathrm{dom}(A) \ni x_n \to x \in X$ and $Ax_n \to y \in Y$ then $x \in \mathrm{dom}(A)$ and $Ax = y$.
   (iii) The space $(\mathrm{dom}(A), \|\cdot\|_A)$ is a Banach space.

b) Prove that each two of the following three assertions together imply the third one:
   (i) A is continuous for the norm of X, i.e., there is $c \geq 0$ such that $\|Ax\| \leq c\|x\|$ for all $x \in \mathrm{dom}(A)$.
   (ii) A is closed.

(iii) $\mathrm{dom}(A)$ is a closed subspace of X.

[Hint: closed graph theorem.]

2.2. Let A, B, C be linear operators on a Banach space X.

a) $(A+B)C = AC + BC$.

b) $C(A+B) \supseteq CA+CB$, with equality if, for instance, $\mathrm{ran}(A) \subseteq \mathrm{dom}(C)$.

c) If A is closed and B is bounded, then AB is closed.

d) If B is closed and A is invertible, then AB is closed.

2.3. Let A and B be linear operators on a Banach space X, $T \in \mathcal{L}(X)$ and $\lambda \in \rho(A) \cap \rho(B)$. Show that the following assertions are equivalent:

(i) For all $x, y \in X$, if $Ax = y$ then $BTx = Ty$;

(ii) $TA \subseteq BT$;

(iii) $TR(\lambda, A) = R(\lambda, B)T$ for one/all $\lambda \in \rho(A) \cap \rho(B)$.

2.4 (Spectral Mapping Theorem for the Inverse). Let A be linear relation on X and let $\lambda \in \mathbb{K} \setminus \{0\}$. Show that

$$\left(\tfrac{1}{\lambda} - A^{-1}\right)^{-1} = \lambda - \lambda^2(\lambda - A)^{-1}.$$

Conclude that

$$\rho(A) \setminus \{0\} \to \rho(A^{-1}) \setminus \{0\}, \qquad \lambda \mapsto \tfrac{1}{\lambda}$$

is a bijection.

2.5 (Multiplication Operator on Continuous Functions). Let (Ω, d) be a metric space. For a continuous function $a \in \mathrm{C}(\Omega)$ let M_a be the multiplication operator defined on $\mathrm{C_b}(\Omega)$ by

$$M_a f := af \quad \text{for} \quad f \in \mathrm{dom}(M_a) := \{f \in \mathrm{C_b}(\Omega) \mid af \in \mathrm{C_b}(\Omega)\}.$$

a) Show that M_a is a closed operator but need not be densely defined.

b) Show that M_a is bounded if and only if a is bounded, and $\|M_a\| = \|a\|_\infty$ in this case.

c) Show the analogue of e) and f) from Theorem 2.1.

d) Show that $\sigma(M_a) = \sigma_{\mathrm{a}}(M_a) = \overline{a(\Omega)}$.

e) Formulate and prove analogues of h) and i) of Theorem 2.1.

2.6. Let $\Omega = (\Omega, \Sigma, \mu)$ be a measure space and $a : \Omega \to \mathbb{C}$ measurable. Show that $a \in \mathrm{essran}(a)$ μ-almost everywhere.

2.7. Let Φ be an $\mathcal{F}$-proto-calculus on a Banach space X, let $e, f \in \mathcal{F}$ such that $\Phi(e), \Phi(ef) \in \mathcal{L}(X)$. Suppose that e has a left inverse e' in $\mathcal{F}$, i.e., $e'e = \mathbf{1}$. Show that $\Phi(f) = \Phi(e)^{-1}\Phi(ef)$ and

$$\Phi(f)x = y \qquad \Longleftrightarrow \qquad \Phi(ef)x = \Phi(e)y$$

for $x,\ y \in X$.

2.8. Let Φ be the $\mathrm{L}^0(K, \nu)$-calculus on $X = \mathrm{L}^p(\Omega)$ for some multiplication operator M_a, $a : \Omega \to \mathbb{C}$. Examine, under which assumptions this calculus has a nontrivial universal regularizer. [A characterization is possible, but not straightforward.]

2.9. Let, as in Exercise 2.5, Ω be a metric space and $a \in \mathrm{C}(\Omega)$. Let $D := a(\Omega) \subseteq \mathbb{C}$. For a continuous function $f : D \to \mathbb{C}$ define

$$\Phi(f) := M_{f \circ a}.$$

Show that $\Phi : \mathrm{C}(D) \to \mathcal{C}(\mathrm{C_b}(\Omega))$ is a functional calculus for M_a, and each $f \in \mathrm{C}(D)$ has a determining regularizer. Can you think of an example showing that in general the given functional calculus is not "maximal" (whatever this means precisely)?

2.10. Let $\Phi : \mathcal{F} \to \mathcal{C}(X)$ be an $\mathcal{F}$-(proto)-calculus.

a) Suppose that $\mathcal{F}$ is commutative. Show that $\mathcal{N} := \{f \in \mathcal{F} \mid \Phi(f) = 0\}$ is an ideal of $\mathcal{F}$ and that by

$$\Phi(f + \mathcal{N}) := \Phi(f)$$

a $\mathcal{F}/\mathcal{N}$-(proto)-calculus is defined.

b) Let $\mathcal{F}'$ be another unital algebra and $\Psi : \mathcal{F}' \to \mathcal{F}$ a unital algebra homomorphism. Show that $\Phi \circ \Psi : \mathcal{F}' \to \mathcal{C}(X)$ is a $\mathcal{F}'$-proto-calculus. If Φ is a calculus and Ψ is surjective, then $\Phi \circ \Psi$ is a calculus as well.

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Chapter 3
Borel Functional Calculus

In this chapter we introduce the concept of a measurable functional calculus on a Hilbert space with its important special case of Borel functional calculus. We study these concepts axiomatically and prove an important uniqueness result.

3.1 Normal Operators on Hilbert Spaces

Due to the presence of the inner product, abstract operator theory is much richer on Hilbert spaces than on general Banach spaces. In particular, it comprises the concepts of *adjoint* and *numerical range* of an operator. As in the last chapter, we avoid going into details here and refer to Appendix A.5 and Exercise 3.1 instead.

The **adjoint** A^* of a linear operator A on a Hilbert space H is defined as a linear relation by

$$(u, v) \in A^* \quad \overset{\text{def.}}{\Longleftrightarrow} \quad (\, x \,|\, v \,) = (\, Ax \,|\, u \,) \quad \text{for all } x \in \operatorname{dom}(A).$$

In the first section of Appendix A.5 it is shown that if A is a densely defined and closed operator, then so is A^*. Moreover, considered as a linear relation in $H \oplus H$, one has

$$A^* = [J(A)]^\perp = J(A^\perp)$$

where the unitary operator J on $H \oplus H$ is given by

$$J : H \oplus H \to H \oplus H, \qquad J(x, y) := (-y, x).$$

The following elementary results will be used frequently in the following.

Lemma 3.1. *Let $A : H \supseteq \operatorname{dom}(A) \to H$ be a closed and densely defined operator. Then for each pair of vectors $u, v \in H$ there are uniquely determined vectors $y \in \operatorname{dom}(A)$ and $x \in \operatorname{dom}(A^*)$ with*

$$x - Ay = u \quad and \quad A^*x + y = v.$$

Moreover, one has $\|x\|^2 + \|A^*x\|^2 + \|y\|^2 + \|Ay\|^2 = \|u\|^2 + \|v\|^2$.

Proof. Note that whenever $x \in \mathrm{dom}(A^*)$ and $y \in \mathrm{dom}(A)$ one has $(x, A^*x) \perp J(y, Ay) = (-Ay, y)$ and

$$(x, A^*x) + J(y, Ay) = (x - Ay, A^*x + y).$$

As A is closed, also $J(A)$ is closed, and hence the first assertion follows from the existence and uniqueness of the (orthogonal) decomposition of a general element (u, v) in $H \oplus H$ into a vector in A^* and a vector in $J(A)$. The second assertion follows by Pythagoras:

$$\begin{aligned}\|x\|^2 + \|A^*x\|^2 + \|Ay\|^2 + \|y\|^2 &= \|(x, A^*x)\|^2 + \|(-Ay, y)\|^2 \\ &= \|(x - Ay, A^*x + y)\|^2 = \|(u, v)\|^2 = \|u\|^2 + \|v\|^2.\end{aligned}$$ □

If we apply the lemma with $u = 0$, we obtain the following result.

Theorem 3.2. *Let A be a closed and densely defined operator on a Hilbert space H. Then the operator* $\mathrm{I} + A^*A$ *is densely defined and invertible and its inverse*

$$T_A := (\mathrm{I} + A^*A)^{-1}$$

is a contraction, i.e., it satisfies $\|T_A\| \leq 1$. *Both operators T_A and A^*A are self-adjoint and positive, and* $\mathrm{dom}(A^*A)$ *is a core for A.*

Proof. Apply the lemma with $u = 0$ and $v \in H$ to obtain elements $y \in \mathrm{dom}(A)$ and $x \in \mathrm{dom}(A^*)$ with $x = Ay$ and $A^*x + y = v$. It follows that $y \in \mathrm{dom}(\mathrm{I} + A^*A)$ and $(\mathrm{I} + A^*A)y = v$. Moreover,

$$\|v\|^2 = \|y\|^2 + 2\|Ay\|^2 + \|A^*Ay\|^2 \geq \|y\|^2. \tag{3.1}$$

In particular, $\mathrm{I} + A^*A$ is surjective.

On the other hand, suppose that $v \perp \mathrm{dom}(A^*A)$ and $y \in \mathrm{dom}(A^*A)$ with $(\mathrm{I} + A^*A)y = v$. Taking the inner product with y yields

$$0 = (\, y \mid v\,) = (\, y \mid y + A^*Ay\,) = \|y\|^2 + (\, y \mid A^*Ay\,) = \|y\|^2 + \|Ay\|^2, \tag{3.2}$$

which implies that $y = 0$ and $v = 0$. The choice of $v = 0$ yields that $\mathrm{I} + A^*A$ is injective and hence bijective. By (3.1), its inverse $T_A := (\mathrm{I} + A^*A)^{-1}$ satisfies $\|T_Av\|^2 \leq \|v\|^2$ for all $v \in H$, i.e., $\|T_A\| \leq 1$.

Furthermore, (3.2) implies that the orthogonal complement of $\mathrm{dom}(A^*A)$ is trivial, hence $\mathrm{dom}(A^*A)$ is dense in H. It is now straightfoward to show that A^*A is symmetric. Therefore, $\mathrm{I} + A^*A$ is symmetric and bijective, hence self-adjoint with self-adjoint inverse T_A, by Lemma A.21.

The postivity of A^*A follows directly from the definitions.It follows that $\mathrm{I} + A^*A$ is positive, hence si is its inverse T_A.

Finally, in order to show that $\operatorname{dom}(A^*A)$ is a core for A it suffices to show that its orthogonal complement in $\operatorname{dom}(A)$ with respect to the graph inner product is zero. So let $v \in \operatorname{dom}(A)$ be such that $v \perp_{\operatorname{dom}(A)} \operatorname{dom}(A^*A)$. This means that

$$0 = (v \,|\, y) + (Av \,|\, Ay) = (v \,|\, y) + (v \,|\, A^*Ay) = (v \,|\, (\mathrm{I} + A^*A)y)$$

for all $y \in \operatorname{dom}(A^*A)$. As $\mathrm{I} + A^*A$ is surjective, this implies that $v = 0$, and the claim is proved. □

A closed, densely defined operator A on a Hilbert space H is **normal** if $A^*A = AA^*$. If A is normal, then $D := \operatorname{dom}(A^*A) = \operatorname{dom}(AA^*)$ is a core for A and for A^*. For $u \in D$ one has

$$\begin{aligned}\|u\|^2 + \|Au\|^2 &= (u \,|\, u) + (Au \,|\, Au) = ((\mathrm{I} + A^*A)u \,|\, u)\\ &= ((\mathrm{I} + AA^*)u \,|\, u) = \|u\|^2 + \|A^*u\|^2.\end{aligned}$$

It follows that the graph norm of A and of A^* coincide on D. Hence,

$$\operatorname{dom}(A) = \operatorname{dom}(A^*).$$

It follows that $D = \operatorname{dom}(A^2) = \operatorname{dom}(A^{*2})$.

Corollary 3.3. *If A is normal then $T_A A \subseteq A T_A$.*

Proof. Let $x \in \operatorname{dom}(A)$ and $y := T_A x \in D$. Then $y + A^*Ay = x$ and hence $A^*Ay = x - y \in \operatorname{dom}(A)$. Applying A and using the normality we obtain

$$Ax = Ay + AA^*Ay = (\mathrm{I} + AA^*)Ay = (\mathrm{I} + A^*A)Ay,$$

which results in $T_A Ax = Ay = AT_A x$. □

3.2 Measurable Functional Calculus

Recall that a **measurable space** is a pair (K, Σ) where K is a set and Σ is a σ-algebra of subsets of K. A function $f : K \to \mathbb{C}$ is **measurable** if it is Σ-to-Borel measurable in the sense of measure theory. The sets

$$\begin{aligned}\mathcal{M}(K, \Sigma) &:= \{f : K \to \mathbb{C} \mid f \text{ measurable}\} \quad \text{and}\\ \mathcal{M}_{\mathrm{b}}(K, \Sigma) &:= \{f \in \mathcal{M}(K, \Sigma) \mid f \text{ bounded}\}\end{aligned}$$

will play an important role in what follows. Note that $\mathcal{M}_{\mathrm{b}}(K, \Sigma)$ is closed under **bp-convergence**, by which is meant that if a sequence $(f_n)_n$ in $\mathcal{M}_{\mathrm{b}}(K, \Sigma)$ converges **b**oundedly (i.e., with $\sup_n \|f_n\|_\infty < \infty$) and **p**ointwise to a function f, then $f \in \mathcal{M}_{\mathrm{b}}(K, \Sigma)$ as well.

If the σ-algebra Σ is understood, we simply write $\mathcal{M}(K)$ and $\mathcal{M}_{\mathrm{b}}(K)$. If K is a separable metric space, then we take per default $\Sigma = \mathrm{Bo}(K)$, the **Borel** σ-algebra on K generated by the family of open subsets (equivalently: closed subsets, open/closed balls).

A **measurable functional calculus** on a measurable space (K, Σ) is a pair (Φ, H) where H is a Hilbert space and

$$\Phi : \mathcal{M}(K, \Sigma) \to \mathcal{C}(H)$$

is a mapping with the following properties ($f,\ g \in \mathcal{M}(K, \Sigma),\ \lambda \in \mathbb{C}$):

(MFC1) $\Phi(\mathbf{1}) = \mathrm{I}$;

(MFC2) $\Phi(f) + \Phi(g) \subseteq \Phi(f+g)$ and $\lambda\Phi(f) \subseteq \Phi(\lambda f)$;

(MFC3) $\Phi(f)\Phi(g) \subseteq \Phi(fg)$ and

$$\mathrm{dom}(\Phi(f)\Phi(g)) = \mathrm{dom}(\Phi(g)) \cap \mathrm{dom}(\Phi(fg));$$

(MFC4) $\Phi(f)$ is densely defined and $\Phi(f)^* = \Phi(\overline{f})$;

(MFC5) $\Phi(f) \in \mathcal{L}(H)$ if f is bounded;

(MFC6) If $f_n \to f$ pointwise and boundedly, then $\Phi(f_n) \to \Phi(f)$ strongly.

Property (MFC6) is called the **bp-continuity** of the mapping Φ.

Remark 3.4. (MFC1)–(MFC3) simply say that a measurable functional calculus is a proto-calculus in the terminology of Section 2.4. By (MFC5), for a given function $f \in \mathcal{M}(K)$ the function $(1 + |f|^2)^{-1}$ is a Φ-regularizer for f. It is Φ-determining for f by Exercise 2.7. Hence, a measurable functional calculus also satisfies Axiom (FC4) and is therefore a (functional) calculus.

Example 3.5. Let $\Omega = (\Omega, \Sigma, \mu)$ be a semi-finite measure space and $a : \Omega \to \mathbb{C}$ an essentially measurable function. Let, as always, $K := \mathrm{essran}(a) = \sigma(M_a)$ and $\nu := a_*\mu$. In Chapter 2 we have constructed a functional calculus

$$\Phi : \mathrm{L}^0(K, \nu) \to \mathcal{C}(H), \qquad \Phi(f) := M_{f \circ a}$$

for $A := M_a$ on $H := \mathrm{L}^2(\Omega)$. If we compose this with the canonical mapping

$$\mathcal{M}(K) \to \mathrm{L}^0(K, \nu)$$

which sends each function to its equivalence class modulo ν-almost everywhere equality, we obtain a measurable functional calculus on $(K, \mathrm{Bo}(K))$. The proof is Exercise 3.2.

Let us examine more closely the consequences of the axioms defining a measurable functional calculus.

Lemma 3.6. *Let $\Phi : \mathcal{M}(K,\Sigma) \to \mathcal{C}(H)$ be a measurable functional calculus on a measurable space (K,Σ). Then the following assertions hold for each $f \in \mathcal{M}(K,\Sigma)$:*

a) *If $f \neq 0$ everywhere then $\Phi(f)$ is injective and $\Phi(f^{-1}) = \Phi(f)^{-1}$.*

b) *$\Phi(\overline{f})\Phi(f) = \Phi(|f|^2)$ and $\Phi(f)$ is normal.*

c) *$\|\Phi(f)\| \le \|f\|_\infty$ if f is bounded.*

d) *$\Phi(f)$ is self-adjoint if f is real-valued.*

Proof. a) If f is nowhere equal to zero, then f^{-1} is also measurable. So the claim follows from Theorem 2.3.
b) It is clear by (MFC3) that $\Phi(\overline{f})\Phi(f) \subseteq \Phi(|f|^2)$. Hence, by (MFC4),

$$\mathrm{I} + \Phi(f)^*\Phi(f) \subseteq \Phi(1 + |f|^2).$$

By a) the operator on the right-hand side is injective while the operator on the left-hand side is surjective (by Theorem 3.2). Hence these operators must coincide. Normality of $\Phi(f)$ follows readily.
c) For the norm inequality, let $f \in \mathcal{M}(K,\Sigma)$ with $|f| \le 1$. Then, with $g := (1 - |f|^2)^{\frac{1}{2}}$,

$$\big((\mathrm{I} - \Phi(|f|^2))x \big| x\big) = \big(\Phi(g^2)x \,\big|\, x\big) = (\Phi(g)^*\Phi(g)x \,|\, x) = \|\Phi(g)x\|^2 \ge 0$$

and hence

$$\|\Phi(f)x\|^2 = (\Phi(f)^*\Phi(f)x \,|\, x) = (\Phi(|f|^2)x \,|\, x) \le (x \,|\, x) = \|x\|^2$$

for each $x \in H$. (Cf. also Exercise 3.3.)
d) is immediate from (MFC4). □

Remark 3.7. A closer look at the proof of c) above shows that (MFC5) is actually a consequence of (MFC1)–(MFC4). The reason for including it in the definition was to ensure without further ado that (MFC6) is meaningful.

3.3 Projection-Valued Measures

If (Φ, H) is a measurable functional calculus on a measurable space (K,Σ), then the mapping

$$\mathrm{E}_\Phi : \Sigma \to \mathcal{L}(H), \qquad \mathrm{E}_\Phi(B) := \Phi(\mathbf{1}_B) \in \mathcal{L}(H) \qquad (B \in \Sigma)$$

is a **projection-valued measure**. That means, it has the following, easy-to-check properties:

1) $\mathrm{E}_\Phi(B)$ is an orthogonal projection on H for each $B \in \Sigma$.

2) $\mathrm{E}_\Phi(K) = \mathrm{I}$.
3) If $B = \bigsqcup_{n=1}^\infty B_n$ with all $B_n \in \Sigma$ then $\sum_{n=1}^\infty \mathrm{E}_\Phi(B_n) = \mathrm{E}_\Phi(B)$ in the strong operator topology.

Conversely, one can show that to each projection-valued measure E on (K, Σ) there exists a unique measurable functional calculus Φ_E such that $\mathrm{E} = \mathrm{E}_{\Phi_\mathrm{E}}$. In the literature, the notation

$$\int_K f(z)\,\mathrm{E}(\mathrm{d}z) = \Phi_\mathrm{E}(f) \tag{3.3}$$

is used frequently. The reason is that for *bounded* functions f one has

$$(\Phi(f)x \,|\, y) = \int_K f \,\mathrm{d}\mu_{x,y} \qquad (x,\ y \in H),$$

where $\mu_{x,y}$ is the complex measure on Σ defined by

$$\mu_{x,y}(B) := (\mathrm{E}(B)x \,|\, y) \qquad (B \in \Sigma).$$

For *unbounded* functions f the interpretation of (3.3) is not as straightforward any more since the description of the domain of this operator is implicit. We leave it at that since these topics are not so important to us in the following.

Null Sets

Let (Φ, H) be a fixed measurable functional calculus on (K, Σ). Then a set $B \in \Sigma$ is called a **Φ-null set** if $\Phi(\mathbf{1}_B) = 0$. The set

$$\mathcal{N}_\Phi := \{B \in \Sigma \mid \Phi(\mathbf{1}_B) = 0\}$$

of Φ-null sets is a σ-ideal of Σ (Exercise 3.4). Similarly to usual measure theory, we say that something happens **Φ-almost everywhere** if it doesn't happen at most on a Φ-null set. For instance, the assertion "$f = g$ Φ-almost everywhere" for two functions $f, g \in \mathcal{M}(K, \Sigma)$ means just that $[\, f \neq g\,] \in \mathcal{N}_\Phi$.

Finally, the **Φ-essential range** of $f \in \mathcal{M}(K, \Sigma)$ is defined by

$$\operatorname{essran}_\Phi(f) := \{\lambda \in \mathbb{C} \mid \forall\, \varepsilon > 0 : [\,|f - \lambda| \le \varepsilon\,] \notin \mathcal{N}_\Phi\}. \tag{3.4}$$

The following theorem lists the most important properties.

Theorem 3.8. *Let (Φ, H) be a measurable functional calculus on (K, Σ), let $f,\ g \in \mathcal{M}(K, \Sigma)$ and $c \ge 0$. Then the following assertions hold:*

a) $\ker(\Phi(f)) = \ker\big(\Phi(\mathbf{1}_{[\,f\neq 0\,]})\big)$.
b) $\Phi(f) = \Phi(g) \quad\Longleftrightarrow\quad f = g$ *Φ-almost everywhere.*
c) $\Phi(f) = 0 \quad\Longleftrightarrow\quad f = 0$ *Φ-almost everywhere.*

d) $\Phi(f)$ *is injective* $\iff$ $f \neq 0$ Φ*-almost everywhere.*

e) $\boldsymbol{\sigma}(\Phi(f)) = \boldsymbol{\sigma}_{\mathrm{a}}(\Phi(f)) = \mathrm{essran}_\Phi(f)$.

f) $f \in \mathrm{essran}_\Phi(f)$ Φ*-almost everywhere.*

g) $\Phi(f) \in \mathcal{L}(H)$, $\|\Phi(f)\| \le c$ $\iff$ $|f| \le c$ Φ*-almost everywhere.*

Proof. a) Since $f = f\mathbf{1}_{[\,f\neq 0\,]}$ one has $\Phi(f) = \Phi(f)\Phi(\mathbf{1}_{[\,f\neq 0\,]})$. This yields the inclusion "$\supseteq$". Next, define $g := \frac{1}{f}\mathbf{1}_{[\,f\neq 0\,]}$. Then $gf = \mathbf{1}_{[\,f\neq 0\,]}$ and hence $\Phi(g)\Phi(f) \subseteq \Phi(\mathbf{1}_{[\,f\neq 0\,]})$. This yields the inclusion "$\subseteq$".

c) By a), $\Phi(f) = 0$ if and only if $\Phi(\mathbf{1}_{[\,f\neq 0\,]}) = 0$, if and only if $f = 0$ Φ-almost everywhere.

b) If $f = g$ Φ-almost everywhere, then $f - g = 0$ Φ-almost everywhere and hence, by c), $\Phi(f - g) = 0$. Since $f = g + (f - g)$, it follows by general functional calculus rules that $\Phi(f) = \Phi(g) + \Phi(f - g) = \Phi(g)$. The converse implication is left as Exercise 3.5.

d) By a), $\Phi(f)$ is injective if and only if $\Phi(\mathbf{1}_{[\,f\neq 0\,]})$ is injective, if and only if $\Phi(\mathbf{1}_{[\,f\neq 0\,]}) = \mathrm{I}$ (since it is an orthogonal projection), if and only if $\Phi(\mathbf{1}_{[\,f=0\,]}) = \mathrm{I} - \Phi(\mathbf{1}_{[\,f\neq 0\,]}) = 0$.

e) This is Exercise 3.6.

f) Abbreviate $M := \mathrm{essran}_\Phi(f)$. For each $\lambda \in \mathbb{C} \setminus M$ there is $\varepsilon_\lambda > 0$ such that $[\,f \in \mathrm{B}(\lambda, \varepsilon_\lambda)\,]$ is a Φ-null set. Since $\mathbb{C} \setminus M$ is open, it is σ-compact (why precisely?) and hence countably many $\mathrm{B}(\lambda, \varepsilon_\lambda)$ suffice to cover $C \setminus M$. Since $\mathcal{N}_\Phi$ is a σ-ideal, it follows that $[\,f \notin M\,]$ is a Φ-null set, and hence that $f \in M$ Φ-almost everywhere. (This is similar to Exercise 2.6.)

g) If $|f| \le c$ Φ-almost everywhere then $f = f\mathbf{1}_{[\,|f|\le c\,]}$ Φ-almost everywhere. Hence, by b),

$$\|\Phi(f)\| = \|\Phi(f\mathbf{1}_{[\,|f|\le c\,]})\| \le \|f\mathbf{1}_{[\,|f|\le c\,]}\|_\infty \le c.$$

Conversely, suppose that $\|\Phi(f)\| \le c$. Then $\mathrm{essran}_\Phi(f) = \boldsymbol{\sigma}(\Phi(f)) \subseteq \mathrm{B}[0, c]$, and hence $|f| \le c$ by f). □

We say that Φ is **concentrated** on a set $B \in \Sigma$ if B^c is a Φ-null set. For $L \subseteq K$ denote by Σ_L the **trace σ-algebra**

$$\Sigma_L := \{L \cap B \mid B \in \Sigma\}.$$

If Φ is concentrated on $L \in \Sigma$ then one can induce a functional calculus on (L, Σ_L) by defining

$$\Phi_L(f) := \Phi(f^L) \qquad (f \in \mathcal{M}(L, \Sigma_L)),$$

where $f^L = f$ on L and $f^L = 0$ on L^c. (Axioms (MFC2)–(MFC6) are immediate, and Axiom (MFC1) holds since Φ is concentrated on L.)

Conversely, if (Φ, H) is a measurable functional calculus on (L, Σ_L) then by

$$\Phi_K(f) := \Phi(f|_L) \qquad (f \in \mathcal{M}(K, \Sigma))$$

one obtains a measurable functional calculus (Φ_K, H) on (K, Σ) concentrated on L. For a measurable set $L \subseteq K$ this establishes a one-to-one correspondence between measurable functional calculi on (L, Σ_L) and measurable functional calculi on (K, Σ) concentrated on L.

3.4 Borel Functional Calculus

A measurable functional calculus on (K, Σ) is called a **Borel functional calculus** if K is a metric space and $\Sigma = \mathrm{Bo}(K)$ is the Borel algebra. If K is actually a subset of $\mathbb{C}$, then the function $\mathbf{z} = (z \mapsto z)$ is measurable. Hence, the terminology of Section 2.5 applies, and Φ is called a (Borel) functional calculus *for* the operator $\Phi(\mathbf{z})$.

Lemma 3.9. *Let (Φ, H) be a Borel functional calculus on a set $K \subseteq \mathbb{C}$ for an operator $A \in \mathcal{C}(H)$. Then the following assertions hold:*

a) $\sigma_{\mathrm{p}}(A) \subseteq K$ *and* $\sigma(A) \subseteq \overline{K}$ *with*

$$R(\lambda, A) = \Phi\Big(\frac{1}{\lambda - \mathbf{z}}\Big) \qquad (\lambda \notin \overline{K}).$$

b) $\Phi(\mathbf{1}_{K \setminus \sigma(A)}) = 0$.

Proof. a) Let $\lambda \in \sigma_{\mathrm{p}}(A)$. Then $\lambda - A = \Phi(\lambda - \mathbf{z})$ is not injective. By Lemma 3.6, the function $\lambda - \mathbf{z}$ must have a zero in K, hence $\lambda \in K$. If $\lambda \notin \overline{K}$ then $(\lambda - \mathbf{z})^{-1}$ is a bounded measurable function on K, so

$$(\lambda - A)^{-1} = \Phi((\lambda - \mathbf{z})^{-1})$$

is a bounded operator. It follows that $\lambda \in \rho(A)$ and $R(\lambda, A) = \Phi((\lambda - \mathbf{z})^{-1})$.
b) Theorem 3.8.e) applied to $f = \mathbf{z}$ yields $\sigma(A) = \mathrm{essran}_\Phi(\mathbf{z})$. Hence, by f) of the very same theorem, $\mathbf{z} \in \sigma(A)$ Φ-almost everywhere. This means that $K \setminus \sigma(A) = [\, \mathbf{z} \notin \sigma(A) \,] \in \mathcal{N}_\Phi$ as claimed. □

Remark 3.10. By the previous result, a Borel functional calculus on a set $K \subseteq \mathbb{C}$ for an operator A on H is concentrated on $K \cap \sigma(A)$. It follows that a Borel calculus for an operator A on $\mathbb{C}$ is essentially the same as a Borel calculus for A on $\sigma(A)$.

Uniqueness

Our next issue is uniqueness of a Borel functional calculus. The following lemma, which is a functional analytic analogue of Dynkin's lemma from measure theory, will be of great help.

Lemma 3.11. *Let $E \subseteq \mathcal{M}_{\mathrm{b}}(\mathbb{C})$ be a set that contains $\mathrm{C}_0(\mathbb{C})$ and is closed under bp-convergence of sequences. Then $E = \mathcal{M}_{\mathrm{b}}(\mathbb{C})$.*

Proof. Let us call a subset $F \subseteq \mathcal{M}_{\mathrm{b}}(\mathbb{C})$ "good" if it contains $\mathrm{C}_0(\mathbb{C})$ and is closed under bp-convergence. Without loss of generality we may suppose that E is the smallest, i.e., the intersection of all good subsets.

For $f \in \mathcal{M}_{\mathrm{b}}(\mathbb{C})$ define

$$E_f := \{ g \in \mathcal{M}_{\mathrm{b}}(\mathbb{C}) \mid f + g \in E \}.$$

If $f \in \mathrm{C}_0(\mathbb{C})$, then E_f is good, hence $E \subseteq E_f$. This means that $\mathrm{C}_0(\mathbb{C}) + E \subseteq E$. In particular, it follows that E_f is good even if $f \in E$. Hence, $E \subseteq E_f$ for each $f \in E$, and therefore $E + E \subseteq E$. For $\lambda \in \mathbb{C}$ the set

$$\{ g \in \mathcal{M}_{\mathrm{b}}(\mathbb{C}) \mid \lambda g \in E \}$$

is good, and hence $\lambda E \subseteq E$. This establishes that E is a linear subspace of $\mathcal{M}_{\mathrm{b}}(\mathbb{C})$. In the same way one can show that if $f,\ g \in E$, then $fg \in E$, so E is an algebra.

Certainly (how?) one can find a sequence of continuous functions φ_n of compact support such that $\varphi_n \to \mathbf{1}$ boundedly and pointwise. Hence $\mathbf{1} \in E$, so E is a unital algebra. It now is an easy exercise to show that the set

$$\Sigma := \{ A \in \mathrm{Bo}(\mathbb{C}) \mid \mathbf{1}_A \in E \}$$

is a σ-algebra (Exercise 3.8). Since bounded measurable functions are uniform limits of simple functions, it follows that $\mathcal{M}_{\mathrm{b}}(\mathbb{C}, \Sigma) \subseteq E$.

Finally, since for each closed ball $B \subseteq \mathbb{C}$ the characteristic function $\mathbf{1}_B$ is a bp-limit of continuous functions with compact support (why?), $\Sigma = \mathrm{Bo}(\mathbb{C})$, and we are done. □

Remark 3.12. We have formulated and proved the lemma just for $\mathcal{M}_{\mathrm{b}}(\mathbb{C})$ only for the sake of convenience. It actually remains true if one replaces $\mathbb{C}$ by any locally compact and separable metric space (and even in more general situations), in particular for $\mathbb{R}^d$. See Exercise 3.9.

Theorem 3.13. *A Borel functional calculus (Φ, H) on $\mathbb{C}$ is uniquely determined by*

a) *its values on all functions $f \in \mathcal{M}_{\mathrm{b}}(\mathbb{C})$;*

b) *its value on the function $\mathbf{z}$;*

c) *its values on the functions $\dfrac{1}{1 + |\mathbf{z}|^2}$ and $\dfrac{\mathbf{z}}{1 + |\mathbf{z}|^2}$;*

d) *its value on the function $\dfrac{\mathbf{z}}{(1 + |\mathbf{z}|^2)^{\frac{1}{2}}}$.*

Proof. a) Let f be any measurable function on $\mathbb{C}$. Then $g := \frac{1}{1+|f|^2}$ and $h := \frac{f}{1+|f|^2}$ are bounded and measurable. Moreover, g is nowhere zero and hence, by Lemma 3.6.a, $\Phi(g)$ is injective and $\Phi(g^{-1}) = \Phi(g)^{-1}$. From $f = g^{-1}h$ it then follows that

$$\Phi(f) = \Phi(g^{-1}h) = \Phi(g^{-1})\Phi(h) = \Phi(g)^{-1}\Phi(h).$$

So a) is proved.
b) Let $t := \frac{1}{1+|\mathbf{z}|^2}$ and $s := \frac{\mathbf{z}}{1+|\mathbf{z}|^2}$. By Lemma 3.6 we have

$$\Phi(t)^{-1} = \Phi(t^{-1}) = \Phi(1+|\mathbf{z}|^2) = \mathrm{I} + \Phi(|\mathbf{z}|^2) = \mathrm{I} + \Phi(\mathbf{z})^*\Phi(\mathbf{z}).$$

So $\Phi(t)$ is determined by $\Phi(\mathbf{z})$. Also, $\mathbf{z}\,t = s$ and $\Phi(t)$ is bounded (due to (MFC5)), hence $\Phi(s) = \Phi(\mathbf{z})\Phi(t)$. Therefore also $\Phi(s)$ is determined by $\Phi(\mathbf{z})$, and we have reduced b) to c).
c) and d) Suppose that Φ and Ψ are Borel calculi on $\mathbb{C}$ and let

$$E := \{f \in \mathcal{M}_{\mathrm{b}}(\mathbb{C}) \mid \Phi(f) = \Psi(f)\}.$$

For the equality $\Phi = \Psi$ it suffices by a) to show that $E = \mathcal{M}_{\mathrm{b}}(\mathbb{C})$. The bp-continuity of a Borel functional calculus implies that E is closed under bp-convergence. So, by Lemma 3.11 it suffices to show that $\mathrm{C}_0(\mathbb{C}) \subseteq E$.

In order to establish this, let $A := E \cap \mathrm{C}_0(\mathbb{C})$ and note that A is a norm-closed and conjugation-invariant subalgebra of $\mathrm{C}_0(\mathbb{C})$. In case c), A contains the functions

$$t := \frac{1}{1+|\mathbf{z}|^2} \quad \text{and} \quad s := \frac{\mathbf{z}}{1+|\mathbf{z}|^2}$$

and hence, by the Stone–Weierstrass theorem, $A = \mathrm{C}_0(\mathbb{C})$. In case d), E contains the function

$$g := \frac{\mathbf{z}}{(1+|\mathbf{z}|^2)^{\frac{1}{2}}},$$

hence also the function

$$h := \mathbf{1} - g\overline{g} = \mathbf{1} - |g|^2 = \frac{1}{1+|\mathbf{z}|^2}.$$

So, in this case, $h,\ gh \in A$ and, again by the Stone–Weierstrass theorem, $A = \mathrm{C}_0(\mathbb{C})$. □

Corollary 3.14. *Let K, L be Borel subsets of $\mathbb{C}$ and let (Φ, H) and (Ψ, H) be Borel functional calculi on K and L, respectively, for the same operator A on H. Then Φ and Ψ are both concentrated on $K \cap L$ and*

$$\Phi_{K\cap L} = \Psi_{K\cap L}.$$

Proof. By Theorem 3.13 one has $\Phi_{\mathbb{C}} = \Psi_{\mathbb{C}}$. Hence

$$\Phi(\mathbf{1}_{K\setminus L}) = \Phi_{\mathbb{C}}(\mathbf{1}_{K\setminus L}) = \Psi_{\mathbb{C}}(\mathbf{1}_{K\setminus L}) = \Psi(0) = 0.$$

The rest is simple. □

3.5 Supplement: Commutation Results

We begin with an interesting application of the holomorphic functional calculus for bounded operators from Chapter 1.

Theorem 3.15 (Fuglede-Putnam-Rosenblum). *Let A, B be bounded normal operators on the Hilbert spaces K, H, respectively, and let $T \in \mathcal{L}(H;K)$ such that*

$$AT = TB.$$

Then $A^*T = TB^*$.

Proof. The proof makes use of the operator-valued exponential function

$$\mathrm{e}^S = \sum_{n=0}^{\infty} \frac{1}{n!} S^n \qquad (S \in \mathcal{L}(H)),$$

in particular of the following properties:

1) if $SQ = QS$ then $\mathrm{e}^{S+Q} = \mathrm{e}^S \mathrm{e}^Q$;
2) $(\mathrm{e}^S)^* = \mathrm{e}^{S^*}$;
3) the function $\mathbb{C} \to \mathbb{C}$, $z \mapsto \left(\mathrm{e}^{zS}x \,\middle|\, y\right)$ is holomorphic for all $x, y \in H$.

Note that from 1) it follows that e^S is invertible and $(\mathrm{e}^S)^{-1} = \mathrm{e}^{-S}$. Hence if Q is a skew-symmetric operator, i.e., if $Q^* = -Q$, then

$$\left(\mathrm{e}^Q\right)^* = \mathrm{e}^{Q^*} = \mathrm{e}^{-Q} = \left(\mathrm{e}^Q\right)^{-1},$$

which implies that e^Q is a unitary operator. Note that if S is any bounded operator then $S - S^*$ is skew-symmetric.

After these preliminaries we turn to the proof of the claim. From the assumption $AT = TB$ we conclude $\mathrm{e}^A T = T\mathrm{e}^B$ and hence, since A and B are both normal operators,

$$\mathrm{e}^{A^*}T\mathrm{e}^{-B^*} = \mathrm{e}^{A^*-A}\mathrm{e}^A T\mathrm{e}^{-B^*} = \mathrm{e}^{A^*-A}T\mathrm{e}^B\mathrm{e}^{-B^*} = \mathrm{e}^{A^*-A}T\mathrm{e}^{B-B^*}.$$

By the remarks from above, the operators e^{A^*-A} and e^{B^*-B} are both unitary, in particular they are contractions. We therefore obtain

$$\|\mathrm{e}^{A^*}T\mathrm{e}^{-B^*}\| \le \|T\|.$$

Now let $z \in \mathbb{C}$ and replace A by $\overline{z}A$ and B by $\overline{z}B$. Then, for fixed $x, y \in H$

$$\left|(\mathrm{e}^{zA^*}T\mathrm{e}^{-zB^*}x\,|\,y)\right| \le \|T\|\|x\|\|y\|.$$

Hence, the entire function $z \mapsto \left(\mathrm{e}^{zA^*}T\mathrm{e}^{-zB^*}x \,\middle|\, y\right)$ is bounded, so by Liouville's theorem it is constant. Since at $z = 0$ it has the value $(Tx\,|\,y)$, we conclude that

$$\mathrm{e}^{zA^*}T = T\mathrm{e}^{zB^*}$$

for all $z \in \mathbb{C}$. Looking at the power series representation we arrive at $A^*T = TB^*$ as desired. □

Corollary 3.16. *Let (Φ, H_1) and (Ψ, H_2) be measurable functional calculi on a measurable space (K, Σ). Let $f \in \mathcal{M}(K, \Sigma)$ and $T \in \mathcal{L}(H_1; H_2)$ such that $T\Phi(f) \subseteq \Psi(f)T$. Then*

$$T\Phi(\overline{f}) \subseteq \Psi(\overline{f})T.$$

Proof. Let $C \in \Sigma$ be such that $f\mathbf{1}_C$ is bounded, let $P := \Phi(\mathbf{1}_C)$ and $Q := \Psi(\mathbf{1}_C)$. Furthermore, let $T_C := QTP \in \mathcal{L}(H_1; H_2)$ and define $A := \Phi(f)$ and $B := \Psi(f)$. The operators $A_C := \Phi(f\mathbf{1}_C) = AP$ and $B_C := \Psi(f\mathbf{1}_C) = BQ$ are bounded normal operators. Moreover,

$$\begin{aligned} T_C A_C &= QTPAP \subseteq QTAP^2 = QTAP \subseteq QBTP \\ &\subseteq BQTP = BQQTP = B_C T_C. \end{aligned}$$

Hence, Theorem 3.15 applies and yields

$$T_C A_C^* = B_C^* T_C.$$

But

$$A_C^* = \Phi(f\mathbf{1}_C)^* = \Phi(\overline{f}\mathbf{1}_C) = \Phi(\overline{f})\Phi(\mathbf{1}_C) \supseteq \Phi(\mathbf{1}_C)\Phi(\overline{f}) = PA^*$$

and, similarly, $B_C^* = B^*Q$. Therefore

$$T_C A^* = T_C P A^* \subseteq T_C A_C^* = B_C^* T_C = B^* Q T_C = B^* T_C.$$

Now let, for $n \in \mathbb{N}$, $C_n := [\,|f| \le n\,]$ and $P_n := \Phi(\mathbf{1}_{C_n})$, $Q_n := Q_{C_n}$, and $T_n := T_{C_n}$. Then, by the bp-continuity of the functional calculi, $T_n \to T$ strongly. Since $T_n A^* \subseteq B^* T_n$ and B^* is closed, it follows that

$$TA^* \subseteq B^*T$$

as claimed. □

With the help of Corollary 3.16 we now find the following commutation result for Borel calculi.

Theorem 3.17 (Intertwining/Commutation). *Let (Φ, H_1) and (Ψ, H_2) be two Borel functional calculi on $\mathbb{C}$ and $T : H_1 \to H_2$ a bounded linear operator. Then the following assertions are equivalent:*

(i) $T\Phi(f) \subseteq \Psi(f)T$ *for all* $f \in \mathcal{M}(\mathbb{C})$;

(ii) $T\Phi(f) = \Psi(f)T$ *for all* $f \in \mathcal{M}_\mathrm{b}(\mathbb{C})$;

(iii) $T\Phi(g) = \Psi(g)T \quad$ *for* $\quad g = \dfrac{\mathbf{z}}{(1+|\mathbf{z}|^2)^{\frac{1}{2}}}$;

(iv) $T\Phi(\mathbf{z}) \subseteq \Psi(\mathbf{z})T$.

Proof. It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv).
(ii) $\Rightarrow$ (i) is left as Exercise 3.10.
(iii) $\Rightarrow$ (ii): Define

$$E := \{f \in \mathcal{M}_\mathrm{b}(\mathbb{C}) \mid T\Phi(f) = \Psi(f)T\}.$$

Then E is a bp-closed, unital subalgebra of $\mathcal{M}_\mathrm{b}(\mathbb{C})$. It is conjugation-invariant by Corollary 3.16. As in the proof of Theorem 3.13 it is now shown that $g \in E$ implies that $E = \mathcal{M}_\mathrm{b}(\mathbb{C})$.
(iv) $\Rightarrow$ (ii): Let E be defined as before, and define $A := \Phi(\mathbf{z})$ and $B := \Psi(\mathbf{z})$. Then (iv) just tells that $TA \subseteq BT$. By Corollary 3.16, $TA^* \subseteq B^*T$ as well. Since $A^* = \Phi(\overline{\mathbf{z}})$ and $B^* = \Psi(\overline{\mathbf{z}})$, one has

$$T\Phi(1+|z|^2) = T(\mathrm{I} + A^*A) \subseteq (\mathrm{I} + B^*B)T = \Psi(1+|\mathbf{z}|^2)T.$$

This implies that $(1+|\mathbf{z}|^2)^{-1} \in E$. Likewise, one has $\mathbf{z}(1+|\mathbf{z}|^2)^{-1} \in E$. Then, as in the proof of Theorem 3.13 it follows that $E = \mathcal{M}_\mathrm{b}(\mathbb{C})$. $\square$

Exercises

3.1. Prove Lemma A.21 from Appendix A.5.

3.2. Let Ω be a semi-finite measure space, $a : \Omega \to \mathbb{C}$ measurable and $K :=$ essran(a). Let $H := \mathrm{L}^2(\Omega)$. Show that the map

$$\Phi : \mathcal{M}(K) \to \mathcal{C}(H), \qquad \Phi(f) := M_{f\circ a}$$

is a measurable functional calculus.

3.3. Let $\mathcal{A}$ be unital $*$-subalgebra of bounded functions on a set K and let

$$\Phi : \mathcal{A} \to \mathcal{L}(H)$$

be a $*$-representation of $\mathcal{A}$ on H. Suppose that for each $0 \le h \in \mathcal{A}$ one has $\sqrt{h} \in \mathcal{A}$ as well. Show that Φ is contractive, i.e., it satisfies

$$\|\Phi(f)\| \le \|f\|_\infty \qquad (f \in \mathcal{A}).$$

3.4. Let $\Phi : \mathcal{M}(K,\Sigma) \to \mathcal{C}(H)$ be a measurable functional calculus. Show that

$$\mathcal{N}_\Phi := \{B \in \Sigma \mid \Phi(\mathbf{1}_B) = 0\}$$

is a **σ-ideal**, i.e., it is closed under countable unions and under taking measurable subsets.

3.5. Let $\Phi : \mathcal{M}(K,\Sigma) \to \mathcal{C}(H)$ be a measurable functional calculus and $f,\ g \in \mathcal{M}(K,\Sigma)$. Complete the proof of Theorem 3.8.b) and show that

$$\Phi(f) = \Phi(g) \quad \Rightarrow \quad f = g \quad \Phi\text{-almost everywhere.}$$

[Hint: Prove this first for bounded functions $f,\ g$.]

3.6. Let $\Phi : \mathcal{M}(K,\Sigma) \to \mathcal{C}(H)$ be a measurable functional calculus. Show, for $f \in \mathcal{M}(K,\Sigma)$, that

$$\boldsymbol{\sigma}(\Phi(f)) = \boldsymbol{\sigma}_{\mathrm{a}}(\Phi(f)) = \mathrm{essran}_\Phi(f).$$

(See (3.4) for the definition of $\mathrm{essran}_\Phi(f)$.)

3.7. Let $\Phi : \mathcal{M}(K,\Sigma) \to \mathcal{C}(H)$ be a measurable functional calculus and $f \in \mathcal{M}(K,\Sigma)$. Show that by

$$\Psi : \mathcal{M}(\mathbb{C}) \to \mathcal{C}(H), \qquad \Psi(g) := \Phi(g \circ f)$$

a Borel calculus is defined for $A = \Phi(f)$.

3.8. Let (K,Σ) be a measurable space and let $E \subseteq \mathcal{M}_{\mathrm{b}}(K,\Sigma)$ be a unital subalgebra, closed with respect to bp-convergence. Show that

$$\mathcal{E} := \{A \subseteq K \mid \mathbf{1}_A \in E\}$$

is a sub-σ-algebra of Σ.

3.9. Let (K,d) be a locally compact and separable metric space. Show that there is a sequence of continuous functions ψ_n with compact support with $0 \le \psi_n \nearrow \mathbf{1}$ pointwise. Conclude that Lemma 3.11 holds *mutatis mutandis* for subsets E of $\mathcal{M}_{\mathrm{b}}(K)$. [Hint: Show first that there are sequences of compact subsets $(K_n)_n$ and open subsets $(U_n)_n$ such that $K_n \subseteq U_n \subseteq K_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup_n K_n = K$.]

3.10. Let $(\Phi; H_1)$ and (Ψ, H_1) be Borel functional calculi on $\mathbb{C}$ and let $T : H_1 \to H_2$ be a bounded operator. Show that the following assertions are equivalent:

(i) T commutes with each $\Phi(f)$, $f \in \mathcal{M}_{\mathrm{b}}(\mathbb{C})$;

(ii) $T\Phi(f) \subseteq \Phi(f)T$ for all Borel measurable functions on $\mathbb{C}$.

Chapter 4
The Spectral Theorem

So far, we have only exploited properties of Borel functional calculi but not seen so many examples of them. With the so-called "spectral theorem" for normal operators on Hilbert spaces, this will change drastically.

4.1 Three Versions of the Spectral Theorem

The spectral theorem actually comes in different versions, three of which shall be presented here. Probably the most striking (and most powerful) is the following.

Theorem 4.1 (Spectral Theorem, Multiplicator Version). *Let A be a normal operator on a Hilbert space H. Then A is unitarily equivalent to a multiplication operator on some* L^2*-space over a semi-finite measure space.*

To wit: To a normal operator A on a Hilbert space H there exists a semi-finite measure space Ω, a measurable function a on Ω and a unitary operator $U : H \to \mathrm{L}^2(\Omega)$ such that

$$A = U^{-1}\, M_a\, U.$$

We shall call this a **multiplication operator representation** of the normal operator A. So Theorem 4.1 can be rephrased as: *Each normal operator on a Hilbert space has a multiplication operator representation.*

In this form, the spectral theorem can be seen as a far-reaching generalization of a well-known theorem about unitary diagonalization of normal matrices or the well-known spectral theorem for compact normal operators [4, Thm. 13.11]. Indeed, the unitary equivalence to a multiplication operator is a kind of continuous diagonalization.

Actually, one can say much more: The measure space can be chosen to be a Radon measure on a locally compact space and the multiplier function is continuous. And even more, one can simultaneously "diagonalize" any family of pairwise commuting (one would have to define what that is) normal operators on a Hilbert space. However, we shall not treat these generalizations here.

From the multiplicator version it is only a small step to the version which is of most interest to us in this course.

Theorem 4.2 (Spectral Theorem, Functional Calculus Version). *Let A be a normal operator on a Hilbert space H. Then A has a unique Borel functional calculus on $\sigma(A)$.*

Proof. Fix a unitary equivalence, say U, to a multiplication operator M_a on an L^2-space. Now carry over the functional calculus for M_a to one for A by

$$\Phi(f) := U^{-1} M_{f \circ a} U.$$

This yields the existence part of Theorem 4.2, the uniqueness has been shown in Theorem 3.13. □

The main advantage of this second formulation of the spectral theorem is that the functional calculus for a normal operator is unique, whereas the multiplication representation is not.

Finally, here is the third version of the spectral theorem.

Theorem 4.3 (Spectral Theorem, Spectral Measure Version). *Let A be a normal operator on a Hilbert space H. Then there is a unique projection-valued measure* E *defined on the Borel subsets of* $\mathbb{C}$ *such that*

$$A = \int_{\mathbb{C}} z \, \mathrm{E}(\mathrm{d}z).$$

This theorem follows from Theorem 4.2 and Theorem 3.13 by the one-to-one correspondence of measurable functional calculi and projection-valued measures that has been mentioned in Chapter 3. We shall not use this version of the spectral theorem.

Working with the Spectral Theorem

We shall prove the spectral theorem below. Before, let us assume its validity and work with it.

From now on, if A is a normal operator on a Hilbert space H then we denote by Φ_A its (unique) Borel functional calculus on $\sigma(A)$. Note that we may as well consider Φ_A a Borel calculus on $\mathbb{C}$ (which is automatically concentrated on $\sigma(A)$, cf. Remark 3.10). For a Borel measurable function f on $\sigma(A)$ we also write

$$f(A) := \Phi_A(f)$$

frequently. The Φ_A-null sets are just called **A-null sets** and abbreviated by

$$\mathcal{N}_A := \mathcal{N}_{\Phi_A} = \{B \in \mathrm{Bo}(\mathbb{C}) \mid \mathbf{1}_B(A) = 0\}.$$

Note that the presence of A-null sets accounts for the fact that the Borel functional calculus for A may be concentrated on proper subsets of $\boldsymbol{\sigma}(A)$.

Example 4.4. It is easy to see that $\lambda \in \mathbb{C}$ is an eigenvalue of the normal operator A if and only if $\{\lambda\}$ is not a Φ_A-null set. In this case, $\mathbf{1}_{\{\lambda\}}(A)$ is the orthogonal projection onto the corresponding eigenspace (Exercise 4.1).

It follows that if A has no eigenvalues then its functional calculus is concentrated on each set $\boldsymbol{\sigma}(A) \setminus \{\lambda\}$ for each $\lambda \in \boldsymbol{\sigma}(A)$.

One can see this phenomenon most clearly in a multiplication operator representation of A: In the situation of Example 3.5, where $H = \mathrm{L}^2(\Omega)$, $A = M_a$ and $K = \mathrm{essran}(a) = \boldsymbol{\sigma}(A)$, the original functional calculus is

$$\mathrm{L}^0(K, \nu) \to \mathcal{C}(H)$$

but the Borel calculus is just the composition

$$\Phi_A : \mathcal{M}(K) \to \mathrm{L}^0(K, \nu) \to \mathcal{C}(H).$$

It is easy to see that the ν-null sets are precisely the Φ_A-null sets, i.e.,

$$\nu(B) = 0 \quad \iff \quad \mathbf{1}_B(A) = 0$$

for each Borel set $B \subseteq \mathbb{C}$. Hence, $\mathrm{L}^0(K, \nu) = \mathcal{M}(\mathbb{C})/\sim_{\mathcal{N}_A}$ (where $\sim_{\mathcal{N}_A}$ denotes the equivalence relation "equality Φ_A-almost everywhere"). Observe that this factor algebra does *not* depend on the multiplication operator representation.

For a normal operator A and a Borel measurable function f on $\mathbb{C}$ we let

$$\mathrm{essran}_A(f) := \{\lambda \in \mathbb{C} \mid \forall\, \varepsilon > 0 : [\,|f - \lambda| < \varepsilon\,] \notin \mathcal{N}_A\}$$

be the **A-essential range** of f, cf. Exercise 3.6.

Theorem 4.5 (Composition Rule and Spectral Mapping Theorem). *Let A be a normal operator on a Hilbert space, let $f \in \mathcal{M}(\mathbb{C})$ and $B := f(A)$. Then B is normal and for each $g \in \mathcal{M}(\mathbb{C})$ one has*

$$g(B) = g(f(A)) = (g \circ f)(A).$$

Moreover,

$$\boldsymbol{\sigma}(f(A)) = \mathrm{essran}_A(f) \subseteq \overline{f(\boldsymbol{\sigma}(A))}. \tag{4.1}$$

If f is continuous, one has even $\boldsymbol{\sigma}(f(A)) = \overline{f(\boldsymbol{\sigma}(A))}$.

Proof. The definition $\Phi(g) := \Phi_A(g \circ f) = (g \circ f)(A)$ for $g \in \mathcal{M}(\mathbb{C})$ yields a Borel functional calculus on $\mathbb{C}$ for the operator B (Exercise 3.7). By uniqueness, $\Phi(g) = \Phi_B(g) = g(f(A))$ for all $g \in \mathcal{M}(\mathbb{C})$.

The identity $\sigma(f(A)) = \mathrm{essran}_A(f)$ was the subject of Exercise 3.6. If $\lambda \notin \overline{f(\sigma(A))}$ then λ has a positive distance $\varepsilon > 0$ to $f(\sigma(A))$. Hence, $\sigma(A) \subseteq [\,|f - \lambda| \ge \varepsilon\,]$ and so $[\,|f - \lambda| < \varepsilon\,]$ is an A-null set. Therefore, $\lambda \notin \mathrm{essran}_A(f)$, which concludes the proof of (4.1).

Finally, suppose that f is continuous, $\lambda \in \mathbb{C}$ and $f(\lambda) \notin \mathrm{essran}_A(f)$. Then there is $\varepsilon > 0$ such that $[\,|f - f(\lambda)| < \varepsilon\,] \in \mathcal{N}_A$. Since f is continuous, there is $\delta > 0$ such that

$$[\,|\mathbf{z} - \lambda| < \delta\,] = \mathrm{B}(\lambda, \delta) \subseteq [\,|f - f(\lambda)| < \varepsilon\,].$$

Since $\mathcal{N}_A$ is closed under taking measurable subsets, $[\,|\mathbf{z} - \lambda| < \delta\,] \in \mathcal{N}_A$. This means that $\lambda \notin \mathrm{essran}_A(\mathbf{z}) = \sigma(A)$, by what we have shown above. □

We note that for the last part of Theorem 4.5 it suffices that the function f is continuous on $\sigma(A)$. Indeed, in this case one can find a continuous function $\tilde{f}$ on $\mathbb{C}$ that coincides with f on $\sigma(A)$ and hence satisfies $\tilde{f}(A) = f(A)$.

Note further that, in the case that $\sigma(A)$ is compact (e.g., if A is bounded) and f is continuous, Theorem 4.5 yields the **spectral mapping identity**

$$\sigma(f(A)) = f(\sigma(A)).$$

See also Exercise 4.2.

Proof of the Spectral Theorem

In the remainder of this chapter we shall present a proof of the spectral theorem. This will happen in four steps. After the first three, the spectral theorem for self-adjoint operators will be established and this is sufficient in many cases. Normal but not self-adjoint operators are treated in an optional supplement section.

Some of the arguments can be greatly simplified when one is willing to use Gelfand theory, in particular the commutative Gelfand–Naimark theorem. This approach can be found, e.g., in our ealier book [3, App.D], see also [2, Chap. 18]. However, for this course we have decided to avoid that theory.

4.2 Proof: Bounded Self-Adjoint Operators

In the first step of the proof we establish a **continuous functional calculus** for a bounded self-adjoint operator. By this we mean a calculus involving continuous (instead of measurable) functions. We closely follow [7, Section 5.1].

Let $A \in \mathcal{L}(H)$ be a bounded, self-adjoint operator on H and let $a, b \in \mathbb{R}$ such that $\sigma(A) \subseteq [a, b]$. Fix $p \in \mathbb{C}[z]$, denote by p^* the polynomial $p^*(z) := \overline{p(\overline{z})}$, and let $q := pp^*$. By the spectral inclusion theorem for polynomials[1],

$$\sigma(q(A)) \subseteq q(\sigma(A)).$$

Now observe that $p(A)^* p(A) = p^*(A) p(A) = q(A)$. Hence, $q(A)$ is self-adjoint and therefore its norm equals its spectral radius. Since $q = |p|^2$ on $\mathbb{R}$,

$$\begin{aligned}\|p(A)\|^2 = \|p(A)^* p(A)\| = \|q(A)\| = r(q(A)) &= \sup\{|\lambda| \mid \lambda \in \sigma(q(A))\} \\ \leq \sup\{|q(\mu)| \mid \mu \in \sigma(A)\} &\leq \|q\|_{\infty,\sigma(A)} \leq \|p\|^2_{\infty,[a,b]}.\end{aligned}$$

It follows that the polynomial functional calculus for A is contractive for the supremum-norm on $[a, b]$. By the Weierstrass approximation theorem, the polynomials are dense in $\mathrm{C}[a, b]$. A standard result from elementary functional analysis now yields a bounded (in fact: contractive) linear map

$$\Phi : \mathrm{C}[a, b] \to \mathcal{L}(H)$$

such that $\Phi(p) = p(A)$ for $p \in \mathbb{C}[z]$. It is easily seen that Φ is a unital $*$-homomorphism. We have established the following result.

Theorem 4.6. *Let A be a bounded self-adjoint operator on a Hilbert space H and $[a, b]$ a real interval containing $\sigma(A)$. Then there is a unique unital $*$-homomorphism*

$$\Phi : \mathrm{C}[a, b] \to \mathcal{L}(H)$$

such that $\Phi(\mathbf{z}) = A$.

Note that each operator $\Phi(f)$ can be approximated in operator norm by operators of the form $p(A)$, where p is a polynomial. This implies the following corollary.

Corollary 4.7. *In the situation of Theorem 4.6, let B be a closed operator on H such that $AB \subseteq BA$. Then $\Phi(f)B \subseteq B\Phi(f)$ for each $f \in \mathrm{C}[a, b]$.*

4.3 Proof: From Continuous Functions to Multiplication Operators

In the next step we start with a continuous functional calculus and construct a multiplication operator representation for it. Although we want to apply it in the case $K = [a, b]$ first, we formulate it in greater generality for later use.

[1] One even has equality here, see Exercise 1.1.

Theorem 4.8. *Let K be a compact metric space and let $\Phi : \mathrm{C}(K) \to \mathcal{L}(H)$ be a unital $*$-homomorphism. Then there is a semi-finite measure space Ω, a unitary operator $U : H \to \mathrm{L}^2(\Omega)$, and a unital $*$-homomorphism $\Psi : \mathrm{C}(K) \to \mathrm{L}^\infty(\Omega)$ such that*

$$M_{\Psi(f)} = U\,\Phi(f)\,U^{-1} \qquad (f \in \mathrm{C}(K)).$$

Proof. The construction of the measure space Ω employs the *Riesz–Markov–Kakutani representation theorem* (RMK theorem) for positive linear functionals on $\mathrm{C}(K)$, see [2, App.E] or [6, Chap. 2] or [5, Chap. IX]. Namely, for each vector $x \in H$ the mapping

$$\mathrm{C}(K) \to \mathbb{C}, \qquad f \mapsto (\,\Phi(f)x \,|\, x\,)$$

is a linear functional. It is positive, since for $f \in \mathrm{C}(K)$

$$\big(\Phi(|f|^2)x \;\big|\; x\big) = (\,\Phi(f)^*\Phi(f)x \,|\, x\,) = \|\Phi(f)x\|^2 \geq 0. \tag{4.2}$$

Hence, the RMK theorem yields a unique positive regular Borel measure μ_x on K that represents this functional, i.e., with

$$(\,\Phi(f)x \,|\, x\,) = \int_K f \,\mathrm{d}\mu_x \qquad (f \in \mathrm{C}(K)). \tag{4.3}$$

Specializing $f = \mathbf{1}$ we obtain

$$\|\mu_x\| = \mu_x(K) = \|x\|^2.$$

Combining (4.2) and (4.3) we find

$$\|f\|^2_{\mathrm{L}^2(K,\mu_x)} = \|\Phi(f)x\|^2$$

for $f \in \mathrm{C}(K)$ and $x \in H$. Hence, for fixed $x \in H$ the map

$$V_x : \mathrm{C}(K) \to H, \qquad f \mapsto \Phi(f)x$$

extends to an *isometric isomorphism* (i.e., to a unitary operator) of Hilbert spaces $V_x : \mathrm{L}^2(\mu_x) \to Z(x)$, where

$$Z(x) = Z(x;\Phi) := \mathrm{cl}\big\{\Phi(f)x \;|\; f \in \mathrm{C}(K)\big\} = \mathrm{cl}\big\{Sx \;|\; S \in \mathrm{ran}(\Phi)\big\}$$

is the **cyclic subspace** (with respect to Φ) generated by $x \in H$. By virtue of the unitary operator V_x, multiplication by f on $\mathrm{L}^2(\mu_x)$ is unitarily equivalent with application of $\Phi(f)$ on $Z(x)$. Indeed, for $f,\, g \in \mathrm{C}(K)$ one has

$$V_x M_f g = V_x(fg) = \Phi(fg)x = \Phi(f)\Phi(g)x = \Phi(f)V_x g.$$

Since $\mathrm{C}(K)$ is dense in $\mathrm{L}^2(\mu_x)$, $V_x M_f = \Phi(f)V_x$.

At this stage we would be done if we could find a vector $x \in H$ with $Z(x) = H$ (called a **cyclic vector** for Φ). However, cyclic vectors need not exist, so we have to refine our argument. Note that $\operatorname{ran}(\Phi)$ is a $*$-subalgebra of $\mathcal{L}(H)$ and $Z(x)$ is $\operatorname{ran}(\Phi)$-invariant. Hence, so is $Z(x)^\perp$. Therefore, one can employ Zorn's lemma to decompose H orthogonally into cyclic subspaces as

$$H = \bigoplus_\alpha Z(x_\alpha) \cong \bigoplus_\alpha \mathrm{L}^2(K, \mu_{x_\alpha}) \tag{4.4}$$

for some family $(x_\alpha)_\alpha$ of unit vectors in H.

Next, we interpret the Hilbert space direct sum of L^2-spaces in (4.4) as an L^2-space of a larger measure space. To this aim, let $K_\alpha := K \times \{\alpha\}$ be a copy of K for each α, so that the K_α are pairwise disjoint. Define $\varOmega := \bigsqcup_\alpha K_\alpha$,

$$\varSigma := \{B \mid B \cap K_\alpha \in \mathrm{Bo}(K_\alpha) \text{ for all } \alpha\}$$

and $\mu := \bigoplus_\alpha \mu_{x_\alpha}$, i.e.,

$$\mu(B) := \sum_\alpha \mu_{x_\alpha}(B \cap K_\alpha) \qquad (B \in \varSigma).$$

Then $\varSigma$ is a σ-algebra and μ is a measure on it. With $\Omega := (\varOmega, \varSigma, \mu)$ we find

$$H = \bigoplus_\alpha Z(x_\alpha) \cong \bigoplus_\alpha \mathrm{L}^2(K, \mu_{x_\alpha}) \cong \mathrm{L}^2\Big(\bigsqcup_\alpha K_\alpha, \bigoplus_\alpha \mu_{x_\alpha}\Big) = \mathrm{L}^2(\Omega).$$

For $f \in \mathrm{C}(K)$ define $\varPsi(f) \in \mathrm{L}^\infty(\Omega)$ by

$$\varPsi(f) := f \quad \text{on} \quad K_\alpha \cong K.$$

Then $\varPsi : \mathrm{C}(K) \to \mathrm{L}^\infty(\Omega)$ is a unital $*$-homomorphism. Moreover, by virtue of the above unitary equivalence the operator $\Phi(f)$ on H is unitarily equivalent with multiplication by $\varPsi(f)$ on $\mathrm{L}^2(\Omega)$. □

Combining Theorem 4.8 with Theorem 4.6 we arrive at the spectral theorem for bounded self-adjoint operators. Indeed, for a bounded self-adjoint operator A on a Hilbert space H we first find the continuous functional calculus on $[a, b]$ such that $\Phi(\mathbf{z}) = A$ and then a semi-finite measure space Ω and a unitary operator $U : H \to \mathrm{L}^2(\Omega)$ such that UAU^{-1} is a multiplication operator.

This proves the spectral theorem (multiplicator version) for bounded self-adjoint operators. Consequently, also the functional calculus version holds for this special case.

4.4 Proof: From Bounded to Unbounded Operators

In this section we suppose the spectral theorem to be known for bounded self-adjoint or normal operators and show how one can derive the result for unbounded operators from it.

Let $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disc and $\mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$ its boundary. The function

$$\eta : \mathbb{C} \to \mathbb{D}, \qquad \eta(w) := \frac{w}{(1+|w|^2)^{\frac{1}{2}}}$$

is bijective, with inverse

$$\eta^{-1} : \mathbb{D} \to \mathbb{C}, \qquad \eta^{-1}(z) = \frac{z}{(1-|z|^2)^{\frac{1}{2}}}.$$

Given a densely defined and closed operator A (on a Hilbert space), one can define

$$Z_A := AT_A^{\frac{1}{2}}, \quad \text{with} \quad T_A = (\mathrm{I} + A^*A)^{-1}.$$

Note that by Theorem 3.2, T_A is a bounded, injective and positive, self-adjoint operator. So $\sigma(T_A) \subseteq \mathbb{R}_+$ and hence the square root $T_A^{\frac{1}{2}}$ is defined via the continuous functional calculus (Theorem 4.6).

The operator Z_A—which could be seen as an *ad hoc* definition of $\eta(A)$—is called the **bounded transform** of A. It goes back to [8] and has been used in our context originally by Schmüdgen, see e.g. [7, Chapter 5] or [1]. The following lemma, which is [7, Lemma 5.8], summarizes the most important properties of the bounded transform.

Lemma 4.9. *Let A be a densely defined and closed operator on H. Then its bounded transform Z_A has the following properties:*

a) *Z_A is a bounded operator with $\|Z_A\| \le 1$ and*

$$T_A = (\mathrm{I} + A^*A)^{-1} = \mathrm{I} - Z_A^* Z_A.$$

b) *A is uniquely determined by Z_A.*

c) *If A is normal then $Z_A^* = Z_{A^*}$.*

d) *If A is self-adjoint or normal, then so is Z_A.*

Proof. We abbreviate $T = T_A$ and $Z = Z_A$.

a) As a product of a closed and a bounded operator, Z is closed. Note that $\mathrm{ran}(T) = \mathrm{dom}(A^*A) \subseteq \mathrm{dom}(A)$. Hence, AT is fully defined and therefore

$$ZT^{\frac{1}{2}} = AT \in \mathcal{L}(H).$$

It follows that $\operatorname{ran}(T^{\frac{1}{2}}) \subseteq \operatorname{dom}(Z)$. But $T^{\frac{1}{2}}$ is an injective bounded self-adjoint operator (its square is injective!), and hence it must have dense range. Therefore, Z is densely defined. Now take $x = T^{\frac{1}{2}}y \in \operatorname{ran}(T^{\frac{1}{2}})$ and observe that

$$\begin{aligned}\|Zx\|^2 &= (\,ATy \,|\, ATy\,) = (\,A^*ATy \,|\, Ty\,) \le (\,(\mathrm{I} + A^*A)Ty \,|\, Ty\,)\\ &= (\,y \,|\, Ty\,) = \left\|T^{\frac{1}{2}}y\right\|^2 = \|x\|^2.\end{aligned}$$

Hence, Z is closed and contractive on a dense subspace and therefore a bounded operator with $\|Z\| \le 1$. Finally, since $Z = AT^{\frac{1}{2}}$ one has $Z^* \supseteq T^{\frac{1}{2}}A^*$, and hence

$$Z^*ZT^{\frac{1}{2}} \supseteq T^{\frac{1}{2}}A^*AT = T^{\frac{1}{2}}(\mathrm{I} + A^*A)T - T^{\frac{1}{2}}T = T^{\frac{1}{2}} - T^{\frac{1}{2}}T = (\mathrm{I} - T)T^{\frac{1}{2}}.$$

Since $T^{\frac{1}{2}}$ has dense range, $Z^*Z = \mathrm{I} - T$ as claimed.

b) Suppose that A and B are densely defined, closed operators on H such that $Z_A = Z_B$. Then, by a),

$$(\mathrm{I} + A^*A)^{-1} = T_A = \mathrm{I} - Z_A^*Z_A = \mathrm{I} - Z_B^*Z_B = T_B = (\mathrm{I} + B^*B)^{-1}.$$

It follows that $A^*A = B^*B$ and $AT = Z_AT^{\frac{1}{2}} = Z_BT^{\frac{1}{2}} = BT$. So $A = B$ on $\operatorname{dom}(A^*A) = \operatorname{dom}(B^*B)$. But this space is a core for both operators (Theorem 3.2), so $A = B$ as claimed.

c) Let A be normal. Then $T_{A^*} = T_A = T$. Also, by Corollary 3.3, $TA \subseteq AT$. Then Corollary 4.7 applied to T (bounded) and A (unbounded) yields

$$T^{\frac{1}{2}}A \subseteq AT^{\frac{1}{2}}.$$

Hence,

$$Z_{A^*} = A^*T^{\frac{1}{2}} = A^*(T^{\frac{1}{2}})^* \stackrel{!}{=} (T^{\frac{1}{2}}A)^* \supseteq (AT^{\frac{1}{2}})^* = Z_A^*$$

by i) and g) of Lemma A.21. Since both operators Z_A^* and Z_{A^*} are bounded, equality follows.

d) If A is self-adjoint then, by c), $Z_A^* = Z_{A^*} = Z_A$ is also self-adjoint. If A is just normal, then by a) and c)

$$Z_A^*Z_A = \mathrm{I} - T_A = \mathrm{I} - T_{A^*} = Z_{A^*}^*Z_{A^*} = Z_AZ_A^*,$$

and so Z_A is normal, too. □

With the help of the bounded transform we obtain a result that allows to establish the spectral theorem for unbounded operators from the spectral theorem for bounded ones.

Corollary 4.10. *Let A be a normal operator on the Hilbert space H. If Z_A has a multiplication operator representation, then so has A.*

Proof. By hypothesis, we find a semi-finite measure space Ω, a bounded measurable function ζ on Ω and a unitary $U : H \to \mathrm{L}^2(\Omega)$ such that $U^{-1}M_\zeta U = Z_A$.

Since $\|Z_A\| \leq 1$, $\mathrm{essran}(\zeta) \subseteq \overline{\mathbb{D}}$. But more is true: Since $\mathrm{I} - Z_A^* Z_A = T$ is injective, the same is true for its multiplication operator equivalent $M_{\mathbf{1}-|\zeta|^2}$. By Theorem 2.1.b) this implies that $|\zeta| < 1$ almost everywhere, hence we can define

$$a = \eta^{-1} \circ \zeta = \frac{\zeta}{(1-|\zeta|^2)^{\frac{1}{2}}} \in \mathrm{L}^0(\Omega).$$

We claim that A is equivalent to M_a via U. To prove this, let the operator B on H be defined by $B := U^{-1}M_a U$. Then B is a normal operator and it is easy to show that

$$Z_B = U^{-1}Z_{M_a}U = U^{-1}M_{\eta\circ a}U = U^{-1}M_\zeta U = Z_A.$$

From Lemma 4.9.b it then follows that $A = B$. □

Combining Corollary 4.10 with the results from the previous section establishes the spectral theorem for all self-adjoint operators.

4.5 Supplement: Normal Operators

In this supplement we establish the spectral theorem for all normal operators. Reviewing the results of the previous sections, it suffices to prove the following analogue of Theorem 4.6.

Theorem 4.11. *Let A be a bounded normal operator on a Hilbert space H. Then for some real interval $[a,b]$ there is a (unique) unital $*$-homomorphism*

$$\Phi : \mathrm{C}([a,b]^2) \to \mathcal{L}(H)$$

such that $\Phi(\mathbf{z}) = A$.

Indeed, this theorem combined with Theorem 4.8 for $K = [a,b]^2$ yields a multiplication operator representation for every bounded normal operator, and Lemma 4.9 together with Corollary 4.10 then yields such a representation for each normal operator.

Proof. Define the bounded self-adjoint operators B and C by

$$B := \frac{1}{2}(A + A^*) \quad \text{and} \quad C := \frac{1}{2\mathrm{i}}(A - A^*).$$

Note that $CB = BC$, since A is normal. The idea for the proof is that $A = B + \mathrm{i}C$ and a functional calculus for A is the same as a *joint functional calculus* for B and C.

Choose $[a,b]$ such that it contains $\boldsymbol{\sigma}(B)$ and $\boldsymbol{\sigma}(C)$. Since B and C are bounded self-adjoint operators, each of them has a measurable functional calculus Φ_B and Φ_C, say, and both are concentrated on $[a,b]$. Since $CB = BC$ one has $\Phi_B(f)\Phi_C(g) = \Phi_C(g)\Phi_B(f)$ for all f and g (Corollary 4.7).

Consider the space

$$\mathcal{A} := \operatorname{span}\{\mathbf{1}_Q \otimes \mathbf{1}_R \mid Q,\ R \text{ subintervals of } [a,b]\}$$

of rectangle step functions on $[a,b]^2$. We define a linear map

$$\Phi : \mathcal{A} \to \mathcal{L}(H)$$

through the requirement that

$$\Phi(\mathbf{1}_Q \otimes \mathbf{1}_R) := \Phi_B(\mathbf{1}_Q)\, \Phi_C(\mathbf{1}_R).$$

(Of course, one has to show that Φ is well defined, see Exercise 4.9.) Obviously, $\mathcal{A}$ is a $*$-algebra and Φ is a unital $*$-homomorphism. (Here one needs that the two calculi commute.)

Now we observe that if $0 \le h \in \mathcal{A}$ then $\sqrt{h} \in \mathcal{A}$. This is a consequence of the fact that h can be written as

$$h = \sum_j c_j \mathbf{1}_{Q_j} \otimes \mathbf{1}_{R_j}$$

where the rectangles $Q_j \times R_j$ are pairwise disjoint. (Recall this from your classes on Lebesgue integration.) Hence, by Exercise 3.3 we conclude that Φ is bounded and one has

$$\|\Phi(f)\| \le \|f\|_\infty$$

for all $f \in \mathcal{A}$.

By elementary functional analysis, Φ has a continuous linear extension to the $\|\cdot\|_\infty$-closure $\mathrm{cl}\mathcal{A}$ of $\mathcal{A}$. This extension, again called Φ, is of course a unital $*$-homomorphism. But $\mathrm{C}([a,b]^2) \subseteq \mathrm{cl}\mathcal{A}$ and so we have found the desired continuous functional calculus.

It remains to show that $\Phi(\mathbf{z}) = A$. Observe that[2]

$$\mathbf{z} = (\mathbf{x} \otimes \mathbf{1}) + \mathrm{i}(\mathbf{1} \otimes \mathbf{x}).$$

Hence,

$$\Phi(\mathbf{z}) = \Phi_B(\mathbf{x})\Phi_C(\mathbf{1}) + \mathrm{i}\Phi_B(\mathbf{1})\Phi_C(\mathbf{x}) = B + \mathrm{i}C = A.$$

And this concludes the proof. $\square$

[2] We denote by $\mathbf{x}$ the *real* coordinate function, i.e., the mapping $\mathbf{x} : \mathbb{R} \to \mathbb{R}$, $x \mapsto x$.

Exercises

4.1. Let A be a normal operator on a Hilbert space H.

a) Show that for $\lambda \in \mathbb{C}$ the operator $\Phi_A(\mathbf{1}_{\{\lambda\}})$ is the orthogonal projection onto $\ker(\lambda - A)$. Conclude that λ is an eigenvalue of A if and only if $\{\lambda\}$ is not a Φ_A-null set.

b) Show that
$$\Phi_A(f)x = f(\lambda)x$$
for all $f \in \mathcal{M}(\mathbb{C})$ and $x \in \ker(\lambda - A)$. Conclude the **spectral inclusion theorem for the point spectrum**
$$f(\sigma_{\mathrm{p}}(A)) \subseteq \sigma_{\mathrm{p}}(f(A)).$$

c) Show that any isolated point of $\sigma(A)$ is an eigenvalue of A.

4.2. Provide examples of normal operators A on a Hilbert space H and measurable functions f on $\mathbb{C}$ such that

a) $f(\sigma(A)) \not\subseteq \sigma(f(A))$;

b) f is continuous and $\sigma(f(A)) \neq f(\sigma(A))$.

4.3 (Compatibility with the Dunford–Riesz Calculus). Let A be a bounded normal operator on a Hilbert space H and let f be a holomorphic function defined on an open set $U \subseteq \mathbb{C}$ containing $\sigma(A)$. Show that $\Phi_A(f)$ coincides with $\Phi(f)$, where $\Phi : \mathrm{Hol}(U) \to \mathcal{L}(H)$ is the Dunford–Riesz calculus for A.

4.4. Let A be a normal operator on a Hilbert space H. Show that if A is bounded, then
$$\|A\| = r(A).$$
[One proof uses the spectral radius formula, another the Borel functional calculus.]

4.5. Let A be a bounded operator on a Hilbert space H. Show that a closed linear subspace F of a Hilbert space H is A^*-invariant if and only if $F^\perp$ is A-invariant. Then work out the Zorn argument mentioned in the proof of Theorem 4.8.

4.6. Show that the function
$$\eta(z) := \frac{z}{(1+|z|^2)^{\frac{1}{2}}}$$
is a bijection $\mathbb{C} \to \mathbb{D}$ with inverse given by
$$\eta^{-1}(w) = \frac{w}{(1-|w|^2)^{\frac{1}{2}}}.$$

4.7. Let $U : H \to K$ be a unitary operator between the Hilbert spaces H and K. Suppose that A and B are closed and densely defined operators on H and K, respectively, satisfying $B = UAU^{-1}$. Show that $B^* = UA^*U^{-1}$ and

$$Z_B = UZ_AU^{-1},$$

where Z_A and Z_B are the respective bounded transforms.

4.8. Let $H := \mathrm{L}^2(0,1)$, $k(t,s) := \min(s,t)$ for $s,t \in [0,1]$ and $A \in \mathcal{L}(H)$ given by

$$Af(t) := \int_0^1 k(t,s)f(s)\,\mathrm{d}s \qquad (f \in H).$$

Show that A is a bounded self-adjoint operator and determine a multiplication operator representation for it. [Hint: A is compact and satisfies $\operatorname{ran}(A) \subseteq \mathrm{C}^1[0,1]$. What are the eigenvalues and eigenspaces of A?]

4.9. (This exercise only regards the supplementary Section 4.5.) Let F and G be linear spaces of scalar-functions on the sets X and Y, respectively. Suppose that $\beta : F \times G \to E$ is a bilinear mapping, where E is any linear space. Show that there is a unique linear mapping

$$B : F \otimes G \to E \quad \text{such that} \quad B(f \otimes g) = \beta(f,g).$$

Here $f \otimes g$ denotes the function

$$f \otimes g : X \times Y \to \mathbb{C}, \qquad (f \otimes g)(x,y) := f(x)\,g(y)$$

and $F \otimes G$ is the linear span (within the functions on $X \times Y$) of all functions $f \otimes g$ with $f \in F$ and $g \in G$.

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Chapter 5
Fourier Analysis

We know already that functional calculus theory is pretty much the same as the Banach space representation theory of certain algebras (but without an a priori restriction to bounded operators). On the other hand, algebra representations are closely related to (semi)group representations, and this is what we are going to explore in this and the next chapter. We shall mostly confine to the additive group $\mathbb{R}^d$ and certain subsemigroups of it. However, many of the treated results have generalizations to all or all Abelian locally compact topological groups and their subsemigroups.

5.1 Strongly Continuous Semigroup Representations

In the following, $\mathbb{S}$ denotes a closed subsemigroup of $\mathbb{R}^d$. This means that $\mathbb{S} \subseteq \mathbb{R}^d$ is closed and $\mathbb{S} + \mathbb{S} \subseteq \mathbb{S}$. In addition we shall suppose always that $0 \in \mathbb{S}$ unless otherwise stated.[1] Note that if $\mathbb{S}$ is a subsemigroup of $\mathbb{R}^d$ then $\mathbb{S} - \mathbb{S}$ is a sub*group*. We are mostly interested in the cases $\mathbb{S} = \mathbb{R}^d_+$ and $\mathbb{S} = \mathbb{R}^d$ ("continuous case") as well as $\mathbb{S} = \mathbb{Z}^d_+$ and $\mathbb{S} = \mathbb{Z}^d$ ("discrete case")[2].

A **representation** of $\mathbb{S}$ (by bounded operators) on a Banach space X is a mapping

$$T : \mathbb{S} \to \mathcal{L}(X), \qquad T = (T_t)_{t \in \mathbb{S}}$$

such that

$$T_0 = \mathrm{I} \quad \text{and} \quad T_{t+s} = T_t T_s \qquad (t,\ s \in \mathbb{S}). \tag{5.1}$$

Two representations T^1, T^2 of $\mathbb{S}$ on the same Banach space X are called **commuting** if

$$T^1_t T^2_s = T^2_s T^1_t \qquad \text{for all } t, s \in \mathbb{S}.$$

---

[1] Algebraists would call $\mathbb{S}$ a *monoid*, but this term is quite uncommon among analysts.

[2] We use $\mathbb{Z}_+$ synonymously with $\mathbb{N}_0$ here.

If $\mathbb{S}$ is actually a subgroup of $\mathbb{R}^d$ then (5.1) implies that each T_t is invertible with $T_t^{-1} = T_{-t}$. Note that we usually prefer index notation but shall switch to the alternative "$T(t)$" whenever it is convenient. Also, instead of "semigroup representation" we shall simply say "semigroup" as long as the meaning is clear.

Remark 5.1. Representations π of $\mathbb{S} = \mathbb{N}_0$ are in one-to-one correspondence with single operators $T \in \mathcal{L}(X)$, via $T = \pi_1$ and $\pi_n = T^n$. Analogously, the representations of $\mathbb{Z}$ correspond bijectively to invertible operators.

In the same fashion, representations of $\mathbb{S} = \mathbb{Z}_+^d$ ($\mathbb{S} = \mathbb{Z}^d$) correspond to d-tuples $(T_1, T_2, \dots, T_d)$ of pairwise commuting, bounded (and invertible) operators.

Representations T of $\mathbb{S} = \mathbb{R}_+^d$, sometimes called ***d*-parameter semigroups**, correspond to d-tuples of pairwise commuting 1-parameter semigroups $T^1, \dots, T^d$ via

$$T(t_1\mathbf{e}_1 + \cdots + t_d\mathbf{e}_d) = T^1(t_1)\cdots T^d(t_d) \qquad (t_1, \dots, t_d \geq 0),$$

where $\mathbf{e}_1 \dots, \mathbf{e}_d$ is the canonical basis of $\mathbb{R}^d$.

Let Ω be any metric (or topological) space and $T : \Omega \to \mathcal{L}(X)$ any mapping. Then the **orbit** of $x \in X$ under T is the mapping

$$T(\cdot)x : \Omega \to X, \qquad t \mapsto T_t x.$$

(Sometimes, also the image of this mapping is called the orbit of x, but this equivocation is unproblematic.) For us, the following continuity notions shall be important:

1) **strong continuity**: each orbit $T(\cdot)x$, $x \in X$, is continuous.
2) **weak* continuity**: $X = Y'$ for some Banach space Y and each "weak* orbit" $\langle T(\cdot)y', y\rangle$, $y' \in Y'$ and $y \in Y$, is continuous.
3) **operator norm continuity**: the mapping $t \mapsto T_t$ is continuous for the operator norm.

Of course, one could consider also weak continuity, but in the cases we are interested in, weak and strong continuity are equivalent (see, e.g., [1, p. I.5.8]). Moreover, for semigroup representations operator norm continuity is far too restrictive. A good and widely applicable theory exists for strongly continuous semigroups, so this will be our standard assumption. Weakly* continuous mappings appear naturally when passing from a strongly continuous mapping $T : \Omega \to \mathcal{L}(X)$ to its dual $T' : \Omega \to \mathcal{L}(X')$, defined by

$$T'_t := (T_t)' \quad \text{for all } t \in \Omega.$$

In the following, we shall concentrate mostly on representations of the group $\mathbb{R}^d$. Proper semigroup theory will appear in the next chapter.

5.2 The Regular Representations

There is a fairly straightforward way to construct representations of $\mathbb{R}^d$. Just take a Banach space X of functions f on $\mathbb{R}^d$ such that for each $t \in \mathbb{R}^d$ the function

$$\tau_t f := f(\cdot - t)$$

is again a member of X. (Such a space is called **shift invariant**.) Then τ_t is a linear mapping on X and $t \mapsto \tau_t$ satisfies the semigroup law. If, in addition, each τ_t is a bounded operator on X, then τ is a representation of $\mathbb{R}^d$. It is called the **regular representation** or the **right shift group** on X. (If $d = 1$ and $t \geq 0$ then the graph of $\tau_t f$ is just the graph of f shifted by t to the right.)

Examples 5.2. Examples of regular representations abound:

1) The space $\mathrm{C_b}(\mathbb{R}^d)$ of bounded and continuous functions is shift invariant and each operator τ_t is an invertible isometry on it. Hence, τ is a uniformly bounded d-parameter group. However, it is *not* strongly continuous.
2) One can ask for the largest shift invariant subspace of $\mathrm{C_b}(\mathbb{R}^d)$ on which the shift group is strongly continuous. This is $\mathrm{UC_b}(\mathbb{R}^d)$ the space of bounded and uniformly continuous functions (Exercise 5.1).
3) The space $\mathrm{C_0}(\mathbb{R}^d)$ of continuous functions vanishing at infinity is a shift invariant subspace of $\mathrm{UC_b}(\mathbb{R}^d)$. Hence τ is a strongly continuous group on it.
4) The space $\mathrm{L}^p(\mathbb{R}^d)$ of Lebesgue-p-integrable functions (modulo null functions) $(1 \leq p < \infty)$ is shift invariant and each τ_t is an invertible isometry on it. The shift group τ is strongly continuous on it (Exercise 5.1).
5) The space $\mathrm{L}^\infty(\mathbb{R}^d)$ of bounded measurable functions modulo Lebesgue null functions is shift invariant and each operator τ_t is an invertible isometry on it. As in Example 1), this shift group τ is not strongly continuous. However, if we view $\mathrm{L}^\infty(\mathbb{R}^d)$ as the dual of $\mathrm{L}^1(\mathbb{R}^d)$, it is weakly* continuous. (It is the dual of the "left shift" group $(\tau_{-t})_{t\in\mathbb{R}^d}$ on $\mathrm{L}^1(\mathbb{R}^d)$.)
6) Examples 1)–4) have versions for Banach space valued functions. That is, if X is a Banach space, the shift group is (well-defined and) isometric and strongly continuous on the spaces $\mathrm{UC_b}(\mathbb{R}^d; X)$ and $\mathrm{C_0}(\mathbb{R}^d; X)$. This is as easy to prove as in the case $X = \mathbb{C}$.
   Also, for $1 \leq p < \infty$ the shift group is (well-defined and) isometric and strongly continuous on the Bochner space $\mathrm{L}^p(\mathbb{R}^d; X)$. (This is mentioned here only for completeness, but will be important from Chapter 11 on.)

The Shift Group on $\mathrm{M}(\mathbb{R}^d)$

Recall from Appendix A.7 that the space $\mathrm{M}(\mathbb{R}^d)$ of complex (regular) Borel measures on $\mathbb{R}^d$ is a Banach space and can, by the Riesz–Markov–Kakutani theorem (Theorem A.41), be isometrically identified with the dual of $\mathrm{C}_0(\mathbb{R}^d)$ via integration. We shall do this, tacitly, henceforth.

The **support** of a measure $\mu \in \mathrm{M}(\mathbb{R}^d)$ is

$$\operatorname{supp}(\mu) = \operatorname{supp}(|\mu|) = \{x \in \mathbb{R}^d \mid \forall \varepsilon > 0 : |\mu| (\mathrm{B}(x, \varepsilon)) > 0\}.$$

This is a closed subset of $\mathbb{R}^d$ and one can easily show that $|\mu| (\mathbb{R}^d \setminus \operatorname{supp}(\mu)) = 0$. The space $\mathrm{M}_c(\mathbb{R}^d)$ of measures with **compact support** is norm-dense in $\mathrm{M}(\mathbb{R}^d)$, see Exercise 5.3.

Each $g \in \mathrm{L}^1(\mathbb{R}^d)$ can be (isometrically) identified with the complex measure $g\lambda$, defined by

$$(g\lambda)(B) := \int_{\mathbb{R}^d} \mathbf{1}_B g \,\mathrm{d}\lambda = \int_B g \,\mathrm{d}\lambda \qquad (B \in \mathrm{Bo}(\mathbb{R}^d)),$$

where λ denotes the Lebesgue measure. Integration with respect to $g\lambda$ is performed according to the formula

$$\int_{\mathbb{R}^d} f \,\mathrm{d}(g\lambda) = \int_{\mathbb{R}^d} fg \,\mathrm{d}\lambda \qquad (f \in \mathcal{M}_b(\mathbb{R}^d)).$$

In the following, the map $\mathrm{L}^1(\mathbb{R}^d) \to \mathrm{M}(\mathbb{R}^d)$, $g \mapsto g\lambda$, is called the **natural embedding**. We shall usually identify g with $g\lambda$ and use the notation $g\lambda$ only in exceptional cases.

For $\mu \in \mathrm{M}(\mathbb{R}^d)$ and $t \in \mathbb{R}^d$ we define $\tau_t\mu \in \mathrm{M}(\mathbb{R}^d)$ by

$$\int_{\mathbb{R}^d} f \,\mathrm{d}(\tau_t\mu) = \int_{\mathbb{R}^d} \tau_{-t} f \,\mathrm{d}\mu \tag{5.2}$$

for all $f \in \mathrm{C}_0(\mathbb{R}^d)$. This just means that τ_t on $\mathrm{M}(\mathbb{R}^d)$ is the Banach space dual operator to the operator τ_{-t} on $\mathrm{C}_0(\mathbb{R}^d)$. As τ is strongly continuous on $\mathrm{C}_0(\mathbb{R}^d)$, it is weakly* continuous on $\mathrm{M}(\mathbb{R}^d)$. As an application of Lemma 3.11 and Remark 3.12 we obtain that (5.2) holds even for all bounded Borel measurable functions $f \in \mathcal{M}_b(\mathbb{R}^d)$ (Exercise 5.2).

The definition of τ_t on $\mathrm{M}(\mathbb{R}^d)$ is compatible with the definition of τ_t on $\mathrm{L}^1(\mathbb{R}^d)$ under the natural embedding. Indeed, for $g \in \mathrm{L}^1(\mathbb{R}^d)$ and $t \in \mathbb{R}^d$:

$$\int_{\mathbb{R}^d} f \,\mathrm{d}(\tau_t(g\lambda)) = \int_{\mathbb{R}^d} \tau_{-t} f \,\mathrm{d}(g\lambda) = \int_{\mathbb{R}^d} (\tau_{-t} f)\, g \,\mathrm{d}\lambda = \int_{\mathbb{R}^d} f\, (\tau_t g) \,\mathrm{d}\lambda$$

for all $f \in \mathrm{C}_0(\mathbb{R}^d)$. And this implies that $\tau_t(g\lambda) = (\tau_t g)\lambda$.

The **reflection operator** $\mathcal{S}$ is defined on functions f by

$$(\mathcal{S}f)(s) = f(-s) \qquad (s \in \mathbb{R}^d).$$

On $\mathrm{M}(\mathbb{R}^d)$, the reflection is defined by its dual action, i.e. by

$$\int_{\mathbb{R}^d} f \,\mathrm{d}(\mathcal{S}\mu) = \int_{\mathbb{R}^d} \mathcal{S}f \,\mathrm{d}\mu \tag{5.3}$$

for $f \in \mathrm{C}_0(\mathbb{R}^d)$. As before, this formula extends to all $f \in \mathcal{M}_\mathrm{b}(\mathbb{R}^d)$ (Exercise 5.2). Note that

$$\tau_t \mathcal{S} = \mathcal{S}\tau_{-t}$$

on functions as on measures. Also, $\overline{\mathcal{S}\mu} = \mathcal{S}\overline{\mu}$ for each $\mu \in \mathrm{M}(\mathbb{R}^d)$. Note further that the definition of $\mathcal{S}\mu$ is compatible with the natural embedding, i.e.

$$(\mathcal{S}g)\lambda = \mathcal{S}(g\lambda)$$

for all $g \in \mathrm{L}^1(\mathbb{R}^d)$. All this is easy to see and the proofs are left to the reader.

If $\mu \in \mathrm{M}(\mathbb{R}^d)$ and $g \in \mathcal{M}_\mathrm{b}(\mathbb{R}^d)$ then their **product** $g\mu \in \mathrm{M}(\mathbb{R}^d)$ is defined by

$$\int_{\mathbb{R}^d} f \,\mathrm{d}(g\mu) := \int_{\mathbb{R}^d} fg \,\mathrm{d}\mu \tag{5.4}$$

for all $f \in \mathrm{C}_0(\mathbb{R}^d)$. Again, this formula remains true for $f \in \mathcal{M}_\mathrm{b}(\mathbb{R}^d)$. Clearly,

$$\|g\mu\|_\mathrm{M} \le \|g\|_\infty \|\mu\|_\mathrm{M}.$$

Also, one has the formulae

$$\mathcal{S}(g\mu) = (\mathcal{S}g)(\mathcal{S}\mu) \quad \text{and} \quad \overline{g\mu} = \overline{g}\,\overline{\mu} \qquad (\mu \in \mathrm{M}(\mathbb{R}^d),\ g \in \mathcal{M}_\mathrm{b}(\mathbb{R}^d)).$$

Again, the simple verification is left to the reader.

5.3 Averaging a Representation and the Convolution of Measures

We now come to one of the central constructions in the theory of functional calculus.

Let $E \subseteq \mathbb{R}^d$ be a closed subset. Functions on E (to some Banach space) can be identified with their extension by 0 to all of $\mathbb{R}^d$, and we shall do this (tacitly) whenever it is convenient. Also, complex measures $\mu \in \mathrm{M}(E)$ are identified with their canonical (zero-)extension to $\mathbb{R}^d$ so that

$$\mathrm{M}(E) \cong \{\mu \in \mathrm{M}(\mathbb{R}^d) \mid \operatorname{supp}(\mu) \subseteq E\}$$

isometrically.

For a bounded and strongly continuous mapping $T : E \to \mathcal{L}(X)$ and a complex measure $\mu \in \mathrm{M}(\mathbb{R}^d)$ we consider the integral

$$T_\mu := \int_E T_t \, \mu(\mathrm{d}t) \in \mathcal{L}(X)$$

defined strongly by

$$T_\mu x = \int_E T_t x \, \mu(\mathrm{d}t) \in X \qquad (x \in X). \tag{5.5}$$

To begin with, (5.5) can be viewed as a weak integral since the orbit $T(\cdot)x$ is bounded and continuous on E as well as on $\mathbb{R}^d \setminus E$, hence in particular weakly integrable. Of course, that does define $T_\mu x$ only as an element of X''. The easiest way to see that in fact $T_\mu x \in X$ is to realize that the orbit $T(\cdot)x$ is Bochner integrable with respect to the finite positive measure $|\mu|$. (See also Exercise 5.4.)

Lemma 5.3. *Let $T : E \to \mathcal{L}(X)$ be strongly continuous and bounded. Then*

$$\|T_\mu\| \le M_T \|\mu\|_{\mathrm{M}} \qquad (\mu \in \mathrm{M}(\mathbb{R}^d)), \tag{5.6}$$

where $M_T := \sup_{t \in E} \|T_t\|$. If also $S : E \to \mathcal{L}(Y)$ is strongly continuous and bounded, and $Q \in \mathcal{L}(X; Y)$ is such that $QT_t = S_t Q$ for all $t \in E$, then $QT_\mu = S_\mu Q$ for all $\mu \in \mathrm{M}(\mathbb{R}^d)$.

Proof. Note that from (5.5) and (A.13) it follows that

$$\begin{aligned} \|T_\mu x\| = \Big\| \int T_t x \, \mu(\mathrm{d}t) \Big\| &\le \int \|T_t x\| \; |\mu| \, (\mathrm{d}t) \\ &\le M_T \|x\| \|x'\| \, |\mu| \, (\mathbb{R}^d) = M_T \|x\| \|x'\| \|\mu\|_{\mathrm{M}}. \end{aligned}$$

This implies the first statement. The second follows easily from (A.2). □

Convolution

We shall now apply the foregoing results to the strongly continuous regular representations from before. The general scheme is the following: If X is a shift invariant Banach space of (maybe vector-valued) functions on $\mathbb{R}^d$ on which the shift group $(\tau_t)_{t \in \mathbb{R}^d}$ is strongly continuous and bounded, then for each $\mu \in \mathrm{M}(\mathbb{R}^d)$ we can form the operator $\tau_\mu \in \mathcal{L}(X)$. For $f \in X$ one usually writes

$$\mu * f := \tau_\mu f$$

and calls this the **convolution** of f with μ. Convolution is a (clearly bilinear) mapping

$$* : \mathrm{M}(\mathbb{R}^d) \times X \to X, \qquad (\mu, f) \mapsto \mu * f.$$

Following this scheme one obtains the convolution products

$$\begin{aligned}
&\mathrm{M}(\mathbb{R}^d) \times \mathrm{C}_0(\mathbb{R}^d) \to \mathrm{C}_0(\mathbb{R}^d),\\
&\mathrm{M}(\mathbb{R}^d) \times \mathrm{UC_b}(\mathbb{R}^d) \to \mathrm{UC_b}(\mathbb{R}^d),\\
&\mathrm{M}(\mathbb{R}^d) \times \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^p(\mathbb{R}^d) \qquad (1 \le p < \infty)
\end{aligned}$$

with

$$\|\mu * f\|_X \le \|\mu\|_{\mathrm{M}} \, \|f\|_X$$

where $X \in \{\mathrm{UC_b}(\mathbb{R}^d),\, \mathrm{L}^p(\mathbb{R}^d)\}$. If $\mu = g\lambda$ for $g \in \mathrm{L}^1(\mathbb{R}^d)$ we simply write $g * f$ instead of $(g\lambda) * f$. Of course, in all that one has to be careful not to use the same notation for different things. E.g., if $f \in \mathrm{L}^p(\mathbb{R}^d) \cap \mathrm{UC_b}(\mathbb{R}^d)$, then $\mu * f$ should mean the same, no matter whether we interpret this convolution as performed in L^p or in $\mathrm{UC_b}$. This is indeed the case, see Exercise 5.5.

Our definition of convolution may be uncommon to many readers. It has the advantage that Fubini's theorem is completely avoided. An apparent drawback is that one does not obtain immediately the common pointwise (almost everywhere) representation formulae like

$$(\varphi * f)(s) = \int_{\mathbb{R}^d} \varphi(t) f(s-t)\, \mathrm{d}t \qquad (\varphi \in \mathrm{L}^1(\mathbb{R}^d)).$$

If $f \in \mathrm{UC_b}(\mathbb{R}^d)$, this formula is easy to prove, as point evaluations are continuous. If $f \in \mathrm{L}^p(\mathbb{R}^d)$, though, one has to work more (see Exercise 5.7). And of course one has to employ Fubini's theorem.

As already noted, the right shift representation τ on $\mathrm{M}(\mathbb{R}^d)$ is weakly* continuous. Hence, for $\mu,\ \nu \in \mathrm{M}(\mathbb{R}^d)$ one can define

$$\mu * \nu := \tau_\mu \nu := \int_{\mathbb{R}^d} \tau_t \nu \, \mu(\mathrm{d}t)$$

as a weakly*-convergent integral. In other words, $\mu * \nu \in \mathrm{M}(\mathbb{R}^d)$ is given by

$$\int_{\mathbb{R}^d} f \, \mathrm{d}(\mu * \nu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f \, \mathrm{d}(\tau_t \nu)\, \mu(\mathrm{d}t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(t+s)\, \nu(\mathrm{d}s)\, \mu(\mathrm{d}t) \tag{5.7}$$

for all $f \in \mathrm{C}_0(\mathbb{R}^d)$. Since this identity is preserved under bp-limits of functions, it even holds for all bounded measurable functions $f \in \mathcal{M}_{\mathrm{b}}(\mathbb{R}^d)$ (Exercise 5.2). (For consistency with the natural embedding see Exercise 5.6.) We can now state the main theorem about convolutions.

Theorem 5.4 (Convolution Algebra). *With respect to the convolution product,* $\mathrm{M}(\mathbb{R}^d)$ *is a commutative Banach algebra with unit element* δ_0. *In addition, the following statements hold:*

a) $\delta_t * \mu = \tau_t \mu$ *for all* $t \in \mathbb{R}^d$.

b) *Under the natural embedding, the space* $\mathrm{L}^1(\mathbb{R}^d)$ *is an algebra ideal of* $\mathrm{M}(\mathbb{R}^d)$ *and one has*

$$(\mu * g)\lambda = \mu * (g\lambda) \qquad \textit{for all } g \in \mathrm{L}^1(\mathbb{R}^d) \textit{ and } \mu \in \mathrm{M}(\mathbb{R}^d).$$

c) *For all* $f \in \mathrm{UC_b}(\mathbb{R}^d)$ *and* $\mu,\ \nu \in \mathrm{M}(\mathbb{R}^d)$*:*

$$\int_{\mathbb{R}^d} f \,\mathrm{d}(\mu * \nu) = \int_{\mathbb{R}^d} (\mathcal{S}\mu) * f \,\mathrm{d}\nu.$$

d) *If* $\mathbb{S} \subseteq \mathbb{R}^d$ *is a closed subsemigroup then* $\mathrm{M}(\mathbb{S})$ *is a unital subalgebra of* $\mathrm{M}(\mathbb{R}^d)$. *And if* $T : \mathbb{S} \to \mathcal{L}(X)$ *is any bounded and strongly continuous representation, then*

$$T_\mu T_\nu = T_{\mu * \nu} \qquad (\mu,\ \nu \in \mathrm{M}(\mathbb{S})).$$

In other words, the mapping $\mathrm{M}(\mathbb{S}) \to \mathcal{L}(X)$, $\mu \mapsto T_\mu$, *is a homomorphism of unital algebras.*

Proof. We first prove d). Let $B \subseteq \mathbb{R}^d \setminus \mathbb{S}$ be any Borel set. Then

$$\int_{\mathbb{R}^d} \mathbf{1}_B \,\mathrm{d}(\mu * \nu) = \int_{\mathbb{S}} \int_{\mathbb{S}} \mathbf{1}_B(t+s)\, \nu(\mathrm{d}s)\, \mu(\mathrm{d}t) = 0$$

whenever $\operatorname{supp}(\mu), \operatorname{supp}(\nu) \subseteq \mathbb{S}$. Hence, $|\mu * \nu|\,(\mathbb{S}^c) = 0$, i.e., $\operatorname{supp}(\mu * \nu) \subseteq \mathbb{S}$. Let $T : \mathbb{S} \to \mathcal{L}(X)$ be any bounded and strongly continuous representation, and let $\mu,\ \nu \in \mathrm{M}(\mathbb{S})$ and $x \in X$. Then

$$\begin{aligned} T_\mu T_\nu x &= \int_{\mathbb{S}} T_t T_\nu x \,\mu(\mathrm{d}t) = \int_{\mathbb{S}} T_t \int_{\mathbb{S}} T_s x \,\nu(\mathrm{d}s)\,\mu(\mathrm{d}t) \\ &= \int_{\mathbb{S}} \int_{\mathbb{S}} T_t T_s x \,\nu(\mathrm{d}s)\,\mu(\mathrm{d}t) = \int_{\mathbb{S}} \int_{\mathbb{S}} T_{t+s} x \,\nu(\mathrm{d}s)\,\mu(\mathrm{d}t), \end{aligned}$$

where (A.2) has been used at the change of the line. Applying linear functionals $x' \in X'$ to this, interpreting μ and ν as measures on observing (5.7) we find

$$T_\mu T_\nu x = T_{\mu * \nu} x$$

as claimed. (One of course has to intepret μ and μ as measures on $\mathbb{R}^d$ and T as a function on $\mathbb{R}^d$.)

Now we show that convolution is associative. From the formula

$$(\tau_\mu f)(0) = (\mu * f)(0) = \int_{\mathbb{R}^d} (\mathcal{S}f)\ \mathrm{d}\mu \qquad (f \in \mathrm{C}_0(\mathbb{R}^d))$$

it follows that $\|\tau_\mu f\|_\infty \geq |\langle \mathcal{S}f, \mu\rangle|$ for all $f \in \mathrm{C}_0(\mathbb{R}^d)$ and hence $\|\tau_\mu\|_{\mathcal{L}(\mathrm{C}_0(\mathbb{R}^d))} \geq \|\mu\|$. (Use that $\mathcal{S}$ is an isometric isomorphism on $\mathrm{C}_0(\mathbb{R}^d)$.) In particular, the mapping

$$\mathrm{M}(\mathbb{R}^d) \to \mathcal{L}(\mathrm{C}_0(\mathbb{R}^d)), \qquad \mu \mapsto \tau_\mu$$

is injective. Since it turns convolution into operator multiplication and the latter is associative, so must be the former.

Commutativity is seen similarly. Fix $\mu, \nu \in \mathrm{M}(\mathbb{R}^d)$ and let τ be as above the regular representation of $\mathbb{R}^d$ on $\mathrm{C}_0(\mathbb{R}^d)$. Since $\mathbb{R}^d$ is commutative, $\tau_t\tau_s = \tau_s\tau_t$ for all $s, t \in \mathbb{R}^d$. By Lemma 5.3, $\tau_t\tau_\mu = \tau_\mu\tau_t$ for all $t \in \mathbb{R}^d$, and so by d) and Lemma 5.3 again,

$$\tau_{\nu*\mu} = \tau_\nu\tau_\mu = \tau_\mu\tau_\nu = \tau_{\mu*\nu}.$$

As before, it follows that $\nu * \mu = \mu * \nu$.

a) and b) are in Exercise 5.6. Assertion c) follows from

$$\begin{aligned}\int (\mathcal{S}\mu) * f\,\mathrm{d}\nu &= \left\langle \int \tau_t f\,(\mathcal{S}\mu)(\mathrm{d}t), \nu \right\rangle = \left\langle \int \tau_{-t} f\,\mu(\mathrm{d}t), \nu \right\rangle \\ &= \int \langle \tau_{-t} f, \nu\rangle\ \mu(\mathrm{d}t) = \int \langle f, \tau_t\nu\rangle\ \mu(\mathrm{d}t) = \langle f, \mu * \nu\rangle\end{aligned}$$

for all $\mu,\ \nu \in \mathrm{M}(\mathbb{R}^d),\ f \in \mathrm{UC_b}(\mathbb{R}^d)$. □

5.4 The Fourier Transform

The last part of Theorem 5.4 can be rephrased by saying that each bounded strongly continuous representation of the *semigroup* $\mathbb{S}$ gives rise to a representation of the *algebra* $\mathrm{M}(\mathbb{S})$, i.e., to a calculus. In order to interpret this calculus as a functional calculus, we need to represent the algebra $\mathrm{M}(\mathbb{S})$ as an algebra of functions. This is where the Fourier transform comes into play.

The **Fourier transform** of a measure $\mu \in \mathrm{M}(\mathbb{R}^d)$ is the function

$$(\mathcal{F}\mu)(s) := \widehat{\mu}(s) := \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}s\cdot t}\,\mu(\mathrm{d}t) \qquad (s \in \mathbb{R}^d). \tag{5.8}$$

Here, $s \cdot t$ denotes the usual inner product of $\mathbb{R}^d$. One easily checks that this definition of the Fourier transform is, under the natural embedding, consistent with the (probably known) notion of Fourier transform of an L^1-function.

Obviously, $\mathcal{F}\mu$ is a bounded function with

$$\|\mathcal{F}\mu\|_\infty \leq \|\mu\|_{\mathrm{M}}. \tag{5.9}$$

A little less obviously, $\mathcal{F}\mu$ turns out to be uniformly continuous. (This is easy if μ has compact support. The general case follows by approximation, by virtue of (5.9).)

Let us abbreviate

$$\mathrm{e}_t \in \mathrm{UC_b}(\mathbb{R}^d), \quad \mathrm{e}_t(s) := \mathrm{e}^{-\mathrm{i}s\cdot t}.$$

Then, *formally*,

$$\widehat{\mu} = \int_{\mathbb{R}^d} \mathrm{e}_t \, \mu(\mathrm{d}t). \tag{5.10}$$

Note that, however, the mapping $t \mapsto \mathrm{e}_t$ is not continuous with respect to the norm topology on $\mathrm{UC_b}(\mathbb{R}^d)$. It is just continuous with respect to the "topology of uniform convergence on compacts", also called "compact convergence". This means that if $t_n \to t$ in $\mathbb{R}^d$ then $\mathrm{e}_{t_n} \to \mathrm{e}_t$ uniformly on each compact subset of $\mathbb{R}^d$. The next results can be seen as an alternative interpretation of the formula 5.10.

Theorem 5.5. *Let X be a Banach space and let $\Phi : \mathrm{UC_b}(\mathbb{R}^d) \to \mathcal{L}(X)$ be a functional calculus with the following continuity property: If $(f_n)_n$ is a bounded sequence in $\mathrm{UC_b}(\mathbb{R}^d)$ converging to a function $f \in \mathrm{UC_b}(\mathbb{R}^d)$ uniformly on compacts, then $\Phi(f_n) \to \Phi(f)$ strongly on X. Define*

$$T_t := \Phi(\mathrm{e}_t) \qquad (t \in \mathbb{R}^d).$$

Then $(T_t)_{t\in\mathbb{R}^d}$ is a bounded and strongly continuous representation and

$$T_\mu = \Phi(\widehat{\mu}) \qquad \textit{for all } \mu \in \mathrm{M}(\mathbb{R}^d).$$

Proof. By virtue of the closed graph theorem and the continuity property of Φ it follows that Φ is bounded. Hence T is bounded, and it is strongly continuous since $t \mapsto \mathrm{e}_t$ is continuous with respect to compact convergence. Since Φ is multiplicative, T is a strongly continuous group.

Now fix $f \in \mathrm{C_0}(\mathbb{R}^d)$. Then the mapping $(t \mapsto \mathrm{e}_t f) : \mathbb{R}^d \to \mathrm{C_0}(\mathbb{R}^d)$ is continuous. Integrating against $\mu \in \mathrm{M}(\mathbb{R}^d)$ yields

$$\Big(\int_{\mathbb{R}^d} \mathrm{e}_t f \, \mu(\mathrm{d}t)\Big)(s) = \int_{\mathbb{R}^d} \mathrm{e}_t(s) f(s) \, \mu(\mathrm{d}t) = \widehat{\mu}(s) f(s) \quad \text{for all } s \in \mathbb{R}^d.$$

Hence, identity (A.2) yields, with $x \in X$,

$$\begin{aligned} T_\mu \Phi(f)x &= \int_{\mathbb{R}^d} T_t \Phi(f)x \, \mu(\mathrm{d}t) = \int_{\mathbb{R}^d} \Phi(\mathrm{e}_t f)x \, \mu(\mathrm{d}t) = \Phi\Big(\int_{\mathbb{R}^d} \mathrm{e}_t f \, \mu(\mathrm{d}t)\Big)x \\ &= \Phi(\widehat{\mu} f)x = \Phi(\widehat{\mu})\Phi(f)x. \end{aligned}$$

Finally, specialize $f = f_n$ for some sequence $(f_n)_n$ in $\mathrm{C_0}(\mathbb{R}^d)$ that converges to $\mathbf{1}$ uniformly on compacts, and apply again the continuity property of Φ. $\square$

Examples 5.6. Theorem 5.5 can be applied to multiplication operator calculi.

1) For $X = \mathrm{C}_0(\mathbb{R}^d)$ or $X = \mathrm{L}^p(\mathbb{R}^d)$, $1 \le p < \infty$, consider

$$\Phi : \mathrm{UC_b}(\mathbb{R}^d) \to \mathcal{L}(X), \qquad \Phi(f) = M_f,\ M_f g = fg.$$

Then Φ satisfies the assumptions of Theorem 5.5. Hence, $T_\mu = M_{\widehat{\mu}}$ in these cases, where $T_t := M_{\mathrm{e}_t}$ for each $t \in \mathbb{R}^d$.

2) Likewise, consider

$$\Phi : \mathrm{UC_b}(\mathbb{R}^d) \to \mathcal{L}(\mathrm{M}(\mathbb{R}^d)), \qquad \Phi(f)\mu := f\,\mu.$$

Then Φ also satisfies the hypotheses of Theorem 5.5. Indeed, if $f_n \to 0$ uniformly on compacts, then $\|f_n\mu\|_{\mathrm{M}} \to 0$ whenever μ has compact support. By density, this extends to arbitrary $\mu \in \mathrm{M}(\mathbb{R}^d)$ if $\sup_n \|f_n\|_\infty < \infty$.

Here is the central theorem about the Fourier transform.

Theorem 5.7 (Fourier transform). *The Fourier transform $\mathcal{F} : \mathrm{M}(\mathbb{R}^d) \to \mathrm{UC_b}(\mathbb{R}^d)$ is an injective homomorphism of unital algebras. In particular,*

$$\mathcal{F}(\mu * \nu) = (\mathcal{F}\mu) \cdot (\mathcal{F}\nu) \qquad \textit{for } \mu,\ \nu \in \mathrm{M}(\mathbb{R}^d).$$

Moreover, the following assertions hold:

a) *For $t \in \mathbb{R}$ and $\mu \in \mathrm{M}(\mathbb{R}^d)$ the diagrams*

$$\begin{tikzcd} \mathrm{M}(\mathbb{R}^d) \arrow[r, "\mathrm{e}^{-\mathrm{i}t\cdot}\,\cdot"] \arrow[d, "\mathcal{F}"'] & \mathrm{M}(\mathbb{R}^d) \arrow[d, "\mathcal{F}"] \\ \mathrm{UC_b}(\mathbb{R}^d) \arrow[r, "\tau_{-t}"'] & \mathrm{UC_b}(\mathbb{R}^d) \end{tikzcd} \qquad \textit{and} \qquad \begin{tikzcd} \mathrm{M}(\mathbb{R}^d) \arrow[r, "\widehat{\mu}\,\cdot"] \arrow[d, "\mathcal{F}"'] & \mathrm{M}(\mathbb{R}^d) \arrow[d, "\mathcal{F}"] \\ \mathrm{UC_b}(\mathbb{R}^d) \arrow[r, "\mathcal{S}\mu\,*\,\cdot"'] & \mathrm{UC_b}(\mathbb{R}^d) \end{tikzcd}$$

are commutative. The latter encodes the identity

$$\mathcal{F}(\widehat{\mu} \cdot \nu) = (\mathcal{S}\mu) * \widehat{\nu} \tag{5.11}$$

for all $\mu,\ \nu \in \mathrm{M}(\mathbb{R}^d)$. In particular, one has $\displaystyle\int_{\mathbb{R}^d} \widehat{\mu}\,\mathrm{d}\nu = \int_{\mathbb{R}^d} \widehat{\nu}\,\mathrm{d}\mu$.

b) *The Fourier transform maps $\mathrm{L}^1(\mathbb{R}^d)$ into $\mathrm{C}_0(\mathbb{R}^d)$ and the space $\mathcal{F}(\mathrm{L}^1) = \{\widehat{\varphi} \mid \varphi \in \mathrm{L}^1(\mathbb{R}^d)\}$ is dense in $\mathrm{C}_0(\mathbb{R}^d)$.*

Proof. Clearly, $\mathcal{F}\delta_0 = \mathbf{1}$. The injectivity is postponed until the very end of this proof. For the multiplicativity we apply Theorem 5.5 to the multiplication operator functional calculus $\Phi : \mathrm{UC_b}(\mathbb{R}^d) \to \mathcal{L}(\mathrm{C}_0(\mathbb{R}^d))$ described in Example 5.6. Then

$$\Phi(\widehat{\mu}\,\widehat{\nu}) = \Phi(\widehat{\mu})\,\Phi(\widehat{\nu}) = T_\mu T_\nu = T_{\mu*\nu} = \Phi(\mathcal{F}(\mu*\nu)).$$

Since Φ is obviously injective, the claim follows. (A more direct proof uses the formula (5.7).)

a) Consider the multiplication operator functional calculus

$$\Phi : \mathrm{UC_b}(\mathbb{R}^d) \to \mathcal{L}(\mathrm{M}(\mathbb{R}^d)), \qquad \Phi(f)\mu := f\,\mu$$

as discussed in Example 5.6.2) and define $T_t := \Phi(\mathrm{e}_t)$ for $t \in \mathbb{R}^d$. Then

$$\mathcal{F}(\mathrm{e}_t\mu)(s) = \int_{\mathbb{R}^d} \mathrm{e}_s\mathrm{e}_t \mathrm{d}\mu = \int_{\mathbb{R}^d} \mathrm{e}_{s+t}\,\mathrm{d}\mu = \widehat{\mu}(s+t) = \big(\tau_{-t}\widehat{\mu}\big)(s)$$

for all $\mu \in \mathrm{M}(\mathbb{R}^d)$ and all $s,\, t \in \mathbb{R}^d$. But this just means that $\mathcal{F}T_t = \tau_{-t}\mathcal{F}$, i.e., the commutativity of the first diagram.

Fix $\mu \in \mathrm{M}(\mathbb{R}^d)$ and abbreviate $S_t := \tau_{-t}$ (the left shift representation). Then clearly $S_\mu = \tau_{\mathcal{S}\mu}$ is convolution with $\mathcal{S}\mu$ and hence, by Lemma 5.3

$$\mathcal{F}T_\mu = S_\mu\mathcal{F} = \tau_{\mathcal{S}\mu}\mathcal{F}.$$

Since by Theorem 5.5, T_μ is nothing but multiplication with $\widehat{\mu}$, the commutativity of the second diagram and hence the formula (5.11) are established. The last claim follows from evaluating at 0 in (5.11).

b) The first assertion is the classical Riemann–Lebesgue–Lemma. It suffices to show that $\mathcal{F}\varphi \in \mathrm{C}_0(\mathbb{R}^d)$ for φ from a dense subset. Since

$$\mathcal{F}(\varphi_1 \otimes \ldots \otimes \varphi_d) = (\mathcal{F}\varphi_1) \otimes \ldots \otimes (\mathcal{F}\varphi_d)$$

for $\varphi_1, \ldots, \varphi_d \in \mathrm{L}^1(\mathbb{R})$, it suffices to look at the case $d = 1$. There one can take $\varphi \in \mathrm{C}^1_\mathrm{c}(\mathbb{R})$ and perform integration by parts to see that the claim is true, cf. [2, Thm. 9.20].

For the second assertion, it suffices for the same reasons as before to consider the case $d = 1$. Obviously, $\mathcal{F}(\mathrm{L}^1)$ is a conjugation-invariant subalgebra of $\mathrm{C}_0(\mathbb{R})$. Since with $\varphi(s) := \mathrm{e}^{-s}\mathbf{1}_{\mathbb{R}_+}(s)$ one has

$$\widehat{\varphi}(t) = \frac{1}{1+\mathrm{i}t} \qquad (t \in \mathbb{R})$$

and this is nowhere zero and separates the points of $\mathbb{R}$, the claim follows from the Stone–Weierstrass theorem.

Finally, the injectivity of $\mathcal{F}$: If $\widehat{\mu} = 0$ then by a)

$$0 = \int_{\mathbb{R}^d} \widehat{\mu}\,\mathrm{d}\nu = \int_{\mathbb{R}^d} \widehat{\nu}\,\mathrm{d}\mu$$

for all $\nu \in \mathrm{M}(\mathbb{R}^d)$. In particular, μ vanishes on $\mathcal{F}(\mathrm{L}^1)$. Since, by b), this is dense in $\mathrm{C}_0(\mathbb{R}^d)$, $\mu = 0$. □

Exercises

5.1. a) Let $E \subseteq \mathbb{R}^d$ be closed and $T : E \to \mathcal{L}(X)$ be a locally bounded mapping. Show that the space

$$X_c := \{x \in X \mid \text{the map } (t \mapsto T_t x) \text{ is continuous}\}$$

is a closed subspace of X. (The space X_c is called the **subspace of strong continuity** of T.)

b) Show that $\mathrm{UC_b}(\mathbb{R}^d)$ is the subspace of strong continuity for the shift group τ on $\mathrm{C_b}(\mathbb{R}^d)$.

c) Show that on $\mathrm{C_c}(\mathbb{R}^d)$ the shift group is strongly continuous with respect to each norm $\|\cdot\|_{\mathrm{L}^p(\mathbb{R}^d)}$. Conclude that the shift group is strongly continuous on $\mathrm{L}^p(\mathbb{R}^d)$ for each $1 \le p < \infty$.

5.2. Let $\mu \in \mathrm{M}(\mathbb{R}^d)$, $t \in \mathbb{R}^d$ and $g \in \mathcal{M}_\mathrm{b}(\mathbb{R}^d)$. Show that the set

$$\{f \in \mathcal{M}_\mathrm{b}(\mathbb{R}^d) \mid (5.2) \text{ holds}\}$$

is closed under bp-convergence and conclude that (5.2) holds even for all $f \in \mathcal{M}_\mathrm{b}(\mathbb{R}^d)$. Do the same for the statements (5.3) and (5.4).

5.3 (Measures with Compact Support). Let $E \subseteq \mathbb{R}^d$ be a closed subset and

$$\mathrm{M_c}(E) := \{\mu \in \mathrm{M}(E) \mid \exists\, K \subseteq E \text{ compact}, \ |\mu|(K^c) = 0\}$$

be the measures with **compact support**. Show that $\mathrm{M}_+(E) \cap \mathrm{M_c}(E)$ is norm dense in $\mathrm{M}_+(E)$. Conclude that $\mathrm{M_c}(E)$ is norm dense in $\mathrm{M}(E)$.

5.4 (Integrals of Vector-Valued Functions). Let $E \subseteq \mathbb{R}^d$ a closed subset, $f \in \mathrm{C_b}(E; X)$ and $\mu \in \mathrm{M}(E)$. Consider the weakly defined integral

$$\langle f, \mu\rangle := \int_E f \,\mathrm{d}\mu \ \in \ X''$$

With this exercise you should convince yourself (in one or the other way) that $\langle f, \mu\rangle \in X$, at least in the cases $E = \mathbb{R}^d$ and $E = \mathbb{R}^d_+$.

a) Show that it suffices to consider the case $\mu \ge 0$.

b) Show that a bounded and continuous function $f : E \to X$ is Bochner integrable with respect to each finite positive measure $\mu \in \mathrm{M}_+(E)$. Conclude that $\langle f, \mu\rangle \in X$. [This settles the problem for people who feel comfortable with Appendix A.6.]

c) Show that $\|\langle f, \mu\rangle\| \le \|f\|_\infty \|\mu\|_\mathrm{M}$. Conclude with the help of Exercise 5.3 that in order to show that $\langle f, \mu\rangle \in X$ it suffices to consider $\mu \in \mathrm{M}_+(E) \cap \mathrm{M_c}(E)$.

d) Let $Q = \prod_{j=1}^{d}[a_j, b_j]$ be a d-dimensional rectangle. Show that a function $f \in \mathrm{C}(Q; X)$ is the uniform limit of rectangle-step functions. Conclude (with the help of c)) that $\langle f, \mu \rangle \in X$ whenever E is the union of an increasing sequence of rectangles, and in particular for $E = \mathbb{R}^d$ and $E = \mathbb{R}^d_+$.

5.5 (Compatible Representations). A **coupling** of two Banach spaces X, Y is any injective and closed operator $J : X \supseteq \mathrm{dom}(J) \to Y$. By calling the pair (X, Y) a **Banach couple** it is meant that there is a (tacit) coupling. With the help of a coupling J, certain elements of X (those contained in $\mathrm{dom}(J)$) are identified with certain elements of Y (those of $\mathrm{ran}(J)$). In this sense, $\mathrm{dom}(J)$ can be regarded as "$X \cap Y$", and this is a Banach space with respect to the graph norm of J. (Note that one obtains an isometrically isomorphic space if one interchanges the roles of X and Y and considers the coupling J^{-1} instead.

Given a coupling J of X and Y, a pair of operators $(S^X, S^Y) \in \mathcal{L}(X) \times \mathcal{L}(Y)$ is called **compatible**, if $S^Y J \subseteq J S^X$. Informally speaking, this just means that S^X and S^Y agree on $X \cap Y$.

Now let $E \subseteq \mathbb{R}^d$ be closed and suppose that one has bounded strongly continuous mappings $T^X : E \to \mathcal{L}(X)$ and $T^Y : E \to \mathcal{L}(Y)$ which consists for each $t \in E$ of compatible operators T^X_t and T^Y_t. Show that for each $\mu \in \mathrm{M}(E)$ the operators T^X_μ and T^Y_μ are compatible as well. Conclude that the notation "$\mu * f$" is unequivocal, no matter whether $f \in \mathrm{UC_b}(\mathbb{R}^d)$ or $f \in \mathrm{L}^p(\mathbb{R}^d)$, $1 \le p < \infty$.

5.6. Let $\mu \in \mathrm{M}(\mathbb{R}^d)$, $g \in \mathrm{L}^1(\mathbb{R}^d)$ and $t \in \mathbb{R}^d$. Show that

$$\delta_t * \mu = \tau_t \mu \quad \text{and} \quad \mu * (g\lambda) = (\mu * g)\lambda.$$

5.7 (Pointwise Representation of Convolutions). Let $\varphi \in \mathrm{L}^1(\mathbb{R}^d)$. We write $\varphi * f$ as an abbreviation for $\varphi\lambda * f$, whenever the latter is meaningful.

a) Show that for $f \in \mathrm{UC_b}(\mathbb{R}^d)$ one has

$$(\varphi * f)(s) = \int_{\mathbb{R}^d} \varphi(t) f(s - t) \,\mathrm{d}t \tag{5.12}$$

for all $s \in \mathbb{R}^d$.

b) Let $f \in \mathrm{L}^p(\mathbb{R}^d)$, $1 \le p < \infty$. Show that for all Borel sets $B \subseteq \mathbb{R}^d$ of finite measure one has

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi(t) f(s - t) \mathbf{1}_B(s)| \,\mathrm{d}t \,\mathrm{d}s < \infty.$$

Conclude that for almost all $s \in \mathbb{R}^d$ the function $t \mapsto \varphi(t) f(s - t)$ is in $\mathrm{L}^1(\mathbb{R}^d)$. Finally, show that (5.12) holds for almost every $s \in \mathbb{R}^d$.

5.8. Let $\Omega = (\Omega, \Sigma, \nu)$ be a measure space and $a : \Omega \to \mathbb{R}^d$ a measurable function. Fix $p \in [1, \infty)$ and abbreviate $X := \mathrm{L}^p(\Omega)$. For $t \in \mathbb{R}^d$ let $T_t \in \mathcal{L}(X)$ be defined by

$$T_t f := \mathrm{e}^{-\mathrm{i}t \cdot a(\cdot)} f \qquad (f \in X).$$

Show that $(T_t)_{t\in\mathbb{R}^d}$ is a bounded, strongly continuous group and

$$T_\mu f = (\widehat{\mu} \circ a)\, f \qquad (f \in X,\ \mu \in \mathrm{M}(\mathbb{R}^d)).$$

5.9. In this exercise we consider the case of the discrete subgroup $\mathbb{S} := \mathbb{Z} \subseteq \mathbb{R}$. Obviously, $\mathrm{M}(\mathbb{Z})$ consists precisely of the measures

$$\mu = \sum_{n\in\mathbb{Z}} \alpha_n \boldsymbol{\delta}_n, \qquad \alpha \in \ell^1(\mathbb{Z})$$

and the correspondence $\mu \leftrightarrow (\alpha_n)_n$ is an isometric isomorphism $\mathrm{M}(\mathbb{Z}) \cong \ell^1(\mathbb{Z})$. Show that the Fourier transform maps $\mathrm{M}(\mathbb{Z})$ injectively onto a dense subspace of the space $\mathrm{C}_{2\pi}(\mathbb{R})$ of all 2π-periodic functions on $\mathbb{R}$.

5.10 (Reflection and Conjugation). Let $\mu,\ \nu \in \mathrm{M}(\mathbb{R}^d)$ and $f \in X \in \{\mathrm{UC_b}(\mathbb{R}^d), \mathrm{L}^p(\mathbb{R}^d)\}$ where $1 \le p < \infty$. Show that

$$\mathcal{S}(\mu * \nu) = (\mathcal{S}\nu) * (\mathcal{S}\mu), \quad \mathcal{S}(\mu * f) = (\mathcal{S}\mu) * (\mathcal{S}f), \quad \overline{\mu * \nu} = \overline{\mu} * \overline{\nu}.$$

One has

$$\mathcal{S}\mathcal{F}\mu = \mathcal{F}\mathcal{S}\mu = \overline{\mathcal{F}\overline{\mu}}.$$

Conclude that $\mathrm{M}(\mathbb{R}^d)$ is a Banach $*$-algebra with respect to the involution

$$\mathrm{M}(\mathbb{R}^d) \to \mathrm{M}(\mathbb{R}^d), \qquad \mu \mapsto \mu^* := \mathcal{S}\overline{\mu}$$

and that $\mathcal{F} : \mathrm{M}(\mathbb{R}^d) \to \mathrm{UC_b}(\mathbb{R}^d)$ is a homomorphism of Banach $*$-algebras.

References

[1] K.-J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*. Vol. 194. Graduate Texts in Mathematics. Berlin: Springer-Verlag, 2000, pp. xxi+586.

[2] M. Haase. *Functional analysis*. Vol. 156. Graduate Studies in Mathematics. An elementary introduction. American Mathematical Society, Providence, RI, 2014, pp. xviii+372.

Chapter 6
Integral Transform Functional Calculi

In this chapter we continue our investigations from the previous one and encounter functional calculi associated with various semigroup representations.

6.1 The Fourier–Stieltjes Calculus

Recall from the previous chapter that the Fourier transform

$$\mathcal{F} : \mathrm{M}(\mathbb{R}^d) \to \mathrm{UC_b}(\mathbb{R}^d)$$

is a contractive and injective unital algebra homomorphism. Hence, it is an isomorphism onto its image

$$\mathrm{FS}(\mathbb{R}^d) := \mathcal{F}(\mathrm{M}(\mathbb{R}^d)) = \{\widehat{\mu} \mid \mu \in \mathrm{M}(\mathbb{R}^d)\}.$$

This algebra, which is called the **Fourier–Stieltjes** algebra[1] of $\mathbb{R}^d$, is endowed with the norm

$$\|\widehat{\mu}\|_{\mathrm{FS}} := \|\mu\|_{\mathrm{M}} \qquad (\mu \in \mathrm{M}(\mathbb{R}^d)),$$

which turns it into a Banach algebra and the Fourier transform $\mathcal{F} : \mathrm{M}(\mathbb{R}^d) \to \mathrm{FS}(\mathbb{R}^d)$ into an isometric isomorphism. (The **Fourier algebra** of $\mathbb{R}^d$ is the closed ideal(!) $\mathrm{A}(\mathbb{R}^d) := \mathcal{F}(\mathrm{L}^1(\mathbb{R}^d))$ of Fourier transforms of L^1-functions.)

Let $\mathbb{S} \subseteq \mathbb{R}^d$ be a closed subsemigroup. Then $\mathrm{M}(\mathbb{S})$ is a Banach subalgebra of $\mathrm{M}(\mathbb{R}^d)$ and

$$\mathrm{FS}_{\mathbb{S}}(\mathbb{R}^d) := \mathcal{F}(\mathrm{M}(\mathbb{S})),$$

[1] The Fourier transform on the space of measures is sometimes called the *Fourier–Stieltjes transform*, hence the name "Fourier–Stieltjes algebra" for its image. In books on Banach algebras one often finds the symbol "$\mathrm{B}(\mathbb{R}^d)$" for it.

called its associated **Fourier–Stieltjes algebra**, is a Banach subalgebra of $\mathrm{FS}(\mathbb{R}^d)$.

Let $T : \mathbb{S} \to \mathcal{L}(X)$ be a strongly continuous and bounded representation on a Banach space X with associated algebra representation

$$\mathrm{M}(\mathbb{S}) \to \mathcal{L}(X), \quad \mu \mapsto T_\mu = \int_{\mathbb{S}} T_t \, \mu(\mathrm{d}t). \tag{6.1}$$

Since the Fourier transform $\mathcal{F} : \mathrm{M}(\mathbb{S}) \to \mathrm{FS}_{\mathbb{S}}(\mathbb{R}^d)$ is an isomorphism, we can compose its inverse with the representation (6.1). In this way a functional calculus

$$\Psi_T : \mathrm{FS}_{\mathbb{S}}(\mathbb{R}^d) \to \mathcal{L}(X), \qquad \Psi_T(\widehat{\mu}) := T_\mu,$$

is obtained, which we call the **Fourier–Stieltjes calculus** for T. Note that by the definition of the norm on $\mathrm{FS}(\mathbb{R}^d)$ and by (5.6) we have

$$\|\Psi_T(f)\| \le M_T \|f\|_{\mathrm{FS}} \qquad (f \in \mathrm{FS}_{\mathbb{S}}(\mathbb{R}^d)), \tag{6.2}$$

where, as always, $M_T = \sup_{t\in\mathbb{S}} \|T_t\|$.

Example 6.1. Let $\Omega = (\Omega, \Sigma, \nu)$ be a measure space and $a : \Omega \to \mathbb{R}^d$ a measurable function. Fix $p \in [1, \infty)$ and abbreviate $X := \mathrm{L}^p(\Omega)$. For $t \in \mathbb{R}^d$ let $T_t \in \mathcal{L}(X)$ be defined by

$$T_t x := \mathrm{e}^{-\mathrm{i}t\cdot a(\cdot)} x \qquad (x \in X).$$

Then $(T_t)_{t\in\mathbb{R}^d}$ is a bounded and strongly continuous group and, by Exercise 5.8,

$$T_\mu x = (\widehat{\mu} \circ a)\, x \qquad (x \in X)$$

for all $\mu \in \mathrm{M}(\mathbb{R}^d)$. This means that the Fourier-Stieltjes calculus for T is nothing but the restriction of the usual multiplication operator functional calculus to the algebra $\mathrm{FS}(\mathbb{R}^d)$.

Example 6.2. Let $T = \tau$ be the regular (right shift) representation of $\mathbb{R}^d$ on $X = \mathrm{L}^1(\mathbb{R}^d)$. Then, for $f = \widehat{\mu} \in \mathrm{FS}(\mathbb{R}^d)$ one has

$$\Psi_\tau(f)x = \mu * x = \mathcal{F}^{-1}(\widehat{\mu} \cdot \widehat{x}) = \mathcal{F}^{-1}(f \cdot \widehat{x}) \qquad (x \in \mathrm{L}^1(\mathbb{R}^d)).$$

That is, the operator $\Psi_\tau(f)$ is the so-called *Fourier multiplier operator* with the *symbol* f: first take the Fourier transform, then multiply with f, finally transform back.

In the following we shall examine the Fourier–Stieltjes calculus for the special cases $\mathbb{S} = \mathbb{Z}$, $\mathbb{S} = \mathbb{Z}_+$, and $\mathbb{S} = \mathbb{R}_+$.

Doubly Power-Bounded Operators

An operator $T \in \mathcal{L}(X)$ is called **doubly power-bounded** if T is invertible and $M_T := \sup_{n\in\mathbb{Z}} \|T^n\| < \infty$. Such operators correspond in a one-to-one fashion to bounded $\mathbb{Z}$-representations on X (cf. Example 5.1). The spectrum of a doubly power-bounded operator T is contained in the torus $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

By Exercise 5.9, $\mathrm{M}(\mathbb{Z}) \cong \ell^1(\mathbb{Z})$ and $\mathrm{FS}_{\mathbb{Z}}(\mathbb{R})$ consists of all functions

$$f = \sum_{n\in\mathbb{Z}} \alpha_n \mathrm{e}^{-in\mathbf{t}} \qquad (\alpha = (\alpha_n)_n \in \ell^1). \tag{6.3}$$

These functions form a subalgebra of $\mathrm{C}_{2\pi}(\mathbb{R})$, the algebra of **$2\pi$-periodic functions** on $\mathbb{R}$.

By the Fourier–Stieltjes calculus, the function f as in (6.3) is mapped to

$$\Psi_T(f) = \sum_{n\in\mathbb{Z}} \alpha_n T^n.$$

Hence, this calculus is basically the same as a Laurent series calculus

$$\Phi_T : \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{z}^n \quad \longmapsto \quad \sum_{n\in\mathbb{Z}} \alpha_n T^n \tag{6.4}$$

where the object on the left-hand side is considered as a function on $\mathbb{T}$. We prefer this latter version of the Fourier–Stieltjes calculus because it works with functions defined on the spectrum of T and has T as its generator. The algebra

$$\mathrm{A}(\mathbb{T}) := \Bigl\{ \sum_{n\in\mathbb{Z}} \alpha_n \mathbf{z}^n \mid \alpha \in \ell^1(\mathbb{Z}) \Bigr\} \subseteq \mathrm{C}(\mathbb{T})$$

is called the **Wiener algebra** and the calculus (6.4) is called the **Wiener calculus**.

In order to make precise our informal phrase "basically the same" from above, we use the following notion from abstract functional calculus theory.

Definition 6.3. An **isomorphism** of two proto-calculi $\Phi : \mathcal{F} \to \mathcal{C}(X)$ and $\Psi : \mathcal{E} \to \mathcal{C}(X)$ on a Banach space X is an isomorphism of unital algebras $\eta : \mathcal{F} \to \mathcal{E}$ such that $\Phi = \Psi \circ \eta$. If there is an isomorphism, the two calculi are called **isomorphic** or **equivalent**.

Let us come back to the situation from above. The mapping $\mathrm{e}^{-\mathbf{i}\mathbf{t}} : \mathbb{R} \to \mathbb{T}$ induces an isomorphism (of unital Banach algebras)

$$\mathrm{C}(\mathbb{T}) \to \mathrm{C}_{2\pi}(\mathbb{R}), \qquad f \mapsto f(\mathrm{e}^{-\mathbf{i}\mathbf{t}}).$$

This restricts to an isomorphism $\eta : \mathrm{A}(\mathbb{T}) \to \mathrm{FS}_{\mathbb{Z}}(\mathbb{R})$ by virtue of which the Fourier–Stieltjes calculus is isomorphic to the Wiener calculus.

Power-Bounded Operators

Power-bounded operators $T \in \mathcal{L}(X)$ correspond in a one-to-one fashion to $\mathbb{Z}_+$-representations on X. By Exercise 5.9, $\mathrm{M}(\mathbb{Z}_+) \cong \ell^1(\mathbb{Z}_+)$ and $\mathrm{FS}_{\mathbb{Z}_+}(\mathbb{R})$ consists of all functions

$$f = \sum_{n=0}^{\infty} \alpha_n \mathrm{e}^{-\mathrm{i}n\mathbf{t}} \qquad (\alpha = (\alpha_n)_n \in \ell^1). \tag{6.5}$$

These functions form a subalgebra of $\mathrm{C}_{2\boldsymbol{\pi}}(\mathbb{R})$. Under the Fourier–Stieltjes calculus, the function f as in (6.5) is mapped to

$$\Psi_T(f) = \sum_{n=0}^{\infty} \alpha_n T^n.$$

This calculus is isomorphic to the power-series calculus

$$\Phi_T : \mathrm{A}^1_+(\mathbb{D}) \to \mathcal{L}(X), \qquad \Phi_T\Big(\sum_{n=0}^{\infty} \alpha_n \mathbf{z}^n\Big) = \sum_{n=0}^{\infty} \alpha_n T^n \tag{6.6}$$

introduced in Chapter 1. The isomorphism of the two calculi is again given by the algebra homomorphism $f \mapsto f(\mathrm{e}^{-\mathrm{i}\mathbf{t}})$. (Observe that for $\alpha \in \ell^1$ the function $f = \sum_{n=0}^{\infty} \alpha_n \mathbf{z}^n$ can be viewed as a function on $\overline{\mathbb{D}}$, or on $\mathbb{D}$ or on $\mathbb{T}$ or on $(0, 1)$ and in either interpretation α is determined by f.)

6.2 Bounded C_0-Semigroups and the Hille–Phillips Calculus

We now turn to the case $\mathbb{S} = \mathbb{R}_+$. A strongly continuous representation of $\mathbb{R}_+$ on a Banach space is often called a **C_0-semigroup**[2]. Operator semigroup theory is a large field and a thorough introduction would require an own course. We concentrate on the aspects connected to functional calculus theory. If you want to study semigroup theory proper, read [2] or [1].

Let $T = (T_t)_{t \geq 0}$ be a bounded C_0-semigroup on a Banach space X. Then we have the Fourier–Stieltjes calculus

$$\Psi_T : \mathrm{FS}_{\mathbb{R}_+}(\mathbb{R}) \to \mathcal{L}(X), \qquad \Psi_T(\widehat{\mu}) = \int_{\mathbb{R}_+} T_t \, \mu(\mathrm{d}t).$$

[2] The name goes back to the important monograph [4, Chap. 10.1] of Hille and Phillips, where different continuity notions for semigroup representations are considered.

However, as in the case of power-bounded operators, we rather prefer working with an isomorphic calculus, which we shall now describe.

The **Laplace transform** (also called **Laplace–Stieltjes transform**) of a measure $\mu \in \mathrm{M}(\mathbb{R}_+)$ is the function

$$\mathcal{L}\mu : \overline{\mathbb{C}_+} \to \mathbb{C}, \qquad (\mathcal{L}\mu)(z) = \int_{\mathbb{R}_+} \mathrm{e}^{-zt}\, \mu(\mathrm{d}t).$$

Here, $\mathbb{C}_+ := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ is the open right half-plane. By some standard arguments,

$$\|\mathcal{L}\mu\|_\infty \le \|\mu\|_{\mathrm{M}} \quad \text{and} \quad \mathcal{L}\mu \in \mathrm{UC_b}(\overline{\mathbb{C}_+}) \cap \mathrm{Hol}(\mathbb{C}_+),$$

and it is easy to see that the **Laplace transform**

$$\mathcal{L} : \mathrm{M}(\mathbb{R}_+) \to \mathrm{UC_b}(\overline{\mathbb{C}_+}), \qquad \mu \mapsto \mathcal{L}\mu$$

is a homomorphism of unital algebras. Moreover,

$$(\mathcal{L}\mu)(\mathrm{i}s) = (\mathcal{F}\mu)(s) \quad \text{for all } s \in \mathbb{R} \tag{6.7}$$

and since the Fourier transform is injective, so is the Laplace transform.

Let us call its range

$$\mathrm{LS}(\mathbb{C}_+) := \{\mathcal{L}\mu \mid \mu \in \mathrm{M}(\mathbb{R}_+)\}$$

the **Laplace-Stieltjes algebra** and endow it with the norm

$$\|\mathcal{L}\mu\|_{\mathrm{LS}} := \|\mu\|_{\mathrm{M}} \qquad (\mu \in \mathrm{M}(\mathbb{R}_+)).$$

Then the mapping

$$\mathrm{LS}(\mathbb{C}_+) \to \mathrm{FS}_{\mathbb{R}_+}(\mathbb{R}), \qquad f \mapsto f(\mathbf{i}\mathbf{s})$$

is an isometric isomorphism of unital Banach algebras. Given a C_0-semigroup $(T_t)_{t\ge 0}$ one can compose its Fourier–Stieltjes calculus Ψ_T with the inverse of this isomorphism to obtain the calculus

$$\Phi_T : \mathrm{LS}(\mathbb{C}_+) \to \mathcal{L}(X), \qquad \Phi_T(\mathcal{L}\mu) := \int_{\mathbb{R}_+} T_t\, \mu(\mathrm{d}t).$$

This calculus is called the **Hille–Phillips calculus**[3] for T. It satisfies the norm estimate

$$\|\Phi_T(f)\| \le M_T \|\mu\|_{\mathrm{M}} \qquad (f = \mathcal{L}\mu,\ \mu \in \mathrm{M}(\mathbb{R}_+)). \tag{6.8}$$

[3] One would expect the name "Laplace–Stieltjes calculus" but we prefer sticking to the common nomenclature.

Similar to the discrete case, we note that an element $f = \mathcal{L}\mu \in \mathrm{LS}(\mathbb{C}_+)$ can be interpreted as a function on $\mathbb{C}_+$, on $\overline{\mathbb{C}_+}$, on $\mathrm{i}\mathbb{R}$, or on $(0, \infty)$ and in either interpretation μ is determined by f.

The function $\mathbf{z}$ is unbounded on the right half-plane and hence it is not contained in the domain of the Hille–Phillips calculus. Nevertheless, this functional calculus has a generator, as we shall show next.

The Generator of a Bounded C_0-Semigroup

Suppose as before that $T = (T_t)_{t\ge 0}$ is a bounded C_0-semigroup on a Banach space X. For $\operatorname{Re}\lambda$, $\operatorname{Re} z > 0$ one has

$$\frac{1}{\lambda + z} = \int_0^\infty \mathrm{e}^{-\lambda t}\mathrm{e}^{-zt}\,\mathrm{d}t.$$

In other words, the function $\frac{1}{\lambda+\mathbf{z}}$ (defined on $\mathbb{C}_+$) is the Laplace transform of the L^1-function $\mathrm{e}^{-\lambda\mathbf{t}}\mathbf{1}_{\mathbb{R}_+}$. As such, $(\lambda + \mathbf{z})^{-1} \in \mathrm{LS}(\mathbb{C}_+)$ for each $\lambda \in \mathbb{C}_+$.

Theorem 6.4. *Let $T = (T_t)_{t\ge 0}$ be a bounded C_0-semigroup on a Banach space X with associated Hille–Phillips calculus Φ_T. Then there is a (uniquely determined) closed operator A such that*

$$(\lambda + A)^{-1} = \Phi_T\Big(\frac{1}{\lambda + \mathbf{z}}\Big)$$

for one/all $\lambda \in \mathbb{C}_+$. The operator A has the following properties:

a) $[\operatorname{Re}\mathbf{z} < 0\,] \subseteq \rho(A)$ *and* $R(-\lambda, A) = -\Phi_T((\lambda + \mathbf{z})^{-1})$ *for all* $\operatorname{Re}\lambda > 0$.

b) $\operatorname{dom}(A)$ *is dense in* X.

c) $\lambda(\lambda + A)^{-1} \to \mathrm{I}$ *strongly as* $0 < \lambda \nearrow \infty$.

d) *For all* $w \in \mathbb{C}$, $t > 0$ *and* $x \in X$*:*

$$\int_0^t \mathrm{e}^{ws} T_s x\,\mathrm{d}s \;\in\; \operatorname{dom}(A) \quad \textit{and} \quad (A - w)\int_0^t \mathrm{e}^{sw} T_s x\,\mathrm{d}s = x - \mathrm{e}^{tw} T_t x.$$

e) $\Phi_T(f)A \subseteq A\Phi_T(f)$ *for all* $f \in \mathrm{LS}(\mathbb{C}_+)$.

Proof. The operator family

$$R(w) := \Phi_T\Big(\frac{1}{w - \mathbf{z}}\Big) = -\Phi_T\Big(\frac{1}{(-w) + \mathbf{z}}\Big), \qquad \operatorname{Re} w < 0,$$

is a pseudo-resolvent. As such, there is a uniquely determined closed linear relation A on X such that $R(w) = (w - A)^{-1}$ for one/all $\operatorname{Re} w < 0$ (Theorem A.13). In order to see that A is an operator and not just a relation, we need to show that $R(w)$ is injective for one (equivalently: all) $\operatorname{Re} w < 0$. As $\ker(R(w))$

does not depend on w (by the resolvent identity), the claim follows as soon as we have proved part c).

a) This holds by definition of A.

b) Let $t > 0$, $x \in X$, and $w, \lambda \in \mathbb{C}$. Then a little computation yields

$$\begin{aligned}(z+\lambda)\int_0^t \mathrm{e}^{ws}\mathrm{e}^{-sz}\,\mathrm{d}s &= (z-w)\int_0^t \mathrm{e}^{ws}\mathrm{e}^{-sz}\,\mathrm{d}s + (w+\lambda)\int_0^t \mathrm{e}^{ws}\mathrm{e}^{-sz}\,\mathrm{d}s\\ &= 1-\mathrm{e}^{wt}\mathrm{e}^{-tz} + (w+\lambda)\int_0^t \mathrm{e}^{ws}\mathrm{e}^{-sz}\,\mathrm{d}s =: f(z) \qquad (\operatorname{Re} z > 0).\end{aligned}$$

That means that $f = \mathcal{L}\mu \in \mathrm{LS}(\mathbb{C}_+)$ with

$$\mu := \boldsymbol{\delta}_0 - \mathrm{e}^{-wt}\boldsymbol{\delta}_t + (w+\lambda)\mathbf{1}_{[0,t]}\mathrm{e}^{ws}\mathrm{d}s.$$

For each $\operatorname{Re}\lambda > 0$ one can divide by $z+\lambda$ and then apply the Hille–Phillips calculus to obtain

$$\int_0^t \mathrm{e}^{ws}T_s x\,\mathrm{d}s = \Phi_T\Big(\int_0^t \mathrm{e}^{sw}\mathrm{e}^{-s\mathbf{z}}\,\mathrm{d}s\Big)x = -R(-\lambda, A)\Phi_T(f)x \;\in\; \operatorname{dom}(A).$$

For $w = 0$ we hence obtain $\int_0^t T_s x\,\mathrm{d}s \in \operatorname{dom}(A)$, and since

$$\frac{1}{t}\int_0^t T_s x\,\mathrm{d}s \to x \quad \text{as } t \searrow 0$$

by the strong continuity of T, we arrive at $x \in \overline{\operatorname{dom}(A)}$.

c) It follows from the norm estimate (6.8) that

$$\|(\lambda+A)^{-1}\| \le M_T\int_0^\infty \mathrm{e}^{-\operatorname{Re}\lambda t}\,\mathrm{d}t \le \frac{M_T}{\operatorname{Re}\lambda}$$

for all $\operatorname{Re}\lambda > 0$. In particular, $\sup_{\lambda>0}\|\lambda(\lambda+A)^{-1}\| < \infty$. Hence, for fixed $\lambda_0 > 0$ the resolvent identity yields

$$\lambda(\lambda+A)^{-1}(\lambda_0+A)^{-1} = \frac{\lambda}{\lambda-\lambda_0}\big((\lambda_0+A)^{-1} - (\lambda+A)^{-1}\big) \to (\lambda_0+A)^{-1}$$

in operator norm as $\lambda \to \infty$. Since $\operatorname{dom}(A) = \operatorname{ran}((\lambda_0+A)^{-1})$ is dense, assertion c) follows.

d) Let $V := \int_0^t \mathrm{e}^{ws}T_s\,\mathrm{d}s$. In b) we have seen that $V = (\lambda+A)^{-1}\Phi_T(f)$, hence

$$(\lambda+A)V = \Phi_T(f) = \mathrm{I} - \mathrm{e}^{wt}T_t + (w+\lambda)V.$$

By adding scalar multiples of V we obtain the identity

$$(\lambda+A)V = \mathrm{I} - \mathrm{e}^{wt}T_t + (w+\lambda)V$$

for *all* $\lambda \in \mathbb{C}$, in particular for $\lambda = -w$.

e) As $\mathrm{LS}(\mathbb{C}_+)$ is commutative, $\Phi_T(f)$ commutes with the resolvent of A, hence with A (Exercise 2.3). $\square$

The operator A of Theorem 6.4 is called the **generator** of the Hille–Phillips calculus Φ_T and one often writes $f(A)$ in place of $\Phi_T(f)$. With this convention we have

$$T_t = \Phi_T(\mathcal{L}\delta_t) = \Phi_T(\mathrm{e}^{-t\mathbf{z}}) = \mathrm{e}^{-tA} \qquad (t \geq 0).$$

A little unconveniently, it is the operator $-A$ (and not A) which is called the **generator of the semigroup** T. One writes $-A \sim (T_t)_{t\geq 0}$ for this. We shall see in Theorem 6.6 below that the semigroup T is uniquely determined by its generator.

6.3 General C_0-Semigroups and C_0-Groups

Suppose now that $T = (T_t)_{t\geq 0}$ is a C_0-semigroup, but not necessarily bounded. Then, by the uniform boundedness principle, T is still operator norm bounded on compact intervals. This implies that T is **exponentially bounded**, i.e., there is $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|T_t\| \leq M\mathrm{e}^{\omega t} \qquad (t \geq 0) \tag{6.9}$$

(see Exercise 6.1). One says that T is of **type** (M, ω) if (6.9) holds. The number

$$\omega_0(T) := \inf\{\omega \in \mathbb{R} \mid \text{there is } M \geq 1 \text{ such that (6.9) holds}\}$$

is called the **(exponential) growth bound** of T. If $\omega_0(T) < 0$, the semigroup is called **exponentially stable**.

For each $\omega \in \mathbb{C}$ one can consider the **rescaled semigroup** T^ω, defined by

$$T^\omega(t) := \mathrm{e}^{-\omega t} T_t \qquad (t \geq 0),$$

which is again strongly continuous. Since T is exponentially bounded, if $\operatorname{Re}\omega$ is large enough, the rescaled semigroup T^ω is bounded and hence has a generator $-A_\omega$, say. The following tells in particular that the operator $A := A_\omega - \omega$ is independent of ω.

Theorem 6.5. *Let $T = (T_t)_{t\geq 0}$ be a C_0-semigroup on a Banach space X and let λ, $\omega \in \mathbb{C}$ such that T^λ and T^ω are bounded semigroups with generators $-A_\omega$ and $-A_\lambda$, respectively. Then*

$$A := A_\omega - \omega = A_\lambda - \lambda. \tag{6.10}$$

Furthermore, the following assertions hold:

a) *A is densely defined.*

b) *$T_t A \subseteq A T_t$ for all $t \geq 0$.*

c) *$\int_0^t T_s \,\mathrm{d}s A \subseteq A \int_0^t T_s \,\mathrm{d}s = \mathrm{I} - T_t$ for all $t \geq 0$.*

d) *For x, $y \in X$ the following assertions are equivalent:*

   (i) $-Ax = y$*;*

   (ii) $y = \lim_{t \searrow 0} \frac{1}{t}(T_t x - x)$*;*

   (iii) $T(\cdot)x \in \mathrm{C}^1(\mathbb{R}_+; X)$ *and* $\frac{\mathrm{d}}{\mathrm{d}t} T_t x = T_t y$ *on* $\mathbb{R}_+$.

Proof. Without loss of generality we may suppose that $\operatorname{Re}\lambda \geq \operatorname{Re}\omega$. Then

$$\begin{aligned}(1 + A_\lambda)^{-1} &= \int_0^\infty \mathrm{e}^{-t}\mathrm{e}^{-\lambda t} T_t \,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-t}\mathrm{e}^{-(\lambda-\omega)t}\mathrm{e}^{-\omega t} T_t \,\mathrm{d}t \\ &= (1 + \lambda - \omega + A_\omega)^{-1}.\end{aligned}$$

This establishes the first claim. Assertion a) is clear and b) holds true since, by construction, each T_t commutes with the resolvent of A (Exercise 2.3). Assertion c) follows directly from d) and e) of Theorem 6.4. For the proof of d) we note that the implication (iii) $\Rightarrow$ (ii) is trivial.

(i) $\Rightarrow$ (iii): If $-Ax = y$ then $(A + \lambda)x = -y + \lambda x =: z$ and hence

$$T_t x = T_t \int_0^\infty \mathrm{e}^{-\lambda s} T_s z \,\mathrm{d}s = \int_0^\infty \mathrm{e}^{-\lambda s} T_{t+s} z \,\mathrm{d}s = \mathrm{e}^{\lambda t} \int_t^\infty \mathrm{e}^{-\lambda s} T_s z \,\mathrm{d}s.$$

By the fundamental theorem of calculus (Theorem A.3) and the product rule, the orbit $T(\cdot)x$ is differentiable with derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} T_t x = \lambda T_t x - \mathrm{e}^{\lambda t}\mathrm{e}^{-\lambda t} T_t z = T_t(\lambda x - z) = T_t y$$

as claimed.

(ii) $\Rightarrow$ (i): This is left as Exercise 6.2. □

If A is as in (6.10), the operator $-A$ is called the **generator** of the semigroup T. By construction,

$$(\lambda + A)^{-1} = \int_0^\infty \mathrm{e}^{-\lambda t} T_t \,\mathrm{d}t$$

for all sufficiently large $\operatorname{Re}\lambda$.

Theorem 6.6. *A C_0-semigroup is uniquely determined by its generator.*

Proof. Suppose that B is the generator of the C_0-semigroups S and T on the Banach space X. Fix $x \in \operatorname{dom}(B)$ and $t > 0$, and consider the mapping

$$f : [0,t] \to X, \qquad f(s) := T(t-s)S(s)x.$$

Then by Lemma A.5, $f'(s) = -BT(t-s)S(s)x + T(t-s)BS(s)x = -T(t-s)BS(s)x + T(t-s)BS(s)x = 0$ for all $s \in [0,t]$. Hence, f is constant and therefore $T(t)x = f(0) = f(t) = S(t)x$. Since $\operatorname{dom}(B)$ is dense, $T(t) = S(t)$. □

Let $-A$ be the generator of a C_0-semigroup $T = (T_t)_{t\geq 0}$ of type (M, ω). Then the operator $-(A+\omega)$ generates the bounded semigroup T^ω and hence $A + \omega$ generates the associated Hille–Phillips calculus Φ_{T^ω}. It is therefore reasonable to define a functional calculus Φ_T for A by

$$\Phi_T(f) := \Phi_{T^\omega}\big(f(\mathbf{z} - \omega)\big) \tag{6.11}$$

for f belonging to the **Laplace–Stieltjes algebra**

$$\mathrm{LS}(\mathbb{C}_+ - \omega) := \{f \mid f(\mathbf{z} - \omega) \in \mathrm{LS}(\mathbb{C}_+)\}.$$

This calculus is called the **Hille–Phillips** calculus for T (on $\mathbb{C}_+ - \omega$). Note the boundedness property

$$\|\Phi_T(f)\| \leq M \|f\|_{\mathrm{LS}(\mathbb{C}_+ - \omega)} \qquad (f \in \mathrm{LS}(\mathbb{C}_+ - \omega))$$

where $\|f\|_{\mathrm{LS}(\mathbb{C}_+ - \omega)} := \|f(\mathbf{z} - \omega)\|_{\mathrm{LS}(\mathbb{C}_+)}$.

Remark 6.7. Since the type of a semigroup is not unique, the above terminology could be ambiguous. To wit, if T is of type (M, ω), it is also of type (M, α) for each $\alpha > \omega$. Accordingly, one has the Hille–Phillips calculi Φ_T^ω on $\mathbb{C}_+ - \omega$ and Φ_T^α on $\mathbb{C}_+ - \alpha$ for A. However, these calculi are compatible in the sense that (by restriction) $\mathrm{LS}(\mathbb{C}_+ - \alpha) \subseteq \mathrm{LS}(\mathbb{C}_+ - \omega)$ and

$$\Phi_T^\alpha(f) = \Phi_T^\omega(f|_{\mathbb{C}_+ - \omega}) \qquad (f \in \mathrm{LS}(\mathbb{C}_+ - \alpha))$$

(Exercise 6.4). We see that a smaller growth bound results in a larger calculus.

C_0-Groups

A C_0**-group** on a Banach space X is just a strongly continuous representation $U = (U_s)_{s\in\mathbb{R}}$ of $\mathbb{R}$ on X. From such a C_0-group, two C_0-semigroups can be derived, the **forward semigroup** $(U_t)_{t\geq 0}$ and the **backward semigroup** $(U_{-t})_{t\geq 0}$. Obviously, each determines the other, as $U_{-t} = U_t^{-1}$ for all $t \geq 0$. The **generator** of the group U is defined as the generator of the corresponding forward semigroup.

Theorem 6.8. *Let B be the generator of a C_0-semigroup $T = (T_t)_{t\geq 0}$. Then the following assertions are equivalent.*

(i) *T extends to a strongly continuous group.*

(ii) *$-B$ is the generator of a C_0-semigroup.*

(iii) *T_t is invertible for some $t > 0$.*

In this case $-B$ generates the corresponding backward semigroup $(T_t^{-1})_{t\geq 0}$.

Proof. (ii) ⇒ (i): Let $-B \sim (S_t)_{t\geq 0}$. Then, for all $x \in \operatorname{dom}(B)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} S(t)T(t)x &= (-B)S(t)T(t)x + S(t)BT(t)x \\ &= -S(t)BT(t)x + S(t)BT(t)x = 0\end{aligned}$$

by Lemma A.5. Since $\operatorname{dom}(B)$ is dense, it follows that $S(t)T(t) = S(0)T(0) = \mathrm{I}$ for all $t \geq 0$. Interchanging the roles of S and T yields $T(t)S(t) = \mathrm{I}$ as well, hence each $T(t)$ is invertible with $T(t)^{-1} = S(t)$. It is now routine to check that the extension of T to $\mathbb{R}$ given by $T(s) := S(-s)$ for $s \leq 0$ is a C_0-group.

(i) ⇒ (iii) is clear.

(iii) ⇒ (ii): Fix $t_0 > 0$ such that $T(t_0)$ is invertible. For general $t > 0$ we can find $n \in \mathbb{N}$ and $r > 0$ such that $t + r = nt_0$. Hence, $T(t)T(r) = T(r)T(t) = T(t_0)^n$ is invertible, and so must be $T(t)$. Define $S(t) := T(t)^{-1}$ for $t \geq 0$. Then S is a semigroup, and strongly continuous because for fixed $\tau > 0$

$$S(t) = S(\tau)T(\tau)T(t)^{-1} = S(\tau)T(\tau - t)T(t)T(t)^{-1} = S(\tau)T(\tau - t)$$

for $0 \leq t \leq \tau$. Let C be the generator of S and $x \in \operatorname{dom}(B)$. Then for $0 < t < \tau$,

$$\frac{S(t)x - x}{t} = S(\tau)\frac{T(\tau - t)x - T(\tau)x}{t} \to -S(\tau)T(\tau)Bx = -Bx$$

as $t \searrow 0$. (Recall (iii) of Theorem 6.5.d).) Hence, $-B \subseteq C$. By symmetry, it follows that $C = -B$. □

A C_0-group $U = (U_s)_{s\in\mathbb{R}}$ is said to be of **type** (M, ω) for some $M \geq 1$ and $\omega \geq 0$ if

$$\|U_s\| \leq M\mathrm{e}^{\omega|s|} \qquad (s \in \mathbb{R}).$$

By the results from above, each C_0-group is of some type (M, ω). The quantity

$$\theta(U) := \inf\{\omega \geq 0 \mid \exists\, M \geq 1 : U \text{ is of type } (M, \omega)\}$$

is called the **group type** of U.

Let B be the generator of a *bounded* group U and let Ψ_U be the associated Fourier–Stieltjes calculus. Define $A := \mathrm{i}B$, so that $B = -\mathrm{i}A$. Then A is the generator of Ψ_U as

$$\Psi_U(\mathrm{e}^{-\mathrm{i}s\mathbf{z}}) = U_{\delta_s} = U_s \qquad (s \in \mathbb{R})$$

and, for $\operatorname{Im}\lambda > 0$,

$$(\lambda - \mathbf{z})^{-1} = \frac{1}{\mathrm{i}} \int_0^\infty \mathrm{e}^{\mathrm{i}\lambda s}\mathrm{e}^{-\mathrm{i}s\mathbf{z}}\,\mathrm{d}s$$

as functions on $\mathbb{R}$.

If U is not bounded, one can still define a functional calculus based on the Fourier transform. However, one has to restrict to a certain subalgebra of measures/functions. See Exercise 6.7.

6.4 Supplement: Continuity Properties and Uniqueness

In this supplementary section we shall present a uniqueness statement for the Fourier-Stieltjes calculus of a bounded strongly continuous representation of $\mathbb{S}$ in the cases $\mathbb{S} = \mathbb{R}^d$ and $\mathbb{S} = \mathbb{R}^d_+$. These statements involve a certain continuity property of the calculus, interesting in its own right. We shall make use of the results b)-d) of Exercise 6.8.

We start with observing that in certain cases the inequality (6.2) is an identity.

Lemma 6.9. *For each $\mu \in \mathrm{M}(\mathbb{R}^d)$ one has*

$$\|\mu\|_{\mathrm{M}} = \|\tau_\mu\|_{\mathcal{L}(\mathrm{L}^1(\mathbb{R}^d))} = \|\tau_\mu\|_{\mathcal{L}(\mathrm{C}_0(\mathbb{R}^d))}.$$

In other words: For $X = \mathrm{L}^1(\mathbb{R}^d)$ and $X = \mathrm{C}_0(\mathbb{R}^d)$ the regular representation $\tau : \mathrm{M}(\mathbb{R}^d) \to \mathcal{L}(X)$, $\mu \mapsto \tau_\mu$, is isometric.

Proof. The inequality $\|\tau_\mu\| \le \|\mu\|$ (both cases) is (6.2). For the case $X = \mathrm{C}_0(\mathbb{R}^d)$ the converse has already been shown in the proof of Theorem 5.4. For the case $X = \mathrm{L}^1(\mathbb{R}^d)$ we employ duality and compute

$$\begin{aligned}\|\tau_\mu\|_{\mathcal{L}(\mathrm{L}^1)} &= \sup_f \sup_g |\langle \mu * f, g\rangle| = \sup_g \sup_f |\langle f, \mathcal{S}\mu * g\rangle| \\ &= \|\tau_{\mathcal{S}\mu}\|_{\mathcal{L}(\mathrm{C}_0)} = \|\mathcal{S}\mu\|_{\mathrm{M}} = \|\mu\|_{\mathrm{M}},\end{aligned}$$

where the suprema are taken over all g in the unit ball of $\mathrm{C}_0(\mathbb{R}^d)$ and all f in the unit ball of $\mathrm{L}^1(\mathbb{R}^d)$. □

Definition 6.10. A sequence $(\mu_n)_n$ in $\mathrm{M}(\mathbb{R}^d)$ **converges strongly** to $\mu \in \mathrm{M}(\mathbb{R}^d)$ if

$$\mu_n * f \to \mu * f \quad \text{in } \mathrm{L}^1\text{-norm for all } f \in \mathrm{L}^1(\mathbb{R}^d).$$

In other words: $\mu_n \to \mu$ strongly if $\tau_{\mu_n} \to \tau_\mu$ strongly in $\mathcal{L}(\mathrm{L}^1(\mathbb{R}^d))$.

By Lemma 6.9 and the uniform boundedness principle, a strongly convergent sequence is uniformly norm bounded. From this it follows easily that the convolution product is simultaneously continuous with respect to strong convergence of sequences. Strong convergence implies weak* convergence (under the identification $\mathrm{M}(\mathbb{R}^d) \cong \mathrm{C}_0(\mathbb{R}^d)'$), see Exercise 6.11. Note also that if $t_n \to t$ in $\mathbb{R}^d$ then $\boldsymbol{\delta}_{t_n} \to \boldsymbol{\delta}_t$ strongly.

A sequence $(\varphi_n)_n$ in $\mathrm{L}^1(\mathbb{R}^d)$ is called an **approximation of the identity** if $\varphi_n \lambda \to \boldsymbol{\delta}_0$ strongly, and a **Dirac sequence**[4] if

$$\int_{\mathbb{R}^d} f\varphi_n \,\mathrm{d}\lambda \to f(0) \qquad (n \to \infty)$$

for each $f \in \mathrm{C_b}(\mathbb{R}^d; X)$ and any Banach space X. We say that $(\varphi_n)_n$ is a Dirac sequence **on** a closed subset $E \subseteq \mathbb{R}^d$ if it is a Dirac sequence and $\operatorname{supp}(\varphi_n) \subseteq E$ for all $n \in \mathbb{N}$. Each Dirac sequence is an approximation of the identity. Observe that Dirac sequences are easy to construct (Exercise 6.9) and that we have already used a special Dirac sequence on $\mathbb{R}_+$ in the proof of Theorem 6.4.

The following result underlines the importance of our notion of “strong convergence”.

Theorem 6.11. *Let $\mathbb{S} \in \{\mathbb{R}^d, \mathbb{R}^d_+\}$ and let $T : \mathbb{S} \to \mathcal{L}(X)$ be a bounded, strongly continuous representation on a Banach space X. Then the associated calculus $\mathrm{M}(\mathbb{S}) \to \mathcal{L}(X)$ has the following continuity property: If $(\mu_n)_n$ is a sequence in $\mathrm{M}(\mathbb{S})$ and $\mu_n \to \mu$ strongly, then $\mu \in \mathrm{M}(\mathbb{S})$ and $T_{\mu_n} \to T_\mu$ strongly in $\mathcal{L}(X)$.*

Proof. As already mentioned, strong convergence implies weak*-convergence. Hence $\operatorname{supp}(\mu) \subseteq \mathbb{S}$, i.e., $\mu \in \mathrm{M}(\mathbb{S})$.

Since the μ_n are uniformly norm bounded, so are the T_{μ_n}. Hence, it suffices to check strong convergence in $\mathcal{L}(X)$ only on a dense set of vectors. Let $(\varphi_m)_m$ be a Dirac sequence on $E = \mathbb{S}$. For each $x \in X$ and $m \in \mathbb{N}$ one has

$$T_{\mu_n} T_{\varphi_m} x = T_{\mu_n * \varphi_m} x \to T_{\mu * \varphi_m} x = T_\mu T_{\varphi_m} x$$

as $n \to \infty$. But $T_{\varphi_m} x \to x$ as $m \to \infty$, and we are done. □

Let us call a sequence $f_n = \widehat{\mu_n} \in \mathrm{FS}_{\mathbb{S}}(\mathbb{R}^d)$ strongly convergent to $f = \widehat{\mu} \in \mathrm{FS}_{\mathbb{S}}(\mathbb{R}^d)$, if $\mu_n \to \mu$ strongly. And let us call a functional calculus

$$\Psi : \mathrm{FS}_{\mathbb{S}}(\mathbb{R}^d) \to \mathcal{L}(X)$$

strongly continuous if whenever $f_n \to f$ strongly in $\mathrm{FS}_{\mathbb{S}}(\mathbb{R}^d)$ then $\Psi(f_n) \to \Psi(f)$ strongly in $\mathcal{L}(X)$. With this terminology, Theorem 6.11 simply tells that in the case of a bounded and strongly continuous representation of $\mathbb{S} = \mathbb{R}^d$ or $\mathbb{S} = \mathbb{R}^d_+$, the associated Fourier–Stieltjes calculus is strongly continuous. The following is the uniqueness result we annouced.

[4] That is our terminology. Different definitions exist in the literature.

Theorem 6.12. *Let $\mathbb{S} = \mathbb{R}^d$ or $\mathbb{S} = \mathbb{R}^d_+$ and let $\Psi : \mathrm{FS}_{\mathbb{S}}(\mathbb{R}^d) \to \mathcal{L}(X)$ be a strongly continuous calculus. Then $T : \mathbb{S} \to \mathcal{L}(X)$, defined by $T_t := \Psi(\mathrm{e}_t)$ for $t \in \mathbb{S}$, is a bounded and strongly continuous representation, and Ψ coincides with the associated Fourier–Stieltjes calculus.*

Proof. The continuity assumption on Ψ implies that Ψ is norm bounded (via the closed graph theorem) and that the representation T is strongly continuous. By the norm boundedness, T is also bounded.

Let $\mathcal{E} := \{f \in \mathrm{FS}_{\mathbb{S}}(\mathbb{R}^d) \mid \Psi(f) = \Psi_T(f)\}$. Then $\mathcal{E}$ is a strongly closed subalgebra of $\mathrm{FS}_{\mathbb{S}}(\mathbb{R}^d)$ containing all the functions e_t, $t \in \mathbb{S}$. Hence, the claim follows from the next lemma. □

Lemma 6.13. *Let $\mathbb{S} = \mathbb{R}^d_+$ or $\mathbb{S} = \mathbb{R}^d$ and let $M \subseteq \mathrm{M}(\mathbb{S})$ be a convolution subalgebra closed under strong convergence of sequences. Then $M = \mathrm{M}(\mathbb{S})$ in each of the following cases:*

1) *M contains δ_t for each $t \in \mathbb{S}$.*
2) *M contains some dense subset of $\mathrm{L}^1(\mathbb{S})$.*

Proof. As M is strongly closed, it is norm closed. Suppose that 2) holds. Then $\mathrm{L}^1(\mathbb{S}) \subseteq M$. As $\mathrm{L}^1(\mathbb{S})$ contains an approximation of the identity and M is strongly closed, $\mathrm{M}(\mathbb{S}) \subseteq M$.

Suppose that 1) holds and consider first the case $d = 1$. Fix $\varphi \in \mathrm{C_c}(\mathbb{R})$ such that $\mathrm{supp}(\varphi) \subseteq [0, 1]$. Then the sequence of measures

$$\mu_n := \frac{1}{n} \sum_{k=1}^{n} \varphi\big(\tfrac{k}{n}\big) \delta_{\frac{k}{n}}$$

converges to $\varphi\lambda$ in the weak*-sense (as functionals on $\mathrm{C_0}(\mathbb{R})$). As the supports of the μ_n are all contained in a fixed compact set, Exercise 6.12 yields that $\mu_n \to \varphi\lambda$ strongly. Obviously, with slightly more notational effort this argument can be carried out for each $\varphi \in \mathrm{C_c}(\mathbb{S})$. As $\mathrm{C_c}(\mathbb{S})$ is dense in $\mathrm{L}^1(\mathbb{S})$, we obtain condition 2) and are done.

For arbitrary dimension $d \in \mathbb{N}$ one can employ a similar argument (with but even more notational effort). □

6.5 The Heat Semigroup on $\mathbb{R}^d$

In this section we shall apply our knowledge of (semi)groups and the corresponding functional calculi in order to become familiar with a special example: the heat semigroup on $\mathbb{R}^d$. We shall first treat the case $d = 1$.

In the following we use the symbol $\mathbf{s}$ for the real coordinate function and $\mathbf{z}$ for the coordinate function in Fourier/Laplace domain. Let

$$g_t := \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-\mathbf{s}^2/4t} \qquad (t > 0).$$

The family $(g_t)_{t>0}$ is called the **heat kernel** on $\mathbb{R}$. Here are its most important properties.

Lemma 6.14. *The following assertions hold:*

a) $g_t = \frac{1}{\sqrt{t}} g_1\big(\frac{\mathbf{s}}{\sqrt{t}}\big)$ *for all* $t > 0$.

b) $\|g_t\|_1 = \int_{\mathbb{R}} g_t = 1$ *for all* $t > 0$*, and* $(g_t)_{t>0}$ *is a (generalized) Dirac sequence as* $t \searrow 0$.

c) $\mathcal{F}(g_t) = \mathrm{e}^{-t\mathbf{z}^2}$ *for all* $t > 0$.

d) $g_t * g_s = g_{s+t}$ *for all* $s,\, t > 0$.

e) *The map* $(t \mapsto g_t) : (0, \infty) \to \mathrm{L}^1(\mathbb{R})$ *is continuous.*

Proof. a) is immediate.

b) Since $g_t \geq 0$, $\|g_t\|_1 = \int_{\mathbb{R}} g_t$. By a) and substitution,

$$\int_{\mathbb{R}} g_t = \int_{\mathbb{R}} g_1 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-s^2/2}\, \mathrm{d}s = 1,$$

which is well known. (For a proof see (A.22) in Appendix A.11.)

c) is also well known. (For a proof see (A.23) in Appendix A.11.)

d) follows from c) since the Fourier transform is injective and turns convolutions into products.

e) Given $0 < a < b < \infty$ one has

$$|g_t| \leq \frac{1}{\sqrt{4\pi a}} \mathrm{e}^{-\mathbf{s}^2/4b} \qquad (a \leq t \leq b).$$

Since obviously $t \mapsto g_t(s)$ is continuous for each fixed $s \in \mathbb{R}$, the claim follows from Lebesgue's theorem. □

Let $-\mathrm{i}A$ be the generator of a bounded C_0-group $(U_s)_{s\in\mathbb{R}}$ on a Banach space X, with associated Fourier–Stieltjes calculus Ψ_U. Further, let $G = (G_t)_{t\geq 0}$ be defined by

$$G_t := \Psi_U(\mathrm{e}^{-t\mathbf{z}^2}) = \begin{cases} \int_{\mathbb{R}} g_t(s)\, U_s\, \mathrm{d}s & \text{for } t > 0, \\ \quad \mathrm{I} & \text{for } t = 0. \end{cases}$$

Then $(G_t)_{t\geq 0}$ is a bounded C_0-semigroup on X. (The strong continuity follows from Lemma 6.14.b) and Exercise 6.9.c).) This semigroup is called the **heat semigroup** or the **Gauss–Weierstrass semigroup** associated with the group $(U_s)_{s\in\mathbb{R}}$.

Let us denote its generator by $-B$ and the associated Hille–Phillips calculus by Φ_G. For the following result one should recall that elements $f \in \mathrm{LS}(\mathbb{C}_+)$ can be interpreted as functions on $\mathbb{R}_+$.

Theorem 6.15. *Let $(U_s)_{s\in\mathbb{R}}$ be a bounded C_0-group on a Banach space X and let $(G_t)_{t\ge 0}$ be its associated Gauss–Weierstrass semigroup. If $f \in \mathrm{LS}(\mathbb{C}_+)$ then $f(\mathbf{z}^2) \in \mathrm{FS}(\mathbb{R})$ and*

$$\Phi_G(f) = \Psi_U(f(\mathbf{z}^2)).$$

Proof. Given a measure $\mu \in \mathrm{M}(\mathbb{R}_+)$ write $\mu = \alpha\boldsymbol{\delta}_0 + \nu$ where $\alpha \in \mathbb{C}$ and $\nu \in \mathrm{M}(0,\infty)$. Then $\mathcal{L}\mu = \alpha\mathbf{1} + \mathcal{L}\nu$ and it suffices to prove the claim for $f = \mathcal{L}\nu$.

By Lemma 6.14.e) and c) and the definition of the norm on $\mathrm{FS}(\mathbb{R})$, the mapping

$$(0,\infty) \to \mathrm{FS}(\mathbb{R}), \qquad t \mapsto \mathrm{e}^{-t\mathbf{z}^2}$$

is bounded and continuous. Hence,

$$\mathcal{L}\nu(\mathbf{z}^2) = \int_0^\infty \mathrm{e}^{-t\mathbf{z}^2}\,\nu(\mathrm{d}t) \;\in\; \mathrm{FS}(\mathbb{R})$$

since the integral converges in $\mathrm{FS}(\mathbb{R})$. (Note that point evaluations are continuous on $\mathrm{FS}(\mathbb{R})$.) Finally,

$$\begin{aligned}\Phi_G(f) &= \int_0^\infty G_t\,\nu(\mathrm{d}t) = \int_0^\infty \Psi_U(\mathrm{e}^{-t\mathbf{z}^2})\,\nu(\mathrm{d}t) = \Psi_U\Big(\int_0^\infty \mathrm{e}^{-t\mathbf{z}^2}\,\nu(\mathrm{d}t)\Big)\\ &= \Psi_U(f(\mathbf{z}^2))\end{aligned}$$

as claimed. □

Theorem 6.15 helps to identify the generator $-B$ of $(G_t)_{t\ge 0}$.

Corollary 6.16. *In the situation from above we have $B = A^2$.*

Proof. Applying the theorem with $f = \frac{1}{1+\mathbf{z}}$ yields

$$\begin{aligned}(\mathrm{I}+B)^{-1} &= \Psi_U\big(\frac{1}{1+\mathbf{z}^2}\big) = \Psi_U\big(\frac{1}{(1+\mathrm{i}\mathbf{z})(1-\mathrm{i}\mathbf{z})}\big)\\ &= (\mathrm{I}+\mathrm{i}A)^{-1}(\mathrm{I}-\mathrm{i}A)^{-1} = \big((\mathrm{I}-\mathrm{i}A)(\mathrm{I}+\mathrm{i}A)\big)^{-1} = (\mathrm{I}+A^2)^{-1}\end{aligned}$$

(see Theorem A.20) from which the claim follows. □

Note that $-B = -A^2 = (-\mathrm{i}A)^2$, so the generator of G is simply the square of the generator of U.

Examples 6.17. We apply these results to various shift groups, in which case one simply speaks of **the** heat or Gauss–Weierstrass semigroup on the respective space.

1) Let $X = \mathrm{UC_b}(\mathbb{R})$ and $U = \tau$ the right shift group. The associated heat semigroup on X is given by

$$G_t f = g_t * f = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f(s) \mathrm{e}^{-(\mathbf{x}-s)^2/4t} \, \mathrm{d}s \qquad (t > 0). \tag{6.12}$$

Its generator is $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with domain

$$\mathrm{UC}^2_\mathrm{b}(\mathbb{R}) = \{ f \in \mathrm{C}^2(\mathbb{R}) \mid f,\ f',\ f'' \in \mathrm{UC_b}(\mathbb{R}) \}.$$

This follows from the simple-to-prove fact that the generator of τ is $-\frac{\mathrm{d}}{\mathrm{d}x}$ with domain

$$\mathrm{UC}^1_\mathrm{b}(\mathbb{R}) = \{ f \in \mathrm{C}^1(\mathbb{R}) \mid f,\ f' \in \mathrm{UC_b}(\mathbb{R}) \}.$$

An analogous result holds for the heat semigroup on $X = \mathrm{C_0}(\mathbb{R})$.

2) Let $X = \mathrm{L}^p(\mathbb{R})$ for $1 \le p < \infty$ and $U = \tau$ the right shift group. Then its associated heat semigroup is again given by (6.12) (recall Exercise 5.7). The generator of U is the closure of $-\frac{\mathrm{d}}{\mathrm{d}x}$ defined on $\mathrm{C}^\infty_\mathrm{c}(\mathbb{R})$, see Exercise 6.5.b). By Exercise 6.5.a), its square—which is the generator of G—is the closure of $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $\mathrm{C}^\infty_\mathrm{c}(\mathbb{R})$. (Its domain is $\mathrm{W}^{2,\mathrm{p}}(\mathbb{R})$, but we do not prove this here.)

3) Fix $1 \le j \le d$ and let U be the shift group in the direction of e_j on $X = \mathrm{UC_b}(\mathbb{R}^d)$ or $X = \mathrm{C_0}(\mathbb{R}^d)$, i.e., $U_s = \tau_{s\mathrm{e}_j}$ for all $s \in \mathbb{R}$. Its generator is $-\frac{\partial}{\partial x_j}$ with domain

$$\Big\{ f \in X \;\Big|\; \frac{\partial f}{\partial x_j} \text{ exists everywhere and yields a function in } X \Big\}.$$

So the associated heat semigroup has generator $\frac{\partial^2}{\partial x_j^2}$ with domain consisting of those $f \in X$ such that $\frac{\partial f}{\partial x_j}$ and $\frac{\partial^2 f}{\partial x_j^2}$ exist and are in X.

4) Fix $1 \le j \le d$ and let U be the shift group in the direction of e_j on $X = \mathrm{L}^p(\mathbb{R}^d)$ for $1 \le p < \infty$. Its generator $-D_j$, say, is the closure of the operator $-\frac{\partial}{\partial x_j}$ defined originally on $\mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^d)$. (It is true that

$$D_j f = g \quad \Longleftrightarrow \quad \frac{\partial}{\partial x_j} f = g \quad \text{in the weak sense}$$

for $f, g \in \mathrm{L}^p(\mathbb{R}^d)$, but we do not prove this here.) It follows that the generator of the associated heat semigroup is the closure of $-\frac{\partial^2}{\partial x_j^2}$ (defined on $\mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^d)$).

The Multidimensional Case

Let us turn to the multidimensional situation. The **d-dimensional heat kernel** is the family $(g_{d,t})_{t>0}$ given by

$$g_{d,t} = g_t \otimes \ldots \otimes g_t = \frac{1}{(4\pi t)^{\frac{d}{2}}} \mathrm{e}^{-|\mathbf{s}|^2/4t} \qquad (t > 0).$$

(The modulus $|\cdot|$ here denotes the Euclidean norm on $\mathbb{R}^d$.) It is easily seen that Lemma 6.14 holds mutatis mutandis: $(g_{d,t})_{t>0}$ is a generalized Dirac sequence and continuous in $t > 0$ and $\mathcal{F}(g_{d,t}) = \mathrm{e}^{-t|\mathbf{z}|^2}$ for all $t > 0$.

If $(U_s)_{s\in\mathbb{R}^d}$ is a bounded, strongly continuous representation of $\mathbb{R}^d$ on a Banach space X with corresponding Fourier–Stieltjes calculus Ψ_U the associated **heat semigroup** (also: **Gauss–Weierstrass** semigroup) is $(G_t)_{t\geq 0}$, defined by

$$G_t := \Psi_U(\mathrm{e}^{-t|\mathbf{z}|^2}) \qquad (t \geq 0).$$

This means that $G_t = \int_{\mathbb{R}^d} g_{d,t}(s) U_s \,\mathrm{d}s$ whenever $t > 0$. Then, with pretty much the same proof, we obtain the following analogue of Theorem 6.15.

Theorem 6.18. *Let $(U_s)_{s\in\mathbb{R}^d}$ be a bounded and strongly continuous group on a Banach space X with associated Fourier–Stieltjes calculus Ψ_U. Let, furthermore, $(G_t)_{t\geq 0}$ be the associated Gauss–Weierstrass semigroup and Φ_G its Hille–Phillips calculus. If $f \in \mathrm{LS}(\mathbb{C}_+)$ then $f(|\mathbf{z}|^2) \in \mathrm{FS}(\mathbb{R}^d)$ and*

$$\Phi_G(f) = \Psi_U(f(|\mathbf{z}|^2)).$$

In the situation of Theorem 6.18, let $-\mathrm{i}A_j$ be the generator of the bounded C_0-group U^j, defined by $U^j_s = U_{s\mathrm{e}_j}$ for $s \in \mathbb{R}$ and $1 \leq j \leq d$. Then for $g \in \mathrm{FS}(\mathbb{R}^d)$ one can think of $\Psi_U(g)$ as

$$\Psi_U(g) = g(A_1, \ldots, A_d)$$

similar to the one-dimensional case. Now if $-B$ denotes the generator of $(G_t)_{t\geq 0}$ then, as in the proof of Corollary 6.16,

$$(\mathrm{I} + B)^{-1} = \Psi_U\Big(\frac{1}{1 + |\mathbf{z}|^2}\Big) = \Big(\frac{1}{1 + \mathbf{z}_1^2 + \cdots + \mathbf{z}_d^2}\Big)(A_1, \ldots, A_d).$$

Hence, we would like to conclude

$$B = (\mathbf{z}_1^2 + \cdots + \mathbf{z}_d^2)(A_1, \ldots, A_d) \stackrel{?}{=} A_1^2 + \cdots + A_d^2. \tag{6.13}$$

The first identity can be justified (e.g. by results of the next chapter), but the second one fails in general. The best one can say here is the following.

Theorem 6.19. *In the situation just described, $B = \overline{A_1^2 + \cdots + A_d^2}$.*

Proof. By Corollary 6.16, the operator $-A_j^2$ is the generator of G^j, the heat semigroup associated with the group U^j. Since the groups U^j are pairwise commuting, so are the semigroups G^j. Now observe (by a little computation) that

$$G(t) = G^1(t) \cdots G^d(t) \qquad (t \geq 0).$$

Hence, Exercise 6.8.d) yields the claim. □

Examples 6.20. Consider the Gauss–Weierstrass semigroup associated with the right shift group on $X = \mathrm{C}_0(\mathbb{R}^d)$ or $X = \mathrm{L}^p(\mathbb{R}^d)$, $1 \leq p < \infty$. Then it follows from Example 6.17.4) and Theorem 6.19 that its generator Δ_X, say, is the closure in X of the operator

$$\Delta := \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} \tag{6.14}$$

on $\mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d)$. It can be shown—but we do not do this here—that Δ_X is a restriction of the distributional Laplacian to X. We shall see later that, due to the boundedness of the so-called Riesz transforms, for $1 < p < \infty$ the domain of Δ_{L^p} is $\mathrm{W}^{2,\mathrm{p}}(\mathbb{R}^d)$.

6.6 Supplement: Subordinate Semigroups

In this section we review our findings from Section 6.5 on the Gauss–Weierstrass semigroups from a more abstract point of view.

A family of measures $(\mu_t)_{t\geq 0}$ in $\mathrm{M}(\mathbb{R}^d)$ is called a **convolution semigroup** if $\mu_0 = \boldsymbol{\delta}_0$ and $\mu_t * \mu_s = \mu_{s+t}$ whenever $s,\, t \geq 0$. It is called **strongly continuous** if $\mu_t \to \boldsymbol{\delta}_0$ strongly (as defined in Section 6.4 above) as $t \searrow 0$.

In the following we suppose that $\mathbb{S} = \mathbb{R}^d$ or $\mathbb{S} = \mathbb{R}_+^d$ and $(\mu_t)_{t\geq 0}$ is a convolution semigroup in $\mathrm{M}(\mathbb{S})$. Then to each bounded and strongly continuous representation $S : \mathbb{S} \to \mathcal{L}(X)$ a semigroup $(T_t)_{t\geq 0}$ on X is given by

$$T_t := \Psi_S(\widehat{\mu_t}) = \int_{\mathbb{S}} S_s \, \mu_t(\mathrm{d}s) \, \in \mathcal{L}(X) \qquad (t \geq 0).$$

The semigroup T is called **subordinate** to the representation S, and the family $(\mu_t)_{t\geq 0}$ is the so-called **subordinator**. One has the following lemma.

Lemma 6.21. *If the convolution semigroup $(\mu_t)_{t\geq 0}$ is strongly continuous then so is the semigroup $(T_t)_{t\geq 0}$.*

Proof. It follows from Theorem 6.11 that T is strongly continuous at $t = 0$. By Exercise 6.14, T is strongly continuous on the whole of $\mathbb{R}_+$. □

For strongly continuous convolution semigroups, the following result is of fundamental importance.

Theorem 6.22. *Let $(\mu_t)_{t\geq 0}$ be any strongly continuous and uniformly bounded convolution semigroup in $\mathrm{M}(\mathbb{S})$. Then there is a uniquely determined continuous function $a : \mathbb{R}^d \to \overline{\mathbb{C}_+}$ such that*

$$\widehat{\mu_t} = \mathrm{e}^{-ta} \qquad (t \geq 0).$$

Moreover, for each $f \in \mathrm{LS}(\mathbb{C}_+)$ one has $f \circ a \in \mathrm{FS}_{\mathbb{S}}(\mathbb{R}^d)$ and

$$\Phi_T(f) = \Psi_S(f \circ a)$$

whenever $S : \mathbb{S} \to \mathcal{L}(X)$ is a strongly continuous and bounded representation with Fourier–Stieltjes calculus Ψ_S, T is the semigroup subordinate to S with respect to the subordinator $(\mu_t)_{t\geq 0}$, and Φ_T is its Hille–Phillips calculus.

For the proof, we need the following auxiliary result, interesting in its own right.

Theorem 6.23. *Let T be a bounded linear operator on $\mathrm{L}^1(\mathbb{R}^d)$. Then the following assertions are equivalent:*

(i) *T commutes with all translations.*

(ii) *$T(\nu * \psi) = \nu * T\psi$ for all $\nu \in \mathrm{M}(\mathbb{R}^d)$ and $\psi \in \mathrm{L}^1(\mathbb{R}^d)$.*

(iii) *$T(\varphi * \psi) = \varphi * T\psi$ for all φ, $\psi \in \mathrm{L}^1(\mathbb{R}^d)$.*

(iv) *There is a function $a : \mathbb{R}^d \to \mathbb{C}$ such that $\mathcal{F}(T\psi) = a\widehat{\psi}$ for all $\psi \in \mathrm{L}^1(\mathbb{R}^d)$.*

(v) *There is $\mu \in \mathrm{M}(\mathbb{R}^d)$ such that $T\psi = \mu * \psi$ for all $\psi \in \mathrm{L}^1(\mathbb{R}^d)$.*

In this case $a = \widehat{\mu}$ and μ is uniquely determined by (v).

Proof. (v) ⇒ (iv): take $a = \widehat{\mu}$.
(iv) ⇒ (ii): For $\nu \in \mathrm{M}(\mathbb{R}^d)$ and $\psi \in \mathrm{L}^1(\mathbb{R}^d)$:

$$\mathcal{F}(T(\nu * \psi)) = a\mathcal{F}(\nu * \psi) = a\widehat{\nu}\widehat{\psi} = \widehat{\nu}a\widehat{\psi} = \widehat{\nu}\mathcal{F}(T\psi) = \mathcal{F}(\nu * T\psi)$$

and hence, by the injectivity of the Fourier transform, $T(\nu * \psi) = \nu * T\psi$.
(ii) ⇒ (i),(iii): For (i) take $\nu = \varphi\lambda$ and for (iii) take $\nu = \boldsymbol{\delta}_t$, $t \in \mathbb{R}^d$.
(i) ⇒ (ii): This follows from Lemma 5.3 applied to the translation group.
(iii) ⇒ (v): Let $(\varphi_n)_{n\in\mathbb{N}}$ be any Dirac sequence in $\mathrm{L}^1(\mathbb{R}^d)$. The sequence $(T\varphi_n)_{n\in\mathbb{N}}$ is bounded and can be regarded as a sequence in $\mathrm{M}(\mathbb{R}^d) \cong \mathrm{C}_0(\mathbb{R}^d)'$. Now let $\psi \in \mathrm{L}^1(\mathbb{R}^d)$ and $f \in \mathrm{C}_0(\mathbb{R}^d)$ be arbitrary. Then $\mathcal{S}\psi * f \in \mathrm{C}_0(\mathbb{R}^d)$ and

$$\langle \mathcal{S}\psi * f, T\varphi_n \rangle = \langle f, \psi * T\varphi_n \rangle = \langle f, \varphi_n * T\psi \rangle \to \langle f, T\psi \rangle$$

as $n \to \infty$, where we used the continuity of T, the commutativity of convolution, and hypothesis (iii). Since elements of the form $\mathcal{S}\psi * f$ are dense in $\mathrm{C}_0(\mathbb{R}^d)$ and the sequence $(T\varphi_n)_n$ is bounded, it follows that it is weakly*-convergent to some $\mu \in \mathrm{M}(\mathbb{R}^d)$. Moreover, we obtain

$$\langle f, \mu * \psi \rangle = \langle \mathcal{S}\psi * f, \mu \rangle = \langle f, T\psi \rangle \qquad (\psi \in \mathrm{L}^1(\mathbb{R}^d),\ f \in \mathrm{C}_0(\mathbb{R}^d)),$$

which implies that $\mu * \psi = T\psi$ for all $\psi \in \mathrm{L}^1(\mathbb{R}^d)$.

Uniqueness follows from Lemma 6.9. □

Remark 6.24. Suppose that in Theorem 6.23 there is a closed subset $E \subseteq \mathbb{R}^d$ such that $\mathrm{L}^1(E)$ contains a Dirac sequence and is invariant under T. Then $\operatorname{supp}(\mu) \subseteq E$.

Indeed, the proof shows that μ is a weak*-limit of functions $T\varphi_n$, where $(\varphi_n)_n$ is a certain subsequence of an arbitrary Dirac sequence in $\mathrm{L}^1(\mathbb{R}^d)$. In particular, one can suppose that $\operatorname{supp}(\varphi_n) \subseteq E$ for all $n \in \mathbb{N}$. Hence, if T leaves $\mathrm{L}^1(E)$ invariant, then $\operatorname{supp}(\mu) \subseteq E$ as well.

We can now prove Theorem 6.22.

Proof of Theorem 6.22. First, consider the multiplication operator semigroup $(T_t)_{t\geq 0}$ on $\mathrm{C}_0(\mathbb{R}^d)$ defined by

$$T_t f = \widehat{\mu_t} f \qquad (f \in \mathrm{C}_0(\mathbb{R}^d),\ t \geq 0).$$

By Example 5.6.1), $T_t = S_{\mu_t}$, where $(S_s)_{s\in\mathbb{R}^d}$ is the bounded strongly continuous group on $\mathrm{C}_0(\mathbb{R}^d)$ given by

$$S_s f = \mathrm{e}_s f \qquad (f \in \mathrm{C}_0(\mathbb{R}^d)).$$

Hence, by Lemma 6.21, $(T_t)_{t\geq 0}$ is strongly continuous. By Exercise 6.15 there is a unique continuous function $a : \mathbb{R}^d \to \overline{\mathbb{C}_+}$ such that $\widehat{\mu_t} = \mathrm{e}^{-ta}$ for all $t \geq 0$.

For the second part we first deal with the case $\mathbb{S} = \mathbb{R}^d$. Let $\nu \in \mathrm{M}(\mathbb{R}_+)$ and $f = \mathcal{L}\nu$. By Theorem 6.23 there is a unique measure $\mu \in \mathrm{M}(\mathbb{R}^d)$ such that

$$\int_0^\infty \mu_t * \psi \, \nu(\mathrm{d}t) = \mu * \psi$$

for all $\psi \in \mathrm{L}^1(\mathbb{R}^d)$. Taking Fourier transforms and inserting $x \in \mathbb{R}^d$ we see that

$$\int_0^\infty \mathrm{e}^{-ta(x)} \widehat{\psi}(x)\, \nu(\mathrm{d}t) = \widehat{\mu}(x)\, \widehat{\psi}(x).$$

This yields $f \circ a = \widehat{\mu} \in \mathrm{FS}(\mathbb{R}^d)$.

Finally, with S and T as in the hypotheses of the theorem and $\psi \in \mathrm{L}^1(\mathbb{R}^d)$,

$$\begin{aligned}\Phi_T(f)S_\psi &= \int_0^\infty T_t\,\nu(\mathrm{d}t)\,S_\psi = \int_0^\infty S_{\mu_t}S_\psi\,\nu(\mathrm{d}t) = \int_0^\infty S_{\mu_t*\psi}\,\nu(\mathrm{d}t)\\ &= S_{\int_0^\infty \mu_t*\psi\,\nu(\mathrm{d}t)} = S_{\mu*\psi} = S_\mu S_\psi = \Psi_S(f\circ a)S_\psi.\end{aligned}$$

As $\psi \in \mathrm{L}^1(\mathbb{R}^d)$ is arbitrary, it follows that $\Phi_T(f) = \Psi_S(f\circ a)$ as claimed.

Now consider the case $\mathbb{S} = \mathbb{R}^d_+$. Then $\operatorname{supp}(\mu_t) \subseteq \mathbb{R}^d_+$ for all $t \ge 0$. By Remark 6.24, $\mu \in \mathrm{M}(\mathbb{R}^d_+)$. The remaining parts of the proof carry over unchanged, save that one has to take $\psi \in \mathrm{L}^1(\mathbb{R}^d_+)$ in the final argument. □

Example 6.25 (Heat Semigroups). Of course, the heat semigroups of Section 6.5 are examples of subordinate semigroups for the special choice $\mu_t = g_{d,t}\lambda$ for $t > 0$.

Remark 6.26. Strong continuity of a convolution semigroup $(\mu_t)_{t\ge 0}$ is not easy to check. However, if all μ_t are *probability* measures, the strong continuity is equivalent with the weak* convergence of μ_t to δ_0 as $t \searrow 0$ (Exercise 6.16). So it should not come as a surprise that subordinate semigroups were first studied by Bochner and Feller in the context of probability theory.

Exercises

6.1 (Growth Bound). Let $T = (T_t)_{t\ge 0}$ be a C_0-semigroup on a Banach space X.

a) Show that

$$\omega_0(T) := \inf_{t>0} \frac{\log \|T_t\|}{t} = \lim_{t\to\infty} \frac{\log \|T_t\|}{t} \in \mathbb{R}\cup\{-\infty\}. \tag{6.15}$$

b) Show that to each $\omega > \omega_0(T)$ there is $M \ge 1$ such that

$$\|T_t\| \le M\mathrm{e}^{\omega t} \qquad (t \ge 0).$$

Then show that $\omega_0(T)$ is actually the infimum of all $\omega \in \mathbb{R}$ with this property.

The number $\omega_0(T)$ is called the **growth bound** of the semigroup T.

6.2. Let $-A$ be the generator of a C_0-semigroup $T = (T_t)_{t\ge 0}$ on a Banach space X and suppose that $x, y \in X$ are such that $y = \lim_{t\searrow 0} \frac{1}{t}(T_t x - x)$.

a) Show that $T(\cdot)x$ is differentiable on $\mathbb{R}_+$ and its derivative is $T(\cdot)y$. [Hint: Show first that the orbit is right differentiable, cf. [1, Lemma II.1.1].]

b) Show that $-Ax = y$. [Hint: For $\lambda > 0$ sufficiently large, compute $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-\lambda t}T_t x$ and integrate over $\mathbb{R}_+$.]

6.3 (Hille–Yosida Estimates). Let for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$ the function $f_n : \mathbb{R} \to \mathbb{C}$ be defined by

$$f_n := \frac{\mathbf{t}^{n-1}}{(n-1)!} \mathrm{e}^{-\lambda \mathbf{t}} \mathbf{1}_{\mathbb{R}_+}.$$

Prove the following assertions.

a) If $\lambda \in \mathbb{C}_+$ then $f_n \in \mathrm{L}^1(\mathbb{R}_+)$ and $f_n = f_1 * \cdots * f_1$ (n times).

b) Let B be the generator of a C_0-semigroup $T = (T_t)_{t\ge 0}$ of type (M, ω) on a Banach space X. Then, for $\operatorname{Re} \lambda > \omega$ and $n \ge 0$,

$$R(\lambda, B)^n = \int_0^\infty \tfrac{t^{n-1}}{(n-1)!} \mathrm{e}^{-\lambda t} T_t \, \mathrm{d}t \quad \text{and}$$
$$\|R(\lambda, B)^n\| \le \frac{M}{(\operatorname{Re} \lambda - \omega)^n}.$$

[Hint: Reduce to the case $\omega = 0$ by rescaling and then employ a).]

6.4. Let $-A$ be the generator of a C_0-semigroup T of type (M, ω) and let $\alpha > \omega$. Prove the following assertions:

a) If $f : \mathbb{R}_+ \to \mathbb{C}$ is such that $\mathrm{e}^{\omega \mathbf{t}} f \in \mathrm{L}^1(\mathbb{R}_+)$, then $g := \int_0^\infty \mathrm{e}^{-\mathbf{z}t} f(t) \, \mathrm{d}t \in \mathrm{LS}(\mathbb{C}_+ - \omega)$ and

$$g(A) = \int_0^\infty f(t) T_t \, \mathrm{d}t.$$

b) Via restriction, the algebra $\mathrm{LS}(\mathbb{C}_+ - \alpha)$ can be considered to be a subalgebra of $\mathrm{LS}(\mathbb{C}_+ - \omega)$, and the Hille-Phillips calculus for A on the larger algebra restricts to the Hille–Phillips calculus for A on the smaller one.

6.5. a) **(Nelson's Lemma)** Let $-A$ be the generator of a C_0-semigroup $T = (T_t)_{t\ge 0}$, let $n \in \mathbb{N}$, and let $D \subseteq \operatorname{dom}(A^n)$ be a subspace which is dense in X and invariant under the semigroup T. Show that D is a core for A^n. (Note that by Theorem A.20 each operator A^n is closed.)

b) **(Coordinate Shifts)** Let $j \in \{1, \dots, d\}$ and consider the shift semigroup $(\tau_{t \mathrm{e}_j})_{t \ge 0}$ in the direction e_j on $X = \mathrm{L}^p(\mathbb{R}^d)$, $1 \le p < \infty$, or $X = \mathrm{C}_0(\mathbb{R}^d)$. Show that its generator B is the closure (as an operator on X) of the operator

$$B_0 = -\frac{\partial}{\partial x_j}$$

defined on the space $D = \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d)$ of smooth functions with compact support.

[Hint: For a) observe first that if $y \in D$ and λ is sufficiently large, then $x_{\lambda,y} := \lambda^n (\lambda + A)^{-n} y$ is contained in the $\|\cdot\|_{A^n}$ closure of D, and second that one can find elements of the form $x_{\lambda,y}$ arbitrarily $\|\cdot\|_{A^n}$-close to any given $x \in \operatorname{dom}(A^n)$, see [1, Proposition II.1.7]. For b) use a).]

6.6 (Right Shift Semigroup on a Finite Interval). Let $\tau = (\tau_t)_{t\ge 0}$ be the right shift semigroup on $X = \mathrm{L}^p(0, 1)$, where $1 \le p < \infty$. This can be described as follows:

$$\tau_t x := \big(\tau_t \widetilde{x}\big)|_{(0,1)} \quad (x \in X,\ t \geq 0),$$

where $\widetilde{x}$ is the extension by 0 to $\mathbb{R}$ of x and τ on the right hand side is the just the regular representation of $\mathbb{R}$ on X. It is easy to see that this yields a C_0-semigroup on X. Let $-A$ be its generator.

a) Show that $\tau_t = 0$ for $t \geq 1$. Conclude that $\sigma(A) = \emptyset$ and A has a Hille–Phillips calculus for $\mathrm{LS}(\mathbb{C}_+{+}\omega)$ for each $\omega \in \mathbb{R}$ (however large).

b) Show that $A^{-1} = V$, the Volterra operator on X (see Section 1.4), and that $\sigma(V) = \{0\}$.

c) Let $r > 0$ and $(\alpha_n)_n$ be a sequence of complex numbers such that $M := \sup_{n \geq 0} |\alpha_n|\, r^{-n} < \infty$. Show that the function $f := \sum_{n=0}^{\infty} \alpha_n \mathbf{z}^{-n}$ is contained in $\mathrm{LS}(\mathbb{C}_+{+}\omega)$ for each $\omega > r$. Show further that

$$f(A) = \sum_{n=0}^{\infty} \alpha_n V^n$$

where $f(A)$ is defined via the Hille–Phillips calculus for A.

d) Find a function f such that $f(A)$ is defined via the Hille–Phillips calculus for A, but $g := f(\mathbf{z}^{-1})$ is not holomorphic at 0, so $g(V)$ is not defined via the Dunford–Riesz calculus for V.

[Hint: for the first part of b) observe that point evaluations are not continuous on $\mathrm{L}^p(0,1)$; for the second cf. Exercise 2.4; for c) cf. Exercise 6.3 and show that the series defining f converges in $\mathrm{FS}(\mathbb{C}_+{+}\omega)$.]

Remark: A closer look would reveal that $A = \frac{\mathrm{d}}{\mathrm{d}t}$ is the *weak derivative* operator with domain

$$\mathrm{dom}(A) = \mathrm{W}^{1,\mathrm{p}}_0(0,1) := \{u \in \mathrm{W}^{1,\mathrm{p}}(0,1) \mid u(0) = 0\}.$$

For the case $p = 2$ this can be found in [3, Section 10.2].

6.7 (Fourier–Stieltjes Calculus for Unbounded C_0-Groups). Let $-\mathrm{i}A$ be the generator of a C_0-group $U = (U_s)_{s \in \mathbb{R}}$ of type (M, ω). Define

$$\mathrm{M}_\omega(\mathbb{R}) := \Big\{\mu \in \mathrm{M}(\mathbb{R}) \;\big|\; \|\mu\|_{\mathrm{M}_\omega} := \int_{\mathbb{R}} \mathrm{e}^{\omega|s|}\, |\mu|\,(\mathrm{d}s) < \infty\Big\}.$$

Show the following assertions:

a) $\mathrm{M}_\omega(\mathbb{R})$ is unital subalgebra of $\mathrm{M}(\mathbb{R})$ and a Banach algebra with respect to the norm $\|\cdot\|_{\mathrm{M}_\omega}$.

b) The mapping

$$\mathrm{M}_\omega(\mathbb{R}) \to \mathcal{L}(X), \qquad \mu \mapsto U_\mu := \int_{\mathbb{R}} U_s\, \mu(\mathrm{d}s)$$

is a unital algebra homomorphism with $\|U_\mu\| \le M\|\mu\|_{\mathrm{M}_\omega}$.

c) The Fourier transform $\widehat{\mu}$ of $\mu \in \mathrm{M}_\omega(\mathbb{R})$ has a unique extension to a function continuous on the strip $\mathrm{St}_\omega = \{z \in \mathbb{C} \mid |\mathrm{Im}\, z| \le \omega\}$ and holomorphic in its interior.

d) The spectrum $\boldsymbol{\sigma}(A)$ of A is contained in the said strip and one has

$$\|R(\lambda, A)\| \le \frac{M}{|\mathrm{Im}\,\lambda| - \omega} \qquad (|\mathrm{Im}\,\lambda| > \omega).$$

[Hint for b): show and use that $\mathrm{M_c}(\mathbb{R})$ is dense in $\mathrm{M}_\omega(\mathbb{R})$ with respect to the norm $\|\cdot\|_{\mathrm{M}_\omega}$.]

The mapping Ψ_U defined on $\{\widehat{\mu} \mid \mu \in \mathrm{M}_\omega(\mathbb{R})\}$ by

$$\Psi_U(\widehat{\mu}) = \int_{\mathbb{R}} U_s\,\mu(\mathrm{d}s) \qquad (\mu \in \mathrm{M}_\omega(\mathbb{R}))$$

is called the **Fourier–Stieltjes** calculus associated with U.

6.8 (Multiparameter C_0-Semigroups). A (strongly continuous) representation $T : \mathbb{R}_+^d \to \mathcal{L}(X)$ is called a d-parameter semigroup (C_0-semigroup). Such d-parameter semigroups T are in one-to-one correspondence with d-tuples $(T^1, \dots, T^d)$ of pairwise commuting 1-parameter semigroups via

$$T(t_1\mathbf{e}_1 + \cdots + t_d\mathbf{e}_d) = T^1(t_1)\cdots T^d(t_d) \qquad (t_1, \dots, t_d \in \mathbb{R}_+),$$

cf. Remark 5.1.

a) Show that a d-parameter semigroup T is strongly continuous if and only if each T^j, $j = 1, \dots, d$, is strongly continuous.

Let T be a d-parameter C_0-semigroup. Then each of the semigroups T^j has a generator $-A_j$, say.

b) Show that T is uniquely determined by the tuple $(A_1, \dots, A_d)$.

c) Show that $\mathrm{dom}(A_1) \cap \cdots \cap \mathrm{dom}(A_d)$ is dense in X.

d) Let $-A$ be the generator of the C_0-semigroup S, defined by

$$S(t) := T(t\mathbf{e}_1 + \cdots + t\mathbf{e}_d) = T^1(t)\cdots T^d(t) \qquad (t \ge 0).$$

Show that $A = \overline{A_1 + \cdots + A_d}$. [Hint: use d) and Exercise 6.5.b).]

If, in addition, T is bounded, we can consider the associated Fourier–Stieltjes calculus. However, as in the case $d = 1$, one rather often works with the **Hille–Phillips calculus** which is based on the "d-dimensional" Laplace transform.

e) Try to give a definition of the Laplace transform of measures in $\mathrm{M}(\mathbb{R}_+^d)$ and built on it a construction of the "Hille–Phillips calculus" for bounded d-parameter C_0-semigroups. What is the connection between this calculus and the invidual calculi for the semigroups T^j?

6.9 (Dirac Sequences). Let $0 \in E \subseteq \mathbb{R}^d$ be closed. A **Dirac sequence** on E is a sequence $(\varphi_n)_n$ in $\mathrm{L}^1(\mathbb{R}^d)$ such that $\operatorname{supp}(\varphi_n) \subseteq E$ for all $n \in \mathbb{N}$ and

$$\int_{\mathbb{R}^d} f\varphi_n \,\mathrm{d}\lambda \to f(0) \qquad (n \to \infty)$$

whenever $f \in \mathrm{C_b}(E; X)$ and X is a Banach space.

a) Let $(\varphi_n)_n$ be a sequence in $\mathrm{L}^1(\mathbb{R}^d)$ with the following properties:
   1) $\operatorname{supp}(\varphi_n) \subseteq E$ for all $n \in \mathbb{N}$.
   2) $\sup_n \|\varphi_n\|_1 < \infty$.
   3) $\lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi_n = 1$.
   4) $\lim_{n\to\infty} \int_{|x|\ge\varepsilon} |\varphi_n(x)| \,\mathrm{d}x \to 0$ for all $\varepsilon > 0$.

   Show that $(\varphi_n)_n$ is a Dirac sequence on E.
b) Let $\varphi \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \varphi = 1$. Define $\varphi_n(x) := n^d\varphi(nx)$ for $x \in \mathbb{R}^d$. Show that $(\varphi_n)_n$ is a Dirac sequence on $E = \overline{\mathbb{R}_+ \operatorname{supp}(\varphi)}$.
c) Let $T : E \to \mathcal{L}(X)$ be bounded and strongly continuous, and let $(\varphi_n)_n$ be a Dirac sequence on E. Show that $T_{\varphi_n} \to \mathrm{I}$ strongly.

6.10 (Self-adjoint Semigroups). Show that for an operator A on a Hilbert space H the following assertions are equivalent:

(i) A is a positive self-adjoint operator.

(ii) $-A$ generates a bounded C_0-semigroup $T = (T_t)_{t\ge0}$ of self-adjoint operators.

Show further that in this case the Borel calculus for A coincides with the Hille–Phillips calculus for A on $\mathrm{LS}(\mathbb{C}_+)$.

Supplementary Exercises

6.11. Show that if $\mu_n \to 0$ strongly in $\mathrm{M}(\mathbb{R}^d)$ then $\mu_n \to 0$ weakly* (under the identification $\mathrm{M}(\mathbb{R}^d) \cong \mathrm{C}_0(\mathbb{R}^d)'$) and $\widehat{\mu_n} \to 0$ uniformly on compacts.

6.12. Let $(\mu_n)_n$ be a sequence in $\mathrm{M}(\mathbb{R}^d)$ such that $\mu_n \to \mu \in \mathrm{M}(\mathbb{R}^d)$ weakly* as functionals on $\mathrm{C}_0(\mathbb{R}^d)$. Show that $\mu_n \to \mu$ strongly if, in addition, there is a compact set $K \subseteq \mathbb{R}^d$ such that $\operatorname{supp}(\mu_n) \subseteq K$ for all $n \in \mathbb{N}$.

6.13. Let H be a Hilbert space and let $U : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d)$ be a strongly continuous representation of $\mathbb{R}^d$ by *unitary* operators on H. Show that the associated Fourier–Stieltjes calculus $\Psi_U : \mathrm{FS}(\mathbb{R}^d) \to \mathcal{L}(H)$ is a $*$-homomorphism. [Hint: recall Exercise 5.10.]

6.14. Let $T = (T_t)_{t\geq 0}$ be an operator semigroup on a Banach space X such that $\lim_{t\searrow 0} T_t = \mathrm{I}$ strongly. Show that T is strongly continuous on $\mathbb{R}_+$. [Hint: Show first that there is $\delta > 0$ such that $\sup_{0\leq t\leq \delta} \|T_t\| < \infty$. See also [1, Prop.I.5.3].]

6.15. Let Ω be a locally compact metric space and let $(e_t)_{t\geq 0}$ be a family of functions in $\mathrm{C_b}(\Omega)$ such that $e_{t+s} = e_t e_s$ for all t, $s \geq 0$ and $e_0 = \mathbf{1}$, and such that the operator family $(T_t)_{t\geq 0}$ defined by $T_t = M_{e_t}$ for all $t \geq 0$ is a bounded strongly continuous semigroup on $X = \mathrm{C_0}(\Omega)$.

a) For $x \in \Omega$ and $t \geq 0$ define $\varphi_x(t) := e_t(x)$. Show that there is a unique $a(x) \in \mathbb{C}$ such that $\varphi_x(t) = \mathrm{e}^{-ta(x)}$ for all $t \geq 0$.

b) Prove that $a(\Omega) \subseteq \overline{\mathbb{C}_+}$ and that a is continuous.

[Hint: For a) note first that φ is continuous; then the result is classical, see [1, Prop. I.1.4]; a different proof proceeds by showing that the Laplace transform of φ_x is $(\lambda + a(x))^{-1}$ for some $a(x) \in \mathbb{C}$ and all $\operatorname{Re}\lambda > 0$. For b) consider the operator $\int_0^\infty \mathrm{e}^{-t} T_t \,\mathrm{d}t$.]

6.16. Let $(\mu_n)_n$ be a sequence of Borel probability measures on $\mathbb{R}^d$ such that $\mu_n \to \delta_0$ weakly* as $n \to \infty$. Show that

$$\int_{\mathbb{R}^d} f \,\mathrm{d}\mu_n \to f(0) \qquad (n \to \infty)$$

whenever $f \in \mathrm{C_b}(\mathbb{R}^d; X)$ and X is any Banach space. Conclude that $\mu_n \to \delta_0$ strongly.

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Chapter 7
More about Abstract Functional Calculus

The Dunford–Riesz calculus from Chapter 1 as well as the Fourier–Stieltjes and Hille–Phillips calculi of the previous chapter are "bounded" calculi in the sense that they are representations of certain algebras in bounded operators. The multiplication operator calculi are so far our only examples of calculi that may really yield unbounded operators. So the question arises: are there interesting other examples that justify our quite general axiomatic notion of an abstract calculus from Chapter 2?

In the following we shall see that the answer to this question is affirmative and that one does not have to go far to find them. Namely, under some very natural assumptions, a given "bounded" calculus can be extended to an "unbounded" calculus for a "larger" algebra.

As an example, take again the Volterra operator V, which featured already in Section 1.4. It was observed there that the Dunford–Riesz calculus Φ for V produces only bounded operators, so the unbounded operator V^{-1}, which could be interpreted as inserting V into the function $\mathbf{z}^{-1}$, is not accessible. The problem would be solved if we could extend the Dunford–Riesz calculus towards an algebra of functions that contains also the function $\mathbf{z}^{-1}$.

7.1 Abstract Functional Calculus (III) — Extension

From now on, $\mathcal{F}$ denotes an algebra with unit $\mathbf{1}$ and $\mathcal{E}$ a subalgebra of $\mathcal{F}$ (not necessarily containing the unit element). Our goal is, eventually, to find conditions on $\mathcal{F}$ such that a given representation $\Phi : \mathcal{E} \to \mathcal{L}(X)$ can be extended to an $\mathcal{F}$-calculus in a unique way.

A first, rather trivial extension is possible if $\mathbf{1} \notin \mathcal{E}$. In this case, $\mathcal{E} \oplus \mathbb{C}\mathbf{1}$ is a unital subalgebra of $\mathcal{F}$ and by

$$\widehat{\Phi}(f) := \Phi(e) + \lambda \mathrm{I}, \qquad f = e + \lambda\mathbf{1},\, e \in \mathcal{E},\, \lambda \in \mathbb{C}$$

a representation $\widehat{\Phi} : \mathcal{E} \oplus \mathbb{C}\mathbf{1} \to \mathcal{L}(X)$ is defined. This extension is obviously the only possible one when the result is supposed to be a proto-calculus. So the real challenge lies in extending beyond $\mathcal{E} \oplus \mathbb{C}\mathbf{1}$.

Idea: Extension by Regularization

The basic idea for a non-trivial extension is (again) multiplicative regularization. Suppose that Φ is an $\mathcal{F}$-calculus and $\mathcal{E} \subseteq \mathcal{F}_\Phi$. (Recall from Chapter 2 that $\mathcal{F}_\Phi = \{h \in \mathcal{F} \mid \Phi(h) \in \mathcal{L}(X)\}$ is the set of Φ-bounded elements of $\mathcal{F}$.) Suppose that $f \in \mathcal{F}$ is such that there is $e \in \mathcal{E}$ with $ef \in \mathcal{E}$ again. Then it is easy to see that the following two assertions are equivalent:

(i) *e is Φ-determining for f;*

(ii) *$\Phi(e)$ is injective and $\Phi(f) = \Phi(e)^{-1}\Phi(ef)$.*

Suppose, conversely, that Φ is only defined on $\mathcal{E}$ (as a representation in $\mathcal{L}(X)$). Then we are tempted to construct an extension $\widehat{\Phi}$ of Φ by *defining*

$$\widehat{\Phi}(f) := \Phi(e)^{-1}\Phi(ef) \tag{7.1}$$

for those $f \in \mathcal{F}$ where one can find a suitable e.

For this approach to work one has to make sure that the operator $\Phi(e)^{-1}\Phi(ef)$ is independent of the chosen element e, that the set $\mathcal{F}'$ of all f which can be treated in this way is a subalgebra of $\mathcal{F}$, and that the definition of $\widehat{\Phi}(f)$ by (7.1) for $f \in \mathcal{F}'$ yields an $\mathcal{F}'$-calculus.

This approach, however, seems to fail without additional commutativity assumptions. In the "classical" situation (considered in [3]), the whole algebra $\mathcal{F}$ is supposed to be commutative. But, as we shall see below, it suffices to require that the regularizer e of f is a member of the **center**

$$\mathcal{Z}(\mathcal{E}) := \{e \in \mathcal{E} \mid \forall\, e' \in \mathcal{E} : ee' = e'e\,\}$$

of the algebra $\mathcal{E}$. (One does *not* need $ef \in \mathcal{Z}(\mathcal{E})$, and this makes our approach here more general than the "classical" one.)

One can generalize the "classical" results also in another respect. Namely, it has been observed for the first time in [4] that it is not necessary to confine oneself to those $f \in \mathcal{F}$ that are Φ-determined by a *single* element e, as above. The price that one pays for this additional amount of generality is that in the technical parts of many proofs one has to work with *anchor sets* instead of single *anchor elements*. The definition of these terms is our next goal.

Anchor Sets and Anchor Elements

Suppose that $\Phi : \mathcal{F} \to \mathcal{C}(X)$ is a proto-calculus. We say that a subset $\mathcal{M} \subseteq \mathcal{F}_\Phi$ is an **anchor set** (with respect to Φ) if

$$\bigcap_{e\in\mathcal{M}} \ker(\Phi(e)) = \{0\}. \tag{7.2}$$

And an element $e \in \mathcal{F}_\Phi$ is an called **anchor element** if the singleton $\{e\}$ is an anchor set, i.e., if $\Phi(e)$ is injective.

Clearly, a superset of an anchor set is an anchor set. Moreover, if $\mathcal{M}, \mathcal{N} \subseteq \mathcal{F}_\Phi$ are both anchor sets, thenso is

$$\mathcal{M} \cdot \mathcal{N} := \{fg \mid f \in \mathcal{M},\ g \in \mathcal{N}\}.$$

This follows easily from the multiplicativity of Φ on $\mathcal{F}_\Phi$. In particular, the product of two anchor elements is an anchor element.

Recall from Lemma 2.4 that each Φ-determining set for an element $f \in \mathcal{F}$ must be an anchor set, but the converse may not always be true. However, there are several technical results asserting that a given anchor set $\mathcal{M} \subseteq \mathrm{Reg}_\Phi(f)$ is actually Φ-determining, see Exercise 7.1 and Lemma 7.26 below.

Let us go back to our standard "extension problem"-situation: $\mathcal{E}$ is a subalgebra of a unital algebra $\mathcal{F}$ and $\Phi : \mathcal{E} \to \mathcal{L}(X)$ is an algebra homomorphism. We do not suppose that $\mathbf{1} \in \mathcal{E}$, but if it is, we assume that $\Phi(\mathbf{1}) = \mathrm{I}$.

Remark 7.1. The assumption that $\Phi(\mathbf{1}) = \mathrm{I}$ is not very restrictive. In any case, $P := \Phi(\mathbf{1})$ is a projection on X and $\Phi(e) = 0$ on $\ker(P)$ for each $e \in \mathcal{E}$. Hence, Φ can be regarded as a unital representation of $\mathcal{E}$ on the closed subspace $Y := \mathrm{ran}(P)$ of X.

The representation $\Phi : \mathcal{E} \to \mathcal{L}(X)$ is called **non-degenerate** if $\mathcal{Z}(\mathcal{E})$ is an anchor set, i.e., if

$$\bigcap_{e\in\mathcal{Z}(\mathcal{E})} \ker(\Phi(e)) = \{0\}.$$

Otherwise, Φ is called **degenerate**. And Φ is called **strictly non-degenerate** if $\mathcal{Z}(\mathcal{E})$ contains an anchor element. If $\mathcal{E}$ is unital and Φ is a unital representation, then Φ is strictly non-degenerate. Recall that one easily can extend a degenerate (and hence non-unital) representation to a unital one.

For $f \in \mathcal{F}$ we introduce the set

$$[f]_\mathcal{E} := \{e \in \mathcal{Z}(\mathcal{E}) \mid ef \in \mathcal{E}\},$$

which is an ideal in the commutative algebra $\mathcal{Z}(\mathcal{E})$. We say that $f \in \mathcal{F}$ is **anchored** in $\mathcal{E}$ (with respect to Φ) if $[f]_\mathcal{E}$ is an anchor set, and **strictly anchored** in $\mathcal{E}$ if $[f]_\mathcal{E}$ contains an anchor element e. (Such an element is then called an **anchor element for** f.) We drop the explicit reference to $\mathcal{E}$ and Φ if no confusion can arise.

Lemma 7.2. *Let $\mathcal{E}$ be a subalgebra of a unital algebra $\mathcal{F}$ and let $f,\, g \in \mathcal{F}$ and $\lambda \in \mathbb{C}$. Then*

$$[f]_{\mathcal{E}} \subseteq [\lambda f]_{\mathcal{E}} \quad \text{and} \quad [f]_{\mathcal{E}} \cdot [g]_{\mathcal{E}} \subseteq [fg]_{\mathcal{E}} \cap [f{+}g]_{\mathcal{E}}.$$

Furthermore, if e_f and e_g are anchor elements for f and g, respectively, then $e_f e_g$ is an anchor element for fg and for $f+g$.

Proof. Let $e_f \in [f]_{\mathcal{E}}$ and $e_g \in [g]_{\mathcal{E}}$. Then $e_f(\lambda f) = \lambda(e_f f) \in \mathcal{E}$, hence $e_f \in [\lambda f]_{\mathcal{E}}$. Furthermore, $e_f e_g \in \mathcal{Z}(\mathcal{E})$ and

$$(e_f e_g)(fg) = (e_g e_f)(fg) = (e_g(e_f f))g = (e_f f)(e_g g) \in \mathcal{E}.$$

Therefore, $e_f e_g \in [fg]_{\mathcal{E}}$. The proof that $e_f e_g \in [f{+}g]_{\mathcal{E}}$ is similar. Finally, if both $\Phi(e_f)$ and $\Phi(e_g)$ are injective, then so is $\Phi(e_f e_g) = \Phi(e_f)\Phi(e_g)$. □

Still in the situation from above, we let

$$\langle \mathcal{E} \rangle_{\Phi} := \{ f \in \mathcal{F} \mid [f]_{\mathcal{E}} \text{ is an anchor set} \}$$

be the set of all anchored elements and

$$\langle\!\langle\, \mathcal{E} \,\rangle\!\rangle_{\Phi} := \{ f \in \mathcal{F} \mid [f]_{\mathcal{E}} \text{ contains an anchor element} \}$$

the set of all strictly anchored elements of $\mathcal{F}$. Obviously, $\langle\!\langle\, \mathcal{E} \,\rangle\!\rangle_{\Phi} \subseteq \langle \mathcal{E} \rangle_{\Phi}$.

Corollary 7.3. *Let $\mathcal{E}$ be a subalgebra of a unital algebra $\mathcal{F}$ and let $\Phi : \mathcal{E} \to \mathcal{L}(X)$ be an algebra homomorphism. Then the following statements hold.*

a) *Φ is non-degenerate if and only if $\langle \mathcal{E} \rangle_{\Phi} \neq \emptyset$, in which case $\langle \mathcal{E} \rangle_{\Phi}$ is a unital subalgebra of $\mathcal{F}$ containing $\mathcal{E}$.*

b) *Φ is strictly non-degenerate if and only if $\langle\!\langle\, \mathcal{E} \,\rangle\!\rangle_{\Phi} \neq \emptyset$, in which case $\langle\!\langle\, \mathcal{E} \,\rangle\!\rangle_{\Phi}$ is a unital subalgebra of $\mathcal{F}$ containing $\mathcal{E}$.*

Proof. If $\langle \mathcal{E} \rangle_{\Phi} \neq \emptyset$ then $\mathcal{Z}(\mathcal{E})$ is an anchor set and, since $[\mathbf{1}]_{\mathcal{E}} = [e]_{\mathcal{E}} = \mathcal{Z}(\mathcal{E})$ for all $e \in \mathcal{E}$, we then have $\mathbf{1} \in \langle \mathcal{E} \rangle_{\Phi}$ and $\mathcal{E} \subseteq \langle \mathcal{E} \rangle_{\Phi}$. Similarly, if $\langle\!\langle\, \mathcal{E} \,\rangle\!\rangle_{\Phi} \neq \emptyset$, then there is an element $e \in \mathcal{Z}(\mathcal{E})$ such that $\Phi(e)$ is injective, and hence e is an anchor element for $\mathbf{1}$ and for each $e' \in \mathcal{E}$.

The remaining assertions follow from Lemma 7.2 and the observation that the set of products of two anchor sets/elements is again an anchor set/element. □

Remark 7.4. Suppose $\Phi : \mathcal{F} \to \mathcal{C}(X)$ is a proto-calculus. Then for each subalgebra $\mathcal{E} \subseteq \mathcal{F}_{\Phi}$ we can restrict Φ to $\mathcal{E}$ and consider the sets $\langle \mathcal{E} \rangle_{\Phi}$ and $\langle\!\langle\, \mathcal{E} \,\rangle\!\rangle_{\Phi}$. If $\mathcal{F}$ is commutative, then for each subalgbra $\mathcal{D}$ one has the implication

$$\mathcal{D} \subseteq \mathcal{E} \quad \Rightarrow \quad \langle \mathcal{D} \rangle_{\Phi} \subseteq \langle \mathcal{E} \rangle_{\Phi} \quad \wedge \quad \langle\!\langle \mathcal{D} \rangle\!\rangle_{\Phi} \subseteq \langle\!\langle \mathcal{E} \rangle\!\rangle_{\Phi}.$$

The same is true under less restrictive assumptions (e.g., if $\mathcal{Z}(\mathcal{D}) \subseteq \mathcal{Z}(\mathcal{E})$, but we doubt that it holds in general without additional assumptions.

The Extension Theorem

We can now state the central result of the present chapter.

Theorem 7.5 (Extension Theorem). *Let $\mathcal{F}$ be a unital algebra and $\mathcal{E} \subseteq \mathcal{F}$ a subalgebra of $\mathcal{F}$. Suppose that X is a Banach space and*

$$\Phi : \mathcal{E} \to \mathcal{L}(X)$$

is a non-degenerate algebra homomorphism. Then there is a unique $\langle \mathcal{E} \rangle_{\Phi}$-calculus

$$\widehat{\Phi} : \langle \mathcal{E} \rangle_{\Phi} \to \mathcal{C}(X)$$

such that $\widehat{\Phi}|_{\mathcal{E}} = \Phi$.
More precisely, for $f \in \langle \mathcal{E} \rangle_{\Phi}$ the operator $\widehat{\Phi}(f)$ is given by ($x,\, y \in X$):

$$\widehat{\Phi}(f)x = y \quad \iff \quad \forall\, e \in \mathcal{M} : \Phi(ef)x = \Phi(e)y, \tag{7.3}$$

where $\mathcal{M} \subseteq [f]_{\mathcal{E}}$ is any anchor set. In particular, if $e \in \mathcal{Z}(\mathcal{E})$ is an anchor element for f, then

$$\widehat{\Phi}(f) = \Phi(e)^{-1}\Phi(ef).$$

We postpone the proof of Theorem 7.5 to the supplementary Section 7.6.

Theorem 7.5 allows to extend any non-degenerate representation Φ of a subalgebra $\mathcal{E}$ of a unital algebra $\mathcal{F}$ to the subalgebra $\langle \mathcal{E} \rangle_{\Phi}$ of $\mathcal{E}$-anchored elements. We shall call this the **canonical extension** of Φ within $\mathcal{F}$, and denote it again by Φ (instead of $\widehat{\Phi}$ as in the theorem).

Is the canonical extension necessarily consistent with an already given calculus? The attempt to answer this question leads to the following notion.

Admissible Subalgebras

Suppose that $\mathcal{F}$ is a unital algebra, $\Phi : \mathcal{F} \to \mathcal{C}(X)$ is a proto-calculus and $\mathcal{E} \subseteq \mathcal{F}_{\Phi}$ is a subalgebra on which Φ is non-degenerate. Then one can restrict Φ to $\mathcal{E}$ and consider its canonical extension $\widehat{\Phi|_{\mathcal{E}}}$ to $\langle \mathcal{E} \rangle_{\Phi} \subseteq \mathcal{F}$. If this coincides with the restriction of the original calculus Φ to $\langle \mathcal{E} \rangle_{\Phi}$, $\mathcal{E}$ is called an **admissible** subalgebra of $\mathcal{F}$.

Lemma 7.6. *In the situation from before, the following assertions are equivalent:*

(i) *$\mathcal{E}$ is admissible, i.e.,* $\widehat{\Phi|_{\mathcal{E}}} = \Phi|_{\langle\mathcal{E}\rangle_\Phi}$.

(ii) *The restriction of Φ to $\langle\mathcal{E}\rangle_\Phi$ is a calculus.*[1]

(iii) *The set* $\mathrm{Reg}_\Phi(f) \cap \langle\mathcal{E}\rangle_\Phi$ *is Φ-determining for f, for each $f \in \langle\mathcal{E}\rangle_\Phi$.*

Proof. The implication (i) $\Rightarrow$ (ii) holds since by Theorem 7.5 the canonical extension is a calculus. The converse follows from the uniqueness part of that theorem. And the equivalence (ii) $\Leftrightarrow$ (iii) follows immediately from the definition of a calculus, since $\mathrm{Reg}_\Phi(f) \cap \langle\mathcal{E}\rangle_\Phi = \mathrm{Reg}_{\Phi|_{\langle\mathcal{E}\rangle_\Phi}}$. □

If $\mathcal{F}$ is commutative and Φ is a calculus (and not just a proto-calculus), then the situation is simple:

Corollary 7.7. *Let $\mathcal{F}$ be a commutative unital algebra, $\Phi : \mathcal{F} \to \mathcal{C}(X)$ an $\mathcal{F}$-calculus on a Banach space X and $\mathcal{E} \subseteq \mathcal{F}_\Phi$ a subalgebra on which Φ is non-degenerate. Then $\mathcal{E}$ is admissible.*

Proof. This is left as Exercise 7.3. □

Whether Corollary 7.7 holds without the assumption of commutativity is dubitable, but we neither have proof nor a counterexample yet.

Related to this is the question what happens if one performs an extension, say from $\mathcal{E}$ to $\langle\mathcal{E}\rangle_\Phi$, then takes a subalgebra $\mathcal{D}$ of Φ-bounded elements of $\langle\mathcal{E}\rangle_\Phi$ and performs another extension, now starting with $\mathcal{D}$. If $\mathcal{F}$ is commutative, then nothing strange can happen, and no "new" functions are included in the domain of the functional calculus (Exercise 7.4). If $\mathcal{F}$ is not commutative, it is unclear (at least to me) what can happen, and only special situations are well understood (Exercise 7.11).

The Generator of a Functional Calculus

Recall from Chapter 2 that an operator A is called the generator of an $\mathcal{F}$-calculus Φ if $\mathcal{F}$ is a space of functions on a set $D \subseteq \mathbb{C}$, the function $\mathbf{z} \in \mathcal{F}$ and $\Phi(\mathbf{z}) = A$. Later on, we extended this terminology towards the situation when $\Phi(\mathbf{z})$ is not, but $\Phi((\lambda - \mathbf{z})^{-1})$ is well defined for some $\lambda \in \mathbb{C}$.

By virtue of the canonical extension, one can unify such auxiliary definitions of a generator, in the following way. Suppose that $\mathcal{F}$ is an algebra of functions on a set $D \subseteq \mathbb{C}$ and $\Phi : \mathcal{E} \to \mathcal{L}(X)$ is a non-degenerate representation, where $\mathcal{E}$ is a subalgebra of $\mathcal{F}$. Then we call the operator A the **generator** of the calculus Φ if $\mathbf{z}$ is anchored in $\mathcal{E}$ with respect to Φ and $\widehat{\Phi}(\mathbf{z}) = A$. That is, A is the generator of the canonical extension of Φ.

In applications, the following situation is typical: there is an element $g \in \mathcal{E}$ and some $\lambda \in \mathbb{C} \setminus D$ such that $e := (\lambda - \mathbf{z})^{-1} g \in \mathcal{E}$ is an anchor element. Since

[1] Cf. Remark 2.5.1).

$$e \cdot \mathbf{z} = -g + \lambda e \in \mathcal{E},$$

e is an anchor element for $\mathbf{z}$. Hence, in such a situation, $\widehat{\Phi}(\mathbf{z}) = \Phi(e)^{-1}\Phi(e \cdot \mathbf{z})$ is the generator of Φ.

In particular, the above happens when $(\lambda - \mathbf{z})^{-1} \in \mathcal{E}$ and $\Phi((\lambda - \mathbf{z})^{-1})$ is injective, since one can then take $e = (\lambda - \mathbf{z})^{-1}$.

Corollary 7.8. *Let $\mathcal{F}$ be an algebra of functions on $D \subseteq \mathbb{C}$, let $\mathcal{E} \subseteq \mathcal{F}$ be a subalgebra and let $\Phi : \mathcal{E} \to \mathcal{L}(X)$ be a representation. Suppose that there is an operator A on X, a number $\lambda \in \rho(A) \setminus D$ and $g \in \mathcal{E}$ such that $\Phi(g)$ is injective, $g \cdot (\lambda - \mathbf{z})^{-1} \in \mathcal{E}$ and*

$$\Phi(g(\lambda - \mathbf{z})^{-1}) = \Phi(g)R(\lambda, A).$$

Then A is the generator of Φ.

Proof. By what we have seen above, with $e := g \cdot (\lambda - \mathbf{z})^{-1}$ we have $e \cdot \mathbf{z} = -g + \lambda e \in \mathcal{E}$ and e is an anchor element for $\mathbf{z}$. It follows that

$$\Phi(e\mathbf{z}) = \Phi(-g + \lambda e) = -\Phi(g) + \lambda \Phi(g)R(\lambda, A) = \Phi(g)[-\mathrm{I} + \lambda R(\lambda, A)].$$

Hence,

$$\begin{aligned} \widehat{\Phi}(\mathbf{z}) &= \Phi(e)^{-1}\Phi(e\mathbf{z}) = (\lambda - A)\Phi(g)^{-1}\Phi(g)[-\mathrm{I} + \lambda R(\lambda, A)] \\ &= (\lambda - A)[-\mathrm{I} + \lambda R(\lambda, A)] = A \end{aligned}$$

as claimed. □

The Extension Theorem will unfold its true power only in coming chapters. However, we can already review the calculi known so far in the light of Theorem 7.5.

7.2 Extension of the Dunford–Riesz Calculus

Let A be a bounded operator on a Banach space X and let $\Phi : \mathrm{Hol}(U) \to \mathcal{L}(X)$ be the Dunford–Riesz calculus, where U is some open superset of $\sigma(A)$. Without loss of generality, we may suppose that $V \cap \sigma(A) \neq \emptyset$ for each connected component V of U. (Make U smaller by discarding all the other components.)

Lemma 7.9. *In the described situation, let $e \in \mathrm{Hol}(U)$. Then for $\lambda \in [\, e = 0 \,]$ one has $\ker(\lambda - A) \subseteq \ker(\Phi(e))$. And $\Phi(e)$ is injective if and only if $[\, e = 0 \,]$ is discrete and contains no eigenvalue of A.*

Proof. If $\lambda \in U$ and $e(\lambda) = 0$ then there is $g \in \mathrm{Hol}(U)$ such that $e = g \cdot (\lambda - \mathbf{z})$. Hence, $\ker(\lambda - A) \subseteq \ker(\Phi(e))$ as claimed.

Suppose that $\Phi(e)$ is injective. Then, as just proved, $[\, e = 0\,]$ contains no eigenvalue of A. If $[\, e = 0\,]$ is not discrete, then $e = 0$ on some connected component V of U. Hence $0 = \Phi(e\mathbf{1}_V) = \Phi(e)\Phi(\mathbf{1}_V)$. As $\Phi(e)$ is injective, $\Phi(\mathbf{1}_V) = 0$. But this cannot happen as $\sigma(A) \cap V \neq \emptyset$. (Note that $\sigma(A) \cap V$ is the spectrum of the restriction of A to $\mathrm{ran}(\Phi(\mathbf{1}_V))$, cf. Exercise 1.8.)

Conversely, suppose that $[\, e = 0\,]$ is discrete. Then $[\, e = 0\,] \cap \sigma(A)$ is finite and hence

$$e = e_1 \prod_{j=1}^{d} (\mathbf{z} - \lambda_j)^{k_j}$$

for some $\lambda_1, \dots, \lambda_d \in U$, $k_1, \dots, k_d \in \mathbb{N}$ and some $e_1 \in \mathrm{Hol}(U)$ such that $\sigma(A) \subseteq [\, e_1 \neq 0\,] =: U_1$. Since e_1 is an invertible element in $\mathrm{Hol}(U_1)$ and the Dunford–Riesz calculi on $\mathrm{Hol}(U)$ and $\mathrm{Hol}(U_1)$ are compatible, $\Phi(e_1)$ is an invertible operator. So if in addition no λ_j is an eigenvalue of A, the operator

$$\Phi(e) = \Phi(e_1) \prod_{j=1}^{d} (A - \lambda_j)^{k_j}$$

is injective. □

In order to extend the Dunford–Riesz calculus on $\mathrm{Hol}(U)$ we choose $\mathcal{F} = \mathrm{Mer}(U)$, the space of **meromorphic** functions on U. Recall that a function f on U is meromorphic if f maps U into the **Riemann sphere** $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ such that the set $P_f := [\, f = \infty\,]$ is discrete, f is holomorphic on $U \setminus P_f$ and each $\lambda \in P_f$ is a pole of f.

A standard (but non-trivial) result of complex analysis is that $\mathrm{Mer}(U)$ consists of quotients g/e where g, e are holomorphic on U with no common zeroes and e has a discrete zero set [9, Section 4.1.5]. We shall employ this in the following characterization.

Theorem 7.10. *Let, as above, $\Phi : \mathrm{Hol}(U) \to \mathcal{L}(X)$ be the Dunford–Riesz calculus of a bounded operator A on a Banach space X, where $U \subseteq \mathbb{C}$ is an open superset of $\sigma(A)$ such that each connected component of U contains a point of $\sigma(A)$. Then the following assertions are equivalent for $f \in \mathrm{Mer}(U)$:*

(i) *f is anchored in $\mathrm{Hol}(U)$, i.e., $f \in \langle \mathrm{Hol}(U) \rangle_\Phi$;*

(ii) *There is $e \in \mathrm{Hol}(U)$ such that $ef \in \mathrm{Hol}(U)$ and $\Phi(e)$ is injective;*

(iii) *No eigenvalue of A is a pole of f.*

Proof. (i) ⇒ (iii): Suppose $\lambda \in U$ is a pole of f and $e \in [f]_{\mathrm{Hol}(U)}$. Then ef is holomorphic, and hence $e(\lambda) = 0$. By Lemma 7.9, $\ker(\lambda - A) \subseteq \ker(\Phi(e))$. But $[f]_{\mathrm{Hol}(U)}$ is an anchor, which implies that $\ker(\lambda - A) = \{0\}$.

(iii) ⇒ (ii): Let $f \in \mathrm{Mer}(U)$ such that $P_f \cap \sigma_{\mathrm{p}}(A) = \emptyset$. Find, by the mentioned characterization, functions $g, e \in \mathrm{Hol}(U)$ such that $[\, e = 0\,]$ is discrete and

$f = \frac{g}{e}$. Then $[\, e = 0\,] \cap \boldsymbol{\sigma}(A)$ is finite. As f has no poles in $\boldsymbol{\sigma}_{\rm p}(A)$, by dividing out from e and g a finite number of linear polynomials we may suppose without loss of generality that $[\, e = 0\,] \cap \boldsymbol{\sigma}_{\rm p}(A) = \emptyset$. By Lemma 7.9, $\Phi(e)$ is injective.

(ii) $\Rightarrow$ (i): This is trivial, since (ii) just says that f is strictly anchored in $\mathrm{Hol}(U)$. □

See Exercise 7.5 for a direct proof of the implication (i) $\Rightarrow$ (ii) that does not use the characterization of $\mathrm{Mer}(U)$ as the ring of quotients of $\mathrm{Hol}(U)$.

From Theorem 7.10 we obtain a good understanding how the canonical extension of the Dunford–Riesz calculus for a bounded operator A looks like. We can insert A into functions f which are meromorphic on a neighborhood of $\boldsymbol{\sigma}(A)$ with no pole of f being an eigenvalue of A. Since f can have only finitely many poles within $\boldsymbol{\sigma}(A)$, one can write it as

$$f = g \cdot \prod_{j=1}^{d} (\mathbf{z} - \lambda_j)^{-k_j}$$

where g is holomorphic on a neighborhood of $\boldsymbol{\sigma}(A)$, the λ_j are pairwise different members of $\boldsymbol{\sigma}(A) \setminus \boldsymbol{\sigma}_{\rm p}(A)$ and each $k_j \ge 0$. Then $f(A)$ is computed as

$$f(A) = \Bigl(\prod_{j=1}^{d} (A - \lambda_j)^{-k_j}\Bigr)\, g(A).$$

7.3 Extensions of the Hille–Phillips and the Fourier–Stieltjes Calculus

Suppose that $-A$ is the generator of a bounded C_0-semigroup $T = (T_t)_{t\ge 0}$ on a Banach space X with associated Hille–Phillips calculus $\Phi_T : \mathrm{LS}(\mathbb{C}_+) \to \mathcal{L}(X)$. We interpret elements of the Laplace-Stieltjes algebra $\mathrm{LS}(\mathbb{C}_+)$ as holomorphic functions on $\mathbb{C}_+$. A natural superalgebra is $\mathrm{Mer}(\mathbb{C}_+)$ the space of meromorphic functions on $\mathbb{C}_+$. The canonical extension of Φ_T within this algebra is called the **extended Hille–Phillips calculus** and denoted (again) with Φ_T. However, we often use the notation "$f(A)$" instead of "$\Phi_T(f)$", and say that $f(A)$ is defined in the extended Hille–Phillips calculus.

Remark 7.11. For which $f \in \mathrm{Mer}(\mathbb{C}_+)$ is this the case? In general, this is a much more difficult question than for the Dunford–Riesz calculus. To understand this, recall that f must be anchored in $\mathrm{LS}(\mathbb{C}_+)$ and this amounts to finding (enough) functions $e \in \mathrm{LS}(\mathbb{C}_+)$ such that $ef \in \mathrm{LS}(\mathbb{C}_+)$ again. So firstly, e has to compensate the singularities of f. But what is more, the function $g := ef$ must be recognized as the Laplace transform of a complex measure. Equivalently, $g \in \mathrm{C_b}(\overline{\mathbb{C}_+}) \cap \mathrm{Hol}(\mathbb{C}_+)$ and $g(\mathbf{i}\mathbf{s})$ is the Fourier transform of a complex measure. (This is a well-known consequence of the classical Paley–Wiener theorems from [8]. For a direct proof see [1, Lemma 1], a larger picture gives [5, p.172].)

Unfortunately, it is quite difficult to decide in general whether a given function is the Fourier transform of a measure. (Exercises 12.6 and 13.2 below are related to this problem.)

Because of the mentioned difficulties, the following statements—although important—are of a rather elementary character.

Lemma 7.12. *Let $-A$ be the generator of a bounded C_0-semigroup on a Banach space X. Then the following assertions hold.*

a) *For each polynomial $p = \sum_{k=0}^{n} \alpha_j z^k \in \mathbb{C}[z]$ the operator $p(A)$ is defined in the extended Hille–Phillips calculus and*

$$p(A) = \sum_{k=0}^{n} \alpha_k A^k.$$

b) *If $\operatorname{Re}\lambda \geq 0$ and $\lambda - A$ is injective, then $(\lambda - \mathbf{z})^{-1}(A)$ is defined in the extended Hille–Phillips calculus and $(\lambda - \mathbf{z})^{-1}(A) = (\lambda - A)^{-1}$.*

Proof. a) By the remarks preceding Corollary 7.8, we know that $\mathbf{z}$ is anchored in $\mathrm{LS}(\mathbb{C}_+)$ and $(\mathbf{z})(A) = A$. Since the anchored functions form an algebra, $p(A)$ is defined for each polynomial p. The remaining part is actually true in any proto-calculus, see Exercise 7.6.

b) The function $f := (\lambda - \mathbf{z})^{-1}$ is anchored in $\mathrm{LS}(\mathbb{C}_+)$ by $e = \frac{\lambda - \mathbf{z}}{1+\mathbf{z}}$. Indeed, the functions

$$e = \frac{\lambda + 1}{1 + \mathbf{z}} - \mathbf{1} \quad \text{and} \quad ef = (1 + \mathbf{z})^{-1}$$

are both in $\mathrm{LS}(\mathbb{C}_+)$, and the operator

$$e(A) = (\lambda + 1)(1 + A)^{-1} - \mathrm{I} = (\lambda - A)(1 + A)^{-1}$$

is injective. The identity $f(A) = (\lambda - A)^{-1}$ follows from Theorem 2.3.c). □

The next result tells that it does not matter whether one starts with the full algebra $\mathrm{LS}(\mathbb{C}_+)$ or with the subalgebra

$$\mathrm{A}(\mathbb{C}_+) := \{\mathcal{L}f \mid f \in \mathrm{L}^1(\mathbb{R}_+)\}$$

as a point of departure for the extension theorem.

Lemma 7.13. *Let $T = (T_t)_{t\geq 0}$ be a bounded C_0-semigroup on a Banach space X, let $\Phi_T : \mathrm{LS}(\mathbb{C}_+) \to \mathcal{L}(X)$ be the associated Hille–Phillips calculus, and let $f \in \mathrm{Mer}(\mathbb{C}_+)$ be such that f is (strictly) anchored in $\mathrm{LS}(\mathbb{C}_+)$ with respect to Φ_T. Then f is (strictly) anchored in $\mathrm{A}(\mathbb{C}_+)$.*

Proof. Let $\gamma \in \mathrm{L}^1(\mathbb{R}_+)$ such that $e := \mathcal{L}\gamma$ is an anchor element. (Such functions exist: one can take, e.g., $\gamma = \mathrm{e}^{-\mathbf{t}}\mathbf{1}_{\mathbb{R}_+}$, so that $\Phi_T(e) = (1 + A)^{-1}$, where $-A$ is the generator of T.) By hypothesis, there is a Φ_T-anchor set $\mathcal{N} \subseteq [f]_{\mathrm{LS}(\mathbb{C}_+)}$. Hence, the product set $\mathcal{M} := \mathcal{L}\gamma \cdot \mathcal{N}$ is an anchor set as well. But $\mathcal{M} \subseteq \mathrm{A}(\mathbb{C}_+)$ since $\mathrm{A}(\mathbb{C}_+)$ is an ideal in $\mathrm{LS}(\mathbb{C}_+)$. If $\mathcal{N}$ is a singleton, then so is $\mathcal{M}$, and the claim is proved. □

Let us turn to bounded strongly continuous d-parameter groups $(U_s)_{s\in\mathbb{R}^d}$ with associated Fourier–Stieltjes calculus Ψ_U. Here, a possible choice of a superalgebra of $\mathrm{FS}(\mathbb{R}^d)$ could be the algebra of formal quotients

$$\frac{f}{g}$$

where $f,\ g \in \mathrm{C}(\mathbb{R}^d)$ and $[\, g \neq 0\,]$ is dense in $\mathbb{R}^d$. Quotients as above can be interpreted as equivalence classes of functions on $\mathbb{R}^d$ modulo equality on a dense subset. Note that functions of the form $\frac{1}{\lambda - \mathbf{z}_j}$ for $\lambda \in \mathbb{R}$ and $1 \le j \le d$ are included in this algebra.

Remark 7.14. Quotient rings are a topic from abstract algebra, see [6, II,§4]. For the construction to work, one needs to know that the set $\{g \in \mathrm{C}(\mathbb{R}^d) \mid [\, g \neq 0\,]$ is dense in $\mathbb{R}^d\}$ is multiplicative, which is a little exercise. The algebra $\mathrm{C}(\mathbb{R}^d)$ embeds injectively into this quotient algebra via $f \mapsto f/\mathbf{1}$, another little exercise.

It is relatively easy to see that the analogue of Lemma 7.12 holds true in this situation. Also, with a completely analogous proof one can establish the following group version of Lemma 7.13.

Lemma 7.15. *Let $U = (U_s)_{s\in\mathbb{R}^d}$ be a bounded C_0-group on a Banach space X, let $\Psi_U : \mathrm{FS}(\mathbb{R}^d) \to \mathcal{L}(X)$ be the associated Fourier–Stieltjes calculus, and let f be (strictly) anchored in* $\mathrm{FS}(\mathbb{R}^d)$ *with respect to Ψ_U. Then f is (strictly) anchored in* $\mathrm{A}(\mathbb{R}^d)$.

Supplement: Universal Anchor Sets

By the next result one can more or less easily identify anchor sets and elements in the group case. (We confine ourselves to one-parameter groups, but a d-dimensional analogue holds.)

Theorem 7.16. *The following assertions are equivalent for a set $E \subseteq \mathrm{A}(\mathbb{R})$:*

(i) *For each $z \in \mathbb{R}$ there is $e \in E$ such that $e(z) \neq 0$.*

(ii) *For each bounded C_0-group $(U_s)_{s\in\mathbb{R}}$ on a Banach space X the set E is an anchor set with respect to Ψ_U.*

Proof. (i) ⇒ (ii): For each $e \in E$ let $\gamma_e \in \mathrm{L}^1(\mathbb{R})$ be such that $\widehat{\gamma_e} = e$. Furthermore, let $x \in X$ such that for all $e \in E$

$$\int_{\mathbb{R}} \gamma_e(s) U_s x \,\mathrm{d}s = \Psi_U(e)x = 0. \tag{7.4}$$

Fix $x' \in X'$ and define $f(s) := \langle U_s x, x' \rangle$ for $s \in \mathbb{R}$. Multiplying (7.4) with U_t and applying x' yields $\mathcal{S}\gamma_e * f = 0$. The set $I := \{\varphi \in \mathrm{L}^1(\mathbb{R}) \mid \mathcal{S}\varphi * f = 0\}$ is a closed convolution ideal in $\mathrm{L}^1(\mathbb{R})$ and it contains all the functions γ_e, $e \in E$. By Wiener's theorem [10, Theorem 9.4], $I = \mathrm{L}^1(\mathbb{R})$. This implies $f = 0$ (use a Dirac sequence, see Exercise 6.9). As $x' \in X'$ was arbitrary, it follows that $x = 0$.

(ii) ⇒ (i): Fix $z \in \mathbb{R}$, let $X = \mathbb{C}$ and $U_s := \mathrm{e}^{-\mathrm{i}sz} \in \mathbb{C} \cong \mathcal{L}(X)$. Then $\Psi_U(e) = e(z)$ for all $e \in \mathrm{FS}(\mathbb{R})$. Since, by hypothesis, E is a anchor set, there must be at least one $e \in E$ such that $e(z) \neq 0$. □

The semigroup analogue of Theorem 7.16 is a little more complicated to formulate.

Theorem 7.17. *The following assertions are equivalent for a set* $E \subseteq \mathrm{A}(\mathbb{C}_+)$*:*

(i) *For each* $z \in \mathbb{C}$ *with* $\operatorname{Re} z \geq 0$ *there is* $e \in E$ *such that* $e(z) \neq 0$*; and for each* $\varepsilon > 0$ *there is* $e \in E$ *such that* $\mathcal{L}^{-1}(e) \neq 0$ *on* $[0, \varepsilon]$.

(ii) *For each bounded* C_0*-semigroup* $(T_s)_{s \geq 0}$ *on a Banach space* X *the set* E *is an anchor set with respect to* Φ_T.

The proof of Theorem 7.17 is analogous to the proof of Theorem 7.16. However, one has to deal with convolution ideals in $\mathrm{L}^1(\mathbb{R}_+)$ and needs *Nyman's theorem*, which is the analogue of Wiener's theorem in that context, see [7] or [2].

7.4 The Spectral Theorem Revisited

With the extension theorem in mind we can now shed new light on the spectral theorem for normal operators.

Theorem 7.18. *Let* (K, Σ) *be a measurable space,* H *a Hilbert space and* $\Phi : \mathcal{M}_{\mathrm{b}}(K, \Sigma) \to \mathcal{L}(H)$ *a unital* $*$*-homomorphism satisfying* (MFC6)*. Then* Φ *extends uniquely to a measurable functional calculus* $\mathcal{M}(K, \Sigma) \to \mathcal{C}(H)$.

Proof. We apply the extension theorem with $\mathcal{E} = \mathcal{M}_{\mathrm{b}}(K, \Sigma)$ and $\mathcal{F} = \mathcal{M}(K, \Sigma)$. First we prove that $\langle \mathcal{E} \rangle_\Phi = \mathcal{F}$. To this end, fix $f \in \mathcal{F}$ and define $e_n := \mathbf{1}_{[\,|f| \leq n\,]} \in \mathcal{E}$. Then $e_n \to \mathbf{1}$ pointwise and boundedly, hence $\Phi(e_n) \to \mathrm{I}$ strongly. In particular $\{e_n \mid n \in \mathbb{N}\} \subseteq [f]_{\mathcal{E}}$ is an anchor set.

Hence, the extension theorem applies and yields the canonical extension $\Phi : \mathcal{F} \to \mathcal{C}(H)$. In order to see that this is a measurable functional calculus, it remains to check (MFC4). For each $f \in \mathcal{F}$ the operator $\Phi(f)$ is densely defined since $\mathrm{dom}(\Phi(f))$ contains the subspace $\bigcup_n \mathrm{ran}(\Phi(e_n))$, which is dense in H.

For the proof that $\Phi(f)^* = \Phi(\overline{f})$ we first note that $x \in \mathrm{dom}(\Phi(f))$ implies $\Phi(e_n)x \in \mathrm{dom}(\Phi(f))$ and $\Phi(f)\Phi(e_n)x = \Phi(fe_n)x = \Phi(e_n)\Phi(f)x$. Since $\Phi(e_n) \to \mathrm{I}$ strongly, one has for all $u, v \in H$:

$$\Phi(f)^*u = v \quad\Leftrightarrow\quad \forall\, n \in \mathbb{N},\, x \in \mathrm{dom}(\Phi(f)) : (\, u \,|\, \Phi(fe_n)x\,) = (\, v \,|\, \Phi(e_n)x\,)\,.$$

Since $e_n f$ and e_n are in $\mathcal{E}$ and Φ is a $*$-homomorphism by hypothesis, $\Phi(fe_n)^* = \Phi(\overline{f}e_n)$ and $\Phi(e_n)^* = \Phi(e_n)$. Hence, we can equivalently write

$$\forall\, n \in \mathbb{N},\, x \in \mathrm{dom}(\Phi(f)) : \big(\Phi(\overline{f})\Phi(e_n)u \,\big|\, x\big) = (\Phi(e_n)v \,|\, x\,)\,.$$

Since $\mathrm{dom}(\Phi(f))$ is dense, this is equivalent to

$$\forall\, n \in \mathbb{N} : \Phi(\overline{f})\Phi(e_n)u = \Phi(e_n)v,$$

which, in turn, is equivalent to $(u, v) \in \Phi(\overline{f})$. (Use again that $\Phi(e_n) \to \mathrm{I}$ strongly.) □

Theorem 7.18 tells that for establishing a measurable functional calculus it suffices to construct its restriction to bounded functions. The rest is canonical.

Remark 7.19. Theorem 7.18 provides in a way the missing step from projection-valued measures to a measurable functional calculus. Namely, it is relatively easy to associate to a projection-valued measure E on a measurable space (K, Σ) a unital $*$-representation $\Phi : \mathcal{M}_\mathrm{b}(K, \Sigma) \to \mathcal{L}(H)$ that satisfies (MFC6). This was indicated in Section 3.3 right after the definition of projection-valued measures. See also Exercise 7.8.

By Theorem 7.18, Φ extends canonically to a measurable functional calculus. In our view, this approach is much more perspicuous than the classical construction as performed, e.g., in [10, 13.22-13.25].

The following example[2] shows that without the assumption of (MFC6) in Theorem 7.18 one can encounter quite degenerate situations.

Supplementary Example 7.20. Let $K = \mathbb{N}$ and $\Sigma = \mathcal{P}(\mathbb{N})$. Then

$$\mathcal{E} := \mathcal{M}_\mathrm{b}(K, \Sigma) = \ell^\infty \quad \textit{and} \quad \mathcal{F} := \mathcal{M}(K, \Sigma) = \mathbb{C}^\mathbb{N},$$

the space of all sequences.

For each strictly increasing mapping ("subsequence") $\pi : \mathbb{N} \to \mathbb{N}$ pick a non-zero multiplicative functional $\Phi_\pi : \ell^\infty \to \mathbb{C}$ which vanishes on the ideal

[2] Thanks to Hendrik Vogt for the core idea!

of sequences $x = (x_n)_n \in \ell^\infty$ such that $\lim_{n\to\infty} x_{\pi(n)} = 0$. This exists: by Zorn's lemma there is a maximal ideal M_π containing this ideal and by the Gelfand–Mazur theorem $\ell^\infty/M_\pi \cong \mathbb{C}$ as Banach algebras. By the commutative Gelfand–Naimark theorem, Φ_π is a unital $*$-homomorphism. (Alternatively one can define Φ_π as the ultrafilter limit with respect to some ultrafilter that contains all the "tails" $\{\pi(k) : k \geq n\}$ for $n \in \mathbb{N}$.)

Now let I be the set of all such subsequences π, let $H := \ell^2(I)$ and define

$$\Phi : \ell^\infty \to \ell^\infty(I) \subseteq \mathcal{L}(H), \qquad \Phi(x) := (\Phi_\pi(x))_\pi,$$

where we identify a bounded function on I with the associated multiplication operator on $H = \ell^2(I)$. Then Φ is a unital $*$-homomorphism.

If $f : \mathbb{N} \to \mathbb{C}$ is any unbounded sequence, then there is a subsequence $\pi \in I$ along which $|f|$ converges to $+\infty$. Hence, if $e \in \ell^\infty$ is such that $ef \in \ell^\infty$ as well, then $e(n)$ converges to zero along π. Consequently, $\Phi_\pi(e) = 0$. Let $\delta_\pi \in H$ be the canonical unit vector which is 1 at π and 0 else. Then $\Phi(e)\delta_\pi = \Phi_\pi(e)\delta_\pi = 0$. This shows that $[f]_\mathcal{E}$ is not an anchor set. It follows that no unbounded function is anchored in ℓ^∞, i.e., $\langle \ell^\infty \rangle_\Phi = \ell^\infty$. In this example, the extension theorem does not lead to a proper extension of the original calculus. Note that (MFC6) fails for Φ.

7.5 Supplement: Approximate Identities in Abstract Functional Calculi

This supplement has nothing to do with the extension problem. Instead, we look at abstract conditions to ensure the formulae

$$\overline{\Phi(f) + \Phi(g)} = \Phi(f+g) \quad \text{and} \quad \overline{\Phi(f)\Phi(g)} = \Phi(fg) \tag{7.5}$$

to hold for an $\mathcal{F}$-(proto)-calculus Φ and $f,\, g \in \mathcal{F}$. The main tool to obtain (7.5) is the following concept.

Definition 7.21. Let $\mathcal{F}$ be a commutative unital algebra, $\Phi : \mathcal{F} \to \mathcal{C}(X)$ a proto-calculus, and $\mathcal{E} \subseteq \mathcal{F}_\Phi$ a subalgebra of Φ-bounded elements. A **(weak) approximate identity** in $\mathcal{E}$ (with respect to Φ) is a sequence $(e_n)_n$ in $\mathcal{E}$ such that $\Phi(e_n) \to \mathrm{I}$ strongly (weakly) as $n \to \infty$.

Note that by the uniform boundedness principle, a weak approximate identity $(e_n)_n$ is uniformly Φ-bounded, i.e., satisfies $\sup_{n\in\mathbb{N}} \|\Phi(e_n)\| < \infty$.

We say that $f \in \mathcal{F}$ **admits** a (weak) approximate identity in $\mathcal{E}$ if $[f]_\mathcal{E}$ contains a (weak) approximate identity. More generally, we say that the elements f of a subset $\mathcal{M} \subseteq \mathcal{F}$ admit a **common** (weak) approximate identity in $\mathcal{E}$, if $\bigcap_{f\in\mathcal{M}} [f]_\mathcal{E}$ contains a (weak) approximate identity.

Lemma 7.22. *In the situation of Definition 7.21 if $f \in \mathcal{F}$ admits a weak approximate identity, then f is anchored in $\mathcal{E}$ and $\Phi(f)$ is densely defined. Moreover, if $(e_n)_n$ is a weak approximate identity in $[f]_{\mathcal{E}}$, then*

$$D := \operatorname{span} \bigcup_{n\in\mathbb{N}} \operatorname{ran}(\Phi(e_n))$$

is dense in X and a core for $\Phi(f)$. If $(e_n)_n$ is an approximate identity then $\Phi(e_n)x \to x$ in the Banach space $\operatorname{dom}(\Phi(f))$ for each $x \in \operatorname{dom}(\Phi(f))$.

Proof. Let $(e_n)_n$ be a weak approximate identity in $[f]_{\mathcal{E}}$. Then, clearly, $\bigcap_{n\in\mathbb{N}} \ker(\Phi(e_n)) = \{0\}$, so $\{e_n \mid n \in \mathbb{N}\}$ is an anchor set. By definition of a weak approximate identity, the space D is weakly dense in X, so by Mazur's theorem (a consequence of Hahn–Banach), D is dense in X.

Note that $\Phi(f)\Phi(e_n) = \Phi(fe_n) \in \mathcal{L}(X)$ and hence $\operatorname{ran}(\Phi(e_n)) \subseteq \operatorname{dom}(\Phi(f))$ for each $n \in \mathbb{N}$. Therefore, $D \subseteq \operatorname{dom}(\Phi(f))$. As D is dense in X, $\Phi(f)$ is densely defined.

Let x, $y \in X$ such that $\Phi(f)x = y$. Then for each $n \in \mathbb{N}$,

$$\Phi(f)\Phi(e_n)x = \Phi(e_n f)x = \Phi(e_n)y.$$

Since $(\Phi(e_n)x, \Phi(e_n)y) \to (x, y)$ weakly, the space $\Phi(f)|_D$—considered as a subspace of $X \oplus X$—is weakly dense in $\Phi(f)$. By Mazur's theorem again, this space is strongly dense, hence D is a core for $\Phi(f)$.

For the final assertion simply re-read the last argument with strong in place of weak convergence. □

Corollary 7.23. *Still in the situation from above, let $f, g \in \mathcal{F}$ and $\lambda \in \mathbb{C}$, and suppose that f admits a weak and g admits a strong approximate identity in $\mathcal{E}$. Then λf, $f + g$, and fg admit a common weak approximate identity and one has*

$$\overline{\Phi(f) + \Phi(g)} = \Phi(f + g) \quad \textit{and} \quad \overline{\Phi(f)\Phi(g)} = \Phi(fg).$$

Proof. The first assertion follows from the fact that if $(e_n)_n$ is a weak and $(e'_n)_n$ is a strong approximate identity, then $(e_n e'_n)_n$ is a weak approximate identity. The remaining assertions follow from the inclusions

$$\operatorname{ran}(\Phi(e_n e'_n)) \subseteq \operatorname{dom}(\Phi(f)) \cap \operatorname{dom}(\Phi(g)) \cap \operatorname{dom}(\Phi(f)\Phi(g))$$

and the next-to-last assertion in Lemma 7.22. □

Example 7.24. If $\Phi : \mathcal{M}(K, \Sigma) \to \mathcal{L}(H)$ is any measurable functional calculus then each $f \in \mathcal{M}(K, \Sigma)$ admits an approximate identity in $\mathcal{E} = \mathcal{M}_{\mathrm{b}}(K, \Sigma)$, for instance $e_n := \frac{n}{n+|f|}$ or $e_n := \mathbf{1}_{[\,|f|\leq n\,]}$, $n \in \mathbb{N}$.

Example 7.25. Suppose that $-A$ generates a bounded C_0-semigroup on a Banach space X and $f \in \operatorname{Mer}(\mathbb{C}_+)$ is such that $f(A)$ is defined in the extended

Hille–Phillips calculus for A. If

$$D_\infty := \bigcap_{n\geq 0} \operatorname{dom}(A^n)$$

is contained in $\operatorname{dom}(f(A))$ then f admits an approximate identity. In particular, D_∞ is a core for $f(A)$ (Exercise 7.7).

7.6 Supplement: Proof of the Extension Theorem

In this supplement we provide a proof of the extension theorem, Theorem 7.5. We start with an auxiliary result.

Lemma 7.26. *Let $\Phi : \mathcal{F} \to \mathcal{C}(X)$ be a proto-calculus and $\mathcal{E} \subseteq \mathcal{F}_\Phi$ a subalgebra. If $f \in \langle\mathcal{E}\rangle_\Phi$ is such that $\operatorname{Reg}_\Phi(f) \cap \langle\mathcal{E}\rangle_\Phi$ is Φ-determining for f, then each anchor set $\mathcal{M} \subseteq [f]_\mathcal{E}$ is Φ-determining for f.*

Proof. Let $\mathcal{M} \subseteq [f]_\mathcal{E}$ be an anchor set and let $x, y \in X$ such that $\Phi(ef)x = \Phi(e)y$ for all $e \in \mathcal{M}$. Let $g \in \operatorname{Reg}_\Phi(f) \cap \langle\mathcal{E}\rangle_\Phi$. Then for each $e \in \mathcal{M}$ and $e_g \in [g]_\mathcal{E}$

$$\begin{aligned}\Phi(e)\Phi(e_g)\Phi(gf)x &= \Phi(ee_g gf)x = \Phi((e_g g)(ef))x = \Phi(e_g g)\Phi(ef)x \\ &= \Phi(e_g g)\Phi(e)y = \Phi((e_g g)e)y = \Phi(ee_g g)y = \Phi(e)\Phi(e_g)\Phi(g)y.\end{aligned}$$

(Here we have used that $e \in \mathcal{Z}(\mathcal{E})$ commutes with $e_g g \in \mathcal{E}$.) Since $\mathcal{M}$ and $[g]_\mathcal{E}$ are both $\mathcal{E}$-anchor sets with respect to Φ, it follows that

$$\Phi(gf)x = \Phi(g)y.$$

As g was an arbitrary element of the Φ-determining set $\operatorname{Reg}_\Phi(f) \cap \langle\mathcal{E}\rangle_\Phi$, it follows that $\Phi(f)x = y$. □

We can now go *in medias res*.

Proof of the Extension Theorem 7.5. Without loss of generality, we may suppose that $\mathcal{F} = \langle\mathcal{E}\rangle_\Phi$.

Uniqueness: Suppose $\widehat{\Phi}$ is an $\mathcal{F}$-calculus that extends Φ, let $f \in \mathcal{F}$ and let $\mathcal{M} \subseteq [f]_\mathcal{E}$ be an anchor set. Then, by (FC4), $\operatorname{Reg}_{\widehat{\Phi}}(f)$ is $\widehat{\Phi}$-determining for f. Hence, Lemma 7.26 applied to $\widehat{\Phi}$ yields that $\mathcal{M}$ is $\widehat{\Phi}$-determining for f. Since $\mathcal{M} \subseteq \mathcal{E}$ and $\widehat{\Phi}|_\mathcal{E} = \Phi$ this means that for $x, y \in X$ one has

$$\widehat{\Phi}(f)x = y \quad \iff \quad \Phi(ef)x = \Phi(e)y \ \text{ for all } e \in \mathcal{M}.$$

This is (7.3).

Existence: We now *define* $\widehat{\Phi}(f)$ through (7.3) with $\mathcal{M} = [f]_{\mathcal{E}}$. Then $\widehat{\Phi}(f)$ is a closed relation (easy) and an operator because $[f]_{\mathcal{E}}$ is an anchor set. For $f \in \mathcal{E}$ one has $\Phi(ef) = \Phi(e)\Phi(f)$ for all $e \in [f]_{\mathcal{E}}$, hence from (7.3) it follows that $\widehat{\Phi}(f) = \Phi(f)$. This yields $\widehat{\Phi}|_{\mathcal{E}} = \Phi$.

Now for the four axioms ($f, g \in \mathcal{F}$, $\lambda \in \mathbb{C}$ and $x, y, z \in X$):

(FC1) By definition,

$$\widehat{\Phi}(\mathbf{1})x = y \qquad \Longleftrightarrow \qquad \forall\, e \in [\mathbf{1}]_{\mathcal{E}} : \Phi(e)x = \Phi(e)y$$

and the latter is equivalent to $x = y$ since $\mathbf{1}$ is anchored. So, $\widehat{\Phi}(\mathbf{1}) = \mathrm{I}$.

(FC2) The inclusion $\lambda\widehat{\Phi}(f) \subseteq \widehat{\Phi}(\lambda f)$ is trivial for $\lambda = 0$, since $\widehat{\Phi}(0) = \Phi(0) = 0$. So suppose that $\lambda \neq 0$ and $\widehat{\Phi}(f)x = y$, and fix $e \in [\lambda f]_{\mathcal{E}}$. Then $e \in [f]_{\mathcal{E}}$, too, so by definition of $\widehat{\Phi}(f)$

$$\Phi(e(\lambda f))x = \lambda\Phi(ef)x = \lambda\Phi(e)y = \Phi(e)(\lambda y).$$

By definition of $\widehat{\Phi}(\lambda f)$, this implies that $\widehat{\Phi}(\lambda f)x = \lambda y = \lambda\widehat{\Phi}(f)x$. Hence, the inclusion $\lambda\widehat{\Phi}(f) \subseteq \widehat{\Phi}(\lambda f)$ is established.

Suppose that $\widehat{\Phi}(f)x = y$ and $\widehat{\Phi}(g)x = z$ and let $e \in [f+g]_{\mathcal{E}}$. Choose any $e_f \in [f]_{\mathcal{E}}$ and $e_g \in [g]_{\mathcal{E}}$. Then

$$\begin{aligned}
\Phi(e_f)\Phi(e_g)\Phi(e(f+g))x &= \Phi(e_f e_g e(f+g))x = \Phi(ee_g e_f f + ee_f e_g g)x \\
&= \Phi(ee_g)\Phi(e_f f)x + \Phi(ee_f)\Phi(e_g g)x \\
&= \Phi(ee_g)\Phi(e_f)y + \Phi(ee_f)\Phi(e_g)z = \dots \\
&= \Phi(e_f)\Phi(e_g)\Phi(e)(y+z).
\end{aligned}$$

Varying e_f and e_g we obtain

$$\Phi(e(f+g))x = \Phi(e)(y+z).$$

Since $e \in [f+g]_{\mathcal{E}}$ was arbitrary, it follows that

$$\widehat{\Phi}(f+g)x = y + z = \widehat{\Phi}(f)x + \widehat{\Phi}(g)x.$$

(FC3) Suppose that $\widehat{\Phi}(g)x = y$ and $\widehat{\Phi}(f)y = z$ and let $e \in [fg]_{\mathcal{E}}$, $e_g \in [g]_{\mathcal{E}}$ and $e_f \in [f]_{\mathcal{E}}$. Then

$$\begin{aligned}
\Phi(e_g)\Phi(e_f)\Phi(e(fg))x &= \Phi(e_g e_f e f g)x = \Phi(ee_g e_f f g)x \\
&= \Phi(e(e_f f)(e_g g))x = \Phi(e)\Phi(e_f f)\Phi(e_g g)x \\
&= \Phi(e)\Phi(e_f f)\Phi(e_g)y = \Phi(e_g)\Phi(e)\Phi(e_f f)y \\
&= \Phi(e_g)\Phi(e)\Phi(e_f)z = \Phi(e_g)\Phi(e_f)\Phi(e)z.
\end{aligned}$$

Varying e_g and e_f yields $\Phi(efg)x = \Phi(e)z$, and hence the inclusion $\widehat{\Phi}(f)\widehat{\Phi}(g) \subseteq \widehat{\Phi}(fg)$. The domain inclusion

$$\operatorname{dom}(\widehat{\Phi}(f)\widehat{\Phi}(g)) \subseteq \operatorname{dom}(\widehat{\Phi}(g)) \cap \operatorname{dom}(\widehat{\Phi}(fg))$$

is now immediate. For the converse inclusion suppose that $x \in \operatorname{dom}(\widehat{\Phi}(g)) \cap \operatorname{dom}(\widehat{\Phi}(fg))$. Then $\widehat{\Phi}(g)x = y$ and $\widehat{\Phi}(fg)x = z$, say. Hence, for all $e_f \in [f]_{\mathcal{E}}$ and $e_g \in [g]_{\mathcal{E}}$:

$$\begin{aligned}\Phi(e_g)\Phi(e_f f)y &= \Phi(e_f f)\Phi(e_g)y = \Phi(e_f f)\Phi(e_g g)x \\ &= \Phi((e_f f)(e_g g))x = \Phi((e_f e_g)(fg))x \\ &= \Phi(e_f e_g)z = \Phi(e_g)\Phi(e_f)z.\end{aligned}$$

Varying e_g yields $\Phi(e_f f)y = \Phi(e_f)z$, and varying e_f shows that $\widehat{\Phi}(f)y = z$. This concludes the proof of Axiom (FC3).

(FC4) Since $\widehat{\Phi}$ extends Φ, $\mathcal{E} \subseteq \mathcal{F}_{\widehat{\Phi}}$. Hence, from the construction of $\widehat{\Phi}$ it follows that for each $f \in \mathcal{F}$ the set $[f]_{\mathcal{E}} \subseteq \operatorname{Reg}_{\widehat{\Phi}}(f)$ is $\widehat{\Phi}$-determining for f. A fortiori also $\operatorname{Reg}_{\widehat{\Phi}}(f)$ is $\widehat{\Phi}$-determining for f. □

Exercises

7.1. Let $\Phi : \mathcal{F} \to \mathcal{C}(X)$ be an $\mathcal{F}$-proto-calculus, let $f, g \in \mathcal{F}$ and $\mathcal{M}, \mathcal{N} \subseteq \mathcal{F}_\Phi$ such that $uv = vu$ for all $u \in \mathcal{M}$ and $v \in \mathcal{N}$. Show the following assertion: If $\mathcal{N} \subseteq \operatorname{Reg}_\Phi(f)$ is Φ-determining for f and $\mathcal{M} \subseteq \operatorname{Reg}_\Phi(f)$ is an anchor set, then $\mathcal{M}$ is Φ-determining for f.

7.2. *[has been removed]*

7.3. Prove Corollary 7.7. [Hint: Exercise 7.1.a)]

7.4. Let $\mathcal{F}$ be a commutative algebra, $\mathcal{E} \subseteq \mathcal{F}$ a subalgebra and $\Phi : \mathcal{E} \to \mathcal{L}(X)$ a non-degenerate representation with the canonical extension to $\langle \mathcal{E} \rangle_\Phi$ being again denoted by Φ.

a) Let $\mathcal{D} \subseteq \langle \mathcal{E} \rangle_\Phi$ be a subalgebra of Φ-bounded operators on which Φ is non-degenerate. Show that $\mathcal{D}$ is admissible and $\langle \mathcal{D} \rangle_\Phi \subseteq \langle \mathcal{E} \rangle_\Phi$. (That means that the extension theorem applied to $\mathcal{D}$ does not lead to anything new. A non-commutative version of this is Exercise 7.11 below.)

b) Suppose further that $f, g \in \mathcal{F}$ are such that $fg = \mathbf{1}$ and f is anchored in $\mathcal{E}$. Show that then g is anchored in $\mathcal{E}$ if and only if $\Phi(f)$ is injective, in which case $\Phi(g) = \Phi(f)^{-1}$. [Hint: compare with Theorem 2.3.]

7.5. Let A be a bounded operator on a Banach space X, let $U \supseteq \boldsymbol{\sigma}(A)$ open and let $\Phi : \operatorname{Hol}(U) \to \mathcal{L}(X)$ be the Dunford–Riesz calculus for A. Suppose in addition that

$$V \cap \sigma(A) \neq \emptyset$$

for each connected component V of U. Let, finally, $\mathcal{M} \subseteq \mathrm{Hol}(U)$ be an ideal of $\mathrm{Hol}(U)$ which is an anchor set with respect to Φ.

a) Show that there are only finitely many connected components V of U.

b) Show that for each eigenvalue λ of A there is $e \in \mathcal{M}$ such that $e(\lambda) \neq 0$.

c) Show that for each connected component V of U there is $e \in \mathcal{M}$ such that $e \neq 0$ on V.

d) Now suppose in addition that $\mathcal{M} = [f]_{\mathrm{Hol}(U)}$ for some meromorphic function f on U. Show (without using that $\mathrm{Mer}(U)$ is the ring of quotients of $\mathrm{Hol}(U)$) that there is $e \in \mathcal{M}$ such that $e \neq 0$ on each connected component of U and $e(\lambda) \neq 0$ for each eigenvalue λ of A. Conclude (with the help of Lemma 7.9) that e is an anchor element of f with respect to Φ.

[Hint: for c) show first that $\Phi(\mathbf{1}_V) \neq 0$, cf. Exercise 1.8; for d) show first that no eigenvalue of A is a pole of f.]
Remark: This yields a direct proof of the implication (i) $\Rightarrow$ (ii) in Theorem 7.10.

7.6 (Polynomials[3]). For an unbounded operator A on a Banach space X and any polynomial $p(z) = \sum_{j=0}^n \alpha_j z^j$ one defines $p(A) := \sum_{j=0}^n \alpha_j A^j$, see also Appendix A.4. In particular, $\mathrm{dom}(p(A)) = \mathrm{dom}(A^{\deg(p)})$ for $p \neq 0$. Note that $p(A)$ need not be closed even if A is.

Let $\mathcal{F}$ be a commutative unital algebra, $\Phi : \mathcal{F} \to \mathcal{C}(X)$ a proto-calculus and $a \in \mathcal{F}$ such that $\frac{1}{\lambda - a} \in \mathcal{F}_\Phi$ for some $\lambda \in \mathbb{C}$. Show that

$$\Phi(p(a)) = p(\Phi(a))$$

for each polynomial $p \in \mathbb{C}[z]$.
[Hint: Induction over $n = \deg(p)$; write $\frac{p(a)}{\lambda - a} = q(a)\frac{a}{\lambda - a} + \alpha_0 \frac{1}{\lambda - a}$ and infer from this that $\mathrm{dom}(\Phi(p(a))) \subseteq \mathrm{dom}(\Phi(a))$; then conclude the proof of the claim.]

7.7. Let $-A$ be the generator of a bounded C_0-semigroup $T = (T_t)_{t \geq 0}$ on a Banach space X.

a) Let $\varphi \in \mathrm{C}_\mathrm{c}^\infty(0, \infty)$ and $e := \mathcal{L}\varphi$. Show that within the extended Hille–Phillips calculus for A the function e is a regularizer for $\mathbf{z}^n$ for each $n \in \mathbb{N}$ and conclude that

$$\mathrm{ran}(e(A)) \subseteq D_\infty := \bigcap_{n \geq 0} \mathrm{dom}(A^n).$$

[3] This exercise could have been posed in Chapter 2.

b) Suppose that $f \in \mathrm{Mer}(\mathbb{C}_+)$ is such that $f(A)$ is defined in the extended Hille–Phillips calculus and satisfies $D_\infty \subseteq \mathrm{dom}(f(A))$. Show that D_∞ is a core for $f(A)$.

[Hint: a) Integration by parts; b) consider a Dirac sequence $(\varphi_n)_n$ in $\mathrm{C}_\mathrm{c}^\infty(0,\infty)$ and cf. Lemma 7.22.]

7.8. Let (K, Σ) be a measurable space and $\mathrm{E} : \Sigma \to \mathcal{L}(H)$ a projection-valued measure on a Hilbert space H in the sense of Section 3.3. Define Φ on Σ-simple functions by

$$\Phi(f) := \sum_{j=1}^{n} \alpha_j \mathrm{E}(B_j) \in \mathcal{L}(H)$$

whenever $f = \sum_{j=1}^{n} \alpha_j \mathbf{1}_{B_j}$, $n \in \mathbb{N}$, $\alpha_j \in \mathbb{C}$ for all j and the $B_j \in \Sigma$ are pairwise disjoint.

a) Show that this is a good definition, i.e., the value $\Phi(f)$ does not depend on the representation of f.

b) Show that $\|\Phi(f)\| \le \|f\|_\infty$ whenever f is a Σ-simple function.

c) By b), Φ has a continuous extension to $\mathcal{M}_\mathrm{b}(K, \Sigma)$; denote this again by Φ. Show that $\Phi : \mathcal{M}_\mathrm{b}(K, \Sigma) \to \mathcal{L}(H)$ is a homomorphism of unital $*$-algebras.

d) Show that for each $x \in H$ the mapping

$$\mu_x : \Sigma \to \mathbb{C}, \qquad \mu_x(B) := (\, \mathrm{E}(B)x \,|\, x \,) \quad (B \in \Sigma)$$

is a finite positive measure on (K, Σ).

e) Show that for each $x \in H$ and $f \in \mathcal{M}_\mathrm{b}(K, \Sigma)$

$$(\, \Phi(f)x \,|\, x \,) = \int_K f \, \mathrm{d}\mu_x.$$

Conclude that the $*$-representation $\Phi : \mathcal{M}_\mathrm{b}(K, \Sigma) \to \mathcal{L}(H)$ satisfies (MFC6).

Supplementary Exercises

7.9. Let $\Phi : \mathcal{F} \to \mathcal{C}(X)$ be a proto-calculus, let $\mathcal{E} \subseteq \mathcal{F}$ a subalgebra. The **commutator** of $\mathcal{E}$ (in $\mathcal{F}$) is

$$\mathcal{E}' := \{ f \in \mathcal{F} \mid ef = fe \text{ for all } e \in \mathcal{E} \}.$$

It is a unital subalgebra of $\mathcal{F}$. For an element $f \in \mathcal{F}$ we define

$$\lfloor f \rfloor_\mathcal{E} := \{ g \in \mathcal{F} \,:\, gf \in \mathcal{E} \},$$

which is a left ideal of $\mathcal{E}$. Finally, let

$$\mathcal{G} := \{g \in \mathcal{F} \mid \lfloor g \rfloor_{\mathcal{E}} \cap \mathcal{F}_{\Phi} \text{ is an anchor set}\}.$$

Let $f \in \mathcal{F}$ and let $\mathcal{M} \subseteq \mathrm{Reg}_{\Phi}(f)$ be an anchor set. Show that each of the following two assumptions imply that $\mathcal{M}$ is Φ-determining for f:

1) $\mathrm{Reg}_{\Phi}(f) \cap \mathcal{E}'$ is Φ-determining for f and $\mathcal{M} \subseteq \mathcal{G}$;
2) $\mathrm{Reg}_{\Phi}(f) \cap \mathcal{G}$ is Φ-determining for f and $\mathcal{M} \subseteq \mathcal{E}'$.

7.10. Show that Lemma 7.26 and Exercise 7.1.a) are special cases of Exercise 7.9.

7.11. Let $\mathcal{E}$ be a subalgebra of a unital algebra $\mathcal{F}$ and $\Phi : \mathcal{E} \to \mathcal{L}(X)$ a non-degenerate representation. Denote its canonical extension to $\langle \mathcal{E} \rangle_{\Phi}$ again by Φ. Now suppose that $\mathcal{D} \subseteq \langle \mathcal{E} \rangle_{\Phi}$ is a subalgebra of Φ-bounded operators on which Φ is non-degenerate. So one can think of taking $\mathcal{D}$ as the basis of another application of the extension theorem. Show that if each element of $\mathcal{Z}(\mathcal{D})$ commutes with each element of $\mathcal{E}$, then $\langle \mathcal{D} \rangle_{\Phi} \subseteq \langle \mathcal{E} \rangle_{\Phi}$ and $\mathcal{D}$ is admissible.

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Chapter 8
The Holomorphic Functional Calculus Approach to Operator Semigroups

In Chapter 6 we have associated with a given C_0-semigroup an operator, called its generator, and a functional calculus, the Hille–Phillips calculus. Here we address the converse problem: which operators are generators? The classical answer to this question and one of our goals in this chapter is the so-called Hille–Yosida theorem, which gives a characterization of the generator property in terms of *resolvent estimates*.

However, instead of following the standard texts like [4], we shall approach the problem via a suitable functional calculus construction. Our method is generic, in that we learn here how to use Cauchy integrals over infinite contours in order to define functional calculi for unbounded operators satisfying resolvent estimates. (The same approach shall be taken later for defining holomorphic functional calculi for sectorial and strip-type operators.) This chapter is heavily based on the first part of [2], which in turn goes back to the unpublished note [5].

8.1 Operators of Half-Plane Type

Recall from Chapter 6 that if B generates a C_0-semigroup of type (M, ω), then $\boldsymbol{\sigma}(B)$ is contained in the closed half-plane $[\,\mathrm{Re}\,\mathbf{z} \le \omega\,]$ and

$$\|R(\lambda, B)\| \le \frac{M}{\mathrm{Re}\,\lambda - \omega} \qquad (\mathrm{Re}\,\lambda > \omega).$$

In particular, if $\omega < \alpha$ then $R(\cdot, A)$ is uniformly bounded for $\mathrm{Re}\,\lambda \ge \alpha$. This "spectral data" is our model for what we shall call below an operator of "left half-plane type".

Recall, however, that the associated Hille–Phillips calculus for the semigroup generated by B is actually a calculus for the *negative generator*

$A := -B$. So in order to stay in line with the Hille–Phillips picture we shall construct our new functional calculus for operators of *right* half-plane type.

Let us turn to the precise definitions. For $\omega \in [-\infty, \infty]$ we let

$$\mathrm{L}_\omega := \{z \in \mathbb{C} \mid \operatorname{Re} z < \omega\} \quad \text{and} \quad \mathrm{R}_\omega := \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega\}$$

be the open left and right half-plane at the **abscissa** ω. (So $\mathrm{R}_\omega = \mathbb{C}_+ + \omega$ in the terminology of Chapter 6.) An operator A on a Banach space X is said to be **of right half-plane type $\boldsymbol{\omega}$** $\in (-\infty, \infty]$ if $\sigma(A) \subseteq \overline{\mathrm{R}_\omega}$ and

$$M(A, \alpha) := \sup\{\|R(z, A)\| \mid \operatorname{Re} z \le \alpha\} \ < \ \infty \qquad \text{for every } \alpha < \omega.$$

We say that A is **of right half-plane type** if A is of right half-plane type ω for some ω. For an operator A of right half-plane type we call

$$\omega_{\mathrm{hp}}(A) := \sup\bigl\{\alpha \bigm| \sup_{\operatorname{Re} z \le \alpha} \|R(z, A)\| < \infty\bigr\}$$

the **abscissa of uniform boundedness** of its resolvent.

An operator B is called of **left half-plane type** ω if $A = -B$ is of right half-plane type $-\omega$. The abscissa of uniform boundedness of the resolvent of such an operator is denoted by

$$s_0(B) := \inf\bigl\{\alpha \bigm| \sup_{\operatorname{Re} \lambda \ge \alpha} \|R(\lambda, B)\| < \infty\bigr\} \in \mathbb{R} \cup \{-\infty\}$$

as it is common in semigroup theory. (See [1, p.342] where this number is called the *pseudo-spectral bound* of B.)

Remark 8.1. The definition of a half-plane type operator may appear complicated and unintuitive at first glance. Why do we not define A to be of right half-plane type ω simply by

$$\sigma(A) \subseteq \overline{\mathrm{R}_\omega}, \qquad \sup_{\operatorname{Re} z < \omega} \|R(z, A)\| < \infty \ ? \tag{8.1}$$

The answer is that, with our definition, A is of right half-plane type $\omega_{\mathrm{hp}}(A)$, whereas this would be false with the alternative definition. See Exercise 8.1.

In the following, when we speak of half-plane type operators, it has to be read as *right* half-plane type, unless explicitly noted otherwise.

8.2 Functional Calculus on Half-Planes

As already announced, we are aiming at defining a functional calculus for operators A of half-plane type. There is a “cheap” way of doing this: pick some $\lambda \in \rho(A)$ and define $A_\lambda := R(\lambda, A)$. Then set up the Dunford-Riesz calculus for A_λ and play it back to A. (See Exercise 8.2 for details.)

This approach, elegant as it appears, does not use at all the resolvent estimates which are characteristic for operators of half-plane type. Moreover, it is principally unsuitable to reach our goal, because the functions $\mathrm{e}^{-t\mathbf{z}}$ have an essential singularity at ∞ and hence are not accessible by that calculus nor by its canonical extension.

So one has to take a different route. The idea is to mimic the construction of the Dunford–Riesz calculus but with contours leading *into* the possible singularity, which here is the point at infinity. Since the spectrum of an operator A of half-plane type may be a whole right half-plane, a straightforward choice for such contours are vertical straight lines, oriented downwards. (Picture this on the Riemann sphere and you will see that this comes as close as possible to “surrounding once the spectrum counterclockwise”.)

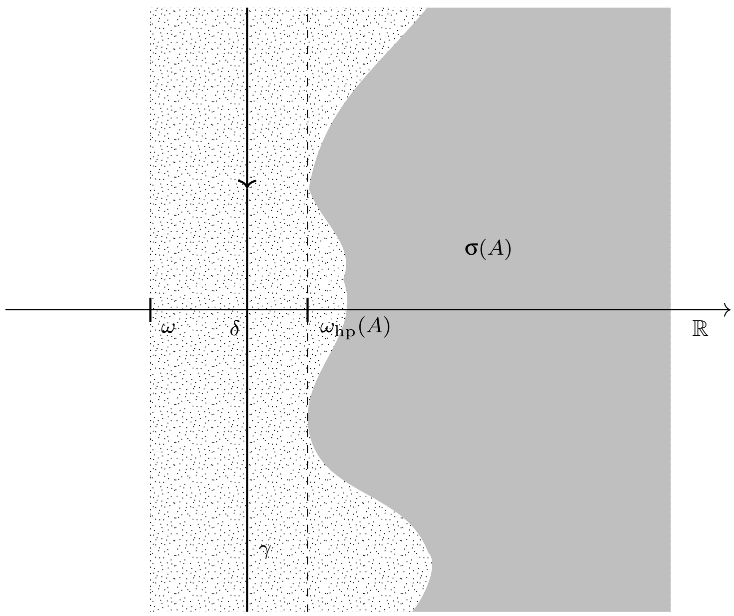

Fig. 8.1 The path γ runs from $\delta + \mathrm{i}\infty$ to $\delta - \mathrm{i}\infty$, where $\omega < \delta < \omega_{\mathrm{hp}}(A)$.

In the following, we shall work out this plan. Let A be an operator of half-plane type on a Banach space X and fix $\omega < \delta < \omega_{\mathrm{hp}}(A)$. We want to define

$$\Phi(f) := \frac{1}{2\pi\mathrm{i}} \int_\gamma f(z) R(z, A) \, \mathrm{d}z$$

for suitable functions $f \in \mathrm{Hol}(\mathrm{R}_\omega)$, where γ is the downwards parameterization of the vertical line $[\,\mathrm{Re}\,\mathbf{z} = \delta\,]$, see Figure 8.1. (We use the alternative expressions

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Re}\, z=\delta} f(z) R(z,A)\,\mathrm{d}z \quad \text{and} \quad \frac{-1}{2\pi \mathrm{i}} \int_{\delta-\mathrm{i}\infty}^{\delta+\mathrm{i}\infty} f(z) R(z,A)\,\mathrm{d}z$$

to denote this integral. Observe that the orientation of the contour is implicit in the first of these.)

Of course, as the contour is infinite, not every holomorphic function f on R_ω will be suitable. Rather, the properties of f and $R(\mathbf{z}, A)$ should match in order to guarantee that the integral is convergent. As the resolvent is assumed to be bounded, it is reasonable to require that f is integrable on vertical lines. This is the reason for the following definition. For $\omega \in \mathbb{R}$ we let[1]

$$\mathcal{E}(\mathrm{R}_\omega) := \big\{ f \in \mathrm{H}^\infty(\mathrm{R}_\omega) \mid \forall\, \delta > \omega : f \in \mathrm{L}^1(\delta + \mathrm{i}\mathbb{R}) \big\}$$

and call it the algebra of **elementary functions** on the half-plane R_ω. By writing $f \in \mathrm{L}^1(\delta + \mathrm{i}\mathbb{R})$ we mean that the function $f(\delta + \mathrm{i}\mathbf{x})$ is in $\mathrm{L}^1(\mathbb{R})$, i.e., that

$$\int_{\mathbb{R}} |f(\delta + \mathrm{i}x)|\,\mathrm{d}x < \infty.$$

It is easily checked that $\mathcal{E}(\mathrm{R}_\omega)$ is a (non-unital) subalgebra of $\mathrm{H}^\infty(\mathrm{R}_\omega)$. For elementary functions we have the following version of Cauchy's integral theorem.

Lemma 8.2. *Let $f \in \mathcal{E}(\mathrm{R}_\omega)$, $\delta > \omega$, and $a \in \mathbb{C}$ with $\mathrm{Re}\, a \neq \delta$. Then*

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Re}\, z=\delta} \frac{f(z)}{z-a}\,\mathrm{d}z = \begin{cases} f(a) & \text{if} \quad \delta < \mathrm{Re}\, a, \\ 0 & \text{if} \quad \delta > \mathrm{Re}\, a. \end{cases} \tag{8.2}$$

Moreover, $f \in \mathrm{C}_0(\overline{\mathrm{R}_\delta})$.

Proof. Consider first the case $\mathrm{Re}\, a < \delta$. Define

$$g(a) := \int_{\mathrm{Re}\, z=\delta} \frac{f(z)}{z-a}\,\mathrm{d}z \qquad (\mathrm{Re}\, a < \delta).$$

Then, by some standard arguments, g is holomorphic and

$$g'(a) = \int_{\mathrm{Re}\, z=\delta} \frac{f(z)}{(z-a)^2}\,\mathrm{d}z.$$

We claim that for fixed a this integral does not depend on $\delta > \mathrm{Re}\, a$. Indeed,

[1] For an open set $O \subseteq \mathbb{C}$ we denote by $\mathrm{H}^\infty(O)$ the space of bounded and holomorphic functions. It is a unital Banach algebra with respect to the sup-norm.

$$0 = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{(z-a)^2} \, dz,$$

by Cauchy's theorem, where γ_R is the positively oriented boundary of the rectangle $[\delta, \delta'] \times [-R, R]$ with $\delta' > \delta$ and $R > 0$. Since f is bounded, the integrals over the horizontal line segments tend to zero as $R \to \infty$, resulting in

$$0 = \frac{1}{2\pi i} \int_{\operatorname{Re} z=\delta} \frac{f(z)}{(z-a)^2} \, dz - \frac{1}{2\pi i} \int_{\operatorname{Re} z=\delta'} \frac{f(z)}{(z-a)^2} \, dz.$$

But now we can let $\delta' \to \infty$ and, again by the boundedness of f, we conclude that $g'(a) = 0$. Hence, g is constant. On the other hand, $g(a) \to 0$ as $\operatorname{Re} a \to -\infty$, which yields $g = 0$.

Consider now the case $\operatorname{Re} a > \delta$. Again we employ Cauchy's theorem with the contour γ_R as above, but now with $\delta < \operatorname{Re} a < \delta'$ and $R > |\operatorname{Im} a|$. This yields

$$f(a) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{z-a} \, dz.$$

Again, the integrals over the horizontal line segments tend to zero as $R \to \infty$. It follows that

$$f(a) = \frac{1}{2\pi i} \int_{\operatorname{Re} z=\delta} \frac{f(z)}{z-a} \, dz - \frac{1}{2\pi i} \int_{\operatorname{Re} z=\delta'} \frac{f(z)}{z-a} \, dz.$$

By what we have proved first, the second integral is equal to zero, and we are done.

For the last assertion, fix $\delta' > \delta$. Then it follows from (8.2) and the dominated convergence theorem that $f(a) \to 0$ as $a \to \infty$ within $\overline{\mathrm{R}_{\delta'}}$. As $\delta' > \delta > \omega$ were arbitrary, the claim follows. □

For $f \in \mathcal{E}(\mathrm{R}_\omega)$ we now let (as announced above)

$$\Phi_A(f) := \frac{1}{2\pi i} \int_{\operatorname{Re} z=\delta} f(z) R(z, A) \, dz, \tag{8.3}$$

where A is the given half-plane type operator and $\omega < \delta < \omega_{\mathrm{hp}}(A)$, see also Figure 8.1. Since the resolvent $R(\cdot, A)$ is bounded on the vertical line $[\operatorname{Re} \mathbf{z} = \delta]$, the integral converges absolutely. By virtue of Cauchy's theorem (for vector-valued functions) and the last assertion of Lemma 8.2, the definition of $\Phi_A(f)$ is independent of the choice of δ.

Theorem 8.3. *The so-defined mapping $\Phi_A : \mathcal{E}(\mathrm{R}_\omega) \to \mathcal{L}(X)$ has the following properties:*

a) *Φ_A is a homomorphism of algebras.*

b) *If $T \in \mathcal{L}(X)$ satisfies $TA \subseteq AT$, then $T\Phi_A(f) = Phi_A(f)T$ for all $f \in \mathcal{E}(\mathrm{R}_\omega)$.*

c) $\varPhi_A(f \cdot (\lambda - \mathbf{z})^{-1}) = \varPhi_A(f)R(\lambda, A)$ *whenever* $\operatorname{Re}\lambda < \omega$ *and* $e \in \mathcal{E}(\mathrm{R}_\omega)$.

d) $\varPhi_A((\lambda - \mathbf{z})^{-1}(\mu - \mathbf{z})^{-1}) = R(\lambda, A)R(\mu, A)$ *whenever* $\operatorname{Re}\lambda, \operatorname{Re}\mu < \omega$.

Proof. a) Additivity is clear. Multiplicativity follows from a combination of Fubini's theorem, the resolvent identity and Lemma 8.2. The computation is the same as for the Dunford–Riesz calculus, see Exercise 8.3.

b) is obvious.

c) By the resolvent identity and Lemma 8.2

$$\begin{aligned}
\varPhi_A(f)R(\lambda, A) &= \frac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} z = \delta} f(z)R(z, A)R(\lambda, A)\,\mathrm{d}z \\
&= \frac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} z = \delta} \frac{f(z)}{\lambda - z}[R(z, A) - R(\lambda, A)]\,\mathrm{d}z \\
&= \frac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} z = \delta} \frac{f(z)}{\lambda - z} R(z, A)\,\mathrm{d}z = \varPhi_A\left(\frac{f}{\lambda - \mathbf{z}}\right).
\end{aligned}$$

d) We only give an informal argument and leave the details to the reader (Exercise 8.3). In the integral

$$\frac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} z = \delta} \frac{1}{(\lambda - z)(\mu - z)} R(z, A)\,\mathrm{d}z$$

we shift the path to the left, i.e., let $\delta \to -\infty$. When one passes the abscissas $\delta = \operatorname{Re}\lambda$ and $\delta = \operatorname{Re}\mu$, the residue theorem yields some additive contributions which sum up to $R(\lambda, A)R(\mu, A)$ by the resolvent identity; if $\delta < \min(\operatorname{Re}\lambda, \operatorname{Re}\mu)$, the integral does not change any more as $\delta \to -\infty$ and hence it is equal to zero. □

The so-defined mapping $\varPhi_A : \mathcal{E}(\mathrm{R}_\omega) \to \mathcal{L}(X)$ is called the **elementary calculus** on R_ω for the half-plane type operator A. Since resolvents are injective, property d) of Theorem 8.3 implies that this calculus is non-degenerate, in the terminology of Chapter 7. We can therefore pass to its canonical extension within the algebra $\mathrm{Mer}(\mathrm{R}_\omega)$ of meromorphic functions on R_ω. The emerging calculus is again denoted by $\varPhi_A$ and called the **extended calculus** for A on R_ω. Its domain, the set of meromorphic functions f such that $\varPhi_A(f)$ is defined in the extended calculus, is denoted by

$$\mathrm{Mer}_A(\mathrm{R}_\omega).$$

Suppose that $f \in \mathrm{Hol}(\mathrm{R}_\omega)$ is such that there is $n \in \mathbb{N}_0$ with

$$|f(z)| \lesssim 1 + |z|^n \qquad (\operatorname{Re} z > \omega).$$

Then f is anchored by the elementary function $e = \frac{1}{(\lambda - \mathbf{z})^{n+2}}$, whenever $\operatorname{Re}\lambda < \omega$. Indeed, $ef \in \mathcal{E}(\mathrm{R}_\omega)$ and, by c) and d) of Theorem 8.3,

$$\Phi_A(e) = R(\lambda, A)^{n+2},$$

which is injective.

Corollary 8.4. *Let A be an operator of half-plane type on a Banach space X, let $\omega < \omega_{\mathrm{hp}}(A)$ and let $\Phi_A : \mathrm{Mer}_A(\mathrm{R}_\omega) \to \mathcal{C}(X)$ be the associated extended calculus as constructed above. Then the following statements hold.*

a) $\mathbf{z} \in \mathrm{Mer}_A(\mathrm{R}_\omega)$ *and* $\Phi_A(\mathbf{z}) = A$.

b) *Whenever* $\lambda \notin \sigma_{\mathrm{p}}(A)$, $(\lambda - \mathbf{z})^{-1} \in \mathrm{Mer}_A(\mathrm{R}_\omega)$ *and* $\Phi_A((\lambda - \mathbf{z})^{-1}) = (\lambda - A)^{-1}$.

c) $\mathrm{H}^\infty(\mathrm{R}_\omega) \subseteq \mathrm{Mer}_A(\mathrm{R}_\omega)$ *and* $(\lambda - \mathbf{z})^{-2}$ *is a universal acnhor element for the whole of* $\mathrm{H}^\infty(\mathrm{R}_\omega)$ *whenever* $\mathrm{Re}\,\lambda < \omega$. *In particular,*

$$\mathrm{dom}(A^2) \subseteq \mathrm{dom}(\Phi_A(f))$$

for all $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$.

Proof. a) follows from Corollary 7.8 with $g = (\lambda - \mathbf{z})^{-2}$ and $\mathrm{Re}\,\lambda < \omega$.
b) This is similar to Lemma 7.12.b), and actually a consequence of a result in abstract functional calculus theory (Exercise 7.4).
c) This is just the case $n = 0$ in the discussion preceeding this corollary. □

Remark 8.5. So far, the elementary calculus still depends on the parameter $\omega < \omega_{\mathrm{hp}}(A)$. This dependence is somehow virtual: For $\omega < \omega'$ the restriction map

$$\mathcal{E}(\mathrm{R}_\omega) \hookrightarrow \mathcal{E}(\mathrm{R}_{\omega'}), \qquad f \mapsto f|_{\mathrm{R}_{\omega'}}$$

is an embedding (by the identity theorem for holomorphic functions) which allows us to regard $\mathcal{E}(\mathrm{R}_\omega)$ as a subalgebra of $\mathcal{E}(\mathrm{R}_{\omega'})$. And the δ-independence of the integral in (8.3) implies that the elementary calculi defined on $\mathcal{E}(\mathrm{R}_\omega)$ and on $\mathcal{E}(\mathrm{R}_{\omega'})$ are compatible with this identification. If desired, one could then pass to the "inductive limit" algebra

$$\tilde{\mathcal{E}}(\mathrm{R}_{\omega_{\mathrm{hp}}(A)}) := \bigcup_{\omega < \omega_{\mathrm{hp}}(A)} \mathcal{E}(\mathrm{R}_\omega)$$

and define Φ_A thereon. (Something similar could have been done with the Dunford–Riesz calculus in Chapter 1.) One could even talk about "function germs" instead of functions here, but at least for our purposes we regard this as terminological overkill.

From now on, we abbreviate

$$f(A) := \Phi_A(f)$$

whenever A is an operator of right half-plane type on a Banach space X and $f \in \mathrm{Mer}_A(\mathrm{R}_\omega)$ with $\omega < \omega_{\mathrm{hp}}(A)$. Also, we say that $f(A)$ **is defined by the (holomorphic) half-plane calculus for** A.

8.3 Convergence Theorems

Let us look at continuity properties of the half-plane calculus.

Continuity with Respect to the Operator

One says that the elements of a set $\mathcal{A}$ of operators on a Banach space X are **uniformly** of right half-plane type $r > -\infty$ if

$$\sup_{A\in\mathcal{A}} M(A,\alpha) = \sup\{\|R(\lambda,A)\| \mid A \in \mathcal{A},\ \operatorname{Re}\lambda \le \alpha\} < \infty$$

for all $\alpha < r$. A subset $\mathcal{F} \subseteq \mathcal{E}(\mathrm{R}_\omega)$ is called **dominated** if for each $\delta > \omega$ sufficiently close to ω the set of functions $\{f(\delta + \mathrm{i}\mathbf{x}) \mid f \in \mathcal{F}\}$ is dominated in $\mathrm{L}^1(\mathbb{R})$.

Lemma 8.6. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of operators on a Banach space X, uniformly of right half-plane type $r \in \mathbb{R}$. Let, furthermore, A be an operator on X such that $\mathrm{L}_r \subseteq \rho(A)$ and*

$$R(\lambda, A_n) \to R(\lambda, A) \quad \textit{strongly on } X \textit{ whenever } \operatorname{Re}\lambda < r.$$

Then A is also of right half-plane type r and $f(A_n) \to f(A)$ strongly on X for all $\omega < r$ and all $f \in \mathcal{E}(\mathrm{R}_\omega)$. Even more, for each $x \in X$ the convergence

$$f(A_n)x \to f(A)x$$

is uniform in f from dominated subsets of $\mathcal{E}(\mathrm{R}_\omega)$.

Proof. It is easy to see that A is of right half-plane type r and $M(A,\alpha) \le \sup_{n\in\mathbb{N}} M(A_n,\alpha)$ for all $\alpha < r$. Let $\omega < r$ and $\mathcal{F} \subseteq \mathcal{E}(\mathrm{R}_\omega)$ be dominated. Choose $\omega < \delta < r$ and $0 \le g \in \mathrm{L}^1(\mathbb{R})$ such that $|f(z)| \le g(\operatorname{Im} z)$ for $z \in \delta + \mathrm{i}\mathbb{R}$ and $f \in \mathcal{F}$. Then

$$f(A_n)x - f(A)x = \frac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} z=\delta} f(z)\big(R(z,A_n)x - R(z,A)x\big)\,\mathrm{d}z.$$

Hence, $\displaystyle\sup_{f\in\mathcal{F}} \|f(A_n)x - f(A)x\|$

$$\le \frac{1}{2\pi} \int_{\operatorname{Re} z=\delta} g(\operatorname{Im} z)\|R(z,A_n)x - R(z,A)x\|\ |\mathrm{d}z| \to 0$$

by the dominated convergence theorem. □

Remark 8.7. An inspection of the proof reveals that Lemma 8.6 holds mutatis mutandis for operator norm convergence. I.e., if one has $R(\lambda, A_n) \to$

$R(\lambda, A)$ in operator norm for each $\operatorname{Re}\lambda < r$, then also $f(A_n) \to f(A)$ in operator norm, uniformly in f from dominated subsets of $\mathcal{E}(\mathrm{R}_\omega)$, $\omega < r$.

The sequence $(A_n)_n$ is called a **half-plane type approximation** of A on R_r if the hypotheses of Lemma 8.6 are satisfied.

Theorem 8.8 (First Convergence Theorem). *Let A be a densely defined operator on a Banach space X and let $(A_n)_n$ be a half-plane type approximation of A on the half-plane R_r. Let $\omega < r$ and $\mathcal{F} \subseteq \mathrm{H}^\infty(\mathrm{R}_\omega)$ be uniformly bounded such that*

$$C := \sup_{n\in\mathbb{N}, f\in\mathcal{F}} \|f(A_n)\| < \infty.$$

Then $\sup_{f\in\mathcal{F}} \|f(A)\| \le C$ and for each $x \in X$

$$f(A_n)x \to f(A)x \qquad (n \to \infty)$$

uniformly in $f \in \mathcal{F}$.

Proof. Fix $\lambda < \omega$ and define $e_f := f \cdot (\lambda - \mathbf{z})^{-2} \in \mathcal{E}(\mathrm{R}_\omega)$ for $f \in \mathcal{F}$. Then, for $x \in \operatorname{dom}(A^2)$,

$$\begin{aligned}\|e_f(A_n)(\lambda - A)^2 x\| &= \|f(A_n)R(\lambda, A_n)^2(\lambda - A)^2 x\| \\ &\le C\|R(\lambda, A_n)^2(\lambda - A)^2 x\|.\end{aligned}$$

Since $R(\lambda, A_n) \to R(\lambda, A)$ and $e_f(A_n) \to e_f(A)$ strongly (by Lemma 8.6),

$$\|f(A)x\| = \|e_f(A)(\lambda - A)^2 x\| \le C\|R(\lambda, A)^2(\lambda - A)^2 x\| = C\|x\|.$$

Since $\operatorname{dom}(A)$ is dense, so is $\operatorname{dom}(A^2)$ (Exercise 8.4), whence it follows that $f(A) \in \mathcal{L}(X)$ with $\|f(A)\| \le C$.
Now abbreviate $y := (\lambda - A)^2 x$ and observe that

$$f(A_n)R(\lambda, A_n)^2 y = e_f(A_n)y \to e_f(A)y = f(A)R(\lambda, A)^2 y = f(A)x$$

uniformly in $f \in \mathcal{F}$. Indeed, this follows from Lemma 8.6 since

$$|e_f| \le \big(\sup_{f\in\mathcal{F}} \|f\|_\infty\big)\ |\lambda - \mathbf{z}|^{-2}$$

for all $f \in \mathcal{F}$, which is why the set $\{e_f \mid f \in \mathcal{F}\} \subseteq \mathcal{E}(\mathrm{R}_\omega)$ is dominated. In addition,

$$\begin{aligned}\|f(A_n)x - f(A_n)R(\lambda, A_n)^2 y\| &= \|f(A_n)R(\lambda, A)^2 y - f(A_n)R(\lambda, A_n)^2 y\| \\ &\le C\|R(\lambda, A)^2 y - R(\lambda, A_n)^2 y\| \to 0\end{aligned}$$

for all $f \in \mathcal{F}$. It follows that $f(A_n)x \to f(A)x$ uniformly in $f \in \mathcal{F}$ for each $x \in \operatorname{dom}(A^2)$. Since $\operatorname{dom}(A^2)$ is dense and the operator family $(f(A_n))_{n\in\mathbb{N}, f\in\mathcal{F}}$ is uniformly bounded, the claim is proved. $\square$

Continuity with Respect to the Function

Now we look at approximating the function and keeping the operator fixed.

Lemma 8.9. *Let $\mathcal{A}$ be a set of operators on a Banach space X, uniformly of half-plane type $r \in \mathbb{R}$, let $\omega < r$ and $(f_n)_n$ be a dominated sequence in $\mathcal{E}(\mathrm{R}_\omega)$ converging pointwise to zero. Then $\|f_n(A)\| \to 0$ uniformly in $A \in \mathcal{A}$.*

Proof. This is left as Exercise 8.5. □

The following result is traditionally named the "convergence lemma".

Theorem 8.10 (Second Convergence Theorem). *Let A be a densely defined operator of half-plane type $r \in \mathbb{R}$ on a Banach space X. Let $\omega < r$ and let $(f_n)_n$ be a sequence in $\mathrm{H}^\infty(\mathrm{R}_\omega)$ converging pointwise and boundedly to some function $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$. If*

$$C := \sup_{n \in \mathbb{N}} \|f_n(A)\| < \infty$$

then $\|f(A)\| \le C$ and $f_n(A) \to f(A)$ strongly on X.

Proof. This is also left as Exercise 8.5. □

Remark 8.11. A complete analogue of Theorem 8.8 (in which one would have uniformity in A from a given set $\mathcal{A}$ of half-plane type operators) seems out of reach here. Such a result will only hold under additional hypotheses for $\mathcal{A}$.

8.4 The Theorems of Trotter–Kato and Hille–Yosida

Suppose that A is an operator of right half-plane type on a Banach space X. For each $t \ge 0$ the function $\mathrm{e}^{-t\mathbf{z}}$ is bounded and holomorphic on each right half-plane R_ω. Hence, we can define

$$e^{-tA} := (\mathrm{e}^{-t\mathbf{z}})(A) = \Phi_A(e^{-t\mathbf{z}}) \qquad (t \ge 0).$$

Recall that $\mathrm{dom}(A^2) \subseteq \mathrm{dom}(e^{-tA})$.

Lemma 8.12. *Let A be an operator of right half-plane type, and let $\omega < \omega_{\mathrm{hp}}(A)$. Then for each $x \in \mathrm{dom}(A^2)$ the function*

$$\mathbb{R}_+ \to X, \qquad t \mapsto e^{-tA}x$$

is continuous and satisfies $\sup_{t \ge 0} \|e^{\omega t} e^{-tA} x\| < \infty$. Its Laplace transform is

$$\int_0^\infty e^{-\lambda t} e^{-tA} x \,\mathrm{d}t = R(\lambda, -A)x = (\lambda + A)^{-1} x \qquad (\mathrm{Re}\,\lambda > -\omega).$$

Proof. Fix $\mu < \omega$. By Lemma 8.9, the mapping $t \mapsto \big(\frac{e^{-t\mathbf{z}}}{(\mu-z)^2}\big)(A)$ is continuous. Hence, so is the mapping

$$t \mapsto e^{-tA}x = \Big(\frac{e^{-t\mathbf{z}}}{(\mu-\mathbf{z})^2}\Big)(A)\,(\mu-A)^2x.$$

The uniform bound is established by a straightfoward estimation. Finally, an application of Fubini's theorem yields (with $y := (\mu - A)^2x$)

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\lambda t}\mathrm{e}^{-tA}x\,\mathrm{d}t &= \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{Re}\,z=\delta}\int_0^\infty \mathrm{e}^{-\lambda t}\mathrm{e}^{-tz}\,\mathrm{d}t\,\frac{R(z,A)y}{(\mu-z)^2}\,\mathrm{d}z\\ &= \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{Re}\,z=\delta}\frac{R(z,A)y}{(\lambda+z)(\mu-z)^2}\,\mathrm{d}z\\ &= (\lambda+A)^{-1}R(\mu,A)^2y = (\lambda+A)^{-1}x\end{aligned}$$

as claimed. □

The following result is the gateway to many so-called "generation theorems", i.e., theorems that assert that operators with certain properties are generators of semigroups.

Theorem 8.13 (Basic Generation Theorem). *Let A be an operator of right half-plane type on a Banach space X. Then $-A$ is the generator of a C_0-semigroup $T = (T_t)_{t\ge0}$ if and only if A is densely defined, e^{-tA} is a bounded operator for all $t \in [0,1]$, and $\sup_{t\in[0,1]}\|e^{-tA}\| < \infty$. In this case, $T(t) = e^{-tA}$ for all $t \ge 0$.*

Proof. Let $-A$ generate a C_0-semigroup $(T(t))_{t\ge0}$. Then A is densely defined, so $\mathrm{dom}(A^2)$ is dense (Exercise 8.4). Lemma 8.12 yields that $R(\cdot,-A)x$ is the Laplace transform of $(t \mapsto e^{-tA}x)$ for $x \in \mathrm{dom}(A^2)$. By the uniqueness of Laplace transforms, $T(t)x = e^{-tA}x$, $t \ge 0$. Since $\mathrm{dom}(A^2)$ is dense and e^{-tA} is a closed operator, $e^{-tA} = T(t)$ is a bounded operator. The uniform boundedness for $t \in [0,1]$ is a standard property of C_0-semigroups.

Conversely, suppose that A is densely defined and $T(t) := e^{-tA}$ is a bounded operator for all $t \ge 0$. Then T is a semigroup (by abstract functional calculus) and $\mathrm{dom}(A^2)$ is dense (see above). From the uniform boundedness $\sup_{t\in[0,1]}\|T(t)\| < \infty$ and the semigroup property one concludes easily that $(T(t))_{t\ge0}$ is uniformly bounded on compact intervals. Lemma 8.12 and the density of $\mathrm{dom}(A^2)$ imply that $(T(t))_{t\ge0}$ is strongly continuous. Its Laplace transform coincides with the resolvent of $-A$ on $\mathrm{dom}(A^2)$ (Lemma 8.12), and hence on X by density. So $-A$ is the generator of T. □

Remark 8.14. The boundedness assumption cannot be omitted from Theorem 8.13. Indeed, Phillips in [9, p.337] has given an example of a densely defined operator A of half-plane type such that e^{-tA} is a bounded operator for all $t \ge 0$ but the map $t \mapsto \mathrm{e}^{-tA}$ is not strongly continuous at $t = 0$. (Note that it is necessarily strongly continuous for $t > 0$.)

Combining the convergence Theorem 8.8 with the generation Theorem 8.13 we obtain the following important result.

Theorem 8.15 (Trotter–Kato). *Suppose that for each $n \in \mathbb{N}$ the operator $-A_n$ is the generator of a C_0-semigroup of common type $(M, -r)$. Suppose further that A is a densely defined operator and for some* $\operatorname{Re}\lambda_0 < r$ *one has $\lambda_0 \in \rho(A)$ and $R(\lambda_0, A_n) \to R(\lambda_0, A)$ strongly. Then $-A$ generates a C_0-semigroup and one has $e^{-tA_n}x \to e^{-tA}x$ uniformly in $t \in [0, \tau]$, for each $x \in X$ and $\tau > 0$.*

Proof. First, observe that the operators A_n, $n \in \mathbb{N}$, are uniformly of half-plane type r. By Exercise 8.6, $\mathrm{L}_r \subseteq \rho(A)$ and $R(\lambda, A_n) \to R(\lambda, A)$ strongly for all $\operatorname{Re}\lambda < r$. So, the claim is a consequence of Theorem 8.8 applied with $\mathcal{F} = \{e^{-t\mathbf{z}} \mid t \in [0, \tau]\}$. □

Remarks 8.16. 1) It is easy to see that the following converse of Theorem 8.15 holds: *If $-A_n$ for each $n \in \mathbb{N}$ and $-A$ generate C_0-semigroups of common type $(M, -r)$ and* $e^{-tA_n} \to e^{-tA}$ *strongly for each $t \geq 0$, then $R(\lambda, A_n) \to R(\lambda, A)$ strongly for each* $\operatorname{Re}\lambda < r$.

2) A common assumption on A_n and A implying that $R(\lambda_0, A_n) \to R(\lambda_0, A)$ strongly is the following: $\lambda_0 - A$ has dense range, and there exists a core D of A such that $A_n x \to Ax$ for all $x \in D$. See Exercise 8.7 and [4, Theorem III.4.9].

From the Trotter–Kato theorem it is just a small step to the most important generation theorem. It should be called after Hille, Yosida, Miyadera and Phillips, but the short “Hille–Yosida” has gained the most popularity.

Theorem 8.17 (Hille–Yosida). *The following assertions are necessary and sufficient for an operator B to be the generator of a C_0-semigroup $T = (T_t)_{t\geq 0}$ on X of type (M, ω):*

1) *B is densely defined;*
2) *$(\omega, \infty) \subseteq \rho(B)$;*
3) $\|R(\lambda, B)^n\| \leq \dfrac{M}{(\lambda - \omega)^n}$ *for all $\lambda > \omega$ and $n \in \mathbb{N}$.*

Proof. The necessity of 1)–3) follow easily from our findings in Chapter 6, see Exercise 6.3. For the proof of sufficiency we can confine ourselves to the case $\omega = 0$ (by passing to $B - \omega$). Define $A = -B$ and let

$$A_n := nA(n + A)^{-1} = n - n^2(n + A)^{-1} \qquad (n \in \mathbb{N})$$

be the **Yosida approximants** of A. These are bounded operators and the respective generated semigroups satisfy

$$\|e^{-tA_n}\| = \|e^{-tn}e^{tn^2(n+A)^{-1}}\| \le e^{-tn}\sum_{k=0}^{\infty}\frac{(tn)^k}{k!}\|n^k(n+A)^{-k}\|$$
$$\le e^{-tn}\sum_{k=0}^{\infty}\frac{(tn)^k}{k!}M = M \qquad \text{for each } n\in\mathbb{N} \text{ and } t\ge 0.$$

(Here we have used that bounded operators are generators of semigroups which are given by the usual power series. See also Exercise 8.9.) Finally, a little computation (Exercise 8.8.a)) shows that $-1 \in \rho(A_n)$ and

$$R(-1, A_n) = \frac{n^2}{(n+1)^2}R\Big(-\frac{n}{n+1}, A\Big) - \frac{1}{n+1} \to R(-1, A)$$

in operator norm as $n\to\infty$. Hence, Theorem 8.15 yields that $-A$ generates a semigroup of type $(M, 0)$. □

Remark 8.18. A more direct proof of the Hille–Yosida theorem, but still using our functional calculus approach, proceeds as follows: Reduce again to $\omega = 0$. Then prove that $A := -B$ is of right half-plane type 0. (Look at the power series for $R(\mathbf{z}, A)$ around $-\lambda$ for $\lambda > 0$; it has radius of convergence at least λ.) Next, consider the rational functions

$$r_{n,t} := \Big(1 + \frac{t}{n}\mathbf{z}\Big)^{-n}$$

and note that $\|r_{n,t}\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)} = \big(1+\frac{t\omega}{n}\big)^{-n}$ for each $\omega < 0$. The Hille–Yosida condition just tells that $\|r_{n,t}(A)\| \le M$ for all $n\in\mathbb{N}$ and $t\ge 0$. Finally, one can apply the second convergence theorem (Theorem 8.10). It follows that

$$\big(\mathrm{I} + \tfrac{t}{n}A\big)^{-n} = r_{n,t}(A) \to e^{-tA}$$

strongly. A more refined version of Theorem 8.10 even yields that this convergence is uniform in t from compact subsets of $\mathbb{R}_+$. For details see [2, Sec.3].

Let us stress the fact that our functional calculus approach to the classical generation theorems can be "tweaked" in order to obtain new generation theorems, e.g., for semigroups that are not strongly continuous at zero. Namely, one can set up Cauchy-integral based holomorphic functional calculi under much weaker hypotheses (on the expense of stronger requirements for the elementary functions, of course.) As soon as the functions $e^{-t\mathbf{z}}$, $t > 0$, are contained in the domain of the canonical extension of the so-constructed calculus, one is in the game.

8.5 Supplement: Compatibility and the Complex Inversion Formula

In this supplement we address the problem of compatibility of the half-plane calculus and the Hille–Phillips calculus for negative generators of semigroups. We start with some considerations about compatibility of abstract functional calculi.

Abstract Functional Calculus (IV) — Compatibility

Let X be a Banach space and $\Phi : \mathcal{F} \to \mathcal{C}(X)$ and $\Psi : \mathcal{F}' \to \mathcal{C}(X)$ be proto-calculi for unital algebras $\mathcal{F}, \mathcal{F}'$. Let $\eta : \mathcal{F} \to \mathcal{F}'$ be a unital homomorphism of algebras. Then the proto-calculi Φ and Ψ are called **compatible** with respect to η if $\Psi \circ \eta = \Phi$. In this case, η is called a **homomorphism** of the two proto-calculi.

The following theorem states that compatibility of non-degenerate representations in bounded operators sometimes implies the compatibility of canonical extensions.

Theorem 8.19. *Let $\eta : \mathcal{F} \to \mathcal{F}'$ be a unital homomorphism of unital algebras, let $\mathcal{E}$ and $\mathcal{E}'$ be subalgebras of $\mathcal{F}$ and $\mathcal{F}'$, respectively, and let $\Phi : \mathcal{E} \to \mathcal{L}(X)$ and $\Phi' : \mathcal{E}' \to \mathcal{L}(X)$ be representations on a Banach space X such that $\eta(\mathcal{E}) \subseteq \mathcal{E}'$ and $\Phi = \Phi' \circ \eta$ on $\mathcal{E}$.*
Suppose that Φ is non-degenerate and $\eta(\mathcal{Z}(\mathcal{E})) \subseteq \mathcal{Z}(\mathcal{E}')$. Then Φ' is also non-degenerate,

$$\eta(\langle \mathcal{E} \rangle_\Phi) \subseteq \langle \mathcal{E}' \rangle_{\Phi'} \quad \text{and} \quad \widehat{\Phi} = \widehat{\Phi'} \circ \eta \text{ on } \langle \mathcal{E} \rangle_\Phi.$$

Proof. Let $f \in \langle \mathcal{E} \rangle_\Phi$ and define $f' := \eta(f)$. We claim that

a) $\eta([f]_\mathcal{E})$ is a Φ'-anchor set and

b) $\eta([f]_\mathcal{E}) \subseteq [f']_{\mathcal{E}'}$.

a) follows easily from the fact that $[f]_\mathcal{E}$ is a Φ-anchor set and $\Phi' \circ \eta = \Phi$ on $\mathcal{E}$. For b), note that if $e \in [f]_\mathcal{E}$, then $e \in \mathcal{Z}(\mathcal{E})$ and $ef \in \mathcal{E}$. Hence, $\eta(e) \in \mathcal{Z}(\mathcal{E}')$ and

$$\eta(e)f' = \eta(e)\eta(f) = \eta(ef) \in \mathcal{E}',$$

by which b) is established. It follows directly from a) and b) that $f' \in \langle \mathcal{E}' \rangle_\Psi$. To prove the identity $\widehat{\Phi'}(f') = \widehat{\Phi}(f)$, recall that $\eta([f]_\mathcal{E})$ (actually, any anchor set contained in $[f']_{\mathcal{E}'}$) is $\widehat{\Phi'}$-determining. So, for $x,\, y \in X$, $\widehat{\;}(f')x = y$ if and only if $\Psi(\eta(e)f')x = \Psi(\eta(e))y$ for all $e \in [f]_\mathcal{E}$. But $\Psi(\eta(e)) = \Phi(e)$ and

$$\Psi(\eta(e)f') = \Psi(\eta(e)\eta(f)) = \Psi(\eta(ef)) = \Phi(ef).$$

Hence $\Psi(f')x = y$ is equivalent with $\Phi(f)x = y$, and the proof is complete. □

Note that if the involved algebras are commutative, the additional hypothesis $\eta(\mathcal{Z}(\mathcal{E})) \subseteq \mathcal{Z}(\mathcal{E}')$ is automatic.

Theorem 8.19 is frequently applied to the case that $\mathcal{F} = \mathcal{F}'$, $\eta = \mathrm{id}$, and $\mathcal{E} \subseteq \mathcal{E}'$.

With Theorem 8.19 at hand we are in the advantageous position that almost always it suffices to show compatibility only on relatively small subalgebras. Our next topic shall illustrate this.

Compatibility of Hille–Phillips and Half-Plane Type Calculus

Suppose that $-A$ generates a bounded C_0-semigroup $(T_t)_{t\geq 0}$. Then we can consider the Hille–Phillips calculus Φ_T on $\mathrm{LS}(\mathbb{C}_+)$ and its canonical extension, again named Φ_T, within $\mathrm{Mer}(\mathbb{C}_+)$.

On the other hand, A is also of half-plane type 0 and for $\omega < 0$ we can form the half-plane type calculus Φ_A on $\mathcal{E}(\mathrm{R}_\omega)$ and its canonical extension, again named Φ_A, within $\mathrm{Mer}(\mathrm{R}_\omega)$. Its domain has been called $\mathrm{Mer}_A(\mathrm{R}_\omega)$ above.

The restriction mapping $\eta : \mathrm{Mer}(\mathrm{R}_\omega) \to \mathrm{Mer}(\mathbb{C}_+)$ is a unital algebra homomorphism. We shall show that the two calculi are compatible with respect to η, i.e., η restricts to a homomorphism of the two calculi. Since all the involved algebras are commutative, by Theorem 8.19 the claim is a consequence of the following theorem.

Theorem 8.20. *Let $-A$ be the generator of a bounded C_0-semigroup $(T_t)_{t\geq 0}$ on a Banach space X. Let $\omega < 0$ and $f \in \mathcal{E}(\mathrm{R}_\omega)$. Then there is a unique $\varphi \in \mathrm{L}^1(\mathbb{R}_+)$ such that $f = \mathcal{L}\varphi$ on $\overline{\mathbb{C}_+}$. Moreover,*

$$\Phi_A(f) = \int_0^\infty \varphi(t) T_t \,\mathrm{d}t.$$

Proof. Let us first try to "guess" φ. Since the claim should hold for all bounded C_0-semigroups, it must in particular be true for the right shift group τ on $\mathrm{C}_0(\mathbb{R})$, whose negative generator is $A_0 = \frac{\mathrm{d}}{\mathrm{d}t}$, the first derivative operator (with maximal domain). But this just means that $\Phi_{A_0}(f) = \tau_\varphi$.
In order to find φ, pick $\delta \in (\omega, 0)$ and consider the formula

$$\Phi_{A_0}(f) = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Re}\, z = \delta} f(z) R(z, A_0) \,\mathrm{d}z.$$

Note that $R(z, A_0) = \tau_{r_z}$ (convolution with r_z), where

$$r_z = -\mathrm{e}^{\mathbf{t}z} \mathbf{1}_{\mathbb{R}_+} \in \mathrm{L}^1(\mathbb{R}_+).$$

By Lemma 6.9, the mapping

$$\mathrm{L}^1(\mathbb{R}_+) \to \mathcal{L}(\mathrm{C}_0(\mathbb{R})), \qquad \varphi \to \tau_\varphi$$

is isometric. It follows that $\Phi_{A_0}(f) = \tau_\varphi$ for

$$\varphi := \frac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} z=\delta} f(z) r_z \,\mathrm{d}z, \tag{8.4}$$

and the integral converges (actually: must converge) in $\mathrm{L}^1(\mathbb{R}_+)$.

It remains to show that φ has the desired properties. Since the integral in (8.4) is convergent in $\mathrm{L}^1(\mathbb{R}_+)$,

$$\begin{aligned}\Phi_T(\mathcal{L}\varphi) &= \int_0^\infty \varphi(t) T_t \,\mathrm{d}t = \frac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} z=\delta} f(z) \int_0^\infty r_z(t) T_t \,\mathrm{d}t \,\mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} z=\delta} f(z) R(z, A) \,\mathrm{d}z = \Phi_A(f).\end{aligned}$$

Finally, fix $\operatorname{Re} a \geq 0$ and specialize $X = \mathbb{C}$, $A = a$ and $T_t = \mathrm{e}^{-ta}$, $t \geq 0$. With this choice,

$$\begin{aligned}(\mathcal{L}\varphi)(a) &= \int_0^\infty \varphi(t) T_t \,\mathrm{d}t = \frac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} z=\delta} f(z) R(z, A) \,\mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} z=\delta} \frac{f(z)}{z-a} \,\mathrm{d}z = f(a)\end{aligned}$$

by Lemma 8.26. □

Evaluating at points $t \geq 0$ in the formula (8.4) we obtain

$$\varphi(t) = \frac{1}{2\pi \mathrm{i}} \int_{\delta - \mathrm{i}\infty}^{\delta + \mathrm{i}\infty} \mathrm{e}^{tz} (\mathcal{L}\varphi)(z) \,\mathrm{d}z \qquad (t \geq 0),$$

which is the **complex inversion formula** for the Laplace transform. (The point evaluation is justified, as the integral (8.4) converges also in $\mathrm{C}_0(\mathbb{R}_+)$.)

The Complex Inversion Formula for Semigroups

Let B be of *left* half-plane type 0. If B generates a semigroup T, then the resolvent $R(\cdot, B)$ is the Laplace transform of T. Hence, one should be able to reconstruct T from $R(\cdot, B)$ by the complex inversion formula from above in some sense. This is true, as the following considerations show.

Lemma 8.21. *Let B be of left half-plane type* 0*. Then for $\delta > 0$ and $x \in \operatorname{dom}(B^2)$*

$$\mathrm{e}^{tB} x = \frac{1}{2\pi \mathrm{i}} \int_{\delta - \mathrm{i}\infty}^{\delta + \mathrm{i}\infty} \mathrm{e}^{tz} R(z, B) x \,\mathrm{d}z \qquad (t > 0), \tag{8.5}$$

where the integral converges in the improper sense, uniformly in t from compact subintervals of $(0, \infty)$.

Proof. Let $x \in \mathrm{dom}(B^2)$ and $\lambda < -\delta =: \delta'$. By Lemma 8.26 below,

$$\int_{\mathrm{Re}\, z=\delta'} \frac{e^{-zt}}{\lambda - z}\, \mathrm{d}z = 0 \tag{8.6}$$

in the improper sense. Next, we let $A := -B$, which is of right half-plane type 0, and employ the formula $(\lambda - A)R(z, A) = (\lambda - z)R(z, A) + \mathrm{I}$ as well as (8.6) twice in computing (with $y := (\lambda - A)^2 x$)

$$\begin{aligned} \mathrm{e}^{-tA}x &= \left(\frac{\mathrm{e}^{-t\mathbf{z}}}{(\lambda - \mathbf{z})^2}\right)(A)y = \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{Re}\, z=\delta'} \frac{\mathrm{e}^{-tz}}{(\lambda - z)^2} R(z, A)(\lambda - A)^2 x \,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{Re}\, z=\delta'} \frac{\mathrm{e}^{-tz}}{\lambda - z} R(z, A)(\lambda - A)x \,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{Re}\, z=\delta'} \mathrm{e}^{-tz} R(z, A)x \,\mathrm{d}z. \end{aligned}$$

The last term is easily seen to be equal to (8.5). The proof of the uniformity statement is left to the reader. □

We remark that if B is actually a generator, we can say more.

Theorem 8.22 (Complex Inversion for Semigroups). *Let B be the generator of a C_0-semigroup $(T_t)_{t\ge 0}$ on a Banach space X. Then for each $\delta > \omega_0(T)$ and $x \in \mathrm{dom}(B)$*

$$T_t x = \frac{1}{2\pi\mathrm{i}} \int_{\delta - \mathrm{i}\infty}^{\delta + \mathrm{i}\infty} \mathrm{e}^{tz} R(z, B)x \,\mathrm{d}z \qquad (t > 0), \tag{8.7}$$

where the integral converges in the improper sense, uniformly in t from compact subintervals of $(0, \infty)$.

Proof. By rescaling we may suppose that $M := \sup_{t\ge 0} \|T_t\| < \infty$ and $\delta > 0$. Since $\mathrm{dom}(B^2)$ is a core for $\mathrm{dom}(B)$, by Lemma 8.21 it suffices to show that for some fixed $\lambda > \delta$ the operators

$$\int_{\delta - \mathrm{i}N}^{\delta + \mathrm{i}N'} \mathrm{e}^{tz} R(z, B)R(\lambda, B)\,\mathrm{d}z \qquad (0 < N,\ N' < \infty)$$

are uniformly bounded. By the resolvent identity and Lemma 8.26 it suffices to show that the operators

$$S_{N,N'} := \int_{\delta - \mathrm{i}N}^{\delta + \mathrm{i}N'} \frac{\mathrm{e}^{tz}}{\lambda - z} R(z, B)\,\mathrm{d}z \qquad (0 < N,\ N' < \infty)$$

are uniformly bounded. For this, fix $x \in X$ and $x' \in X'$ and estimate

$$\begin{aligned}|\langle S_{N,N'}x,x'\rangle| &\le \int_{\delta-\mathrm{i}N}^{\delta+\mathrm{i}N'} \frac{\mathrm{e}^{\delta t}}{|\lambda - z|}\,|\langle R(z,B)x,x'\rangle|\;|\mathrm{d}z| \\ &\le \mathrm{e}^{\delta t}\left(\int_{\delta-\mathrm{i}\infty}^{\delta+\mathrm{i}\infty}\frac{|dz|}{|\lambda - z|^2}\right)^{\frac{1}{2}}\left(\int_{\delta-\mathrm{i}\infty}^{\delta+\mathrm{i}\infty}|\langle R(z,B)x,x'\rangle|^2\;|\mathrm{d}z|\right)^{\frac{1}{2}}.\end{aligned}$$

By Plancherel's theorem [7, Thm. 9.33],

$$\begin{aligned}\int_{\delta-\mathrm{i}\infty}^{\delta+\mathrm{i}\infty}|\langle R(z,B)x,x'\rangle|^2\;|\mathrm{d}z| &= 2\pi\int_0^\infty \mathrm{e}^{-2\delta s}\,|\langle T_s x,x'\rangle|^2\;\mathrm{d}s \\ &\le \frac{\pi}{\delta}M^2\|x\|^2\|x'\|^2,\end{aligned}$$

which implies that $\|S_{N,N'}\| \le \frac{\pi \mathrm{e}^{\delta t}M}{\sqrt{\delta(\lambda-\delta)}}$. The claim follows. □

Remark 8.23. The complex inversion formula (8.7) is a classical result from semigroup theory, see for instance [8, Thm.11.6.1]. Clearly, it immediately suggests to use it in the other direction as a starting point for generation theorems. So it is of no suprise that this has been done again and again since the early days of semigroup theory, see for instance [8, Thm.12.6.1] for a classical and [3, Thm.2.4] for a more recent example. Lemma 8.21 from above reveals that also our approach to generation theorems is just the complex inversion formula in disguise. However, there is a decisive difference: we have embedded the definition of the semigroup in a whole functional calculus; and we do not employ closures of operators in our definitions, hence tedious approximation arguments are avoided.

8.6 Supplement: Some Results from Complex Analysis

In this section we prove some results of complex analysis. The first regards the function classes $\mathcal{E}(\mathrm{R}_\omega)$ that were used to define the half-plane functional calculus. Namely, one may wonder about how natural our choice of elementary functions actually is. Integrability along a vertical line is certainly important, as well as to have Lemma 8.2. But how about uniform boundedness on a half-plane? The following theorem and its corollary (in combination with Remark 8.5) tells that we could not have done much better as we actually did.

Theorem 8.24. *Let $\omega \in \mathbb{R}$ and $u \in \mathrm{L}^1(\omega + \mathrm{i}\mathbb{R})$ such that*

$$\int_{\operatorname{Re} z=\omega}\frac{u(z)}{z-a}\,\mathrm{d}z = 0 \qquad (\operatorname{Re} a < \omega).$$

Let the function $f : \mathrm{R}_\omega \to \mathbb{C}$ be given by

$$f(a) := \frac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} z = \omega} \frac{u(z)}{z - a} \,\mathrm{d}z \qquad (\operatorname{Re} a > \omega).$$

Then the following statements hold:

a) *f is holomorphic on R_ω.*

b) *$f \in \mathrm{C}_0(\overline{\mathrm{R}_\delta})$ for each $\delta > \omega$.*

c) *$f \in \mathrm{L}^1(\delta + \mathrm{i}\mathbb{R})$ with $\|f\|_{\mathrm{L}^1(\delta+\mathrm{i}\mathbb{R})} \le \|u\|_{\mathrm{L}^1(\omega+\mathrm{i}\mathbb{R})}$ for all $\delta > \omega$.*

Proof. Without loss of generality $\omega = 0$. The proof of a) is standard: one uses Lebesgue's theorem to show that f is continuous; and then Morera's theorem in combination with an interchanging of integrals to show that f is holomorphic.

b) This is an elementary estimate and left as exercise.

c) We fix $\delta > 0$ and $a = \delta + \mathrm{i}t$, for $t = \operatorname{Im} a \in \mathbb{R}$. Then $\operatorname{Re}(-\overline{a}) < 0$ and hence

$$\begin{aligned} f(a) &= \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{i}\mathbb{R}} \frac{u(z)}{z-a}\,\mathrm{d}z = \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{i}\mathbb{R}} \frac{u(z)}{z-a}\,\mathrm{d}z - \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{i}\mathbb{R}} \frac{u(z)}{z+\overline{a}}\,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{i}\mathbb{R}} \frac{2\delta}{(z-a)(z+\overline{a})}\, u(z)\,\mathrm{d}z. \end{aligned}$$

Hence,

$$\begin{aligned} \|f\|_{\mathrm{L}^1(\delta+\mathrm{i}\mathbb{R})} &= \int_{\mathbb{R}} |f(\delta+\mathrm{i}t)|\,\mathrm{d}t \le \frac{\delta}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(\mathrm{i}x)|}{|(\mathrm{i}x - \delta - \mathrm{i}t)\,(\mathrm{i}x + \delta - \mathrm{i}t)|}\,\mathrm{d}t\,\mathrm{d}x \\ &= \frac{\delta}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(\mathrm{i}x)|}{\delta^2 + (x-t)^2}\,\mathrm{d}t\,\mathrm{d}x = \frac{\delta}{\pi} \|u\|_{\mathrm{L}^1(\mathrm{i}\mathbb{R})} \int_{\mathbb{R}} \frac{\mathrm{d}t}{\delta^2 + t^2} \\ &= \|u\|_{\mathrm{L}^1(\mathrm{i}\mathbb{R})}. \end{aligned}$$ □

Corollary 8.25. *Suppose that $f \in \operatorname{Hol}(\mathrm{R}_\alpha)$ is such that for some $\alpha < \omega$ one has $f \in \mathrm{L}^1(\omega + \mathrm{i}\mathbb{R})$ and*

$$\frac{1}{2\pi\mathrm{i}} \int_{\operatorname{Re} z = \omega} \frac{f(z)}{z-a} = \begin{cases} f(a), & \text{if} \quad \operatorname{Re} a > \omega, \\ 0 & \text{if} \quad \operatorname{Re} a < \omega. \end{cases}$$

Then $f \in \mathcal{E}(\mathrm{R}_\delta)$ for each $\delta > \omega$.

The next lemma has been used in the proofs of Lemma 8.21 and Theorem 8.22. It is a classic from elementary complex analysis and is usually proved with the help of "Jordan's lemma". Here we give a different proof that builds on Lemma 8.2.

Lemma 8.26. *Let $\operatorname{Re} \lambda < \delta \in \mathbb{R}$. Then*

$$\int_{\operatorname{Re} z = \delta} \frac{e^{-zt}}{\lambda - z}\,\mathrm{d}z = 0 \quad (t > 0),$$

where the integral converges in the improper sense, uniformly in t from compact subintervals of $(0, \infty)$.

Proof. Note first that

$$\begin{aligned}\frac{e^{-zt}}{\lambda - z} &= \frac{e^{-zt}}{t(\lambda - z)^2} - \left(\frac{e^{-zt}}{t(\lambda - z)}\right)' \\ &= \frac{2e^{-zt}}{t^2(\lambda - z)^3} - \left(\frac{e^{-zt}}{t^2(\lambda - z)^2}\right)' - \left(\frac{e^{-zt}}{t(\lambda - z)}\right)'.\end{aligned}$$

From the first line of the computation above and the Cauchy criterion for the existence of improper integrals, it follows that the improper integral exists uniformly in t from compact subintervals. To determine its value, look at the second line of the computation. The first summand is of the form $\frac{g(z)}{\lambda - z}$ for $g \in \mathcal{E}(\mathrm{R}_\omega)$, $\omega < \delta$, so the integral over this term equals zero by Lemma 8.2. The improper integrals over the two remaining terms is zero by the fundamental theorem of calculus. □

Exercises

8.1. Let A be an operator on a Banach space X such that the straight line $[\operatorname{Re} \mathbf{z} = \omega]$ is contained in the resolvent set and

$$M := \sup_{\operatorname{Re} z = \omega} \|R(z, A)\| < \infty.$$

Show that there is $r = r(M) > 0$ such that $[\,|\operatorname{Re} \mathbf{z} - \omega| < r\,] \subseteq \rho(A)$ and for each $0 < r' < r$ one has

$$\sup_{|\operatorname{Re} z - \omega| \le r'} \|R(z, A)\| < \infty.$$

Conclude that $\omega_{\mathrm{hp}}(A) > \omega$ for an operator A satisfying (8.1). [Hint: Power series representation of $R(\cdot, A)$.]

8.2 (Dunford–Riesz Calculus for Unbounded Operators). In this exercise we let $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and consider it as a (simple) Riemann surface. In particular, a function $f : U \to \mathbb{C}$ defined on an open subset $U \subseteq \mathbb{C}_\infty$ is called holomorphic, if f is holomorphic on $U \cap \mathbb{C}$ and $f(\mathbf{z}^{-1})$ is holomorphic on $U \setminus \{0\}$.

Let A be any operator on a Banach space with $\rho(A) \neq \emptyset$. Define

$$\tilde{\sigma}(A) := \begin{cases} \sigma(A) & \text{if } A \text{ is bounded} \\ \sigma(A) \cup \{\infty\} & \text{if } A \text{ is not bounded.} \end{cases}$$

Fix an open set $U \subseteq \mathbb{C}_\infty$ such that $\tilde{\sigma}(A) \subseteq U$. For $\lambda \in \rho(A)$ let $U_\lambda := \{\frac{1}{\lambda - z} : z \in U\}$ and for $f \in \mathrm{Hol}(U)$ let $f_\lambda := f(\lambda - \mathbf{z}^{-1}) \in \mathrm{Hol}(U_\lambda)$. Moreover, let $A_\lambda := R(\lambda, A)$. Then define

$$\Phi_A(f) := \Phi_{A_\lambda}(f_\lambda) \qquad (f \in \mathrm{Hol}(U))$$

where $\Phi_{A_\lambda} : \mathrm{Hol}(U_\lambda) \to \mathcal{L}(X)$ is the Dunford–Riesz calculus for A_λ. (Recall the spectral mapping theorem for the inverse from Exercise 2.4 and cf. also [6, Sec. A.3].)

Show that $\Phi_A(f)$ does not depend on the choice of $\lambda \in \rho(A)$ and that $\Phi_A : \mathrm{Hol}(U) \to \mathcal{L}(X)$ is a unital algebra homomorphism that maps the function $(\lambda - \mathbf{z})^{-1}$ to $R(\lambda, A)$ whenever $\lambda \in \rho(A)$. Finally, show that Φ_A coincides with the Dunford–Riesz calculus for A if A is bounded.

[Hint: For the proof of the independence of λ note that A_λ and A_μ are linked by a Möbius transformation. One way to proceed is to use Remark 1.9. Alternatively one can compute directly with the integrals employing Exercise 2.4.]

8.3. Let A be an operator of right half-plane type on a Banach space X, let $\omega < \omega_{\mathrm{hp}}(A)$ and $\Phi_A : \mathcal{E}(\mathrm{R}_\omega) \to \mathcal{L}(X)$ defined by (8.3) with $\omega < \delta < \omega_{\mathrm{hp}}(A)$. Show that $\Phi_A(fg) = \Phi_A(f)\Phi_A(g)$ for all $f, g \in \mathcal{E}(\mathrm{R}_\omega)$ and

$$\Phi_A\Big(\frac{1}{(\lambda - \mathbf{z})(\mu - \mathbf{z})}\Big) = R(\lambda, A)R(\mu, A)$$

whenever $\mathrm{Re}\,\mu$, $\mathrm{Re}\,\lambda < \omega$. [This completes the proof of Theorem 8.3.]

8.4. Let A be an operator with non-empty resolvent set on a Banach space X. Show that if A is densely defined, then so is A^n for each $n \in \mathbb{N}$. [Hint: induction.]

8.5. Prove Lemma 8.9 and the "second convergence theorem" (Theorem 8.10).

8.6. Let $(A_n)_n$ be a sequence of linear operators on a Banach space X. Let Ω be a subset of $\bigcap_{n\in\mathbb{N}} \rho(A_n)$ such that

$$M := \sup_{n\in\mathbb{N}, \lambda\in\Omega} \|R(\lambda, A_n)\| < \infty.$$

Finally, let A be an operator on X and consider

$$\Lambda := \{\lambda \in \Omega \cap \rho(A) \mid R(\lambda, A_n) \to R(\lambda, A) \text{ strongly}\}.$$

Prove that a) $\mathrm{B}(\lambda, \frac{1}{M}) \cap \Omega \subseteq \Lambda$ for all $\lambda \in \Lambda$ and that b) $\Omega \cap \overline{\Lambda} \subseteq \Lambda$. Conclude that if Ω is connected and $\Lambda \neq \emptyset$, then $\Lambda = \Omega$.
[Hint: for a) use the power series expansion of the resolvent, for b) use a).]

8.7. Let $(A_n)_n$ be a sequence of operators on a Banach space X and let $\lambda \in \bigcap_n \rho(A_n)$ with $\sup_n \|R(\lambda, A_n)\| < \infty$. Let, furthermore, A be a closed operator on X and $D \subseteq X$ a subspace with the following properties:

1) D is a core for A;
2) $D \subseteq \bigcap_n \mathrm{dom}(A_n)$ and $A_n x \to Ax$ for all $x \in D$;
3) $\mathrm{ran}(\lambda - A)$ is dense in X.

Show that then $\lambda \in \rho(A)$ and $R(\lambda, A_n) \to R(\lambda, A)$ strongly.

8.8. Let A be an operator on a Banach space X with $(-\infty, 0) \subseteq \rho(A)$. Define its **Yosida approximants** by

$$A_\lambda := \lambda A(\lambda + A)^{-1} \qquad (\lambda \geq 1).$$

Prove the following assertions.

a) If $\lambda \neq \mu \in \mathbb{C}$ is such that $\frac{\lambda\mu}{\lambda-\mu} \in \rho(A)$, then $\mu \in \rho(A_\lambda)$ and

$$R(\mu, A_\lambda) = \frac{\lambda^2}{(\lambda - \mu)^2} R\Big(\frac{\lambda\mu}{\lambda - \mu}, A\Big) - \frac{1}{\lambda - \mu}.$$

b) If A is of right half-plane type 0, then the A_λ for $\lambda \geq 1$ are uniformly of right half-plane type 0 and actually a half-plane approximation of A on $\mathbb{C}_+$.

8.9 (Compatibility with the Dunford–Riesz calculus). Let A be a bounded operator on a Banach space X and let $r := \inf\{\mathrm{Re}\,\lambda \mid \lambda \in \sigma(A)\}$.

a) Show that $-A$ generates an operator norm continuous group $(U_s)_{s\in\mathbb{R}}$ defined by

$$U_s = \sum_{n=0}^{\infty} \frac{s^n}{n!}(-A)^n.$$

b) Show that A is of right half-plane type r.

c) Fix $\omega < r$ and $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$. Show that the definitions of $f(A)$ within the Dunford–Riesz calculus of Chapter 1 and within the (extended) half-plane calculus of the current chapter coincide.

[Hint: for a) use the “baby version” of the Dunford–Riesz calculus; for c) observe that $\int_{\mathrm{Re}\, z=\delta} f(z)R(z, A)\,\mathrm{d}z = 0$ for large $\delta \in \mathbb{R}$ and $f \in \mathcal{E}(\mathrm{R}_\omega)$.]

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Chapter 9
The Holomorphic Functional Calculus for Sectorial Operators

9.1 Sectorial Operators

For $0 < \omega \le \pi$ and $0 < \omega'$ we consider the **open sector** and the **open strip**

$$\mathrm{S}_\omega := \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \omega\},$$
$$\mathrm{St}_{\omega'} := \{z \in \mathbb{C} \mid |\mathrm{Im}\, z| < \omega'\}.$$

So, S_ω is symmetric about the positive real axis with opening angle 2ω. For $\omega = \pi/2$ we have $\mathrm{S}_{\pi/2} = \mathbb{C}_+$. (Note that $2\omega > \pi$ is allowed here.) The strip $\mathrm{St}_{\omega'}$ extends horizontally, symmetric about the real axis. We also define

$$\mathrm{S}_0 := (0, \infty) \quad \text{and} \quad \mathrm{St}_0 := \mathbb{R}$$

and consider them as a degenerate sector and strip, respectively. For $0 \le \omega < \pi$, the exponential function is a biholomorphic mapping (conformal equivalence)

$$\mathrm{e}^{\mathbf{z}} : \mathrm{St}_\omega \to \mathrm{S}_\omega \quad \text{with inverse} \quad \log \mathbf{z} : \mathrm{S}_\omega \to \mathrm{St}_\omega,$$

the (principal branch of the) **logarithm**. Note that the additive group $\mathbb{R}$ acts by translations on each strip, whereas the multiplicative group $(0, \infty)$ acts by multiplication on each sector.

An operator A on a Banach space X is called **sectorial of angle** $\omega \in [0, \pi)$ if $\sigma(A) \subseteq \overline{\mathrm{S}_\omega}$ and for each $\alpha \in (\omega, \pi)$

$$M(A, \alpha) := \sup\{\|\lambda R(\lambda, A)\| \mid \lambda \in \mathbb{C} \setminus \overline{\mathrm{S}_\alpha}\} < \infty.^1$$

An operator A is simply called **sectorial** if it is sectorial of angle ω for some $\omega \in [0, \pi)$. In this case,

[1] In these notes, the meaning of the symbol "$M(A, \alpha)$" is heavily depending on the context, cf. Section 8.1.

$$\omega_{\mathrm{se}}(A) := \min\{\omega \in [0, \pi) \mid A \text{ sectorial of angle } \omega\}$$

is called the **sectoriality angle** of A. Analogously to the half-plane case, we say that a set $\mathcal{A}$ of operators is **uniformly sectorial of angle** $\omega < \pi$ if

$$\sup_{A \in \mathcal{A}} M(A, \alpha) < \infty$$

for all $\alpha \in (\omega, \pi)$.

Examples 9.1. (See Exercise 9.1).

1) Let Ω be a semi-finite measure space and $a : \Omega \to \mathbb{C}$ a measurable mapping. The multiplication operator M_a on $\mathrm{L}^p(\Omega)$, $1 \le p \le \infty$, is sectorial of angle ω if and only if $a \in \overline{\mathrm{S}_\omega}$ almost everywhere.
2) Let H be a Hilbert space and A a closed operator on H with numerical range $\mathrm{W}(A) \subseteq \overline{\mathrm{S}_\omega}$, $\omega \le \pi/2$, and such that $\mathrm{ran}(\mathrm{I} + A) = H$. Then A is sectorial of angle ω. In particular, a positive, self-adjoint operator is sectorial of angle 0.
3) Suppose that the resolvent of an operator A satisfies an estimate of the form
$$\|R(\lambda, A)\| \le \frac{M}{|\mathrm{Re}\,\lambda|} \qquad \text{for all } \mathrm{Re}\,\lambda < 0.$$
Then A is sectorial of angle $\pi/2$. In particular, this is the case if $-A$ generates a bounded C_0-semigroup.
4) Suppose that the resolvent of an operator A satisfies an estimate of the form
$$\|R(\lambda, A)\| \le \frac{M}{|\mathrm{Im}\,\lambda|} \qquad \text{for all } \lambda \in \mathbb{C} \setminus \mathbb{R}.$$
Then A^2 is sectorial of angle 0. In particular, this is the case if $-\mathrm{i}A$ generates a bounded C_0-group.

Let us list some elementary properties of sectorial operators.

Theorem 9.2. *An operator A on a Banach space X is sectorial if and only if $(-\infty, 0) \subseteq \rho(A)$ and $M := \sup_{t>0} \|t(t + A)^{-1}\| < \infty$. Moreover, if A is sectorial, the following assertions hold:*

a) $x \in \overline{\mathrm{dom}}(A)$ *if and only if* $\lim_{t\to\infty} t(t + A)^{-1}x = x$;
   $x \in \overline{\mathrm{ran}}(A)$ *if and only if* $\lim_{t\to 0} t(t + A)^{-1}x = 0$.
b) $\overline{\mathrm{ran}}(A) \cap \ker(A) = \{0\}$.
c) *If X is reflexive then* $\mathrm{dom}(A)$ *is dense and* $X = \ker(A) \oplus \overline{\mathrm{ran}}(A)$.

Proof. Note that, trivially, if A is sectorial then the stated criterion holds. The converse is proved in Exercise 9.2, as well as the assertions in a) and b).

Let X be reflexive and let $x \in X$. Then the bounded sequence $n(n + A)^{-1}x$ has a subsequence which is weakly convergent to some y. Since $\mathrm{dom}(A)$ is a subspace, by Mazur's theorem we have $y \in \overline{\mathrm{dom}}(A)$. On the other hand,

$$(1+A)^{-1}n(n+A)^{-1} = \frac{n}{n-1}\big((1+A)^{-1} - (n+A)^{-1}\big) \to (1+A)^{-1}$$

in operator norm. It follows that $(1+A)^{-1}x = (1+A)^{-1}y$ and hence $x = y \in \overline{\mathrm{dom}}(A)$.

The proof that $X = \ker(A) \oplus \overline{\mathrm{ran}}(A)$ is similar. First, for $x \in X$ find a weak limit y of a subsequence of $(A(\frac{1}{n}+A)^{-1}x)_{n\in\mathbb{N}}$. Then, by Mazur's theorem, $y \in \overline{\mathrm{ran}}(A)$. Finally, observe that $x - y \in \ker(A(1+A)^{-1}) = \ker(A)$. □

Remark 9.3. In dealing with sectorial operators and functions on sectors, the following observations are frequently helpful. Each sector S_ω is invariant under inversion $\mathbf{z}^{-1}$ and under multiplication by positive scalars. Moreover,

$$\mathbb{C} \setminus \overline{\mathrm{S}_\omega} = -\mathrm{S}_{\pi-\omega}.$$

If $\omega + \omega' \le \pi$ then

$$\mathrm{S}_\omega \cdot \mathrm{S}_{\omega'} = \mathrm{S}_{\omega+\omega'} \quad \text{and} \quad \mathrm{S}_\omega + \mathrm{S}_{\omega'} = \mathrm{S}_{\max(\omega,\omega')}.$$

If $0 < \omega < \alpha < \pi$ then

$$\sup_{z,\lambda} \left|\frac{\lambda}{\lambda - z}\right| + \left|\frac{z}{\lambda - z}\right| < \infty, \tag{9.1}$$

where the supremum runs over all $z \in \mathrm{S}_\omega$ and $\lambda \in \mathbb{C}\setminus\mathrm{S}_\alpha$. (The reason is that for these choices of z and λ one has $z/\lambda, \lambda/z \in \mathbb{C} \setminus \mathrm{S}_{\alpha-\omega}$ and hence these fractions have a uniform distance to 1.)

In accordance with these properties of sectors, sectorial operators enjoy certain permanence properties, see Exercise 9.3.

9.2 Elementary Functions on Sectors and Strips

Quite analogously to the half-plane case in the previous chapter, we introduce the algebra of **elementary functions**[2] on S_ω, $0 < \omega \le \pi$, by

$$\mathcal{E}(\mathrm{S}_\omega) := \Big\{ f \in \mathrm{H}^\infty(\mathrm{S}_\omega) \;\Big|\; \int_0^\infty \big|f(r\mathrm{e}^{\mathrm{i}\alpha})\big| \,\frac{\mathrm{d}r}{r} < \infty \text{ for all } |\alpha| < \omega \Big\}.$$

So, $f : \mathrm{S}_\omega \to \mathbb{C}$ is elementary if f is holomorphic and bounded and, for each $0 \le \alpha < \omega$, f is integrable over $\partial\mathrm{S}_\alpha$ with respect to the measure $\frac{|\mathrm{d}z|}{|z|}$. We

[2] Actually $\mathrm{S}_{\pi/2} = \mathbb{C}_+ = \mathrm{R}_0$ and hence the symbol $\mathcal{E}(\mathbb{C}_+)$ is ambiguous. To avoid confusion, the elementary functions on the *sector* $\mathbb{C}_+$ will be denoted by $\mathcal{E}(\mathrm{S}_{\pi/2})$, while the elementary functions on the *half-plane* $\mathbb{C}_+$ are denoted by $\mathcal{E}(\mathrm{R}_0)$.

shall write $f \in \mathrm{L}^1_*(\partial \mathrm{S}_\alpha)$ for this. The boundary $\partial \mathrm{S}_\alpha$ shall be so oriented that the points of S_α are to its left. That means that we use the formula

$$\int_{\partial \mathrm{S}_\alpha} f(z) \frac{\mathrm{d}z}{z} = \int_0^\infty f(x\mathrm{e}^{-\mathrm{i}\alpha}) \frac{\mathrm{d}x}{x} - \int_0^\infty f(x\mathrm{e}^{\mathrm{i}\alpha}) \frac{\mathrm{d}x}{x}$$

for integration along $\partial \mathrm{S}_\alpha$.

For technical reasons here, and for later use, we also introduce the algebra of **elementary functions** on the strip St_ω, $\omega > 0$, by

$$\mathcal{E}(\mathrm{St}_\omega) := \Big\{ f \in \mathrm{H}^\infty(\mathrm{St}_\omega) \;\Big|\; \int_{-\infty}^\infty |f(r + \mathrm{i}\alpha)| \,\mathrm{d}r < \infty \text{ for all } |\alpha| < \omega \Big\}.$$

In other words, a function $f : \mathrm{St}_\omega \to \mathbb{C}$ is elementary if f is holomorphic and bounded and, for each $0 \le \alpha < \omega$, f is integrable over $\partial \mathrm{St}_\alpha$ with respect to arclength measure. We shall write $f \in \mathrm{L}^1(\partial \mathrm{St}_\alpha)$ for this. The boundary $\partial \mathrm{St}_\alpha$ shall be so oriented that the points of St_α are to its left, so that we have the formula

$$\int_{\partial \mathrm{St}_\alpha} f(z) \,\mathrm{d}z = \int_{\mathbb{R}} f(x - \mathrm{i}\alpha) \,\mathrm{d}x - \int_{\mathbb{R}} f(x + \mathrm{i}\alpha) \,\mathrm{d}x,$$

for integration along $\partial \mathrm{St}_\alpha$. Note that if $0 < \alpha < \omega < \pi$,

$$f \in \mathcal{E}(\mathrm{S}_\omega) \qquad \Longleftrightarrow \qquad f(\mathrm{e}^{\mathbf{z}}) \in \mathcal{E}(\mathrm{St}_\omega)$$

with

$$\int_{\partial \mathrm{S}_\alpha} f(z) \frac{\mathrm{d}z}{z} = \int_{\partial \mathrm{St}_\alpha} f(\mathrm{e}^z) \,\mathrm{d}z,$$

The following is the analogue of Lemma 8.2 for elementary functions on strips and sectors.

Lemma 9.4. a) *Let* $0 < \delta < \omega$, $f \in \mathcal{E}(\mathrm{St}_\omega)$, *and* $a \in \mathbb{C} \setminus \partial \mathrm{St}_\delta$. *Then*

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{St}_\delta} \frac{f(z)}{z - a} \,\mathrm{d}z = \begin{cases} f(a) & \text{if} \quad |\operatorname{Im} a| < \delta, \\ 0 & \text{if} \quad |\operatorname{Im} a| > \delta. \end{cases}$$

Moreover, $f \in \mathrm{C}_0(\overline{\mathrm{St}_\delta})$ *and* $\int_{\partial \mathrm{St}_\delta} f(z) \,\mathrm{d}z = 0$.

b) *Let* $0 < \delta < \omega \le \pi$, $f \in \mathcal{E}(\mathrm{S}_\omega)$ *and* $a \in \mathbb{C} \setminus \partial \mathrm{S}_\delta$. *Then*

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\delta} \frac{f(z)}{z - a} \,\mathrm{d}z = \begin{cases} f(a) & \text{if} \quad |\arg a| < \delta, \\ 0 & \text{if} \quad |\arg a| > \delta. \end{cases}$$

Moreover, $f \in \mathrm{C}_0(\overline{\mathrm{S}_\delta} \setminus \{0\})$ *and* $\int_{\partial \mathrm{S}_\delta} f(z) \,\frac{\mathrm{d}z}{z} = 0$.

Proof. a) For $R > 0$ let γ_R be the positively oriented boundary of the rectangle $[-R, R] \times [-\delta, \delta]$. Then

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{f(z)}{z-a}\,\mathrm{d}z \;=\; \int_{\partial\mathrm{St}_\delta} \frac{f(z)}{z-a}\,\mathrm{d}z$$

since f is uniformly bounded on $\overline{\mathrm{St}_\delta}$. If $a \notin \overline{\mathrm{St}_\delta}$ then

$$\int_{\gamma_R} \frac{f(z)}{z-a}\,\mathrm{d}z = 0$$

for all $R > 0$, by Cauchy's theorem. And if $a \in \mathrm{St}_\delta$ then $a \in \mathrm{Int}(\gamma_R)$ for all sufficiently large $R > 0$ and hence

$$\int_{\gamma_R} \frac{f(z)}{z-a}\,\mathrm{d}z = 2\pi\mathrm{i}\, f(a),$$

again by Cauchy's theorem.

For the next assertion, fix $0 < \delta' < \delta$. By the dominated convergence theorem and the first, already proved, assertion, it follows that $f(a) \to 0$ as $a \to \infty$ within $\overline{\mathrm{St}_{\delta'}}$. As $0 < \delta < \omega$ is arbitrary, the claim is proved. For the last claim we take γ_R as above and observe that, since $f \in \mathrm{C}_0(\overline{\mathrm{St}_\delta})$,

$$0 = \int_{\gamma_R} f(z)\,\mathrm{d}z \to \int_{\partial\mathrm{St}_\delta} f(z)\,\mathrm{d}z \qquad (R \to \infty).$$

b) We let $b := \log a$ and $g := f(\mathrm{e}^{\mathbf{z}})$ and note that

$$h := \frac{\mathrm{e}^{\mathbf{z}}}{\mathrm{e}^{\mathbf{z}} - \mathrm{e}^{b}} - \frac{1}{\mathbf{z} - b} \in \mathrm{H}^\infty(\mathrm{St}_\omega).$$

In particular, $gh \in \mathcal{E}(\mathrm{St}_\omega)$ and hence, by the last assertion of a),

$$\int_{\partial\mathrm{S}_\delta} \frac{f(z)}{z-a}\,\mathrm{d}z = \int_{\partial\mathrm{St}_\delta} \frac{\mathrm{e}^z g(z)}{\mathrm{e}^z - \mathrm{e}^b}\,\mathrm{d}z = \int_{\partial\mathrm{St}_\delta} \frac{g(z)}{z-b}\,\mathrm{d}z.$$

Now all claims in b) follow from a). □

Remark 9.5 (The class $\mathrm{H}_0^\infty(\mathrm{S}_\omega)$). Fix $0 < \omega < \pi$ and let $f \in \mathrm{Hol}(\mathrm{S}_\omega)$ be such that for some $s > 0$ and some $C \ge 0$ one has

$$|f(z)| \le C \min\{|z|^s\,,\ |z|^{-s}\} \qquad (z \in \mathrm{S}_\omega). \tag{9.2}$$

The set of functions with this property was introduced by McIntosh in his groundbreaking article [5] and is usually denoted by $\mathrm{H}_0^\infty(\mathrm{S}_\omega)$. It still features in many texts dealing with the functional calculus for sectorial operators.

Clearly $\mathrm{H}_0^\infty(\mathrm{S}_\omega) \subseteq \mathcal{E}(\mathrm{S}_\omega)$ and one has

$$\sup_{|\alpha|\le\delta} \int_0^\infty \left|f(r\mathrm{e}^{\mathrm{i}\alpha})\right| \frac{\mathrm{d}r}{r} < \infty \qquad (0 \le \delta < \omega) \tag{9.3}$$

for each $f \in \mathrm{H}_0^\infty(\mathrm{S}_\omega)$. (One can show that (9.3) holds actually for each $f \in \mathcal{E}(\mathrm{S}_\omega)$, see Exercise 9.11.)

9.3 The Sectorial Functional Calculus

Let A be a sectorial operator and let $\omega_{\mathrm{se}}(A) < \omega < \pi$. Then, for $f \in \mathcal{E}(\mathrm{S}_\omega)$ we define

$$\Phi_A(f) := \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\delta} f(z) R(z, A) \, \mathrm{d}z, \tag{9.4}$$

where $\omega_{\mathrm{se}}(A) < \delta < \omega$. The integral does not depend on δ (by Cauchy's theorem) because $f \in \mathrm{C}_0(\overline{\mathrm{S}_{\omega'}} \setminus \{0\})$ for each $0 < \omega' < \omega$. (Note that there is a problem not only at ∞ but also at 0, and one needs f to vanish also at 0 in order to prove the independence of δ.)

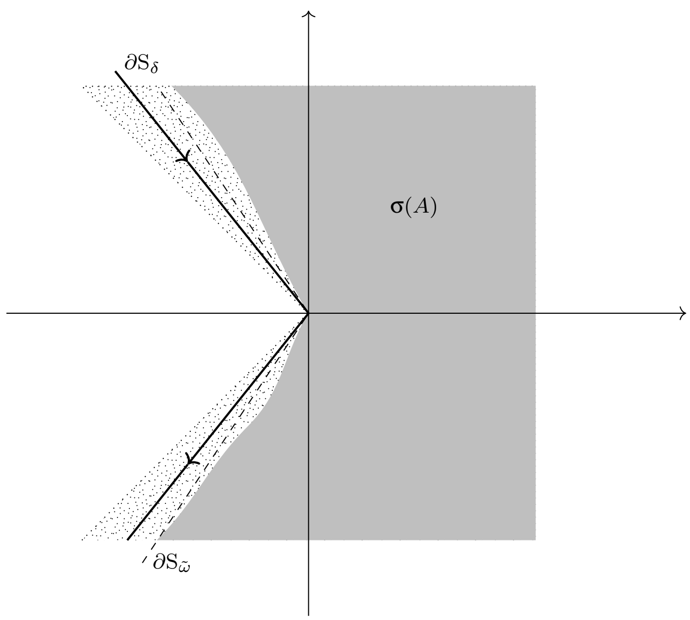

Fig. 9.1 The integration contour $\partial \mathrm{S}_\delta$ lies within the domain S_ω of the function f and outside the sector $\mathrm{S}_{\tilde{\omega}}$, where $\tilde{\omega} = \omega_{\mathrm{se}}(A) > \pi/2$ is the sectoriality angle of A.

Theorem 9.6. *The so-defined mapping $\Phi_A : \mathcal{E}(\mathrm{S}_\omega) \to \mathcal{L}(X)$ has the following properties ($f \in \mathcal{E}(\mathrm{S}_\omega)$):*

a) *Φ_A is a homomorphism of algebras.*

b) *If $T \in \mathcal{L}(X)$ satisfies $TA \subseteq AT$, then $\Phi_A(f)T = T\Phi_A(f)$.*

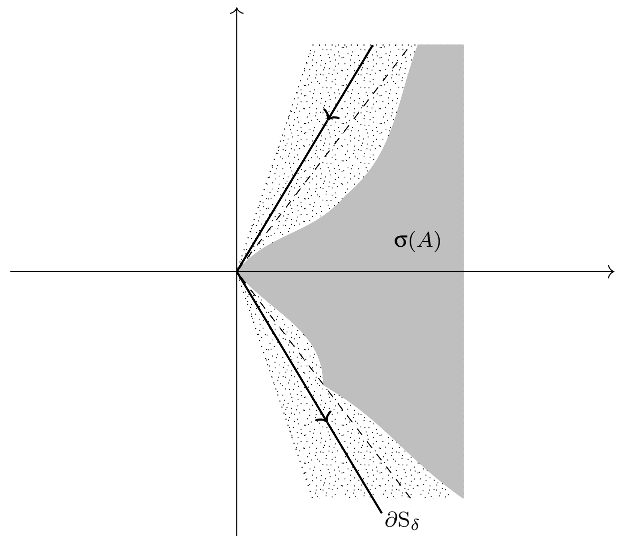

Fig. 9.2 An example where $\omega_{\mathrm{se}}(A) < \delta < \omega < \frac{\pi}{2}$.

c) $\Phi_A(f \cdot (\lambda - \mathbf{z})^{-1}) = \Phi_A(f) R(\lambda, A)$ *whenever* $\lambda \in \mathbb{C} \setminus \overline{\mathrm{S}_\omega}$.

d) $\Phi_A(\mathbf{z}(\lambda - \mathbf{z})^{-1}(\mu - \mathbf{z})^{-1}) = AR(\lambda, A) R(\mu, A)$ *whenever* $\lambda,\ \mu \in \mathbb{C} \setminus \overline{\mathrm{S}_\omega}$.

e) $\sup_{r>0} \|\Phi_A(f(r\mathbf{z}))\| \le \inf\Big\{ \dfrac{M(A,\delta)}{2\pi} \|f\|_{\mathrm{L}^1_*(\partial \mathrm{S}_\delta)} \;\Big|\; \omega_{\mathrm{se}}(A) < \delta < \omega \Big\} < \infty.$

Proof. The proof of a)–c) is analogous to the proof of Theorem 8.3.

d) Fix $\omega_{\mathrm{se}}(A) < \delta < \omega$ and for $R > 0$ let γ_R be the positively oriented boundary of the "cake piece" $\mathrm{B}[0, R] \setminus \mathrm{S}_\delta$. If R is large enough, the points λ and μ are contained in the interior of γ_R. Since $\mathbf{z}R(\mathbf{z}, A)$ is bounded on γ_R uniformly in $R > 0$, it follows that

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\delta} \frac{z}{(z-\lambda)(z-\mu)} R(z, A)\, \mathrm{d}z = \lim_{R\to\infty} \frac{-1}{2\pi \mathrm{i}} \int_{\gamma_R} \frac{z}{(z-\lambda)(z-\mu)} R(z, A)\, \mathrm{d}z.$$

By the residue theorem, the right-hand side equals

$$-\frac{\lambda}{\lambda - \mu} R(\lambda, A) - \frac{\mu}{\mu - \lambda} R(\mu, A)$$

which is equal to $AR(\mu, A) R(\lambda, A)$ by an elementary computation.

e) We use the definition of Φ_A to estimate

$$\|\Phi_A(f(r\mathbf{z}))\| \le \frac{1}{2\pi} \int_{\partial \mathrm{S}_\delta} |f(rz)|\, \|R(z, A)\|\, |\mathrm{d}z| \le \frac{M(A,\delta)}{2\pi} \|f\|_{\mathrm{L}^1_*(\partial \mathrm{S}_\delta)}.$$

This yields the claim. □

Corollary 9.7. *In the situation from before, Φ_A is non-degenerate if and only if A is injective.*

Proof. If A is injective, then so is $AR(\lambda, A)R(\mu, A)$. On the other hand, for $x \in \ker(A)$ and $e \in \mathcal{E}(\mathrm{S}_\omega)$ we have

$$\Phi_A(e)x = \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\delta} e(z)R(z, A)x \,\mathrm{d}z = \Big(\frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\delta} e(z) \frac{\mathrm{d}z}{z} \Big) x = 0$$

by Lemma 9.4. Hence, $\ker(A) \subseteq \ker(\Phi(e))$. □

So, if A is not injective one has to extend the calculus Φ_A in order to render it non-degenerate. The extension to $\mathcal{E}(\mathrm{S}_\omega) \oplus \mathbb{C}\mathbf{1}$ as described in Section 7.1 would do, but it has the unpleasant feature that no "resolvent function" $(\lambda - \mathbf{z})^{-1}$ is anchored in that algebra. (Look at limits at 0 and at ∞.) Therefore, one rather takes the larger algebra

$$\mathcal{E}_e(\mathrm{S}_\omega) := \mathcal{E}(\mathrm{S}_\omega) \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}(1 + \mathbf{z})^{-1}.$$

It follows from Theorem 9.6.c) that this extension, which is again denoted by Φ_A, is a non-degenerate algebra homomorphism, see Exercise 9.12. Note that for $\lambda \in \mathbb{C} \setminus \overline{\mathrm{S}_\omega}$ one has $(\lambda - \mathbf{z})^{-1} \in \mathcal{E}_e(\mathrm{S}_\omega)$, see Exercise 9.6.

The domain within $\mathrm{Mer}(\mathrm{S}_\omega)$ of the canonically extended calculus is denoted by $\mathrm{Mer}_A(\mathrm{S}_\omega)$. As the operator A is clearly the generator of this calculus, we write $f(A)$ in place of $\Phi_A(f)$ and say that $f(A)$ **is defined by**[3] **the sectorial calculus for** A.

Remark 9.8 (Compatibility with the Hille–Phillips Calculus). Suppose that $-A$ generates a bounded C_0-semigroup $(T_t)_{t \ge 0}$ on a Banach space X. Then A is sectorial of angle $\pi/2$ (Example 9.1.3). By essentially the same argument as in the proof of Theorem 8.20 one can show that each $f \in \mathcal{E}(\mathrm{S}_\omega)$, $\omega > \pi/2$, is the Laplace transform $f = \mathcal{L}\varphi$ of a function $\varphi \in \mathrm{L}^1(\mathbb{R}_+)$, and that

$$f(A) = \int_0^\infty \varphi(t) T_t \,\mathrm{d}t,$$

see [3, Lemma 3.3.1]. Hence, the Hille–Phillips calculus, which takes the right-hand side as the definition of "$f(A)$" is an extension of the sectorial calculus.

Injective vs. Non-Injective Sectorial Operators

The sectorial calculus is much nicer for injective operators than for non-injective ones. The reason is that for an injective sectorial operator A the

[3] or: within

$\mathcal{E}(\mathrm{S}_\omega)$-calculus is already non-degenerate and every $f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$ is anchored in $\mathcal{E}(\mathrm{S}_\omega)$ (e.g., by $e = \mathbf{z}(1+\mathbf{z})^{-2}$). In particular, the function $(1+\mathbf{z})^{-1}$ is anchored, and hence $\mathcal{E}_e \subseteq \langle \mathcal{E} \rangle_{\Phi_A}$. By Exercise 7.4, this means that $\langle \mathcal{E} \rangle_{\Phi_A} = \langle \mathcal{E}_e \rangle_{\Phi_A}$, and one does not need the algebra $\mathcal{E}_e$ in this case.

On the other side, the fact that a sectorial operator A is not injective has unpleasant consequences. First of all, one has to use the algebra $\mathcal{E}_e(\mathrm{S}_\omega)$ instead of $\mathcal{E}(\mathrm{S}_\omega)$ as a basis for the canonical extension. As a result, proofs become usually clumsier than under the injectivity assumption.

Next, the class of functions f for which $f(A)$ is defined in the sectorial calculus is more restricted, due to a necessary condition at $z = 0$ which is not needed in the injective case. (See Exercise 9.7 for a precise formulation of this condition.)

As a consequence, if f does not have a limit at $z = 0$ it is impossible to define $f(A)$ if A is not injective. (For instance, the **imaginary powers** $(\mathbf{z}^{\mathrm{i}s})(A)$ of A for $s \neq 0$ are *not* defined, cf. Section 10.3 below.)

Furthermore, there are functions f which do have a limit at $z = 0$ and for which $f(A)$ can be reasonably defined by other means, but not by the sectorial calculus. This is the topic of the supplementary Section 9.5 below.

This being said, one should recall that on reflexive spaces X one has the decomposition $X = \ker(A) \oplus \overline{\mathrm{ran}}(A)$, by which one can reduce problems for general sectorial operators to injective ones. So the unpleasant features of non-injective operators can be avoided in this case.

Convergence Theorems

Similar to the half-plane case, one can coin the notion of a **sectorial approximation** $(A_n)_{n\in\mathbb{N}}$ of a sectorial operator A and prove analogues of Lemma 8.6 and Theorem 8.8 for the sectorial calculus [3, Sec.2.1.2]. However, we shall not do this here, but rather turn to the approximations of functions. We say that a subset $\mathcal{F} \subseteq \mathcal{E}(\mathrm{S}_\omega)$ is **dominated** if for each $0 < \delta < \omega$ sufficiently close to ω the set $\{f|_{\partial \mathrm{S}_\delta} \mid f \in \mathcal{F}\}$ is dominated in $\mathrm{L}^1_*(\partial \mathrm{S}_\delta)$. Clearly, if $\mathcal{F} \subseteq \mathcal{E}(\mathrm{S}_\omega)$ is dominated then

$$\sup_{f\in\mathcal{F}} \|f(A)\| < \infty$$

for each sectorial operator A with $\omega_{\mathrm{se}}(A) < \omega$.

Lemma 9.9. *Let A be sectorial of angle $\omega_{\mathrm{se}}(A) < \omega$ and $(e_n)_n$ a sequence in $\mathcal{E}(\mathrm{S}_\omega)$ converging to* 0 *pointwise and boundedly on S_ω. Then the following assertions hold:*

a) *If $\{e_n \mid n \in \mathbb{N}\}$ is dominated, then $e_n(A) \to 0$ in norm.*

b) *If $e_n(A) \to T$ strongly for some bounded operator T, then $T = 0$.*

Proof. a) This is a consequence of Lebesgue's theorem like in Lemma 8.9.

b) Let $e := \frac{\mathbf{z}}{(1+\mathbf{z})^2}$. By a), $(ee_n)(A) \to 0$ in norm. On the other hand,

$$(ee_n)(A) = e(A)e_n(A) \to e(A)T = A(1+A)^{-2}T$$

strongly. Hence, $\operatorname{ran}((1+A)^{-2}T) \subseteq \ker(A)$. On the other hand, $(1+\mathbf{z})^{-2}e_n \in \mathcal{E}(\mathrm{S}_\omega)$ and hence, by Exercise 9.5,

$$(1+A)^{-2}Tx = \lim_{n\to\infty} \big((1+\mathbf{z})^{-2}e_n\big)(A)x \in \overline{\operatorname{ran}}(A) \qquad (x \in X).$$

Since $\ker(A) \cap \overline{\operatorname{ran}}(A) = \{0\}$ (Theorem 9.2), it follows that $(1+A)^{-2}T = 0$, which implies that $T = 0$. □

Theorem 9.10 ("Convergence Lemma"). *Let $0 < \omega < \pi$ and let $(f_n)_n$ be a sequence in $\mathrm{H}^\infty(\mathrm{S}_\omega)$ converging pointwise and boundedly on S_ω to some $f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$. Let, furthermore, A be an injective sectorial operator on a Banach space X with $\omega_{\mathrm{se}}(A) < \omega$. Suppose that $f_n(A) \in \mathcal{L}(X)$ for all $n \in \mathbb{N}$. Then the following assertions hold:*

a) *If $f_n(A) \to T \in \mathcal{L}(X)$ strongly, then $f(A) = T$.*

b) *If $\operatorname{dom}(A) \cap \operatorname{ran}(A)$ is dense in X and $\sup_n \|f_n(A)\| < \infty$ then $f(A) \in \mathcal{L}(X)$, $f_n(A) \to f(A)$ strongly, and $\|f(A)\| \le \liminf_{n\to\infty} \|f_n(A)\|$.*

Proof. Let $e = \mathbf{z}(1+\mathbf{z})^{-2}$. For the proof of a), apply Lemma 9.9.b) with $e_n := e(f_n - f)$. This yields $e(A)T = (ef)(A)$. Hence,

$$f(A) = (e^{-1})(A)(ef)(A) = e(A)^{-1}e(A)T = T$$

as claimed.

For the proof of b), apply Lemma 9.9.a) to conclude that $\lim_{n\to\infty} f_n(A)x$ exists for all $x \in \operatorname{ran}(e(A)) = \operatorname{ran}(A) \cap \operatorname{dom}(A)$. Uniform boundedness in combination with the density yields that $(f_n(A))_{n\in\mathbb{N}}$ converges strongly, hence to $f(A)$ by a). The rest is standard. □

An analogue for non-injective sectorial operators is treated in the supplementary Section 9.5 below.

9.4 Holomorphic Semigroups

In this section we shall see the sectorial calculus "at work".

Observe that the function $\mathrm{e}^{-\mathbf{z}}$ is an element of $\mathcal{E}_e(\mathrm{S}_{\pi/2})$ since the function

$$f := \mathrm{e}^{-\mathbf{z}} - \frac{1}{1+\mathbf{z}} \in \mathcal{E}(\mathrm{S}_{\pi/2}) \tag{9.5}$$

satisfies (9.2) with $s = 1$ for each $\omega < \pi/2$. Consequently, for each $\lambda \in \mathrm{S}_{\pi/2}$ the function

$$\mathrm{e}^{-\lambda \mathbf{z}} = f(\lambda \mathbf{z}) + \frac{1}{1+\lambda \mathbf{z}}$$

is contained in $\mathcal{E}_e(\mathrm{S}_\omega)$, where $\omega = \pi/2 - |\arg \lambda|$. In particular,

$$\mathrm{e}^{-\lambda A} := (\mathrm{e}^{-\lambda \mathbf{z}})(A)$$

is defined for each sectorial operator A of angle $\omega_{\mathrm{se}}(A) < \pi/2 - |\arg \lambda|$.

Let us abbreviate $\omega_A := \omega_{\mathrm{se}}(A)$ and $\theta_A := \pi/2 - \omega_A$. Then $\mathrm{e}^{-\lambda A}$ is defined for each $\lambda \in \mathrm{S}_{\theta_A}$. Moreover, functional calculus rules imply that

$$\mathrm{e}^{-\lambda A}\mathrm{e}^{-\mu A} = \mathrm{e}^{-(\lambda+\mu)A} \qquad (\lambda, \mu \in \mathrm{S}_{\theta_A}).$$

The operator family $(\mathrm{e}^{-\lambda A})_{\lambda \in \mathrm{S}_{\theta_A}}$ is called the **holomorphic semigroup generated by** $-A$. The reason for this terminology is the following classical result, see [1, Section II.4].

Theorem 9.11. *Let A be a sectorial operator of angle $\omega_A < \pi/2$ and let $\theta_A := \pi/2 - \omega_A$. Then the mapping*

$$\mathrm{S}_{\theta_A} \to \mathcal{L}(X), \qquad \lambda \mapsto \mathrm{e}^{-\lambda A}$$

is holomorphic with

$$\sup_{\lambda \in \mathrm{S}_\theta} \|\mathrm{e}^{-\lambda A}\| < \infty \quad \textit{for each } 0 < \theta < \theta_A. \tag{9.6}$$

The derivatives are given by

$$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\mathrm{e}^{-\lambda A} = (-A)^n \mathrm{e}^{-\lambda A} \in \mathcal{L}(X) \qquad (n \in \mathbb{N}). \tag{9.7}$$

For each $\mu \in \mathrm{S}_{\pi/2}$

$$(\mu + A)^{-1} = \int_0^\infty \mathrm{e}^{-\mu t}\mathrm{e}^{-tA}\,\mathrm{d}t. \tag{9.8}$$

Finally, $x \in \overline{\mathrm{dom}}(A)$ if and only if $\mathrm{e}^{-\lambda A}x \to x$ as $\lambda \to 0$ within S_θ for one/each $0 < \theta < \theta_A$.

Proof. Fix $0 < \theta < \theta_A$, let $\omega := \pi/2 - \theta$ and $f := \mathrm{e}^{-\mathbf{z}} - (1+\mathbf{z})^{-1} \in \mathcal{E}(\mathrm{S}_{\pi/2})$. Since

$$(1+\lambda \mathbf{z})^{-1}(A) = -\lambda^{-1}R(-\lambda^{-1}, A),$$

and this is holomorphic in λ and uniformly bounded for $\lambda \in \mathrm{S}_\theta$ (even for $\lambda \in \mathrm{S}_{\pi/2}$), it suffices to consider the function $\lambda \mapsto f(\lambda A)$. Fix $\omega_A < \delta < \omega$ and define

$$F_n(\lambda) := \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_n} f(\lambda z)R(z, A)\,\mathrm{d}z \qquad (n \in \mathbb{N},\ \lambda \in \mathrm{S}_\theta),$$

where Γ_n is just $\partial \mathrm{S}_\delta$, but restricted to the region $[\,1/n \le |\mathbf{z}| \le n\,]$. By standard arguments, F_n is holomorphic on S_θ. Moreover, $F_n(\lambda) \to f(\lambda A)$ pointwise and boundedly on S_θ since

$$\|F_n(\lambda)\| \lesssim \int_{\partial \mathrm{S}_\delta} |f(\lambda z)| \frac{|\mathrm{d}z|}{|z|} \le \sup_{|\alpha|\le \delta+\theta} \int_0^\infty \left|f(r\mathrm{e}^{\mathrm{i}\alpha})\right| \frac{\mathrm{d}r}{r}$$

for all $\lambda \in \mathrm{S}_\theta$ (cf. Remark 9.5). The uniform bound (9.6) follows readily.

Since $\mathbf{z}^n \mathrm{e}^{-\lambda \mathbf{z}} \in \mathcal{E}(\mathrm{S}_{\pi/2})$, it follows that $A^n \mathrm{e}^{-\lambda A} \in \mathcal{L}(X)$ for all $\lambda \in \mathrm{S}_\theta$. In particular, $\mathrm{e}^{-\lambda A}$ maps into $\mathrm{dom}(A^n)$ for each $n \in \mathbb{N}$. Hence, if $\mathrm{e}^{-\lambda A}x \to x$ as $\lambda \to 0$, then $x \in \overline{\mathrm{dom}}(A)$.

For the converse implication note that it follows from Theorem 9.2.a) and the uniform boundedness of $(t(t+A)^{-1})_{t>0}$ that

$$\overline{\mathrm{dom}}(A) = \overline{\mathrm{dom}}(A^2).$$

So by (9.6) it suffices to suppose that $x \in \mathrm{dom}(A^2)$. For $\lambda \in \mathrm{S}_\theta$ consider the function

$$f_\lambda := \frac{1}{(1+\mathbf{z})^2}(1-\mathrm{e}^{-\lambda \mathbf{z}}) = \lambda \frac{1-\mathrm{e}^{-\lambda \mathbf{z}}}{\lambda \mathbf{z}} \frac{\mathbf{z}}{(1+\mathbf{z})^2} \in \mathcal{E}(\mathrm{S}_\omega).$$

Since the function ${}^{(1-\mathrm{e}^{-\mathbf{z}})}/_{\mathbf{z}}$ is bounded on $\mathrm{S}_{\pi/2}$, the functions f_λ/λ are dominated in $\mathcal{E}(\mathrm{S}_\omega)$. It follows that

$$(\mathrm{I} - \mathrm{e}^{-\lambda A})(1+A)^{-2} = f_\lambda(A) \to 0$$

in norm as $0 \leftarrow \lambda \in \mathrm{S}_\theta$.

For the computation of the derivatives it suffices to compute right derivatives since we already know that $\lambda \mapsto \mathrm{e}^{-\lambda A}$ is holomorphic. Fix $\lambda \in \mathrm{S}_\theta$ and note that

$$\frac{1}{t}\big(\mathrm{e}^{-\lambda A} - \mathrm{e}^{-(\lambda+t)A}\big) - A\mathrm{e}^{-\lambda A} = \Big[\Big(\frac{1-\mathrm{e}^{-t\mathbf{z}}}{t\mathbf{z}} - 1\Big)\mathbf{z}\mathrm{e}^{-\lambda \mathbf{z}}\Big](A) =: g_t(A).$$

Since the function $\mathbf{z}\mathrm{e}^{-\lambda \mathbf{z}}$ is contained in $\mathcal{E}(\mathrm{S}_\omega)$ and the term in big round brackets converges to 0 pointwise and boundedly on S_ω, Lemma 9.9.a) yields that $g_t(A) \to 0$ in operator norm as $t \searrow 0$. The claim about higher derivatives follows easily.

Finally, fix $\mu \in \mathrm{S}_{\pi/2}$ and compute

$$
\begin{aligned}
(1+A)^{-2}\Big(\mu\int_0^\infty \mathrm{e}^{-\mu t}\mathrm{e}^{-tA}\,\mathrm{d}t-1\Big) &= \int_0^\infty \Big(\frac{\mathrm{e}^{-t\mathbf{z}}-1}{(1+\mathbf{z})^2}\Big)(A)\,\mu\mathrm{e}^{-\mu t}\,\mathrm{d}t\\
&= \int_0^\infty \frac{1}{2\pi\mathrm{i}}\int_{\partial\mathrm{S}_\delta}\frac{\mathrm{e}^{-tz}-1}{tz}\,\frac{zR(z,A)}{(1+z)^2}\,\mathrm{d}z\,\mu t\,\mathrm{e}^{-\mu t}\,\mathrm{d}t\\
&= \frac{1}{2\pi\mathrm{i}}\int_{\partial\mathrm{S}_\delta}\int_0^\infty(\mathrm{e}^{-tz}-1)\mu\mathrm{e}^{-\mu t}\,\mathrm{d}t\,\frac{R(z,A)}{(1+z)^2}\,\mathrm{d}z\\
&= \frac{1}{2\pi\mathrm{i}}\int_{\partial\mathrm{S}_\delta}\frac{-z}{\mu(\mu+z)}\,\frac{R(z,A)}{(1+z)^2}\,\mathrm{d}z = -(1+A)^{-2}A(\mu+A)^{-1}.
\end{aligned}
$$

(We have used that $(\mathrm{e}^{-t\mathbf{z}}-1)/(1+\mathbf{z})^2\in\mathcal{E}(\mathrm{S}_{\pi/2})$ and that Fubini's theorem is applicable since $\mathbf{t}\mathrm{e}^{-\mu\mathbf{t}}\in\mathrm{L}^1(\mathbb{R}_+)$.) It follows that

$$
\mu\int_0^\infty \mathrm{e}^{-\mu t}\mathrm{e}^{-tA}\,\mathrm{d}t = \mathrm{I}-A(\mu+A)^{-1} = \mu(\mu+A)^{-1}
$$

and hence (9.8). □

Suppose that A is as above and, in addition, densely defined. Then, by Theorem 9.11, $(\mathrm{e}^{-tA})_{t\ge 0}$ is a bounded C_0-semigroup with generator $-A$. Moreover, this semigroup has a holomorphic extension to the sector S_{θ_A} and is uniformly bounded on each smaller sector.

Conversely, suppose that an operator $-A$ generates a C_0-semigroup $(T_t)_{t\ge 0}$ which for some $0<\theta_0\le \pi/2$ has a holomorphic extension to S_{θ_0}, uniformly bounded on each smaller sector. Then A is sectorial of angle $\pi/2-\theta_0$. (The proof of this claim is Exercise 9.8.)

9.5 Supplement: A Topological Extension of the Sectorial Calculus

Let A be a non-injective sectorial operator and $\omega\in(\omega_{\mathrm{se}}(A),\pi)$. As we have observed above, there are bounded and holomorphic functions f such that $f(A)$ is not defined. This is unavoidable, since if $f\in\mathrm{H}^\infty(\mathrm{S}_\omega)$ is such that $f(A)$ is defined, the limit $f(0):=\lim_{z\searrow 0}f(z)$ must exist.

However, there is a more serious shortcoming. Suppose that $\mu\in\mathrm{M}(0,\infty)$ is a bounded complex measure on $(0,\infty)$ and f is given by

$$
f(z)=\int_0^\infty \frac{t}{t+z}\,\mu(\mathrm{d}t)\quad\text{for}\quad z\in\mathbb{C}\setminus(-\infty,0].
$$

Then $f\in\mathrm{H}^\infty(\mathrm{S}_\omega)$ for each $0<\omega<\pi$ and $\lim_{z\to 0,z\in\mathrm{S}_\omega}f(z)$ exists. Moreover, one clearly expects the formula

$$
f(A)=\int_0^\infty t(t+A)^{-1}\,\mu(\mathrm{d}t)\in\mathcal{L}(X). \tag{9.9}
$$

However, there are examples of measures μ such that f is not anchored in $\mathcal{E}_e(\mathrm{S}_\omega)$ [2, Example 5.2]. That is, $f(A)$ is not defined within the sectorial calculus for A even if the right-hand side of (9.9) is a perfectly well-defined expression.

The best one can say is that *if* $f(A)$ is defined in the sectorial calculus for A, then (9.9) holds. This is a consequence of the following analogue of Theorem 9.10.a), see also Exercise 9.13.

Theorem 9.12. *Let A be a non-injective sectorial operator on a Banach space X, let $\omega \in (\omega_{\mathrm{se}}(A), \boldsymbol{\pi})$ and let $(f_n)_n$ be a sequence in $\mathrm{H}^\infty(\mathrm{S}_\omega)$ and $f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$ such that all $f_n(A)$ and $f(A)$ are defined within the sectorial calculus for A.*

Suppose in addition that $f_n \to f$ pointwise and boundedly on $\mathrm{S}_\omega \cup \{0\}$, that $f_n(A) \in \mathcal{L}(X)$ for all $n \in \mathbb{N}$ and that $f_n(A) \to T \in \mathcal{L}(X)$ strongly. Then $f(A) = T$.

Proof. Let $e := (1+\mathbf{z})^{-1}$, so that $e(A) = (1+A)^{-1}$. By passing to $f_n - f_n(0)e$ and $f - f(0)e$ we may suppose that $f_n(0) = f(0) = 0$. By Exercise 9.7, $ef_n,\, ef \in \mathcal{E}(\mathrm{S}_\omega)$. Now apply Lemma 9.9.b) to $e_n := e(f_n - f)$. This yields $e(A)T = (ef)(A)$ and hence

$$f(A) = e(A)^{-1}(ef)(A) = e(A)^{-1}e(A)T = T$$

as claimed. □

In order to cover all instances of (9.9) it is necessary to extend the sectorial calculus again, but now in a topological way. There is an abstract framework for this.

Abstract Functional Calculus (V) — Topological Extensions

Let $\mathcal{F}$ be an algebra. A **sequential convergence structure** on $\mathcal{F}$ is a mapping

$$\tau : \mathcal{F}^{\mathbb{N}} \supseteq \operatorname{dom}(\tau) \to \mathcal{F}$$

with the following properties:

1) $\operatorname{dom}(\tau)$ is a subalgebra of $\mathcal{F}^{\mathbb{N}}$ and τ is an algebra homomorphism.
2) For each $f \in \mathcal{F}$ the constant sequence $(f)_n$ is in $\operatorname{dom}(\tau)$ and $\tau\big((f)_n\big) = f$.
3) $\tau L = \tau$, where L is the left shift on $\mathcal{F}^{\mathbb{N}}$.

One writes $f_n \xrightarrow{\tau} f$ in place of $f = \tau((f_n)_n)$. From now on we suppose that $\mathcal{F}$ is endowed with a fixed sequential convergence structure τ.

Let $\mathcal{E}$ be a subalgebra of $\mathcal{F}$ and $\Phi : \mathcal{E} \to \mathcal{L}(X)$ a representation. Then the set

$$\mathcal{E}^\tau := \{f \in \mathcal{F} \mid \exists\, (e_n)_n \text{ in } \mathcal{E},\, T \in \mathcal{L}(X) : e_n \xrightarrow{\tau} f,\; \Phi(e_n) \to T \text{ strongly}\}$$

is a subalgebra of $\mathcal{F}$ containing $\mathcal{E}$. Suppose in addition that Φ is **closable with respect to** τ, which means that

$$(e_n)_n \in \mathcal{E}^{\mathbb{N}},\, T \in \mathcal{L}(X),\, e_n \xrightarrow{\tau} 0,\, \Phi(e_n) \to T \text{ strongly} \quad \Rightarrow \quad T = 0. \tag{9.10}$$

Then one can define the **τ-extension** $\Phi_\tau : \mathcal{E}^\tau \to \mathcal{L}(X)$ of Φ by

$$\Phi^\tau(f) := \lim_{n\to\infty} \Phi(e_n)$$

whenever $(e_n)_n \in \mathcal{E}^{\mathbb{N}}$, $e_n \xrightarrow{\tau} f$, and $\lim_{n\to\infty} \Phi(e_n)$ exists (strongly) in $\mathcal{L}(X)$. (Indeed, (9.10) just guarantees that Φ^τ is well-defined, i.e., does not depend on the chosen τ-approximating sequence $(e_n)_n$.) The following theorem is straightforward.

Theorem 9.13. *The so-defined mapping $\Phi^\tau : \mathcal{E}^\tau \to \mathcal{L}(X)$ is an algebra homomorphism which extends Φ.*

Now suppose in addition that $\Phi : \mathcal{E} \to \mathcal{L}(X)$ is non-degenerate, so that we can speak of its canonical (algebraic) extension. Since, as easily seen, $\mathcal{Z}(\mathcal{E}) \subseteq \mathcal{Z}(\mathcal{E}^\tau)$, Theorem 8.19 yields

$$\langle \mathcal{E} \rangle_\Phi \subseteq \langle \mathcal{E}^\tau \rangle_{\Phi_\tau} \quad \text{and} \quad \widehat{\Phi^\tau}|_{\langle \mathcal{E} \rangle_\Phi} = \widehat{\Phi}$$

for the canonical (algebraic) extensions of Φ and Φ^τ within $\mathcal{F}$.

Topological Extension of the Sectorial Calculus

Let A be a sectorial operator on a Banach space X, let $\omega_{\mathrm{se}}(A) < \omega < \pi$ and

$$\Phi_A : \mathcal{E}_e(\mathrm{S}_\omega) \to \mathcal{L}(X)$$

the sectorial calculus for A. As an immediate consequence of Theorems 9.10 and Theorem 9.12 (with $f_n \in \mathcal{E}_e(\mathrm{S}_\omega)$ and $T = 0$) we obtain:

Theorem 9.14. *The sectorial calculus Φ_A on $\mathcal{E}_e(\mathrm{S}_\omega)$ is closable in $\mathrm{H}^\infty(\mathrm{S}_\omega)$ with respect to bounded and pointwise convergence on $\mathrm{S}_\omega \cup \{0\}$.*

Applying Theorem 9.13 yields the **bp-extension** Φ_A^{bp} of the sectorial calculus Φ_A for A. Note that if A is injective then, by Theorem 9.10.a), Φ_A on $\mathcal{E}_e(\mathrm{S}_\omega)$ is even closed (and not just closable) in $\mathrm{H}^\infty(\mathrm{S}_\omega)$ with respect to bp-convergence on S_ω. Hence, the bp-extension of the sectorial calculus does not lead to a larger calculus in this case.

This is different for non-injective sectorial operators, as the following example shows.

Example 9.15 (Hirsch Functional Calculus). Suppose that $\mu \in \mathrm{M}(0,\infty)$ and

$$f(z) := \int_0^\infty \frac{t}{t+z}\,\mu(\mathrm{d}t) \qquad (z \in \mathbb{C} \setminus (-\infty, 0]). \tag{9.11}$$

Functions of this form are the core of the so-called **Hirsch calculus**, see [4, Chap.4]. It is easy to see that f is bounded and holomorphic on S_ω with limits

$$f(0) = \int_0^\infty \mathbf{1}\,\mathrm{d}\mu \quad \text{and} \quad f(\infty) = 0$$

for each $\omega \in (0, \pi)$. The approximants

$$f_n(z) = \int_{[1/n, n]} \frac{t}{t+z}\,\mu(\mathrm{d}t)$$

converge to f uniformly on each such sector S_ω. Moreover, one can show that $f_n \in \mathcal{E}_e(\mathrm{S}_\omega)$ and, by an application of Fubini's theorem, that

$$f_n(A) = \int_{[1/n, n]} t(t+A)^{-1}\,\mu(\mathrm{d}t)$$

for every sectorial operator A. It follows that $f(A)$ is defined in the bp-extension of the sectorial calculus for A and

$$f(A) = \int_0^\infty t(t+A)^{-1}\,\mu(\mathrm{d}t)$$

as expected. (Cf. also Exercise 9.13).

As mentioned in the beginning of this section, there are examples of measures μ such that the function f defined by (9.11) is not covered by the sectorial calculus for non-injective operators. Hence, in this case, the bp-extension of the sectorial calculus is strictly larger than the sectorial calculus.

Exercises

9.1 (Examples of Sectorial Operators).

a) Let Ω be a semi-finite measure space and $a : \Omega \to \mathbb{C}$ a measurable mapping. Show that the multiplication operator M_a on $\mathrm{L}^p(\Omega)$, $1 \le p \le \infty$, is sectorial of angle ω if and only if $a \in \overline{\mathrm{S}_\omega}$ almost everywhere.

b) Let H be a Hilbert space and A a closed operator on H with numerical range $\mathrm{W}(A) \subseteq \overline{\mathrm{S}_\omega}$ and such that $\mathrm{ran}(\mathrm{I}+A) = H$. Show that A is sectorial of angle ω.

c) Suppose that the resolvent of an operator A satisfies an estimate of the form
$$\|R(\lambda, A)\| \le \frac{M}{|\operatorname{Re}\lambda|} \qquad \text{for all } \operatorname{Re}\lambda < 0.$$
Show that A is sectorial of angle $\pi/2$.

d) Suppose that the resolvent of an operator A satisfies an estimate of the form
$$\|R(\lambda, A)\| \le \frac{M}{|\operatorname{Im}\lambda|} \qquad \text{for all } \lambda \in \mathbb{C} \setminus \mathbb{R}.$$
Show thay A^2 is sectorial of angle 0.

[Hints: For a) observe that $\|R(\lambda, A)\| = 1/\operatorname{dist}(\lambda, S_\omega)$ and use (9.1); for b) use Theorem A.23 and the same argument as in a); for d) prove first that $\|\lambda R(\lambda^2, A^2)\| \le M/|\operatorname{Im}\lambda|$.]

9.2. Let A be an operator on a Banach space X such that $(-\infty, 0) \subseteq \rho(A)$ and $M := \sup_{t>0} \|t(t+A)^{-1}\| < \infty$. Show that $M \ge 1$ and that A is sectorial of angle $\omega_{\mathrm{se}}(A) \le \pi - \arcsin(\frac{1}{M})$. Moreover, show for $x \in X$ the equivalences
$$x \in \overline{\operatorname{dom}}(A) \quad \text{if and only if} \quad \lim_{t\to\infty} t(t+A)^{-1}x = x$$
$$x \in \overline{\operatorname{ran}}(A) \quad \text{if and only if} \quad \lim_{t\to 0} t(t+A)^{-1}x = 0.$$
Conclude that $\ker(A) \cap \overline{\operatorname{ran}}(A) = \{0\}$.
[Hint: For the first statement, fix $\omega > \pi - \arcsin(\frac{1}{M})$ and $\lambda \in \mathbb{C} \setminus \overline{S_\omega}$ and pick $a < 0$ such that that the triangle with vertices 0, a and λ has a right angle at λ. Then use the Taylor expansion of $R(\mathbf{z}, A)$ at $z = a$ to estimate $\lambda R(\lambda, A)$.]

9.3. Let A be a sectorial operator of angle $\omega \in (0, \pi)$. Show that

a) for each $r > 0$ the operator rA is sectorial of angle ω with
$$M(rA, \alpha) = M(A, \alpha) \qquad (\omega < \alpha < \pi);$$

b) if A is injective then A^{-1} is sectorial of angle ω with
$$M(A^{-1}, \alpha) \le 1 + M(A, \alpha) \qquad (\omega < \alpha < \pi);$$

c) for each $|\theta| < \pi - \omega$ the operator $e^{i\theta}A$ is sectorial of angle $\omega + |\theta|$ with
$$M(e^{i\theta}A, \alpha') \le M(A, \alpha' - |\theta|) \qquad (\omega + |\theta| < \alpha' < \pi);$$

d) for each $\mu \in \mathbb{C} \setminus \{0\}$ with $|\arg \mu| =: \omega' < \pi - \omega$ the operator $A + \mu$ is sectorial of angle $\max(\omega, \omega')$ with
$$M(A + \mu, \alpha') \le \frac{1}{\sin\bigl(\min(\alpha' - \omega', \pi/2)\bigr)} \cdot M(A, \alpha),$$
where $\alpha = \min(\alpha', \pi - \omega')$ and $\max(\omega, \omega') < \alpha' < \pi$.

[Hint for b) cf. Exercise 2.4.]

9.4. Show that for a closed operator A on a Banach space X and a number $0 < \theta_0 \le \pi/2$ the following assertions are equivalent:

a) $\mathrm{e}^{\pm \mathrm{i}\theta_0} A$ are sectorial of angle $\pi/2$.

b) A is sectorial of angle $\pi/2 - \theta_0$.

[Hint: For the implication b) $\Rightarrow$ a) note Exercise 9.3.c).]

9.5. Let A be a sectorial operator and let $f \in \mathcal{E}(\mathrm{S}_\omega)$ for some $\omega \in (\omega_{\mathrm{se}}(A), \pi)$. Show that

$$\operatorname{ran}\bigl(f(A)\bigr) \subseteq \overline{\operatorname{dom}}(A) \cap \overline{\operatorname{ran}}(A).$$

Moreover, show that the following assertions are equivalent for $x \in X$:

(i) $x \in \overline{\operatorname{dom}}(A) \cap \overline{\operatorname{ran}}(A)$;

(ii) $x \in \overline{\operatorname{dom}(A) \cap \operatorname{ran}(A)}$;

(iii) $n^2 A(1 + nA)^{-1}(n + A)^{-1}x \to x$ as $n \to \infty$.

9.6 (Functions with polynomial limits). Let $0 < \omega < \pi$. Show that

$$(\lambda - \mathbf{z})^{-1} \in \mathcal{E}_e(\mathrm{S}_\omega) \qquad (\lambda \in \mathbb{C} \setminus \overline{\mathrm{S}_\omega}).$$

More generally, let $f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$ such that

$$f(z) - c = O(|z|^s) \quad (z \to 0) \quad \text{and} \quad f(z) - d = O(|z|^{-s'}) \quad (z \to \infty)$$

for some $c,\ d \in \mathbb{C}$ and $s,\ s' > 0$. Show that

$$f - d\mathbf{1} - \frac{c - d}{1 + \mathbf{z}} \in \mathcal{E}(\mathrm{S}_\omega)$$

and hence $f \in \mathcal{E}_e(\mathrm{S}_\omega)$. (One says that f has **polynomial limit** c at 0 and d at ∞.) Finally, show that f has a polynomial limit at 0 if f has a holomorphic extension to a neighborhood of 0.

9.7. Let A be a non-injective sectorial operator of angle ω on a Banach space X and let $f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$, $\omega \in (\omega_{\mathrm{se}}(A), \pi)$. Show that the following assertions are equivalent:

(i) $f(A)$ is defined in the sectorial calculus for A;

(ii) $f(0) := \lim_{z \searrow 0} f(z)$ exists and $(1 + \mathbf{z})^{-1}(f - f(0)) \in \mathcal{E}(\mathrm{S}_\omega)$;

(iii) $(1 + \mathbf{z})^{-1}$ is an anchor element for f w.r.t. the sectorial calculus for A.

9.8. Let $-A$ be the generator of a bounded C_0-semigroup $T : \mathbb{R}_+ \to \mathcal{L}(X)$ and suppose that T has a holomorphic extension (again denoted by T) to S_{θ_0} for some $0 < \theta_0 \le \pi/2$ such that T is uniformly bounded on each sector S_θ, $0 < \theta < \theta_0$. For $|\theta| < \theta_0$ define

$$T^\theta(t) := T(t\mathrm{e}^{\mathrm{i}\theta}), \qquad t \geq 0.$$

a) Show that $T(z+w) = T(z)T(w)$ for all $z,\ w \in \mathrm{S}_{\theta_0}$.

b) Show that

$$\int_0^\infty \mathrm{e}^{-\lambda t} T^\theta(t)\,\mathrm{d}t = (\lambda + \mathrm{e}^{\mathrm{i}\theta}A)^{-1} \qquad (\lambda \in \mathrm{S}_{\pi/2-\theta},\ |\theta| < \theta_0).$$

c) Conclude from b) that T^θ is a bounded C_0-semigroup and $-\mathrm{e}^{\mathrm{i}\theta}A$ is its generator ($|\theta| < \theta_0$).

d) Show that A is sectorial of angle $\pi/2 - \theta_0$ and that

$$T(\lambda) = \mathrm{e}^{-\lambda A} \qquad (\lambda \in \mathrm{S}_{\theta_0}).$$

[Hints: For b) consider $\int_{\mathrm{e}^{\mathrm{i}\theta}\mathbb{R}_+} T(z)\mathrm{e}^{(\mathrm{e}^{-\mathrm{i}\theta}\lambda z)}\,\mathrm{d}z$ and apply Cauchy's theorem. For the strong continuity in c) note that it suffices to consider elements of the form $x = (\lambda + \mathrm{e}^{\mathrm{i}\theta}A)^{-1}y$ for $y \in X$ and $\lambda > 0$. For the first part of d) see Exercise 9.4, for the second use the uniqueness of the Laplace transform.]

9.9. Let $-\mathrm{i}A$ be the generator of a bounded C_0-group $U = (U_s)_{s\in\mathbb{R}}$ on a Banach space X. Show that A^2 is sectorial of angle 0 and the holomorphic semigroup generated by $-A^2$ is given by

$$\mathrm{e}^{-\lambda A^2} = \frac{1}{\sqrt{4\pi\lambda}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{s^2}{4\lambda}} U_s\,\mathrm{d}s \qquad (\operatorname{Re}\lambda > 0).$$

[See also Exercise 9.1.d) and Exercise 9.8.]

9.10. Let A be a normal operator on a Hilbert space H and let $\omega \in (0, \pi)$.

a) Show that A is sectorial of angle ω if and only if $\sigma(A) \subseteq \overline{\mathrm{S}_\omega}$.

b) Suppose that A is sectorial of angle ω and $f \in \mathrm{Mer}(\mathrm{S}_\omega)$ is such that $f(A)$ is defined in the sectorial calculus for A. Show that the set $[\, f = \infty\,]$ of poles of f is an A-null set and that $f(A) = \Psi(g)$, where Ψ is the Borel calculus for A and g is any Borel function on $\mathbb{C}$ such that $[\, g \neq f\,]$ is an A-null set.

[Hint for a): Exercise 9.1.a).]

Supplementary Exercises

9.11 (Cauchy–Gauss Representation). Let $\omega > 0$. Show that

$$\mathrm{e}^{-(a-\mathbf{z})^2} \in \mathcal{E}(\mathrm{St}_\omega) \quad \text{for each } a \in \mathbb{C}.$$

Conclude that for each $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$ one has the **Cauchy–Gauss representation**

$$f(a) = \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{St}_\delta} f(z) \frac{\mathrm{e}^{-(z-a)^2}}{z-a} \, \mathrm{d}z \qquad (a \in \mathrm{St}_\delta,\ 0 < \delta < \omega). \tag{9.12}$$

Use this to show that for each $f \in \mathcal{E}(\mathrm{St}_\omega)$

$$\sup_{|\alpha| \le \delta} \int_{\mathbb{R}} |f(r + \mathrm{i}\alpha)| \, \mathrm{d}r < \infty \qquad (0 < \delta < \omega).$$

In the case $\omega \le \pi$ conclude that

$$\sup_{|\alpha| \le \delta} \int_0^\infty \left| f(r\mathrm{e}^{\mathrm{i}\alpha}) \right| \frac{\mathrm{d}r}{r} < \infty \qquad (0 < \delta < \omega)$$

for each $f \in \mathcal{E}(\mathrm{S}_\omega)$.

9.12. Let A be a sectorial operator on a Banach space X, let $\omega_{\mathrm{se}}(A) < \omega < \pi$ and $\Phi_A : \mathcal{E}(\mathrm{S}_\omega) \to \mathcal{L}(X)$ the associated sectorial calculus. Show that

$$\mathcal{E}_e(\mathrm{S}_\omega) := \mathcal{E}(\mathrm{S}_\omega) \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\frac{1}{1+\mathbf{z}}$$

is an algebra and that $\Psi : \mathcal{E}_e(\mathrm{S}_\omega) \to \mathcal{L}(X)$ given by $(c,\ d \in \mathbb{C},\ e \in \mathcal{E}(\mathrm{S}_\omega))$

$$\Psi(e + c\mathbf{1} + d(1+\mathbf{z})^{-1}) := \Phi_A(e) + c\mathrm{I} + d(1+A)^{-1}$$

is a well-defined algebra homomorphism. Show further that

$$\sup_{r>0} \|\Psi(f(r\mathbf{z}))\| < \infty$$

for each $f \in \mathcal{E}_e(\mathrm{S}_\omega)$ and that b) of Theorem 9.6 holds for Ψ instead of Φ_A. (We usually write again Φ_A instead of Ψ.)

9.13. Let $\mu \in \mathrm{M}(0,\infty)$ and define

$$f(z) := \int_0^\infty \frac{t}{t+z} \, \mu(\mathrm{d}t) \qquad (z \in \mathbb{C} \setminus (-\infty, 0]).$$

Moreover, for $n \in \mathbb{N}$ define

$$f_n(z) := \int_{[1/n, n]} \frac{t}{t+z} \, \mu(\mathrm{d}t) \qquad (z \in \mathbb{C} \setminus (-\infty, 0])$$

Fix $0 < \omega < \pi$ and a sectorial operator A of angle $\omega_{\mathrm{se}}(A) < \omega$ on a Banach space X. Prove the following assertions:

a) $f_n = \frac{f_n(0)}{1+\mathbf{z}} + h_n$, where $\mathbf{z} \cdot h_n$ and $\mathbf{z}^{-1} \cdot h_n$ are bounded on S_ω. Conclude that $f_n \in \mathcal{E}_e(\mathrm{S}_\omega)$.

b) $f_n(A) = \displaystyle\int_{[1/n,n]} t(t+A)^{-1} \, \mathrm{d}t \to \int_0^\infty t(t+A)^{-1} \, \mathrm{d}t$ in operator norm.

c) $f_n \to f$ uniformly on S_ω.

d) If $f(A)$ is defined within the sectorial calculus for A, then

$$f(A) = \int_0^\infty t(t+A)^{-1}\,\mu(\mathrm{d}t). \tag{9.13}$$

e) $f(A)$ is defined in the bp-extension of the sectorial calculus for A, and (9.13) holds.

[Hint: d) is a direct consequence of Theorem 9.12, but also of e).]

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Chapter 10
Fractional Powers and the Logarithm

10.1 Fractional Powers

Let A be a sectorial operator on a Banach space X. Then for each $\alpha \in \mathbb{C}_+$ the operator

$$A^\alpha := (\mathbf{z}^\alpha)(A) := (\mathrm{e}^{\alpha \log \mathbf{z}})(A)$$

is defined within the sectorial calculus for A. Indeed, if $n > \operatorname{Re}\alpha$ then

$$\frac{\mathbf{z}^\alpha}{(1+\mathbf{z})^n} \in \mathcal{E}(\mathrm{S}_\omega)$$

for each $0 < \omega < \boldsymbol{\pi}$ and hence the function $(1+\mathbf{z})^{-n}$ is an anchor element for the function $\mathbf{z}^\alpha$ with respect to the sectorial calculus for A.

Theorem 10.1. *For a sectorial operator A on a Banach space X, the following assertions hold ($\alpha, \beta \in \mathbb{C}$):*

a) *If $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$, then*
$$A^\alpha A^\beta = A^{\alpha+\beta}.$$

b) *If $A \in \mathcal{L}(X)$ and $\operatorname{Re}\alpha > 0$, then $A^\alpha \in \mathcal{L}(X)$.*

c) *If $0 < \operatorname{Re}\alpha < \operatorname{Re}\beta$, then $\operatorname{dom}(A^\beta) \subseteq \operatorname{dom}(A^\alpha)$ and the mapping*
$$A^{\mathbf{z}}x : [\, 0 < \operatorname{Re}\mathbf{z} < \operatorname{Re}\beta \,] \to X$$
is holomorphic for each $x \in \operatorname{dom}(A^\beta)$.

d) *For each $\varepsilon > 0$ and $\operatorname{Re}\alpha > 0$ the operator $(A+\varepsilon)^\alpha$ is invertible with inverse*
$$\big((A+\varepsilon)^\alpha\big)^{-1} = \frac{1}{(\mathbf{z}+\varepsilon)^\alpha}(A).$$

e) *For each $\varepsilon > 0$ and $\operatorname{Re}\alpha > 0$ one has $\operatorname{dom}(A+\varepsilon)^\alpha = \operatorname{dom}(A^\alpha)$.*

f) *For each $\operatorname{Re}\alpha > 0$ one has $\operatorname{ran}(A^\alpha) \subseteq \overline{\operatorname{ran}}(A)$ and $\operatorname{dom}(A^\alpha) \subseteq \overline{\operatorname{dom}}(A)$.*

Proof. a) The inclusion $A^\alpha A^\beta \subseteq A^{\alpha+\beta}$ follows from functional calculus rules. Fix $x \in \mathrm{dom}(A^{\alpha+\beta})$ and $n > \mathrm{Re}\,\alpha, \mathrm{Re}\,\beta$. For $\mathrm{Re}\,\gamma < n$ define $f_\gamma := \mathbf{z}^\gamma(1+\mathbf{z})^{-n}$ and let $u := f_\beta(A)x$. Then

$$A^n(1+A)^{-n}u = \left[\frac{\mathbf{z}^{n+\beta}}{(1+\mathbf{z})^{2n}}\right](A)x = (1+A)^{-n} f_{n-\alpha}(A)A^{\alpha+\beta}x,$$

which yields $A^n(1+A)^{-n}u \in \mathrm{dom}(A^n)$. This implies first that $(1+A)^{-n}u \in \mathrm{dom}(A^{2n})$, which in turn yields $u \in \mathrm{dom}(A^n)$. So there is $y \in X$ such that $(1+A)^{-n}y = u$, i.e.,

$$\left[\frac{\mathbf{z}^\beta}{(1+\mathbf{z})^n}\right](A)x = \left[\frac{1}{(1+\mathbf{z})^n}\right](A)y.$$

Since $(1+\mathbf{z})^{-n}$ is an anchor element for $\mathbf{z}^\beta$, $x \in \mathrm{dom}(A^\beta)$ and $y = A^\beta x$. As $\mathrm{dom}(A^\alpha A^\beta) = \mathrm{dom}(A^\beta) \cap \mathrm{dom}(A^{\alpha+\beta})$ by general rules, the claim is proved.

b) Fix $n > \mathrm{Re}\,\alpha$. Then $A^{n-\alpha}A^\alpha = A^n$, which is bounded. So, $\mathrm{dom}(A^\alpha) = X$, hence A^α is bounded as well.

c) The first assertion follows from a). The second is left as an exercise.

d) For each $0 < \omega < \boldsymbol{\pi}$ the function

$$f := \frac{1}{(\mathbf{z}+\varepsilon)^\alpha} \in \mathrm{H}^\infty(\mathrm{S}_\omega)$$

has "polynomial limits" at 0 and at ∞, so $f \in \mathcal{E}_e(\mathrm{S}_\omega)$, see Exercise 9.6. In particular, $(\mathbf{z}+\varepsilon)^{-\alpha}(A)$ is defined and a bounded operator.

e) Fix $\varepsilon > 0$ and $\omega \in (\omega_{\mathrm{se}}(A), \boldsymbol{\pi})$. For $\mathrm{Re}\,\alpha \in (0,1)$ the assertion follows from Exercise 10.1, where it is shown that there is a bounded operator T such that $(A+\varepsilon)^\alpha = A^\alpha + T$. Actually, the proof shows even more: there is a function $f \in \mathcal{E}_e(\mathrm{S}_\omega)$ such that $(\mathbf{z}+\varepsilon)^\alpha = \mathbf{z}^\alpha + f$. For $n \in \mathbb{N}$ one therefore has

$$(\mathbf{z}+\varepsilon)^{n\alpha} = (\mathbf{z}^\alpha + f)^n = \mathbf{z}^{n\alpha} + \sum_{j=0}^{n-1} \binom{n}{j} f^{n-j} \cdot \mathbf{z}^{j\alpha}.$$

Since $\mathrm{dom}(A^{n\alpha}) \subseteq \mathrm{dom}(A^{j\alpha})$ for all $0 \le j < n$, it follows that $\mathrm{dom}(A^{n\alpha}) \subseteq \mathrm{dom}((A+\varepsilon)^{n\alpha})$.

The converse inclusion follows similarly. Alternatively one can argue as follows. Fix $\mathrm{Re}\,\alpha > 0$ and observe that

$$g := \frac{\mathbf{z}^\alpha}{(\mathbf{z}+\varepsilon)^\alpha} \in \mathcal{E}_e(\mathrm{S}_\omega).$$

Writing $(A+\varepsilon)^{-\alpha} := [(A+\varepsilon)^\alpha]^{-1} = (\mathbf{z}+\varepsilon)^{-\alpha}(A)$ (cf. d)) we obtain

$$A^\alpha(A+\varepsilon)^{-\alpha} = g(A) \in \mathcal{L}(X),$$

which implies that $\mathrm{dom}(A+\varepsilon)^\alpha = \mathrm{ran}((A+\varepsilon)^{-\alpha}) \subseteq \mathrm{dom}(A^\alpha)$.

f) By a) we may suppose that $\mathrm{Re}\,\alpha < 1$. Let $x \in \mathrm{dom}(A^\alpha)$ and $y := A^\alpha x$. Then

$$(1+A)^{-1}y = \Big(\frac{\mathbf{z}^\alpha}{1+\mathbf{z}}\Big)(A)x \in \overline{\mathrm{ran}}(A)$$

by Exercise 9.5. Hence,

$$y = (1+A)^{-1}y + A(1+A)^{-1}y \in \overline{\mathrm{ran}}(A) + \mathrm{ran}(A) \subseteq \overline{\mathrm{ran}}(A)$$

as claimed. For the second assertion we use e) and write with $u := (1+A)^\alpha x$

$$x = (1+\mathbf{z})^{-\alpha}(A)(1+A)^\alpha x = (1+A)^{-\alpha}u.$$

Since the function $f := (1+\mathbf{z})^{-\alpha} - (1+\mathbf{z})^{-1}$ is elementary, by Exercise 9.5 we have $\mathrm{ran}(f(A)) \subseteq \overline{\mathrm{dom}}(A)$ and hence

$$x = (1+A)^{-\alpha}u = f(A)u + (1+A)^{-1}u \in \overline{\mathrm{dom}}(A) + \mathrm{dom}(A) \subseteq \overline{\mathrm{dom}}(A)$$

as claimed. □

Remark 10.2. There is a certain ambiguity in the term $(A+\varepsilon)^\alpha$. Above, we have always read this as $(\mathbf{z}+\varepsilon)^\alpha(A)$. However, since $A+\varepsilon$ is also a sectorial operator, we could also read it as $(\mathbf{z}^\alpha)(A+\varepsilon)$. This ambiguity is virtual, as one can show that both operators are equal. More generally, one can prove a **composition rule** of the form

$$f(\mathbf{z}+\varepsilon)(A) = f(A+\varepsilon)$$

for (many) functions defined on S_ω. And this is just an instance of a more general composition rule of the form

$$(f \circ g)(A) = f(g(A)),$$

which holds under certain conditions on g and f. See [3, Section 2.4] for a more detailed discussion.

Remark 10.3. One can show that if A is a sectorial operator and $\alpha > 0$ is such that $\alpha\omega_{\mathrm{se}}(A) < \boldsymbol{\pi}$, then A^α is sectorial with

$$\omega_{\mathrm{se}}(A^\alpha) = \alpha\omega_{\mathrm{se}}(A).$$

Moreover, a composition rule of the form

$$f(A^\alpha) = f(\mathbf{z}^\alpha)(A)$$

holds in the sense that the left-hand side is defined in the sectorial calculus for A^α if and only if the right-hand side is defined in the sectorial calculus for A. It follows that for such α and all $\beta \in \mathbb{C}_+$ one has

$$(A^\alpha)^\beta = A^{\alpha\beta}.$$

See [3, 3.1.2–3.1.5] for details.

Fractional Powers of Bounded Operators

If A is a bounded sectorial operator, then each fractional power A^α of A with $\operatorname{Re}\alpha > 0$ is again bounded, by Theorem 10.1.b). By a) and c) of that theorem, the mapping

$$\mathrm{S}_{\pi/2} \to \mathcal{L}(X), \qquad \alpha \mapsto A^\alpha$$

is a holomorphic semigroup (representation). By Exercise 10.4, this semigroup is bounded for $\alpha \to 0$ within each proper subsector S_φ of $\mathrm{S}_{\pi/2}$.

Since $\operatorname{ran}(A^\alpha) \subseteq \overline{\operatorname{ran}}(A)$ by Theorem 10.1, one has the implication "$\Leftarrow$" in the equivalence

$$x \in \overline{\operatorname{ran}}(A) \quad \Longleftrightarrow \quad A^\alpha x \to x \quad (\mathrm{S}_\varphi \ni \alpha \to 0), \tag{10.1}$$

where $0 < \varphi < \pi/2$. The remaining implication is proved in Exercise 10.4.

Fractional Powers with Negative Real Part

Suppose that A is an *injective* sectorial operator on a Banach space X and fix $\omega \in (\omega_{\mathrm{se}}(A), \pi)$. For each $\alpha \in \mathbb{C}$ one can find $n \in \mathbb{N}$ such that

$$\frac{\mathbf{z}^n}{(1+\mathbf{z})^{2n}} \mathbf{z}^\alpha \in \mathcal{E}(\mathrm{S}_\omega).$$

Hence

$$A^\alpha := (\mathbf{z}^\alpha)(A)$$

is defined in the sectorial calculus for A. By a general composition rule (Exercise 10.5) we find

$$A^{-\alpha} = (A^\alpha)^{-1} = (A^{-1})^\alpha.$$

This allows to apply Theorem 10.1 and we find, for instance, that

$$A^\alpha A^\beta = A^{\alpha+\beta} \tag{10.2}$$

whenever $\operatorname{Re}\alpha, \operatorname{Re}\beta < 0$. Note, however, that (10.2) is true for *all* $\alpha, \beta \in \mathbb{C}$ if and only if A is bounded and invertible. (Indeed, it would follow that $\mathrm{I} = AA^{-1} = A^{-1}A$.)

If A is invertible (and not just injective), then $(A^{-\alpha})_{\operatorname{Re}\alpha>0}$ is a holomorphic semigroup of bounded operators. Its space of strong continuity is $\overline{\operatorname{ran}}(A^{-1}) = \overline{\operatorname{dom}}(A)$. Its generator is described in the following section.

10.2 The Logarithm and Operators of Strip Type

From now on we shall suppose that A is an *injective* sectorial operator on a Banach space X. Then the **operator logarithm**

$$\log A := (\log \mathbf{z})(A)$$

is defined in the sectorial calculus. Indeed, $\log \mathbf{z}$ has only a mild growth at 0 and at ∞ and is anchored by $e := \mathbf{z}(1+\mathbf{z})^{-2}$. By Exercise 10.5 one has

$$\log A^{-1} = -\log A. \tag{10.3}$$

If A is invertible and has dense domain, then $(A^{-s})_{s\geq 0}$ is a C_0-semigroup and $-\log A$ is its generator (Exercise 10.6).

Recall that $\log \mathbf{z}$ maps the sector S_ω (for $\omega \in [0, \pi]$) biholomorphically onto the strip St_ω. Since A has its spectrum in the sector of angle $\omega_{\mathrm{se}}(A)$, one could imagine that $\log A$ has spectrum in the strip of height $\omega_{\mathrm{se}}(A)$. A first result in this direction is the following theorem of Nollau from [5].

Theorem 10.4 (Nollau). *Let A be an injective sectorial operator on a Banach space X. Then $\sigma(\log A) \subseteq [\,|\operatorname{Im}\mathbf{z}| \leq \pi\,]$ and*

$$R(\lambda, \log A) = \int_0^\infty \frac{-1}{(\lambda - \log t)^2 + \pi^2}(t + A)^{-1}\,\mathrm{d}t$$

for all $|\operatorname{Im}\lambda| > \pi$.

Sketch of Proof. In a first step one has to show that the formula is true if $X = \mathbb{C}$ and $A = a \in \mathbb{C} \setminus (-\infty, 0]$ is a number. This can be done by standard path deformation arguments (Exercise 10.9) and yields

$$\frac{1}{\lambda - \log z} = \int_0^\infty \frac{-1}{(\lambda - \log t)^2 + \pi^2}(t + z)^{-1}\,\mathrm{d}t.$$

Since

$$\frac{-1}{\mathbf{t}\big((\lambda - \log \mathbf{t})^2 + \pi^2\big)} \in \mathrm{L}^1(\mathbb{R}_+),$$

the claim follows from Exercise 9.13.d). □

From Nollau's theorem and (10.3) it follows immediately that

$$\operatorname{dom}(\log A) \subseteq \overline{\operatorname{dom}}(A) \cap \overline{\operatorname{ran}}(A). \tag{10.4}$$

By combining Nollaus result with the scaling technique from Remark 10.3, one can prove that $\boldsymbol{\sigma}(\log A) \subseteq [\,|\operatorname{Im} \mathbf{z}| \leq \omega_{\mathrm{se}}(A)\,]$. Actually, one has the following stronger result.

Theorem 10.5. *Let A be an injective sectorial operator on a Banach space X and let $B := \log(A)$. Then for $\omega = \omega_{\mathrm{se}}(A)$ the following assertions hold:*

a) *The operator B has spectrum in the strip $\overline{\mathrm{St}_\omega}$.*

b) *For each $\alpha > \omega$ there is $M_\alpha > 0$ such that*

$$\|R(\lambda, B)\| \leq \frac{M_\alpha}{|\operatorname{Im} \lambda| - \alpha} \qquad (\lambda \in \mathbb{C} \setminus \overline{\mathrm{St}_\alpha}).$$

Moreover, $\omega_{\mathrm{se}}(A)$ is the smallest number ω such that a) *and* b) *hold.*

For a proof see [3, Prop.3.5.2 and Thm.4.3.1].

An operator B that satisfies a) and b) of Theorem 10.5 is said to be **of strong strip type** ω. The minimal ω that is possible here is denoted by

$$\omega_{\mathrm{st}}(B)$$

and is called **the strip type** of B. So Theorem 10.5 tells that if A is an injective sectorial operator then $\log(A)$ is of strong strip type $\omega_{\mathrm{st}}(\log(A)) = \omega_{\mathrm{se}}(A)$.

Remark 10.6 (Functional Calculus for Strip Type Operators). In the same way as for sectorial operators one can construct a holomorphic calculus for strong strip type operators. For $\omega > \omega_{\mathrm{st}}(B)$ and an elementary function $f \in \mathcal{E}(\mathrm{St}_\omega)$ one defines

$$\Phi_B(f) := \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{St}_\delta} f(z) R(z, B)\, \mathrm{d}z \in \mathcal{L}(X)$$

where $\omega_{\mathrm{st}}(B) < \delta < \omega$ is arbitrary. (Note that $R(\mathbf{z}, B)$ is bounded on $\partial \mathrm{St}_\delta$.) Thanks to Lemma 9.4.a) one obtains an analogue of Theorem 9.6, with a completely analogous proof.

It follows that

$$\Phi_B : \mathcal{E}(\mathrm{St}_\omega) \to \mathcal{L}(X)$$

is a non-degenerate representation, hence it has a canonical extension within $\mathrm{Mer}(\mathrm{St}_\omega)$, again denoted by Φ_B. The domain of this extension is denoted by $\mathrm{Mer}_B(\mathrm{St}_\omega)$ (by abuse of notation). If $f \in \mathrm{Mer}_B(\mathrm{St}_\omega)$ we write (as usual) $f(B)$ instead of $\Phi_B(f)$ and say that $f(B)$ is defined in the strip calculus for B.

The logarithm $\log A$ of an injective sectorial operator A on X is of strong strip type $\omega_{\mathrm{se}}(A)$. One can show that

$$f(\log A) = f(\log \mathbf{z})(A) \tag{10.5}$$

for each $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$, $\omega > \omega_{\mathrm{se}}(A)$. As a consequence, one obtains that an injective sectorial operator is completely determined by its logarithm (Exercise 10.11).

Furthermore, if A is of strong strip type then one can show that e^A is defined in the (extended) strip calculus. This operator need not be sectorial, see Exercise 12.6 below. However, *if* e^A is sectorial, then one has

$$\log(\mathrm{e}^A) = A$$

as one would expect. Detailed proofs of all these statements can be found in [3, Sec.4.2].

10.3 The Purely Imaginary Powers

The **(purely) imaginary powers** of an injective sectorial operator A on a Banach space X is the family of operators

$$A^{-\mathrm{i}s}, \qquad s \in \mathbb{R}.$$

Note that the holomorphic function $\mathbf{z}^{-\mathrm{i}s}$ is bounded on each sector S_φ with $0 < \varphi \le \pi$ since

$$\left|z^{-\mathrm{i}s}\right| = \left|\mathrm{e}^{-\mathrm{i}s\log z}\right| = \mathrm{e}^{s\arg z} \le \mathrm{e}^{\varphi|s|} \qquad (z \in \mathrm{S}_\varphi).$$

One says that A has **bounded imaginary powers (BIP)**, if the conditions (i)–(iii) of the following theorem are satisfied.

Theorem 10.7. *For an injective sectorial operator A on a Banach space X the following assertions are equivalent:*

(i) *A has dense domain and range and $A^{\mathrm{i}s} \in \mathcal{L}(X)$ for all $s \in \mathbb{R}$.*

(ii) *The family $(A^{\mathrm{i}s})_{s\in\mathbb{R}}$ is a C_0-group of bounded operators on X.*

(iii) *The operator $\mathrm{i}\log A$ is the generator of a C_0-group on X.*

In this case, $\mathrm{i}\log A$ is the generator of the C_0-group $(A^{\mathrm{i}s})_{s\in\mathbb{R}}$.

Proof. Abbreviate $e := \mathbf{z}(1+\mathbf{z})^{-2}$, so that $\mathrm{ran}(e(A)) = \mathrm{dom}(A) \cap \mathrm{ran}(A)$. Observe that by Lemma 9.9 the function

$$\mathbb{R} \to \mathcal{L}(X), \qquad s \mapsto A^{\mathrm{i}s}e(A) = \Big(\frac{\mathbf{z}^{\mathrm{i}s+1}}{(1+\mathbf{z})^2}\Big)(A)$$

is continuous. This will be used several times in the following.

(i) $\Rightarrow$ (ii): Fix $x = e(A)y \in \mathrm{dom}(A) \cap \mathrm{ran}(A)$. Then the mapping

$$\mathbb{R} \to X, \qquad s \mapsto A^{\mathrm{i}s}x = A^{\mathrm{i}s}e(A)y$$

is continuous. Since A has dense range and domain, $\mathrm{dom}(A) \cap \mathrm{ran}(A)$ is dense in X (Exercise 9.5). Consequently, for each $x \in X$ its orbit $A^{\mathbf{is}}x$ is strongly measurable. By a classical result of Hille and Phillips [4, Thm.10.2.3], the orbit is continuous for $s > 0$. But as $(A^{\mathrm{i}s})_{s\in\mathbb{R}}$ is a group, the orbit is continuous on the whole of $\mathbb{R}$.

(ii) ⇒ (iii): Fix $\theta \in (\omega_{\mathrm{se}}(A), \boldsymbol{\pi})$, $\mathrm{Re}\,\lambda > \theta$ and let $r_\lambda := (\lambda - \mathrm{i}\log \mathbf{z})^{-1} \in \mathrm{H}^\infty(\mathrm{S}_\theta)$. Then, by using the definition of $A^{\mathrm{i}s}e(A) = (e \cdot \mathbf{z}^{\mathrm{i}s})(A)$ as a Cauchy integral and interchanging the order of integration, we obtain

$$\int_0^\infty \mathrm{e}^{-\lambda s} A^{\mathrm{i}s} e(A)\,\mathrm{d}s = \Big(\frac{e}{\lambda - \mathrm{i}\log \mathbf{z}}\Big)(A) = (er_\lambda)(A).$$

Hence, with B denoting the generator of $(A^{\mathrm{i}s})_{s\in\mathbb{R}}$ and if $\mathrm{Re}\,\lambda$ is large enough,

$$e(A)R(\lambda, B) = e(A)\int_0^\infty \mathrm{e}^{-\lambda s} A^{\mathrm{i}s}\,\mathrm{d}s = \int_0^\infty \mathrm{e}^{-\lambda s} A^{\mathrm{i}s} e(A)\,\mathrm{d}s = (er_\lambda)(A).$$

It follows that $r_\lambda(A) = R(\lambda, B)$. By general functional calculus arguments we conclude that $B = \mathrm{i}\log A$.

(iii) ⇒ (ii): Suppose that $\mathrm{i}\log A$ generates a C_0-group $(U_s)_{s\in\mathbb{R}}$. Then

$$\int_0^\infty \mathrm{e}^{-\lambda s} e(A) U_s\,\mathrm{d}s = e(A)R(\lambda, \mathrm{i}\log A) = (er_\lambda)(A) = \int_0^\infty \mathrm{e}^{-\lambda s} A^{\mathrm{i}s} e(A)\,\mathrm{d}s$$

for large $\mathrm{Re}\,\lambda > 0$. By the uniqueness of the Laplace transform, it follows that

$$e(A)U_s = (e\mathbf{z}^{\mathrm{i}s})(A),$$

which implies that $U_s = A^{\mathrm{i}s}$ for each $s \in \mathbb{R}$.

(iii) ⇒ (i) follows from (10.4). □

Corollary 10.8 (Prüss–Sohr). *Let A be a sectorial operator with bounded imaginary powers and $\theta \geq 0$ such that there is $K \geq 0$ with*

$$\|A^{\mathrm{i}s}\| \leq K\mathrm{e}^{|s|\theta} \qquad (s \in \mathbb{R}). \tag{10.6}$$

Then $\omega_{\mathrm{se}}(A) \leq \theta$. In other words: the group type of $(A^{\mathrm{i}s})_{s\in\mathbb{R}}$ is always larger than the sectoriality angle of A.

Proof. We know that $B := \mathrm{i}\log(A)$ is the generator of the group $(A^{\mathrm{i}s})_{s\in\mathbb{R}}$. Hence, the Hille–Yosida estimates yield

$$\|R(\lambda, B)\| \leq \frac{K}{|\mathrm{Re}\,\lambda| - \theta}$$

for all $\lambda \in \mathbb{C}$ with $|\mathrm{Re}\,\lambda| > \theta$. But this just says that $\log(A) = -\mathrm{i}B$ is of strong strip type θ. By Theorem 10.5, $\omega_{\mathrm{se}}(A) \le \theta$. □

Exercise 10.7 shows that the domain/range-condition cannot be dropped from (i) of Theorem 10.7.

The imaginary powers of a sectorial operator are notoriously mysterious objects. Even if A is bounded, the powers $A^{\mathrm{i}s}$ need not be bounded (unless $s = 0$ of course). There are examples, where $A^{\mathrm{i}s}$ is bounded for infinitely many, but not all, $s \in \mathbb{R}$, see [6]. We shall encounter non-trivial examples of operators with and without BIP in the following chapters.

A natural question is of course which C_0-groups are the imaginary powers of an injective sectorial operator. Or, more generally, which strip type operators are logarithms of sectorial operators. We shall see soon that the shift group on $\mathrm{L}^1(\mathbb{R})$ is bad in this respect. Eventually, we shall encounter *Monniaux*'s theorem, which states that each C_0-group U of group type strictly less than π on a so-called UMD-Banach space is the group of imaginary powers of a sectorial operator.

10.4 Two Examples

Fractional Integrals and the Riemann–Liouville Semigroup

Let $-A$ be the generator of the right shift semigroup $\tau = (\tau_s)_{s\ge 0}$ on $X = \mathrm{L}^p(0,1)$ as in Exercise 6.6. From that exercise we know already that $\sigma(A) = \emptyset$ and $A^{-1} = V$, the Volterra operator, given by

$$(A^{-1}x)(s) = (Vx)(s) = \int_0^s x(t)\,\mathrm{d}t \qquad (x \in X).$$

The domain of A, viz. the range of V, is

$$\mathrm{dom}(A) = \mathrm{ran}(V) = \mathrm{W}_0^{1,\mathrm{p}}(0,1)$$

and $A = \frac{\mathrm{d}}{\mathrm{d}t}$ in the sense of weak derivatives, but we shall not need these facts in the following.

Since $-A$ generates a bounded C_0-semigroup, A is a densely defined sectorial operator of angle $\pi/2$. As A is injective, we can form the fractional powers A^α for all $\alpha \in \mathbb{C}$. Since A is invertible, we obtain the holomorphic C_0-semigroup

$$A^{-\alpha} = V^\alpha \qquad (\mathrm{Re}\,\alpha > 0),$$

which is sometimes called the **Riemann–Louville** semigroup. The individual operators V^α are the so-called Riemann–Liouville **fractional integral** operators. The inverse operators A^{α} are the (Riemann–Liouville) **fractional differentiation** operators.

Theorem 10.9. *For* $x \in \mathrm{L}^p(0,1)$ *and* $\operatorname{Re}\alpha > 0$*,*

$$(A^{-\alpha}x)(s) = (V^{\alpha}x)(s) = \frac{1}{\Gamma(\alpha)}\int_0^s (s-t)^{\alpha-1}x(t)\,\mathrm{d}t \qquad (s \in (0,1)). \quad (10.7)$$

Sketch of Proof. For $z > 0$ one has (by definition of the Gamma function)

$$\Gamma(\alpha) = \int_0^{\infty} s^{\alpha-1}\mathrm{e}^{-s}\,\mathrm{d}s = z^{\alpha}\int_0^{\infty} s^{\alpha-1}\mathrm{e}^{-sz}\,\mathrm{d}s.$$

By the uniqueness theorem for holomorphic functions,

$$z^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} s^{\alpha-1}\mathrm{e}^{-sz}\,\mathrm{d}s$$

for all $z \in \mathbb{C}_+$. This means that the function $\mathbf{z}^{-\alpha}$ is the Laplace transform of the function $\mathbf{s}^{\alpha-1}$. Now observe that $-A$ is the generator of the nilpotent, in particular exponentially decaying, semigroup τ. So τ admits a Hille–Phillips calculus on each half-plane $\mathbb{C}_+ + \varepsilon$, $\varepsilon > 0$, as described in (6.11). One can show that this calculus is compatible with the sectorial calculus (cf. also Remark 9.8). Hence,

$$A^{-\alpha} = \int_0^{\infty} s^{\alpha-1}\tau_s\,\mathrm{d}s.$$

Applying both sides to $x \in \mathrm{L}^p(0,1)$ and performing a change of variables yields (10.7). (See [3, Cor.3.3.6] for a full proof rather than a sketch.) □

The Poisson Semigroup

Suppose that $-A$ generates a bounded C_0-semigroup $(T_s)_{s\ge 0}$. Then A is sectorial of angle $\pi/2$ and one can form the square root $\sqrt{A} := A^{1/2}$ through the sectorial calculus. By Remark 10.3, $\sqrt{A}$ is sectorial of angle $\pi/4$ and hence $-\sqrt{A}$ generates an analytic semigroup of that angle, namely $(\mathrm{e}^{-t\sqrt{A}})_{t\ge 0}$.

This semigroup is also accessible by the Hille–Phillips calculus for A. Indeed, define

$$\varphi_t(s) := \frac{t\mathrm{e}^{-t^2/4s}}{\sqrt{4\pi}s^{3/2}} \qquad (s > 0)$$

for $t > 0$. By (A.24) from Appendix A.11,

$$(\mathcal{L}\varphi_t)(z) = \int_0^{\infty} \frac{t\mathrm{e}^{-t^2/4s}}{\sqrt{4\pi}s^{3/2}}\,\mathrm{e}^{-sz}\,\mathrm{d}s = \mathrm{e}^{-t\sqrt{z}}$$

for all $\operatorname{Re} z > 0$. (Observe that letting $z \searrow 0$ shows that $\varphi_t \in \mathrm{L}^1(\mathbb{R}_+)$.) Since the Hille–Phillips calculus and the sectorial calculus for A are compatible (Remark 9.8), we obtain

$$\mathrm{e}^{-t\sqrt{A}} = \int_0^\infty \varphi_t(s) T_s \, \mathrm{d}s.$$

Now let us specialize the above situation to the case that $-A$ is an "abstract Laplace operator", i.e., $(T_s)_{s\geq 0} = (G_s)_{s\geq 0}$ is the heat semigroup associated with some strongly continuous and bounded representation $(U_r)_{r\in\mathbb{R}^d}$. Then Theorem 6.18 yields

$$\mathrm{e}^{-t\sqrt{A}} = \Psi_U(\mathrm{e}^{-t|\mathbf{z}|}) \qquad (t > 0),$$

where Ψ_U is the Fourier–Stieltjes calculus for U. The semigroup $(\mathrm{e}^{-t\sqrt{A}})_{t\geq 0}$ is called the **Poisson semigroup** associated with the group U and is often denoted by $(P_t)_{t\geq 0}$.

Theorem 10.10. *Let $U = (U_s)_{s\in\mathbb{R}^d}$ be a bounded and strongly continuous representation on a Banach space X. Then the associated Poisson semigroup $(P_t)_{t\geq 0}$ is given by*

$$P_t = \int_{\mathbb{R}^d} p_t(x) U_x \, \mathrm{d}x \qquad (t > 0),$$

where

$$p_t(x) = \frac{\Gamma\left(\frac{d+1}{2}\right) t}{\pi^{\frac{d+1}{2}} \left(t^2 + |x|^2\right)^{\frac{d+1}{2}}} \tag{10.8}$$

for $t > 0$ and $x \in \mathbb{R}^d$. In particular,

$$\mathcal{F} p_t = \mathrm{e}^{-t|\mathbf{z}|} \quad (t > 0). \tag{10.9}$$

Proof. Recall from the proof of Theorem 6.15 that the mapping

$$(0, \infty) \to \mathrm{FS}(\mathbb{R}^d), \quad s \mapsto \mathrm{e}^{-s|\mathbf{z}|^2}$$

is bounded and continuous. So

$$\mathrm{e}^{-t|\mathbf{z}|} = \int_0^\infty \varphi_t(s) \mathrm{e}^{-s|\mathbf{z}|^2} \, \mathrm{d}s$$

as an integral in $\mathrm{FS}(\mathbb{R}^d)$. It follows that $\mathrm{e}^{-t|\mathbf{z}|} = \mathcal{F} p_t$, where

$$\begin{aligned} p_t(x) &= \int_0^\infty \varphi_t(s) g_s(x)\,\mathrm{d}s = \int_0^\infty \frac{t\mathrm{e}^{-t^2/4s}}{\sqrt{4\pi}s^{3/2}} \frac{1}{(4\pi s)^{\frac{d}{2}}} \mathrm{e}^{-|x|^2/4s}\,\mathrm{d}s \\ &= \frac{t}{(4\pi)^{\frac{d+1}{2}}} \int_0^\infty s^{-\frac{d+3}{2}} \mathrm{e}^{-(t^2+|x|^2)/4s}\,\mathrm{d}s \\ &= \frac{t}{(4\pi)^{\frac{d+1}{2}} (t^2+|x|^2)^{\frac{d+1}{2}}} \int_0^\infty s^{-\frac{d+1}{2}} \mathrm{e}^{-1/4s}\,\frac{\mathrm{d}s}{s} \end{aligned}$$

and

$$\int_0^\infty s^{-\frac{d+1}{2}} \mathrm{e}^{-1/4s}\,\frac{\mathrm{d}s}{s} \overset{u=1/4s}{=} 4^{\frac{d+1}{2}} \int_0^\infty u^{\frac{d+1}{2}} \mathrm{e}^{-u}\,\frac{\mathrm{d}u}{u} = 4^{\frac{d+1}{2}} \Gamma\big(\tfrac{d+1}{2}\big).$$

This establishes (10.8). The first claim now follows from the definition of Ψ_U since $P_t = \Psi_U(\mathrm{e}^{-t|\mathbf{z}|})$. □

The Poisson semigroup associated with the right shift group τ on $\mathrm{L}^p(\mathbb{R}^d)$ or $\mathrm{C}_0(\mathbb{R}^d)$ is the classical Poisson semigroup. Its generator is (in a sense) the operator $-\sqrt{-\Delta}$.

Exercises

10.1. Let A be a sectorial operator on a Banach space X and $0 < \operatorname{Re}\alpha < 1$. Show that for each $\varepsilon > 0$ there is a bounded operator $T_\varepsilon \in \mathcal{L}(X)$ such that $(A+\varepsilon)^\alpha = A^\alpha + T_\varepsilon$ and

$$\|T_\varepsilon\| = O(\varepsilon^\alpha) \qquad (\varepsilon > 0)$$

[Hint: Consider the function $\varepsilon^\alpha f(\mathbf{z}/\varepsilon)$ with $f := (\mathbf{z}+1)^\alpha - \mathbf{z}^\alpha - (1+\mathbf{z})^{-1}$.]

10.2. Let A be a sectorial operator on a Banach space X. Show that for each $\operatorname{Re}\alpha > 0$

$$\ker(A^\alpha) = \ker(A).$$

[Hint: reduce the proof of "$\subseteq$" to the case that $\alpha = n \in \mathbb{N}$; then use Theorem 9.2 to reduce the proof to the case $\alpha = n-1$ (in case $n \geq 2$).]

10.3. Fix $0 < \operatorname{Re}\alpha < 1$. In this exercise we take the formula

$$\int_0^\infty \frac{t^{-\alpha}}{1+t}\,\mathrm{d}t = \frac{\pi}{\sin\alpha\pi} \tag{10.10}$$

for granted. (See Exercise 10.10 below or any standard textbook on complex analysis, for instance [2, Prop.III.7.12] or [1, Example V.2.12].)

a) Show from (10.10) that

$$z^\alpha = \frac{\sin \alpha\pi}{\pi} \int_0^\infty t^{-\alpha} \frac{z}{1+tz}\, \mathrm{d}t \tag{10.11}$$

for all $z \in \mathbb{C} \setminus (-\infty, 0]$.

b) Let A be a sectorial operator on a Banach space X. Show that

$$A^\alpha (1+A)^{-1} = \frac{\sin \alpha\pi}{\pi} \int_0^\infty t^{-\alpha} A(1+tA)^{-1}(1+A)^{-1}\, \mathrm{d}t$$

as an absolutely convergent integral.

[Hint: Reduce a) to the case $z > 0$ and perform a suitable change of variables. For b), imitate the proof of Exercise 9.13.d); or rewrite (10.11) as

$$\frac{\pi}{\sin \alpha\pi} z^\alpha = \frac{1}{\alpha} - \int_0^\infty t^{\alpha-1} \mathbf{1}_{[\mathbf{t}<1]} \frac{t}{t+z}\, \mathrm{d}t + z \int_0^\infty t^{\alpha-2} \mathbf{1}_{[\mathbf{t}>1]} \frac{t}{t+z}\, \mathrm{d}t$$

and apply Exercise 9.13.d).]

10.4. Let A be a bounded and sectorial operator on a Banach space X.

a) Show that for $\operatorname{Re} \alpha \in (0,1)$

$$A^\alpha = \frac{\sin \alpha\pi}{\pi} \int_0^\infty t^{-\alpha} A(1+tA)^{-1}\, \mathrm{d}t.$$

b) Conclude from a) that for some constant $K \geq 0$

$$\|A^\alpha\| \leq K \frac{|\alpha|}{\operatorname{Re} \alpha} \qquad \text{for} \quad 0 < \operatorname{Re} \alpha < 1/2,\ |\alpha| \leq 1.$$

c) Let $\varphi \in (0, \pi/2)$ and $x \in \overline{\operatorname{ran}}(A)$. Show that

$$\lim_{\mathrm{S}_\varphi \ni \alpha \to 0} A^\alpha x = x.$$

[Hint: For a) use Exercise 10.3.b). For b) observe that $\sup_{t>0} \|(1+t)A(1+tA)^{-1}\| < \infty$.]

10.5. Let A be an injective sectorial operator on a Banach space X and let $\omega \in (\omega_{\mathrm{se}}(A), \pi)$.

a) Let $f \in \mathcal{E}(\mathrm{S}_\omega)$. Show that $f(\mathbf{z}^{-1}) \in \mathcal{E}(\mathrm{S}_\omega)$ and

$$f(A^{-1}) = f(\mathbf{z}^{-1})(A). \tag{10.12}$$

Cf. also Exercise 9.3.b).

b) Let $f \in \mathrm{Mer}_{A^{-1}}(\mathrm{S}_\omega)$. Show that $f(\mathbf{z}^{-1}) \in \mathrm{Mer}_A(\mathrm{S}_\omega)$ with (10.12).

[Hint: b) follows from a) and Theorem 8.19.]

10.6. Let A be an invertible sectorial operator on a Banach space X. By Exercise 10.4, the holomorphic semigroup $(A^{-t})_{t>0}$ is bounded for $0 < t \le 1/2$. It follows that there is $M \ge 1$ and $\omega > 0$ such that $\|A^t\| \le M\mathrm{e}^{\omega t}$ for all $t > 0$. Let $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > \omega$ and $|\operatorname{Im}\lambda| > \pi$.

a) Show that

$$\int_a^b \mathrm{e}^{-\lambda t} A^{-t}\,\mathrm{d}t = \mathrm{e}^{-\lambda a} A^{-a}(\lambda + \log A)^{-1} - \mathrm{e}^{-\lambda b} A^{-b}(\lambda + \log A)^{-1}$$

for all $0 < a < b < \infty$.

b) Show that

$$(\lambda + \log A)^{-1} = \int_0^\infty \mathrm{e}^{-\lambda t} A^{-t}\,\mathrm{d}t$$

c) Conclude from b) that, in the case that $\overline{\operatorname{dom}}(A) = X$, the operator $-\log A$ is the generator of the C_0-semigroup $(A^{-s})_{s\ge 0}$.

[Hint: a) Fix $b < n \in \mathbb{N}$ and multiply by $A^n(1 + A)^{-n}$. b) Use (10.4) and Exercise 10.4.c) for the limit as $a \searrow 0$.]

10.7 (Sectorial Multiplication Operators). Let Ω be a metric space and $a : \Omega \to \mathbb{C}$ a continuous function such that $a(\Omega) \subseteq \overline{\mathrm{S}_\theta}$ for some $\theta \in (0, \pi)$. Let $X = \mathrm{C_b}(\Omega)$ and $A = M_a$ the multiplication operator associated with a (with maximal domain) on X as in Exercise 2.5. Finally, fix $\varphi \in (\theta, \pi)$.

a) Show that A is sectorial with $\omega_{\mathrm{se}}(A) \le \theta$.

b) Show that $f(A) = M_{f\circ a}$ for each $f \in \mathcal{E}_e(\mathrm{S}_\varphi)$.

c) Show that $\lambda \in \sigma_{\mathrm{p}}(A)$ if and only if $[\, a = \lambda\,]$ has non-empty interior.

d) Let $f \in \operatorname{Hol}(\mathrm{S}_\varphi) \cap \mathrm{C}(\overline{\mathrm{S}_\theta})$ such that $f(A)$ is defined in the sectorial calculus for A. Show that $f(A) = M_{f\circ a}$. Conclude that

$$A^\alpha = M_{a^\alpha} \qquad (\operatorname{Re}\alpha > 0).$$

e) Suppose that A is injective and let $f \in \mathrm{H}^\infty(\mathrm{S}_\varphi)$. Show that $f(A) = M_{f\circ a}$, where the operator on the right-hand side is defined by

$$(x, y) \in M_{f\circ a} \quad \overset{\text{def.}}{\Longleftrightarrow} \quad (f \circ a)x = y \quad \text{on } [\, a \ne 0\,].$$

f) Still in the situation from e), show that $f(A) \in \mathcal{L}(X)$ if and only if $f \circ a$ has a (necessarily unique) continuous extension to the whole of Ω.

g) Give an example of a metric space Ω and a continuous function $a : \Omega \to \mathbb{C}$ such that $A := M_a$ is sectorial on $X = \mathrm{C_b}(\Omega)$ and $\sup_{s\in\mathbb{R}} \|A^{-\mathrm{i}s}\| < \infty$, but neither $\operatorname{dom}(A)$ nor $\operatorname{ran}(A)$ is dense.

10.8. Let A be a sectorial operator on a Banach space X and let $Y := \overline{\operatorname{dom}}(A) \cap \overline{\operatorname{ran}}(A)$. Let B be the part of A in Y, i.e., $B := A \cap (Y \oplus Y)$. Show

that B is sectorial of angle $\omega_{\mathrm{se}}(B) \le \omega_{\mathrm{se}}(A)$ and $R(\lambda, B) = R(\lambda, A)|_Y$ for all $\lambda \in \rho(A)$. Also, show that B has dense domain and dense range.
[Hint: Theorem 9.2.a).]

Supplementary Exercises

10.9. Let $a \in \mathbb{C} \setminus (-\infty, 0]$ and $\lambda \in \mathbb{C}$ with $|\mathrm{Im}\,\lambda| > \pi$. Show that

$$\frac{1}{\lambda - \log a} = \int_0^\infty \frac{-1}{(\lambda - \log t)^2 + \pi^2}(t+a)^{-1}\,\mathrm{d}t.$$

[Hint: Start by writing

$$\frac{1}{\lambda - \log a} = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{\mathrm{d}z}{(\lambda - \log z)(z-a)},$$

where Γ is a "keyhole contour" with the cut on the negative real axis and then pass to the limit so that the circular contour vanishes and one is left with an integral from ∞ to 0 on $[\arg \mathbf{z} = \pi]$ and an integral from 0 to ∞ on $[\arg \mathbf{z} = -\pi]$.]

10.10. Let $0 < \mathrm{Re}\,\alpha < 1$. Show that

$$\int_{\mathbb{R}} \frac{\mathrm{e}^{\alpha x}}{1 + \mathrm{e}^x}\,\mathrm{d}x = \frac{\pi}{\sin \alpha\pi}, \tag{10.13}$$

for instance by shifting the contour from $\mathbb{R}$ to $\mathbb{R} + 2\pi\mathrm{i}$. Then derive (10.10) from (10.13).

10.11. Let A be an injective sectorial operator on a Banach space X, let $\omega \in (\omega_{\mathrm{se}}(A), \pi)$ and $\lambda \in \mathbb{C} \setminus \overline{\mathrm{S}_\omega}$. Then $(\lambda - \mathrm{e}^{\mathbf{z}})^{-1} \in \mathrm{H}^\infty(\mathrm{St}_\omega)$. Take the existence of the strip calculus and (10.5) from Remark 10.6 for granted and show that

$$(\lambda - \mathrm{e}^{\mathbf{z}})^{-1}(\log A) = R(\lambda, A).$$

Conclude that $(\mathrm{e}^{\mathbf{z}})(\log A)$ is defined within the strip calculus for $\log A$, and

$$(\mathrm{e}^{\mathbf{z}})(\log(A)) = A.$$

(This shows in particular that an injective sectorial operator is uniquely determined by its logarithm.)

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Chapter 11
Bounded H^∞-Calculus for Hilbert Space Operators

11.1 Bounded H^∞-Calculus

Let $D \subseteq \mathbb{C}$ be some (non-empty) domain in the complex plane. We say that an operator A on a Banach space X has a **bounded** H^∞**-calculus on** D if there is a bounded representation

$$\Phi : \mathrm{H}^\infty(D) \to \mathcal{L}(X)$$

such that its canonical extension within $\mathrm{Mer}(D)$ has A as its generator (in the sense that $\Phi(\mathbf{z}) = A$). Obviously, the case $D = \mathbb{C}$ is pretty uninteresting since $\mathrm{H}^\infty(\mathbb{C}) = \mathbb{C}\mathbf{1}$ by Liouville's theorem. So in the following we take $D \neq \mathbb{C}$ as a standing assumption.

In general, to say that A "has a bounded $\mathrm{H}^\infty(D)$-calculus" does not tell much. In particular, considered in this generality there is no reason to think that such a calculus must be unique.

However, if we suppose even more that $\overline{D} \neq \mathbb{C}$, then for each $\lambda \in \mathbb{C} \setminus \overline{D}$ one has $r_\lambda := (\lambda - \mathbf{z})^{-1} \in \mathrm{H}^\infty(D)$, and hence $\Phi(r_\lambda) = R(\lambda, A)$. In this case Φ is completely determined at least on the space of rational functions bounded on D and then, by boundedness of Φ, on the closure of this space with respect to the uniform norm

$$\|f\|_{\infty,D} := \sup\{|f(z)| \mid z \in D\}.$$

For certain domains D one can then use results from complex approximation theory to infer that some special, explicitly constructed, H^∞-calculus Φ_A for A is bounded. (The reason for this is usually that a general function $f \in \mathrm{H}^\infty(D)$ can be approximated by a bp-convergent sequence of bounded rational functions, and the calculus Φ_A is bp-continuous in a certain sense.) In such a situation, A having "a" bounded H^∞-calculus on D is equivalent to the concretely given Φ_A being bounded on $\mathrm{H}^\infty(D)$, and the latter is what is usually intended when one uses the former terminology.

To wit, this all holds for sectorial operators with dense domain and range, as the following result shows.

Theorem 11.1. *Let A be a sectorial operator with dense domain and range on a Banach space X. Then the following assertions are equivalent for $\omega \in (\omega_{\mathrm{se}}(A), \pi)$ and $C \geq 0$:*

(i) *The sectorial calculus for A is bounded on $\mathrm{H}^\infty(\mathrm{S}_\omega)$ with*

$$\|f(A)\| \leq C\|f\|_{\infty,\mathrm{S}_\omega} \tag{11.1}$$

for all $f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$.

(ii) *The estimate* (11.1) *holds for each rational function $f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$ with $f(0) = 0 = f(\infty)$.*

(iii) *The estimate* (11.1) *holds for each elementary function $f \in \mathcal{E}(\mathrm{S}_\omega)$.*

Sketch of Proof. The implication (i) ⇒ (ii) is clear.

Now define

$$\mathcal{A} := \{f \in \mathrm{H}^\infty(\mathrm{S}_\omega) \mid (11.1) \text{ holds}\}.$$

It follows from the convergence lemma (Theorem 9.10) that if $\mathcal{A} \ni f_n \to f$ pointwise and $\liminf_{n\to\infty} \|f_n\|_\infty \leq \|f\|_\infty$, then $f \in \mathcal{A}$.

(ii) ⇒ (iii): Denote by $K := \overline{\mathrm{S}_\omega} \cup \{\infty\} \subseteq \mathbb{C}_\infty$ and let $\mathcal{R}_0(\mathrm{S}_\omega)$ be the algebra of rational functions f bounded on S_ω and with $f(0) = f(\infty) = 0$. Using [4, II, Thm.10.4] it can be shown that $\mathcal{R}_0(\mathrm{S}_\omega)$ is sup-norm dense in the algebra

$$\mathcal{B} := \{f \in \mathrm{C}(K) \cap \mathrm{H}^\infty(\mathrm{S}_\omega) \mid f(0) = 0 = f(\infty)\}$$

(cf. the proof of [6, Prop.F.3]). So (ii) implies $\mathcal{B} \subseteq \mathcal{A}$. Given $f \in \mathcal{E}(\mathrm{S}_\omega)$ define $f_n := f(\mathbf{z}^{\alpha_n})$, where $\alpha_n := 1 - 1/n$, $n \in \mathbb{N}$. By Lemma 9.4.b), $f_n \in \mathcal{B}$, so $f_n \in \mathcal{A}$. But $f_n \to f$ pointwise on S_ω and $\sup_{n\in\mathbb{N}} \|f_n\|_\infty \leq \|f\|_\infty$, hence $f \in \mathcal{A}$ as claimed.

(iii) ⇒ (i): Let $f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$ and define $f_n := f \cdot \mathbf{z}^{1/n}(1+\mathbf{z})^{-2/n}$ for $n \in \mathbb{N}$. Then $f_n \in \mathcal{E}(\mathrm{S}_\omega)$ and $f_n \to f$ pointwise on S_ω. Moreover, $\|f_n\|_\infty \leq \|f\|_\infty c^{\frac{1}{n}}$, where $c = \|\mathbf{z}(1+\mathbf{z})^{-2}\|_{\infty,\mathrm{S}_\omega}$. It follows that $f \in \mathcal{A}$ as desired. □

Remark 11.2. A result analogous to Theorem 11.1 holds for operators of right half-plane type and for strong strip type operators (Exercise 11.10). See Exercise 11.1 for a similar theorem in the context of bounded operators.

An estimate of the form $\|f(A)\| \leq C\|f\|_\infty$ is usually the best one can expect since one has equality (with $C = 1$) for multiplication operators. To understand the situation better, suppose that A is a densely defined operator of right half-plane type 0. For such an operator one has the holomorphic calculus on each open half-plane which contains $\overline{\mathbb{C}_+}$. But as $-A$ need not be the generator of a C_0-semigroup, the operators $(\mathrm{e}^{-t\mathbf{z}})(A)$ need not be bounded. A fortiori, A cannot have a bounded H^∞-calculus on any half-plane.

But now suppose that $-A$ generates a bounded C_0-semigroup $(T_t)_{t\geq 0}$. Then one has the Hille–Phillips calculus on $\mathbb{C}_+$ for A, with the estimate

$$\|f(A)\| \leq (\sup_{t\geq 0} \|T_t\|)\, \|f\|_{\mathrm{LS}(\mathbb{C}_+)} \qquad (f \in \mathrm{LS}(\mathbb{C}_+)).$$

The Hille–Phillips calculus is already "better" than the holomorphic half-plane calculus. However, the Laplace-Stieltjes norm is strictly stronger than the uniform norm, and the example of the shift semigroup on $\mathrm{L}^1(\mathbb{R})$ shows that one cannot expect a better estimate in general (Lemma 6.9).

This indicates that a supremum norm estimate for a functional calculus (and in particular a bounded H^∞-calculus) is quite special and one may wonder whether there are interesting non-trivial situations where such an estimate holds. We shall see soon that the answer to this question is yes.

11.2 Plancherel's Theorem

The most satisfying results on H^∞-estimates for functional calculi hold for Hilbert space operators. The spectral theorem (Chapter 4) is the most impressive witness for this claim. In the remainder of this chapter we shall see what can be said when the operators in question are not normal anymore.

One of the most important "gateways" to H^∞-boundedness results on Hilbert spaces, at least in the context of semigroups, is a vector-valued version of Plancherel's theorem.

For any Banach space X the **vector-valued Fourier transform** is defined as for scalar functions by

$$(\mathcal{F}f)(t) := \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}t\cdot s} f(s)\,\mathrm{d}s \qquad (t \in \mathbb{R}^d,\ f \in \mathrm{L}^1(\mathbb{R}^d; X)).$$

(We shall freely use the results of Appendix A.6 on Bochner spaces from now on.) In Appendix A.9 we have collected important information about the Fourier transform for vector-valued functions. In particular, one can find a proof of the Fourier inversion theorem (Theorem A.47).

Note that if H is a Hilbert space, then so is $\mathrm{L}^2(\mathbb{R}^d; H)$, with the scalar product being

$$(\, f \,|\, g\,)_{\mathrm{L}^2(\mathbb{R}^d;H)} = \int_{\mathbb{R}^d} (\, f(s) \,|\, g(s)\,)_H \,\mathrm{d}s$$

for $f,\ g \in \mathrm{L}^2(\mathbb{R}^d; H)$. For a proof of the following result, see Theorem A.50 in Appendix A.9. (Recall that $\mathcal{S}$ is the reflection operator, defined in Section 5.2.)

Theorem 11.3 (Plancherel). *Let H be a Hilbert space. Then the Fourier transform maps $\mathrm{L}^1(\mathbb{R}^d; H) \cap \mathrm{L}^2(\mathbb{R}^d; H)$ into $\mathrm{L}^2(\mathbb{R}^d; H)$ and extends to a* *bounded operator (again denoted by $\mathcal{F}$) on $\mathrm{L}^2(\mathbb{R}^d;H)$ in such a way that $\mathcal{F}^2=(2\pi)^d\mathcal{S}$ and the operator*

$$(2\pi)^{-\frac{d}{2}}\mathcal{F}:\mathrm{L}^2(\mathbb{R}^d;H)\to\mathrm{L}^2(\mathbb{R}^d;H)$$

is unitary. Moreover, for each $\mu\in\mathrm{M}(\mathbb{R}^d)$ and $f\in\mathrm{L}^2(\mathbb{R}^d;H)$

$$\mathcal{F}(\mu * f)=\widehat{\mu}\cdot\widehat{f}. \tag{11.2}$$

The Gearhart–Greiner–Prüss Theorem

Plancherel's theorem is useful in the context of semigroup theory through the following observation. Let B be the generator of a C_0-semigroup $T=(T_t)_{t\geq 0}$ on a Hilbert space H, let $\omega>\omega_0(T)$, the exponential growth bound of T. Then, for $s\in\mathbb{R}$ and $x\in X$ we have

$$R(\omega+\mathrm{i}s,B)x=\int_0^\infty \mathrm{e}^{-(\omega+\mathrm{i}s)t}T_t x\,\mathrm{d}t=\int_0^\infty \mathrm{e}^{-\mathrm{i}st}T_t^\omega x\,\mathrm{d}t.$$

(Recall the definition of T^ω from page 84.) Hence,

$$R(\omega+\mathrm{i}\mathbf{s},B)x=\mathcal{F}(T_{\mathbf{t}}^\omega x)\qquad(x\in H).$$

(We view T as a function on $\mathbb{R}$ by setting $T(t)=0$ for $t<0$.) Plancherel's theorem then implies that

$$\|R(\omega+\mathrm{i}\mathbf{s},B)x\|_{\mathrm{L}^2(\mathbb{R};H)}=\sqrt{2\pi}\|T_{\mathbf{t}}^\omega x\|_{\mathrm{L}^2(\mathbb{R};H)}\leq C_\omega\|x\|$$

for some C_ω independent of $x\in H$ (Exercise 11.2). The following important result of semigroup theory is a nice application.

Theorem 11.4 (Gearhart–Greiner–Prüss). *Let B be the generator of a C_0-semigroup $T=(T_t)_{t\geq 0}$ on a Hilbert space. Then $R(\mathbf{z},B)$ is not bounded on $[\,\mathrm{Re}\,\mathbf{z}>\omega_0(T)\,]$.*

Proof. By rescaling we may suppose without loss of generality that $\omega_0(T)=0$. Suppose towards a contradiction that $M':=\sup_{\mathrm{Re}\,z>0}\|R(z,B)\|<\infty$. For each $0<\alpha<\omega$ the resolvent identity yields

$$R(\alpha+\mathrm{i}\mathbf{s},B)=\big(\mathrm{I}+(\omega-\alpha)R(\alpha+\mathrm{i}\mathbf{s},B)\big)\,R(\omega+\mathrm{i}\mathbf{s},B).$$

Hence, there is $C\geq 0$ independent of α such that

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-2\alpha t}\|T_t x\|^2\,\mathrm{d}t&=(2\pi)^{-1}\|R(\alpha+\mathrm{i}\mathbf{s},B)x\|_{\mathrm{L}^2}^2\\&\leq(2\pi)^{-1}(1+M'(\omega-\alpha))^2\|R(\omega+\mathrm{i}\mathbf{s},B)x\|_{\mathrm{L}^2}^2\leq C^2\|x\|^2\end{aligned}$$

for all $x \in H$. Letting $\alpha \searrow 0$ yields

$$\int_0^\infty \|T_t x\|^2 \,\mathrm{d}t \le C^2 \|x\|^2 \qquad (x \in H).$$

Now fix $M \ge 0$ such that $\|T_s\| \le M\mathrm{e}^{\omega s}$ for all $s \ge 0$. Then

$$\Bigl(\int_0^t \mathrm{e}^{-2\omega s}\,\mathrm{d}s\Bigr)\|T_t x\|^2 = \int_0^t \mathrm{e}^{-2\omega s}\,\|T_s T_{t-s} x\|^2 \,\mathrm{d}s \le M^2 C^2 \|x\|^2.$$

It follows that $K := \sup_{t\ge 0}\|T_t\| < \infty$. Therefore,

$$t\|T_t x\|^2 = \int_0^t \|T_{t-s} T_s x\|^2 \,\mathrm{d}s \le K^2 C^2 \|x\|^2.$$

This implies that $\|T_t\| \to 0$ and then, by Exercise 6.1.a), that $\omega_0(T) < 0$. $\square$

Remark 11.5. Theorem 11.4 tells that if the resolvent of a semigroup generator B is bounded on some right half plane R_ω, then $\omega_0(T) < \omega$. (In still other words: $s_0(B) = \omega_0(T)$, where $s_0(B)$ is the left half-plane type of B, cf. page 128.) Note that the identity $s_0(B) = \omega_0(T)$ may fail on a general Banach space [3, Comments V.1.12].

Recall from Chapter 6 that if $(U_s)_{s\in\mathbb{R}}$ is a C_0-group on a Banach space X then

$$\theta(U) = \inf\{\theta \ge 0 \mid \sup_{s\in\mathbb{R}} \mathrm{e}^{-\theta|s|}\|U_s\| < \infty\}$$

is the **group type** of U.

Corollary 11.6. *Let $-\mathrm{i}A$ be the generator of a C_0-group U on a Banach space X Then A is an operator of strong strip type $\omega_{\mathrm{st}}(A) \le \theta(U)$. If $X = H$ is a Hilbert space, then $\omega_{\mathrm{st}}(A) = \theta(U)$.*

Proof. The first part follows from the first order Hille–Yosida estimates (Exercise 6.3) and the fact that $\mathrm{i}A$ generates the semigroup $(U_{-s})_{s\ge 0}$ (Theorem 6.8). The second part follows from the Gearhart–Greiner–Prüss Theorem. See Exercise 11.7. $\square$

11.3 Von Neumann's Inequality

Von Neumann's inequality is without doubt one of the most important non-trivial H^∞-boundedness results in the theory of functional calculus. In its simplest, discrete, form it reads as follows. (Recall that $\mathbb{D} := [\,|\mathbf{z}| < 1\,]$ is the open unit disc.)

Theorem 11.7 (Von Neumann). *Let T be a linear contraction on a Hilbert space H. Then*

$$\|f(T)\| \le \|f\|_{\infty,\mathbb{D}} \tag{11.3}$$

for each polynomial $f \in \mathbb{C}[z]$.

Theorem 11.7 is called **von Neumann's inequality**. By Exercise 11.1 it is equivalent to the assertion that *a contraction on a Hilbert space has a contractive H^∞-calculus on $\mathbb{D}_r$ for each $r > 1$.*

So, von Neumann's inequality is in fact a result about bounded H^∞-calculus, even if it does *not* assert that each contraction T has a bounded H^∞-functional calculus on $\mathbb{D}$. This, namely, cannot be true: T may have eigenvalues on $\mathbb{T} = \partial\mathbb{D}$, but a generic H^∞-function on $\mathbb{D}$ cannot be meaningfully evaluated in those eigenvalues.

At the end of this section we shall deduce von Neumann's inequality from the following continuous version of it. A **contraction semigroup** is a semigroup $(T_t)_{t\ge 0}$ with $\|T_t\| \le 1$ for all $t \ge 0$. By (a special case of) the so-called Lumer–Phillips theorem, the negative generators of contraction semigroups are precisely the so-called **m-accretive** operators. See Appendix A.8 for the precise definition of accretivity and the Lumer–Phillips theorem.

Theorem 11.8. *Let $-A$ be the generator of a strongly continuous contraction semigroup $(T_t)_{t\ge 0}$ on a Hilbert space H. Then*

$$\Big\| \int_{\mathbb{R}_+} T_t \,\mu(\mathrm{d}t) \Big\| \le \|\mathcal{L}\mu\|_\infty \quad \textit{for all} \quad \mu \in \mathrm{M}(\mathbb{R}_+). \tag{11.4}$$

Equivalently, $\quad \|f(A)\| \le \|f\|_{\infty,\mathbb{C}_+} \quad$ *for all* $\quad f \in \mathrm{LS}(\mathbb{C}_+)$.

Our proof of Theorem 11.8 is an elaboration of [12, Cor. 3.5].

Without loss of generality one may suppose that T is **exponentially stable**, i.e., satisfies $\omega_0(T) < 0$. Indeed, if Theorem 11.8 is true for exponentially stable contraction semigroups, then it is true for $T^\varepsilon = \mathrm{e}^{-\varepsilon \mathbf{t}} T_{\mathbf{t}}$ for each $\varepsilon > 0$, and hence for T: simply let $\varepsilon \searrow 0$ in the inequality

$$\Big\| \int_{\mathbb{R}_+} \mathrm{e}^{-\varepsilon t} T_t \,\mu(\mathrm{d}t) \Big\| \le \|\mathcal{L}\mu\|_\infty .$$

An exponentially stable semigroup T is **strongly stable**, by which it is meant that $\lim_{t\to\infty} \|T_t x\| = 0$ for all $x \in H$. Also, the generator of an exponentially stable semigroup is invertible.

Lemma 11.9 (Zwart). *Let $-A$ be the generator of a contraction semigroup $(T_t)_{t\ge 0}$ on a Hilbert space H. If T is strongly stable and A is invertible, then there is an operator $C : \mathrm{dom}(A) \to H$ such that*

$$\int_0^\infty \|CT_t x\|^2 \,\mathrm{d}t = \|x\|^2 \qquad \textit{for all } x \in \mathrm{dom}(A). \tag{11.5}$$

Proof. Since A is invertible, $S := A^{-1} + (A^{-1})^*$ is bounded and self-adjoint. The orbit $T_t x$ of $x \in \operatorname{dom}(A)$ is differentiable with derivative $-AT_t x$. Since each T_t is a contraction, $\|T_t x\|$ is decreasing. Hence, for $x \in \operatorname{dom}(A)$,

$$\begin{aligned} 0 \le -\frac{\mathrm{d}}{\mathrm{d}t}\|T_t x\|^2 &= (\, AT_t x \,|\, T_t x\,) + (\, T_t x \,|\, AT_t x\,) \\ &= \big(\, AT_t x \,\big|\, A^{-1}AT_t x\,\big) + \big(\, A^{-1}AT_t x \,\big|\, AT_t x\,\big) \\ &= \big(\, (A^{-1})^* AT_t x \,\big|\, AT_t x\,\big) + \big(\, A^{-1}AT_t x \,\big|\, AT_t x\,\big) \\ &= (\, SAT_t x \,|\, AT_t x\,)\,. \end{aligned}$$

Inserting $t = 0$ yields $(\, SAx \,|\, Ax\,) \ge 0$ for all $x \in \operatorname{dom}(A)$, and since $\operatorname{ran}(A) = H$, it follows that $S \ge 0$. Therefore,

$$-\frac{\mathrm{d}}{\mathrm{d}t}\|T_t x\|^2 = (\, SAT_t x \,|\, AT_t x\,) = \|S^{1/2}AT_t x\|^2.$$

It follows that

$$\|x\|^2 = \lim_{r\to\infty}\big(\|x\|^2 - \|T_r x\|^2\big) = \int_0^\infty -\frac{\mathrm{d}}{\mathrm{d}t}\|T_t x\|^2\,\mathrm{d}t = \int_0^\infty \|S^{1/2}AT_t x\|^2\,\mathrm{d}t,$$

which is (11.5) with $C := S^{1/2}A$. □

By Lemma 11.9 and the density of $\operatorname{dom}(A)$ in H we obtain an isometric operator

$$J : H \to \mathrm{L}^2(\mathbb{R}; H),$$

defined for $x \in \operatorname{dom}(A)$ by

$$(Jx)(s) := \begin{cases} CT_{-s}x & s \le 0 \\ 0 & s > 0. \end{cases}$$

The range $\operatorname{ran}(J)$ of J is a closed subspace of $\mathrm{L}^2(\mathbb{R}; H)$. Define

$$P : \mathrm{L}^2(\mathbb{R}; H) \to H, \qquad P := J^{-1}Q,$$

where Q is the orthogonal projection onto $\operatorname{ran}(J)$.

Lemma 11.10. *Let the invertible operator $-A$ be the generator of a strongly stable contraction semigroup $(T_t)_{t\ge0}$ on a Hilbert space H, and let the operators J and P be defined as above. Then*

$$P \circ \tau_t \circ J = T_t \qquad (t \ge 0),$$

where $(\tau_t)_{t\in\mathbb{R}}$ is the right shift group on $\mathrm{L}^2(\mathbb{R}; H)$. Equivalently, the diagram

$$\begin{tikzcd} \mathrm{L}^2(\mathbb{R};H) \arrow[r, "\tau_t"] & \mathrm{L}^2(\mathbb{R};H) \arrow[d, "P"] \\ H \arrow[u, "J"] \arrow[r, "T_t"] & H \end{tikzcd} \tag{11.6}$$

is commutative for each $t \ge 0$.

Proof. Note first that $\mathrm{ran}(J)$ is contained in the closed subspace

$$K = \{f \in \mathrm{L}^2(\mathbb{R};H) \mid \mathbf{1}_{\mathbb{R}_-} f = f\}.$$

The orthogonal projection onto K is multiplication by $\mathbf{1}_{\mathbb{R}_-}$, hence $Pf = P(\mathbf{1}_{\mathbb{R}_-} f)$ for each $f \in \mathrm{L}^2(\mathbb{R};H)$. Now let $x \in \mathrm{dom}(A)$. Then for $s \le 0$ and $t \ge 0$ one has $(Jx)(s-t) = CT_{t-s}x = CT_{-s}T_t x = (JT_t x)(s)$. Consequently,

$$\mathbf{1}_{\mathbb{R}_-}\tau_t Jx = \mathbf{1}_{[\,\mathbf{s}\le 0\,]} \cdot (Jx)(\mathbf{s}-t) = JT_t x.$$

Applying P yields $P\tau_t Jx = T_t x$, hence the claim by density of $\mathrm{dom}(A)$. □

Proof of Theorem 11.8. Let $\mu \in \mathrm{M}(\mathbb{R}_+)$. Then after integrating against μ, (11.6) becomes

$$\begin{tikzcd} \mathrm{L}^2(\mathbb{R};H) \arrow[r, "\tau_\mu"] & \mathrm{L}^2(\mathbb{R};H) \arrow[d, "P"] \\ H \arrow[u, "J"] \arrow[r, "T_\mu"] & H \end{tikzcd} \tag{11.7}$$

from which it follows that $\|T_\mu\| \le \|\tau_\mu\|$. But τ_μ is convolution with μ, hence

$$\tau_\mu f = \mathcal{F}^{-1}(\widehat{\mu}\cdot\widehat{f}).$$

By Plancherel's theorem, this implies $\|\tau_\mu\| \le \|\widehat{\mu}\|_{\infty,\mathbb{R}} \le \|\mathcal{L}\mu\|_{\infty,\mathbb{C}_+}$. (One even has equality in the last step as a consequence of the maximum principle.) This completes the proof of Theorem 11.8. □

Remarks 11.11. 1) Actually, an appeal to Plancherel's theorem here is not necessary if we assume the spectral theorem to be known. Indeed, the right shift group on the Hilbert space $\mathrm{L}^2(\mathbb{R};H)$ is unitary, so its generator B, say, is skew-symmetric. Since the Borel calculus for B is compatible with the Fourier–Stieltjes calculus (this is similar to Exercise 6.10), we obtain the desired norm estimate.

2) The commutativity of the diagram (11.6) means that the pair (J,P) is a *dilation* of the original semigroup T to the unitary group τ. A famous theorem of Szökefalvi-Nagy and Foiaş [10, Sec.I.8] states that *each contraction semigroup on a Hilbert space has a dilation to a unitary group on another Hilbert space.*

Theorem 11.8 can be rephrased in terms of H^∞-calculi as follows.

Theorem 11.12. *Let $-A$ be the generator of a contraction C_0-semigroup on a Hilbert space H. Then the inequality*

$$\|f(A)\| \le \|f\|_{\infty,\mathbb{C}_+} \tag{11.8}$$

holds in the following cases:

1) *$f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$, $\omega < 0$, and $f(A)$ is defined within the half-plane calculus for A.*
2) *$f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$, $\omega \in (\pi/2, \pi)$, A is injective, and $f(A)$ is defined in the sectorial calculus for A.*

Proof. We give a proof in the case of 2) and leave case 1) as Exercise 11.3. So suppose that the hypotheses of 2) are satisfied. Take first $f \in \mathcal{E}(\mathrm{S}_\omega)$. Then by Remark 9.8 there is $\varphi \in \mathrm{L}^1(\mathbb{R}_+)$ such that $f = \mathcal{L}(\varphi)$ and $f(A) = \int_0^\infty \varphi(t) T_t \,\mathrm{d}t$. Hence, (11.8) is true in this case, by Theorem 11.8.

For general $f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$ let $f_n := e_n f$, where $e_n := n^2\mathbf{z}(n+\mathbf{z})^{-1}(1+n\mathbf{z})^{-1}$ for $n \in \mathbb{N}$. Then $\|e_n f\|_{\infty,\mathbb{C}_+} \le \|f\|_{\infty,\mathbb{C}_+}$ and $e_n f \to f$ pointwise and boundedly on S_ω. Since A is injective and H is reflexive, A has dense range (and anyway dense domain). By the convergence lemma, $f(A) \in \mathcal{L}(H)$ with

$$\|f(A)\| \le \liminf_n \|f_n(A)\| \le \liminf_n \|f_n\|_{\infty,\mathbb{C}_+} \le \|f\|_{\infty,\mathbb{C}_+}$$

as desired. □

Proof of Von Neumann's Inequality

Von Neumann's inequality (Theorem 11.7) is a consequence of Theorem 11.8 in the following way. By passing to rT for $r < 1$ and then letting $r \nearrow 1$ it suffices to consider contractions T such that $\mathrm{I} - T$ is invertible. Now let

$$A := (\mathrm{I} + T)(\mathrm{I} - T)^{-1} \in \mathcal{L}(X).$$

It is a simple exercise to show that $\mathrm{Re}\,(Ax \,|\, x) \ge 0$ for all $x \in H$. By Theorem A.23 it follows that $[\,\mathrm{Re}\,\mathbf{z} < 0\,] \subseteq \rho(A)$ and

$$\|R(\lambda, A)\| \le \frac{1}{\mathrm{dist}(\lambda, \mathbb{C}_+)} = \frac{1}{|\mathrm{Re}\,\lambda|} \qquad (\mathrm{Re}\,\lambda < 0).$$

Since A is bounded, it is densely defined. So the Hille–Yosida theorem with $\omega = 0$ and $M = 1$ yields that $-A$ generates a contraction semigroup.

We can write $A = f(T)$, where $f = (1 + \mathbf{z})(1 - \mathbf{z})^{-1}$ is a Möbius transformation which maps the unit disc to the right half-plane. Its inverse is $c := (\mathbf{z} - 1)(\mathbf{z} + 1)^{-1}$, called the **Cayley transform mapping**, and it is another simple exercise to show that $A + \mathrm{I}$ is invertible and $c(A) = T$. (Observe that $c \in \mathrm{LS}(\mathbb{C}_+)$.) Hence, if $p \in \mathbb{C}[z]$ is any polynomial,

$$p(T) = (p \circ c)(A).$$

Since c maps $\mathbb{C}_+$ into $\mathbb{D}$, it follows from Theorem 11.8 that

$$\|p(T)\| = \|(p \circ c)(A)\| \le \|p \circ c\|_{\infty,\mathbb{C}_+} \le \|p\|_{\infty,\mathbb{D}}.$$

This concludes the proof of Theorem 11.7. (See also Exercise 11.5.) □

11.4 Strongly Continuous Groups on Hilbert Spaces

In this section we shall prove that generators of C_0-groups on Hilbert spaces have a bounded H$^\infty$-calculus on strips. The best result here holds for *bounded* C_0-groups.

Theorem 11.13 (Szökefalvi-Nagy). *Let $-\mathrm{i}A$ be the generator of a bounded C_0-group $(U_s)_{s\in\mathbb{R}}$ on a Hilbert space H. Then the group U is unitary with respect to some equivalent scalar product*[1] *on H, and A admits a bounded $\mathcal{M}_\mathrm{b}(\mathbb{R})$-calculus.*

Proof. If U is unitary, then by Stone's theorem (Theorem A.45), A is self-adjoint. By the spectral theorem, A has a Borel calculus on $\mathbb{R}$, in particular a contractive $\mathcal{M}_\mathrm{b}(\mathbb{R})$-calculus. Hence, if U is unitary merely with respect to some equivalent scalar product, A still admits a bounded (but not necessarily contractive) $\mathcal{M}_\mathrm{b}(\mathbb{R})$-calculus. So it suffices to prove the first assertion.

To this end, let $M := \sup_{s\in\mathbb{R}} \|U_s\|$. Then

$$M^{-2}\|x\|^2 \le \|U_s x\|^2 \le M^2\|x\|^2 \qquad (s \in \mathbb{R},\, x \in H). \tag{11.9}$$

Let p be a *an invariant mean* on $\ell^\infty(\mathbb{R};\mathbb{C})$, i.e. a positive, shift invariant linear functional with $p(\mathbf{1}) = 1$.[2] Define the sesquilinear form

$$\alpha : H \times H \to \mathbb{C}, \qquad \alpha(x,y) := p\big((U_\mathbf{s} x \,|\, U_\mathbf{s} y)\big).$$

Since p is positive, it is monotone. Applying p to (11.9) therefore yields

$$M^{-2}\|x\|^2 \le \alpha(x,x) \le M^2\|x\|^2 \qquad (x \in H).$$

Hence, α is an equivalent scalar product on H. Since p is shift invariant,

$$\alpha(U_t x, U_t x) = p\big((U_\mathbf{s} U_t x \,|\, U_\mathbf{s} U_t x)\big) = p\big((U_{\mathbf{s}+t} x \,|\, U_{\mathbf{s}+t} x)\big) = \alpha(x,y)$$

---

[1] An **equivalent scalar product** on a Hilbert space H is a scalar product that induces an equivalent norm on H.

[2] Such an object exists by an application of the Markov-Kakutani fixed point theorem ([2, Thm.10.1], [9, Thm.5.11]), which shows that $\mathbb{R}$ is an *amenable* group. Most textbooks cover the discrete version, under the name *Banach limit*. Lax [7, Chap.4] derives the result from a version of the Hahn–Banach theorem.

for all $t \in \mathbb{R}$ and $x, y \in H$. Hence, U is a unitary group with respect to α. □

An alternative proof of Theorem 11.13 is provided in Exercise 11.6.

The Boyadzhiev–deLaubenfels Theorem

Let us now turn to unbounded C_0-groups.

Theorem 11.14 (Boyadzhiev–de Laubenfels). *Let A be a densely defined operator on a Hilbert space H. Then the following assertions are equivalent:*

(i) $-\mathrm{i}A$ *generates a C_0-group $(U_s)_{s\in\mathbb{R}}$ on H;*

(ii) *A has a bounded H^∞-calculus on some strip St_ω, $\omega > 0$.*

In this case, for each $\omega > \theta(U)$ there is $C \geq 0$ such that

$$\|f(A)\| \leq C\|f\|_{\infty,\mathrm{St}_\omega} \qquad (f \in \mathrm{H}^\infty(\mathrm{St}_\omega)),$$

where $f(A)$ is defined in the canonical extension of the Fourier–Stieltjes calculus[3] *for A.*

Recall from Exercise 6.7 that the Fourier–Stieltjes calculus Ψ_U for a C_0-group U of type (M, ω) on a Banach space X is defined by

$$\Psi_U(f) = \int_{\mathbb{R}} U_s\,\mu(\mathrm{d}s), \qquad f = \mathcal{F}\mu,\ \mu \in \mathrm{M}_\omega(\mathbb{R}),$$

where

$$\mathrm{M}_\omega(\mathbb{R}) = \Big\{\mu \in \mathrm{M}(\mathbb{R}) \;\Big|\; \int_{\mathbb{R}} \mathrm{e}^{\omega|s|}\,|\mu|\,(\mathrm{d}s) < \infty\Big\}.$$

We denote by $\mathrm{L}^1_\omega(\mathbb{R}) := \mathrm{M}_\omega(\mathbb{R}) \cap \mathrm{L}^1(\mathbb{R})$.

Proof of (ii) ⇒ (i) in Theorem 11.14. Let $\omega > 0$ and $\Phi : \mathrm{H}^\infty(\mathrm{St}_\omega) \to \mathcal{L}(H)$ be a bounded representation. Let $U_s := \Phi(\mathrm{e}^{-\mathrm{i}s\mathbf{z}})$ for $s \in \mathbb{R}$. Then $(U_s)_{s\in\mathbb{R}}$ is an operator group satisfying

$$\|U_s\| \leq \|\Phi\|\,\|\mathrm{e}^{-\mathrm{i}s\mathbf{z}}\|_{\infty,\mathrm{St}_\omega} = \|\Phi\|\mathrm{e}^{\omega|s|} \qquad (s \in \mathbb{R}).$$

Pick $\lambda \in \mathbb{C} \setminus \overline{\mathrm{St}_\omega}$ and note that the function

$$\mathbb{R} \to \mathrm{H}^\infty(\mathrm{St}_\omega), \qquad s \mapsto \frac{\mathrm{e}^{-\mathrm{i}s\mathbf{z}}}{(\lambda - \mathbf{z})^2}$$

is continuous. Hence,

[3] We shall see below in Theorem 11.15 that $f(A)$ *is* indeed defined in that way.

$$s \mapsto U_s R(\lambda, A)^2 = \Phi\Big(\frac{\mathrm{e}^{-\mathrm{i}s\mathbf{z}}}{(\lambda - \mathbf{z})^2}\Big)$$

is continuous in operator norm. Since $\mathrm{dom}(A^2)$ is dense and U is locally bounded, U is strongly continuous. For $\alpha > \omega$ we have

$$\begin{aligned} R(\lambda, A)^2 \int_0^\infty \mathrm{e}^{-\alpha s} U_s \,\mathrm{d}s &= \int_0^\infty \mathrm{e}^{-\alpha s} U_s R(\lambda, A)^2 \,\mathrm{d}s \\ &= \Phi\Big(\int_0^\infty \mathrm{e}^{-\alpha s} \frac{\mathrm{e}^{-\mathrm{i}s\mathbf{z}}}{(\lambda - \mathbf{z})^2} \,\mathrm{d}s\Big) = \mathrm{i}\Phi\big((\mathrm{i}\alpha - \mathbf{z})^{-1}(\lambda - \mathbf{z})^{-2}\big) \\ &= \mathrm{i}R(\lambda, A)^2 R(\mathrm{i}\alpha, A), \end{aligned}$$

because the integral in the second line converges within $\mathrm{H}^\infty(\mathrm{St}_\omega)$. Hence,

$$\int_0^\infty \mathrm{e}^{-\alpha s} U_s \,\mathrm{d}s = \mathrm{i}R(\mathrm{i}\alpha, A) = (\alpha + \mathrm{i}A)^{-1}.$$

The concludes the proof of the implication (ii) $\Rightarrow$ (i) in Theorem 11.14. □

The proof of the implication (i) $\Rightarrow$ (ii) is more complicated. In order to facilitate the reasoning it is convenient to switch from the Fourier–Stieltjes calculus to the strip calculus explained in Remark 10.6. However, in our situation we do not start with an arbitrary (strong) strip type operator (for which one has to define the strip calculus in the first place) but with a group generator, where a functional calculus is already at hand. What we need is the following representation theorem (which of course expresses the compatibility of strip calculus and Fourier–Stieltjes calculus).

Theorem 11.15. *Let $\omega > 0$ and $f \in \mathcal{E}(\mathrm{St}_\omega)$. Then the function $\varphi : \mathbb{R} \to \mathbb{C}$ defined by*

$$\varphi(s) := \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}sz} f(z) \,\mathrm{d}z \qquad (s \in \mathbb{R})$$

has the following properties:

1) *φ is continuous and $\mathrm{e}^{\alpha|\mathbf{s}|}\varphi$ is bounded for each $0 \le \alpha < \omega$.*
2) *$\varphi \in \mathrm{L}^1_\alpha(\mathbb{R})$ for each $0 \le \alpha < \omega$.*
3) *$\mathcal{F}\varphi = f|_{\mathbb{R}}$.*
4) *Whenever $-\mathrm{i}A$ generates a C_0-group $(U_s)_{s\in\mathbb{R}}$ with $\theta(U) < \omega$ on a Banach space X,*

$$\Psi_U(f) = \frac{1}{2\pi\mathrm{i}} \int_{\partial \mathrm{St}_\delta} f(z) R(z, A) \,\mathrm{d}z$$

for each $\delta \in (\theta(U), \omega)$.

Proof. 1) Since f is elementary, it is integrable over $\mathbb{R}$, so $\varphi \in \mathrm{C_b}(\mathbb{R})$. Let $0 \le \alpha < \omega$. Then, since $\mathrm{e}^{\mathrm{i}s\mathbf{z}} f$ is elementary we can shift the contour onto $[\,\mathrm{Im}\,\mathbf{z} = \pm\alpha\,]$ and obtain

$$2\pi\, \varphi(s) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}sz} f(z)\,\mathrm{d}z = \int_{\mathbb{R}\pm\mathrm{i}\alpha} \mathrm{e}^{\mathrm{i}sz} f(z)\,\mathrm{d}z = \mathrm{e}^{\mp\alpha s} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}st} f(t \pm \mathrm{i}\alpha)\,\mathrm{d}t.$$

It follows that $\mathrm{e}^{\pm\alpha\mathbf{s}}\varphi$ is bounded and this yields 1). 2) follows from 1).
For 3) and 4) we use the observation from above to write

$$\begin{aligned}
2\pi \int_{\mathbb{R}} \varphi(s) U_s\,\mathrm{d}s &= \int_0^\infty \int_{\mathbb{R}+\mathrm{i}\delta} \mathrm{e}^{\mathrm{i}sz} f(z)\,\mathrm{d}z\, U_s\,\mathrm{d}s + \int_{-\infty}^0 \int_{\mathbb{R}-\mathrm{i}\delta} \mathrm{e}^{\mathrm{i}sz} f(z)\,\mathrm{d}z\, U_s\,\mathrm{d}s \\
&= \int_{\mathbb{R}+\mathrm{i}\delta} f(z) \int_0^\infty \mathrm{e}^{\mathrm{i}sz} U_s\,\mathrm{d}s\,\mathrm{d}z + \int_{\mathbb{R}-\mathrm{i}\delta} f(z) \int_{-\infty}^0 \mathrm{e}^{\mathrm{i}sz} U_s\,\mathrm{d}s\,\mathrm{d}z \\
&= \int_{\mathbb{R}+\mathrm{i}\delta} f(z) R(-\mathrm{i}z, -\mathrm{i}A)\,\mathrm{d}z + \int_{\mathbb{R}-\mathrm{i}\delta} f(z) R(\mathrm{i}z, \mathrm{i}A)\,\mathrm{d}z \\
&= (-\mathrm{i}) \int_{\partial \mathrm{St}_\delta} f(z) R(z, A)\,\mathrm{d}z.
\end{aligned}$$

Here we used Fubini's theorem and that $-\mathrm{i}A$ is the generator of $(U_s)_{s\geq 0}$ and $\mathrm{i}A$ is the generator of $(U_{-s})_{s\geq 0}$. Applying the result to $X = \mathbb{C}$ and $A = a \in \mathbb{R}$ we obtain

$$\mathcal{F}\varphi(a) = \frac{1}{2\pi\mathrm{i}} \int_{\partial \mathrm{St}_\delta} \frac{f(z)}{z-a}\,\mathrm{d}z = f(a)$$

by Lemma 9.4.b). This completes the proof. □

Theorem 11.15 shows in particular that $\mathcal{E}(\mathrm{St}_\omega)$ is in the domain of the Fourier–Stieltjes calculus for U. It allows us to compute and estimate with the functions f directly rather than with their Fourier pre-images.

Square Function Estimates

At this point of the proof, Plancherel's theorem enters the scene.

Lemma 11.16. *Let $-\mathrm{i}A$ be the generator of a C_0-group U on a Hilbert space H. Then, for each $\omega > \theta(U)$ and $g \in \mathcal{E}(\mathrm{St}_\omega)$,*

$$\int_{\mathbb{R}} \|g(A+t)x\|^2 \mathrm{d}t \lesssim \|x\|^2 \qquad (x \in H). \tag{11.10}$$

Remark: The notation

$$F(x) \lesssim G(x) \qquad (x \in \mathcal{M})$$

is shorthand for: there is a number $C \geq 0$ such that $F(x) \leq C\, G(x)$ for all $x \in \mathcal{M}$. (So, C may depend on F and G, but not on x.) We use this notion if the size of the precise constant is unimportant and keeping track of these constants would render the presentation awkward.

Proof. Find φ to $f := g$ as in Theorem 11.15. Then, as $-\mathrm{i}(A+t)$ generates the group $\mathrm{e}^{-\mathrm{i}t\mathbf{s}}U_{\mathbf{s}}$,

$$g(A+t)x = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}ts}\varphi(s)U_s x \,\mathrm{d}s \qquad (x \in H).$$

Since $\varphi\mathrm{e}^{\theta|\mathbf{s}|} \in \mathrm{L}^2(\mathbb{R})$ and $\mathrm{e}^{-\theta|\mathbf{s}|}U_{\mathbf{s}}$ is uniformly bounded for $\theta(U) < \theta < \omega$, the claim follows from Plancherel's theorem. $\square$

We call (11.10) a **square function estimate**. Note that the operator $-\mathrm{i}A^*$ generates the C_0-group $(U^*_{-s})_{s\in\mathbb{R}}$, so Lemma 11.16 can be applied again and yields a **dual square function estimate**

$$\int_{\mathbb{R}} \|h(A^*+t)x\|^2 \mathrm{d}t \lesssim \|x\|^2 \qquad (x \in H) \tag{11.11}$$

for any $h \in \mathcal{E}(\mathrm{St}_\omega)$. We shall see below that a square function estimate and a dual square function estimate together imply the boundedness of the H^∞-calculus. To this aim, we need another auxiliary result. It will be convenient to write

$$f_t := f(\mathbf{z}+t)$$

for $t \in \mathbb{R}$ and $f \in \mathrm{Hol}(\mathrm{St}_\omega)$.

Lemma 11.17. *Let φ, $\psi \in \mathcal{E}(\mathrm{St}_\omega)$ such that $\int_{\mathbb{R}} \psi(t)\,\mathrm{d}t = 1$. Then the following assertions hold:*

a) $\displaystyle\int_{\mathbb{R}} (\psi_t\varphi)(A)\,\mathrm{d}t = \varphi(A)$ *as an absolutely convergent integral in $\mathcal{L}(H)$.*

b) *There is a constant $C \geq 0$ such that*

$$\sup_{t\in\mathbb{R}} \|(f\varphi_t)(A)\| \leq C\|f\|_{\infty,\mathrm{St}_\omega}$$

for all $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$.

Proof. We leave a) as Exercise 11.8. For b) let $\delta < \omega$ be so close to ω that we have

$$(f\varphi_t)(A) = \frac{1}{2\pi\mathrm{i}} \int_{\partial\mathrm{St}_\delta} f(z)\varphi(z+t)R(z,A)\,\mathrm{d}z.$$

Taking norms we obtain

$$\|(f\varphi_t)(A)\| \lesssim \|f\|_\infty \int_{\partial\mathrm{St}_\delta} |\varphi(z+t)|\;|\mathrm{d}z| = \|f\|_\infty \int_{\partial\mathrm{St}_\delta} |\varphi(z)|\;|\mathrm{d}z|\,.$$

This yields b). $\square$

The following important theorem shows that square function estimates imply a bounded H^∞-calculus. The hypotheses on A are as before, but see also Remark 11.19 below.

Theorem 11.18. *Suppose that g, $h \in \mathcal{E}(\mathrm{St}_\omega) \setminus \{0\}$ and one has a square function estimate* (11.10) *for g and a dual square function estimate* (11.11) *for h. Then there is $C \geq 0$ such that*

$$\|f(A)x\| \leq C\|f\|_{\infty,\mathrm{St}_\omega} \|x\| \tag{11.12}$$

for all $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$ and $x \in H$.

Proof. Since both g and h are non-zero and holomorphic, gh can vanish only on a discrete set, so $c := \|gh\|_{\mathrm{L}^2(\mathbb{R})} \neq 0$. For a function u on S_ω we write $u^* := \overline{u(\overline{\mathbf{z}})}$. With this notation, let $\varphi := c^{-2} hg^*$ and $\psi := \varphi h^* g$. Then

$$\int_{\mathbb{R}} \psi(t)\,\mathrm{d}t = c^{-2} \int_{\mathbb{R}} |h(t)g(t)|^2\,\mathrm{d}t = 1.$$

Take any $e \in \mathcal{E}(\mathrm{St}_\omega)$ and $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$. By Lemma 11.17.a)

$$(fe)(A) = \int_{\mathbb{R}} (\psi_t fe)(A)\,\mathrm{d}t = \int_{\mathbb{R}} h_t^*(A)\,(f\varphi_t)(A)\,g_t(A)\,e(A)\,\mathrm{d}t. \tag{11.13}$$

Inserting $x \in H$ and taking the inner product with $y \in H$ we obtain

$$(\,(fe)(A)x \,|\, y\,) = \int_{\mathbb{R}} \big((f\varphi_t)(A)g_t(A)e(A)x \,\big|\, h_t(A^*)y\big)\,\mathrm{d}t$$

since $h_t(A^*) = \big(h_t^*(A)\big)^*$ (Exercise 11.9). The Cauchy–Schwarz inequality together with the square function estimates (11.10) and (11.11) then yield a constant C' such that

$$|(\,(fe)(A)x \,|\, y\,)| \leq C' \sup_{t\in\mathbb{R}} \|(f\varphi_t)(A)\|\,\|e(A)x\|\,\|y\|.$$

Lemma 11.17.b) then yields $C \geq 0$ with

$$\|(fe)(A)x\| \leq C\,\|f\|_{\infty,\mathrm{St}_\omega} \|e(A)x\| \qquad (x \in H).$$

Specializing $e := (\lambda - \mathbf{z})^{-2}$ for some $\lambda \in \mathbb{C} \setminus \overline{\mathrm{St}_\omega}$ we obtain the estimate (11.12) for $x \in \mathrm{dom}(A^2)$. But this space is dense in H and $f(A)$ is a closed operator, so (11.12) must hold for all $x \in H$. □

This concludes the proof of the Boyadzhiev–deLaubenfels Theorem 11.14.

Supplementary Remark 11.19. Theorem 11.18 holds under the more general condition that A is a strong strip type operator on H with $\omega_{\mathrm{st}}(A) < \omega$. Indeed: the proof works only with Cauchy integrals and the group U did not play a role at any point. So (i) and (ii) of Theorem 11.14 are equivalent to

(iii) *The operator A is of strong strip type and admits a square function estimate* (11.10) *and a dual square function estimate* (11.11) *for some functions $g, h \in \mathcal{E}(\mathrm{St}_\omega) \setminus \{0\}$ and some $\omega > \omega_{\mathrm{st}}(A)$.*

11.5 Supplement: Sectorial Operators with BIP

Boyadzhiev-deLaubenfels' theorem from [1] was a strip analogue of an earlier result by McIntosh from the seminal paper [8]. Analogously to the strip situation, an estimate of the form

$$\int_0^\infty \|g(tA)x\|^2 \frac{\mathrm{d}t}{t} \lesssim \|x\|^2 \qquad (x \in H) \tag{11.14}$$

for a function $g \in \mathcal{E}(\mathrm{S}_\omega)$ is called a **square function estimate** for the sectorial operator A; and

$$\int_0^\infty \|h(tA^*)x\|^2 \frac{\mathrm{d}t}{t} \lesssim \|x\|^2 \qquad (x \in H) \tag{11.15}$$

is called a **dual square function estimate** for A.

Theorem 11.20 (McIntosh). *Let A be an injective sectorial operator on a Hilbert space H. Then the following assertions are equivalent:*

(i) *A has a bounded H^∞-calculus on S_ω for some $\omega \in (\omega_{\mathrm{se}}(A), \pi)$.*

(ii) *A has BIP.*

(iii) *A admits a square function estimate* (11.14) *and a dual square function estimate* (11.15) *for some $\omega \in (\omega_{\mathrm{se}}(A), \pi)$ and some $g, h \in \mathcal{E}(\mathrm{S}_\omega) \setminus \{0\}$.*

In this case, the following assertions hold:

a) *The group type of the group $(A^{\mathrm{i}s})_{s\in\mathbb{R}}$ equals the sectoriality angle $\omega_{\mathrm{se}}(A)$ of A.*

b) *A has a bounded H^∞-calculus on S_ω for all $\omega \in (\omega_{\mathrm{se}}(A), \pi)$.*

c) *A admits square function estimates* (11.14) *and dual square function estimates* (11.15) *for all $\omega \in (\omega_{\mathrm{se}}(A), \pi)$ and all $g, h \in \mathcal{E}(\mathrm{S}_\omega)$.*

Proof. First note that an injective sectorial operator on a Hilbert space has dense domain and range. So (i) $\Rightarrow$ (ii) follows from Theorem 10.7. Suppose that (ii) holds and let $B := \log A$. Then $-\mathrm{i}B$ generates the group $U := A^{-\mathbf{i}\mathbf{s}}$. By the Boyadzhiev–deLaubenfels theorem, B has bounded $\mathrm{H}^\infty(\mathrm{St}_\omega)$-calculus for each $\omega > \theta(U)$. By Corollary 11.6 and Theorem 10.5, $\theta(U) = \omega_{\mathrm{st}}(B) = \omega_{\mathrm{se}}(A)$. Moreover, by Remark 10.6 and, in particular, identity (10.5), we have

$$f(A) = f(\mathrm{e}^{\mathbf{z}})(B) \tag{11.16}$$

for each $f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$. This yields (i) as well as assertion b).

By (11.16), square and dual square function estimates for A and B are in one-to-one correspondence. So the equivalence (i), (ii) $\Leftrightarrow$ (iii) and assertion c) follow from our proof of the Boyadzhiev–deLaubenfels theorem and Remark 11.19. $\square$

Exercises

11.1. Show that the set of polynomial functions is dense in the Banach algebra $\mathrm{H}^\infty(\mathbb{D}) \cap \mathrm{C}(\overline{\mathbb{D}})$. Then show that the following statements are equivalent for a bounded operator $T \in \mathcal{L}(X)$ on a Banach space X and a number $C \geq 0$:

(i) $\|p(T)\| \leq C\|p\|_{\infty,\mathbb{D}}$ for all $p \in \mathbb{C}[z]$;

(ii) T has a bounded $\mathrm{H}^\infty(\mathbb{D}) \cap \mathrm{C}(\overline{\mathbb{D}})$-calculus of norm at most C;

(iii) T has, for each $r > 1$, a bounded $\mathrm{H}^\infty(\mathbb{D}_r)$-calculus of norm at most C.

Suppose that (i)–(iii) hold and show the following assertions:

a) T is power–bounded.

b) The bounded $\mathrm{H}^\infty(\mathbb{D}) \cap \mathrm{C}(\overline{\mathbb{D}})$-calculus asserted in (ii) is unique and its restriction to $\mathrm{A}^1_+(\mathbb{D})$ coincides with the calculus for power–bounded operators defined in Section 1.2.

c) For each $r > 1$ the Dunford–Riesz calculus for T on $\mathbb{D}_r$ restricts to a bounded representation $\mathrm{H}^\infty(\mathbb{D}_r) \to \mathcal{L}(X)$ of norm at most C.

11.2. Let $T = (T_t)_{t\geq 0}$ be a C_0-semigroup on a Banach space X and let $\omega > \omega_0(T)$. Show that for each $1 \leq p \leq \infty$ there is a constant $C_p \geq 0$ such that

$$\|T^\omega_{\mathbf{t}} x\|_{\mathrm{L}^p(\mathbb{R}_+;X)} \leq C_p\|x\|.$$

11.3. Let $-A$ be the generator of a contraction C_0-semigroup on a Hilbert space H and $\omega < 0$. Show that

$$\|f(A)\| \leq \|f\|_{\infty,\mathbb{C}_+} \qquad (f \in \mathrm{H}^\infty(\mathrm{R}_\omega)),$$

where $f(A)$ is defined within the half-plane calculus for A.
[Hint: Imitate the proof of Case 2) in Theorem 11.12; you'll need the compatibility Theorem 8.20 and the convergence lemma (Theorem 8.10).]

11.4. Let A be a closed and densely defined operator on a Hilbert space. Prove the following assertions.

a) If A is of right half-plane type $\omega \in \mathbb{R}$ then so is A^*, and $M(A, \alpha) = M(A^*, \alpha)$ for all $\alpha < \omega$.

b) If A is sectorial of angle $\omega \in (0, \pi)$ then so is A^*, and $M(A, \alpha) = M(A^*, \alpha)$ for all $\alpha \in (\omega, \pi)$.

c) If $-A$ generates a C_0-semigroup $(T_t)_{t\geq 0}$ then $(T^*_t)_{t\geq 0}$ is a C_0-semigroup and $-A^*$ is its generator.

[Hint for a) and b): Corollary A.22. Hint for c): prove first, e.g. by using the Hille–Yosida theorem, that $-A^*$ is a generator; then prove that the generated semigroup coincides with $(T^*_t)_{t\geq 0}$. Note that A^* is densely defined (see the last statement before the rubric "The Numerical Range" in Appendix A.5).]

11.5 (Von Neumann's Inequality). Let T be a contraction on a Hilbert space such that $T^n \to 0$ strongly.

a) Show that there is an operator $C \in \mathcal{L}(H)$ such that

$$\sum_{n=0}^{\infty} \|CT^n x\|^2 = \|x\|^2 \qquad (x \in H).$$

b) Construct an isometric embedding $\eta : H \to \ell^2(\mathbb{Z}; H)$ and a contraction $P : \ell^2(\mathbb{Z}; H) \to H$ such that

$$T^n = P \circ \tau^n \circ \eta \qquad (n \in \mathbb{N}_0)$$

where τ is the right shift on $\ell^2(\mathbb{Z}; H)$.

c) Prove von Neumann's inequality for T. Then prove von Neumann's inequality for an arbitrary contraction on a Hilbert space.

[Hint: One can take $C := (\mathrm{I} - T^*T)^{1/2}$.]

11.6. Let $-\mathrm{i}A$ be the generator of a C_0-group $(U_s)_{s\in\mathbb{R}}$ on a Hilbert space H, and let $M \geq 1$ and $\theta \geq 0$ such that $\|U_s\| \leq M\mathrm{e}^{\theta|s|}$ for all $s \in \mathbb{R}$. For each $\omega > \theta$ let

$$Q_\omega := \int_0^\infty \mathrm{e}^{-2\omega|s|} U_s^* U_s \,\mathrm{d}s$$

as a weak integral.

a) Show that Q_ω is a self-adjoint operator satisfying

$$\tfrac{M^{-2}}{\omega+\theta}\,\mathrm{I} \;\leq\; Q_\omega \;\leq\; \tfrac{M^2}{\omega-\theta}\,\mathrm{I}.$$

b) Conclude that by $(\, x \,|\, y \,)_\omega := (\, Q_\omega x \,|\, y \,)$ an equivalent scalar product on H is given. Prove that

$$\|U_s\|_\omega \leq \mathrm{e}^{\omega|s|} \qquad (s \in \mathbb{R}).$$

c) Suppose in addition that U is bounded, i.e., that $\theta = 0$. Then the sequence $({}^1\!/\!{}_n Q_{1/n})_{n\in\mathbb{N}}$ is bounded. Since closed norm-balls in $\mathcal{L}(H)$ are compact in the weak operator topology, the sequence has a cluster point Q, say. Prove that by $\alpha(x, y) := (\, Qx \,|\, y \,)$ an equivalent scalar product is given in H with respect to which U is unitary.

Remarks: c) provides an alternative proof of Theorem 11.13. The main idea is from Zwart's paper [11].

From b) one can prove that the operator A can be written as $A = B + \mathrm{i}C$, where B and C are self-adjoint with respect to the equivalent scalar product $(\,\cdot\,|\,\cdot\,)_\omega$ and C is bounded with $\|C\| \leq \omega$. In particular: *Each generator of a C_0-group on a Hilbert space is a bounded perturbation of a generator of a bounded C_0-group.* See [5] or [6, Sec.7.2] for details.

11.7. Let $-\mathrm{i}A$ be the generator of a C_0-group $(U_s)_{s\in\mathbb{R}}$ on a Banach space X. Show that A is of strong strip type $\omega_{\mathrm{st}}(A) \le \theta(U)$. If $X = H$ is a Hilbert space, show that $\theta(U) = \omega_{\mathrm{st}}(A)$.

11.8. Let A be an operator of strong strip type on a Banach space X and let $\omega > \omega_{\mathrm{st}}(A)$. Let $\varphi, \psi \in \mathcal{E}(\mathrm{St}_\omega)$ and define $c := \int_{\mathbb{R}} \psi(t)\,\mathrm{d}t$. For a function g on St_ω and $t \in \mathbb{R}$ we abbreviate $g_t := g(\mathbf{z}+t)$. Take the strip calculus (Remark 10.6) for granted and prove the following assertions:

a) For each $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$ the function $t \mapsto (f\psi_t)(A)$ is continuous on $\mathbb{R}$.

b) $\displaystyle\sup_{s>0} \int_{\mathbb{R}} \|(\psi_t\varphi_s)(A)\|\,\mathrm{d}t < \infty$.

c) $\displaystyle\int_{\mathbb{R}} (\psi_t\varphi)(A)\,\mathrm{d}t = c\varphi(A)$.

11.9. Let A be a densely defined operator of strong strip type on a Hilbert space H. Show that also A^* is of strong strip type with $\omega_{\mathrm{st}}(A) = \omega_{\mathrm{st}}(A^*)$ and

$$h^*(A) = h(A^*)^*$$

for each $h \in \mathcal{E}(\mathrm{St}_\omega)$, $\omega > \omega_{\mathrm{st}}(A)$.

Supplementary Exercises

11.10. Suppose that A is a densely defined operator of strong strip type on a Banach space X. Let $\omega > \omega_{\mathrm{st}}(A)$ and take the existence of the strip calculus on $\mathcal{E}(\mathrm{St}_\omega)$ and its natural extension to $\mathrm{H}^\infty(\mathrm{St}_\omega)$ for granted.[4] Prove the following statements:

a) **(Convergence Lemma)** Let $(f_n)_n$ be a sequence in $\mathrm{H}^\infty(\mathrm{St}_\omega)$ which converges pointwise and boundedly on St_ω to some function $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$. If $\sup_n \|f(A)\| < \infty$ then $f(A) \in \mathcal{L}(X)$ and $f_n(A) \to f(A)$ strongly.

b) Let there be $C \ge 0$ such that

$$\|f(A)\| \le C\|f\|_{\infty,\mathrm{St}_\omega}$$

for all $f \in \mathcal{E}(\mathrm{St}_\omega)$. Then this holds for all $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$.

[Hint: For a) imitate the proof of Theorem 9.10. For b) imitate the proof of the implication (iii) ⇒ (i) of Theorem 11.1. One can take $f_n(z) = f(z)\mathrm{e}^{-\frac{1}{n}z^2}$ as an approximation of f.]

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[4] If $-\mathrm{i}A$ generates a C_0-group U with $\theta(U) < \omega$, the strip calculus can be found as a subcalculus of the Fourier–Stieltjes calculus, by Theorem 11.15.

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Chapter 12
Fourier Multipliers and Elliptic Operators on $\mathrm{L}^p(\mathbb{R}^d)$

In the previous chapter we have seen that on Hilbert spaces "many" operators admit a bounded H^∞-calculus in one or the other way. This changes drastically when one leaves the regime of Hilbert spaces. However, on "good" spaces (among which are L^p-spaces for $1 < p < \infty$) one still can prove interesting and non-trivial results.

We shall postpone the clarification of what precisely is meant by "good" here to the next (and final) chapter. Instead, in the present chapter we shall devise an important class of sectorial operators with a bounded H^∞-calculus. As a background, we have to introduce a central notion of harmonic analysis.

12.1 Fourier Multiplier Operators

Let X be a Banach space. For $m \in \mathrm{L}^\infty(\mathbb{R}^d)$ define

$$T_m f := \mathcal{F}^{-1}(m \cdot \widehat{f})$$

for functions $f \in \mathrm{E}(\mathbb{R}^d; X)$, where

$$\mathrm{E}(\mathbb{R}^d; X) := \{f \mid f, \widehat{f} \in \mathrm{L}^1(\mathbb{R}^d; X) \cap \mathrm{C}_0(\mathbb{R}^d; X)\}.$$

Here, $\mathcal{F}^{-1}$ is the inverse Fourier transform given by $\mathcal{F}^{-1} = (2\pi)^{-d}\mathcal{S}\mathcal{F}$ (see Theorem A.47). The operator

$$T_m : \mathrm{E}(\mathbb{R}^d; X) \to \mathrm{C}_0(\mathbb{R}^d; X)$$

is called a **Fourier multiplier operator** with **symbol** m. By Corollary A.48, the space $\mathrm{E}(\mathbb{R}^d; X)$ is dense in $\mathrm{L}^p(\mathbb{R}^d; X)$ for each $1 \le p < \infty$. In particular, this is true for $p = 1$, which implies that m is uniquely determined by T_m.

Now fix $1 \le p < \infty$. The function $m \in \mathrm{L}^\infty(\mathbb{R}^d)$ is called an $\mathrm{L}^p(X)$-**multiplier** if the associated Fourier multiplier operator T_m extends to a bounded operator on $\mathrm{L}^p(\mathbb{R}^d; X)$. By density of $\mathrm{E}(\mathbb{R}^d; X)$, this extension—which is again denoted by T_m—is unique. We let

$$\mathcal{M}_p^X(\mathbb{R}^d) := \{m \in \mathrm{L}^\infty(\mathbb{R}^d) \mid m \text{ is an } \mathrm{L}^p(X)\text{-multiplier}\}$$

and endow it with the norm

$$\|m\|_{\mathcal{M}_p^X} := \|T_m\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R}^d;X))}. \tag{12.1}$$

If $X = \mathbb{C}$ we abbreviate $\mathcal{M}_p(\mathbb{R}^d) := \mathcal{M}_p^{\mathbb{C}}(\mathbb{R}^d)$.

In the following we shall highlight some of the most important properties of L^p-Fourier multipliers. However, we shall not give proofs here but refer to Appendix A.10 for details, in particular to Theorem A.56.

Theorem 12.1. *The space $\mathcal{M}_p^X(\mathbb{R}^d)$ is a subalgebra of $\mathrm{L}^\infty(\mathbb{R}^d)$ and a Banach algebra with respect to the norm (12.1). Moreover, it contains $\mathrm{FS}(\mathbb{R}^d)$ and both inclusions*

$$\mathrm{FS}(\mathbb{R}^d) \subseteq \mathcal{M}_p^X(\mathbb{R}^d) \subseteq \mathrm{L}^\infty(\mathbb{R}^d)$$

are contractive. Finally, the mapping

$$\mathcal{M}_p^X(\mathbb{R}^d) \to \mathcal{L}(\mathrm{L}^p(\mathbb{R}^d; X)), \qquad m \mapsto T_m \tag{12.2}$$

is an isometric and unital algebra homomorphism onto a closed unital subalgebra of $\mathcal{L}(\mathrm{L}^p(\mathbb{R}^d; X))$. It is an extension of the Fourier–Stieltjes calculus for the shift group on $\mathrm{L}^p(\mathbb{R}^d; X)$.

Each operator T_m, $m \in \mathcal{M}_p^X(\mathbb{R}^d)$, is **translation invariant**, i.e., commutes with all translation operators $(\tau_s)_{s\in\mathbb{R}^d}$ (Exercise 12.1). On the other hand, for $X = \mathbb{C}$ one can show that each translation invariant operator on $\mathrm{L}^p(\mathbb{R}^d)$ must already be of the form T_m for some $m \in \mathcal{M}_p(\mathbb{R}^d)$ [1, p.143]. So, in that case, the functional calculus (12.2) is in a certain sense the maximal bounded calculus extending the Fourier–Stieltjes calculus. (This is false for $X \neq \mathbb{C}$, as one can also consider Fourier multiplier operators with *operator-valued* symbols. However, we do not pursue this topic further here.)

Are there good criteria that help deciding whether a certain function m is an L^p-Fourier multiplier or not? Well, if $p = 2$ and $X = H$ is a Hilbert space, then $\mathcal{M}_2^H(\mathbb{R}^d) = \mathrm{L}^\infty(\mathbb{R}^d)$. This is a straightforward consequence of Plancherel's theorem, see Theorem A.56.e). If $p = 1$, then we know from Theorem 6.23 that $\mathcal{M}_1(\mathbb{R}^d) = \mathrm{FS}(\mathbb{R}^d)$, and this stays true also in the vector-valued situation (Theorem A.56.d)).

For $1 < p < \infty$ (and $p \neq 2$ if X is a Hilbert space) things are more interesting (and difficult). There are non-trivial results if the Banach space X is "good" (recall our introductory remarks to this chapter). For now, however, we restrict ourselves to the scalar case $X = \mathbb{C}$.

The Mikhlin Multiplier Theorem

Let $k_d := \lfloor d/2 \rfloor + 1$ be the least integer strictly bigger than $d/2$. A function $m \in \mathrm{C}^{k_d}(\mathbb{R}^d \setminus \{0\})$ is called **Mikhlin function** if its **Mikhlin norm**

$$\|m\|_{\mathrm{Mi}} := \max_{|\alpha| \le k_d} \left\| |\mathbf{t}|^{|\alpha|} \, \mathrm{D}^\alpha m(\mathbf{t}) \right\|_{\infty, \mathbb{R}^d \setminus \{0\}}$$

is finite. We denote by $\mathrm{Mi}(\mathbb{R}^d \setminus \{0\})$ the space of all Mikhlin functions. Note that each Mikhlin function is bounded and determines an element of $\mathrm{L}^\infty(\mathbb{R}^d)$, with

$$\|m\|_{\mathrm{L}^\infty} \le \|m\|_{\mathrm{Mi}}.$$

The following is one of the central results of harmonic analysis.

Theorem 12.2 (Mikhlin). *For each $p \in (1, \infty)$ the space $\mathrm{Mi}(\mathbb{R}^d \setminus \{0\})$ of Mikhlin functions embeds continuously into $\mathcal{M}_p(\mathbb{R}^d)$. In other words: for each $1 < p < \infty$ there is a constant C_p such that each Mikhlin function $m \in \mathrm{Mi}(\mathbb{R}^d \setminus \{0\})$ is an $\mathrm{L}^p(\mathbb{R}^d)$-multiplier with*

$$\|m\|_{\mathcal{M}_p} \le C_p \, \|m\|_{\mathrm{Mi}}.$$

We take this result for granted and refer to [1, Thm.5.2.7] for a proof.

12.2 Elliptic Operators on L^p

In this section we shall define elliptic, constant coefficient differential operators on $\mathrm{L}^p(\mathbb{R}^d)$, $1 < p < \infty$, and show that they are sectorial and admit a bounded H^∞-calculus on sectors. However, we shall take a slightly unusual route, as we shall define the functional calculus first and then identify its "generator" as the operator we are aiming at.

The Polynomial

A polynomial $a : \mathbb{R}^d \to \mathbb{C}$ is called **homogeneous** of degree $m \in \mathbb{N}_0$ if it is of the form[1]

$$a = \sum_{|\alpha| = m} a_\alpha \mathrm{i}^{|\alpha|} \mathbf{t}^\alpha \tag{12.3}$$

for certain $a_\alpha \in \mathbb{C}$ ($\alpha \in \mathbb{N}_0^d$, $|\alpha| = m$). As a result, one has

$$a(\lambda t) = \lambda^m a(t) \qquad (t \in \mathbb{R}^d, \ \lambda \ge 0).$$

[1] The presence of the factors $\mathrm{i}^{|\alpha|}$ in (12.3) is motivated by our wish to obtain the formulae (12.6) and (12.7) below, where D^α is an unscaled product of partial derivative operators.

A homogenous polynomial of degree $m \in \mathbb{N}$ is called **elliptic** if $-1 \notin a(\mathbb{R}^d)$ and there is $c > 0$ such that

$$|a(t)| \geq c\ |t|^m \qquad \text{for all } t \in \mathbb{R}^d; \tag{12.4}$$

and it is called **strongly elliptic** if there is $c > 0$ such that even

$$\operatorname{Re} a(t) \geq c\ |t|^m \qquad \text{for all } t \in \mathbb{R}^d \tag{12.5}$$

holds.

Lemma 12.3. *Let a be a homogeneous and elliptic polynomial. Then $a(t) = 0$ if and only if $t = 0$. Moreover, $a(\mathbb{R}^d) \subseteq \overline{\mathrm{S}_{\omega_a}}$ for some (minimal) $\omega_a \in [0, \pi)$. If the polynomial a is strongly elliptic, then $\omega_a < \pi/2$.*

Proof. By (12.4), $a(t) = 0$ if and only if $t = 0$. Moreover, $|a(t)| \to \infty$ if $|t| \to \infty$. Since a is continuous, $a(\mathbb{R}^d)$ is closed. By homogeneity, $a(\mathbb{R}^d) \subseteq \overline{\mathrm{S}_{\omega_a}}$ where $\omega_a \in [0, \pi]$ is chosen minimally with this property. Since $-1 \notin a(\mathbb{R}^d)$, we have $\omega_a < \pi$. Finally, suppose that a is strongly elliptic. Then $a(\mathbb{R}^d) \cap [\operatorname{Re} \mathbf{z} \leq 0] = \{0\}$ and hence $\omega_a < \pi/2$. □

Examples 12.4. In dimension $d = 1$ the 1-homogeneous polynomial $a = \mathrm{i}\mathbf{t}$ is elliptic with $\omega_a = \pi/2$. In any dimension, the 2-homogeneous polynomial $a = |\mathbf{t}|^2 = \sum_{j=1}^d \mathbf{t}_j^2$ is strongly elliptic with $\omega_a = 0$.

From now on, fix an elliptic homogeneous polynomial a as in (12.3) with $m \in \mathbb{N}$. We associate with a the differential operator

$$A := \sum_{|\alpha|=m} a_\alpha \mathrm{D}^\alpha. \tag{12.6}$$

The reason for this is that

$$Af = \mathcal{F}^{-1}(a \cdot \widehat{f}) \qquad (f \in \mathcal{S}(\mathbb{R}^d)), \tag{12.7}$$

where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space[2] (Theorem A.51). In order to find a "realization" A_p of A on $\mathrm{L}^p(\mathbb{R}^d)$ for $1 < p < \infty$, we shall first prove that a gives rise to a bounded H^∞-Fourier multiplier calculus.

The Calculus

We start with an auxiliary result.

Lemma 12.5. *Let $\omega \in (0, \pi)$, $n \in \mathbb{N}$ and $0 < \varphi < \omega$. Then there is a constant $C \geq 0$ such that*

[2] If you do not know much about the Schwartz space, you can safely ignore this remark at this point. It will be important only in Theorem 12.7 below.

$$\|\mathbf{z}^n f^{(n)}(\mathbf{z})\|_{\infty,\mathrm{S}_\varphi} \le C\|f\|_{\infty,\mathrm{S}_\omega}$$

for all $f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$.

Proof. By Cauchy's integral formula there is a constant C' such that

$$z^n f^{(n)}(z) = C' \int_{\partial \mathrm{S}_\delta} \frac{z^n f(w)}{(w-z)^{n+1}}\,\mathrm{d}w$$

for all $\delta \in (\varphi, \omega)$ and $z \in \mathrm{S}_\varphi$. Taking the modulus yields

$$\begin{aligned}\left|z^n f^{(n)}(z)\right| &\le C' \int_{\partial \mathrm{S}_\delta} \frac{|z|^n\,|f(w)|}{|w-z|^{n+1}}\,|\mathrm{d}w| \le C'\|f\|_{\infty,\mathrm{S}_\delta} \int_{\partial \mathrm{S}_\delta} \frac{|z|^n}{|w-z|^{n+1}}\,|\mathrm{d}w| \\ &\le C'\,\|f\|_{\infty,\mathrm{S}_\omega} \int_{\partial \mathrm{S}_\delta} \frac{|\mathrm{d}w|}{|w-(z/|z|)|^{n+1}}.\end{aligned}$$

Hence, the claim holds with

$$C = C' \sup_\lambda \int_{\partial \mathrm{S}_\delta} \frac{|\mathrm{d}w|}{|w-\lambda|^{n+1}} < \infty,$$

where the supremum is taken over all $\lambda \in \mathrm{S}_\varphi$ with $|\lambda| = 1$. □

Lemma 12.5 is the key to proving that if $f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$ for $\omega > \omega_a$ then $f \circ a$ is a Mikhlin function. More precisely, we have the following.

Lemma 12.6. *Let a be a homogeneous and elliptic polynomial on $\mathbb{R}^d$, and let $\omega_a \in (0, \pi)$ be defined as in Lemma 12.3. Let $\omega \in (\omega_a, \pi)$. Then composition with the function a yields a bounded operator*

$$\mathrm{H}^\infty(\mathrm{S}_\omega) \to \mathrm{Mi}(\mathbb{R}^d \setminus \{0\}), \qquad f \mapsto f \circ a.$$

Proof. Let a be homogeneous of degree $m \in \mathbb{N}$. By Exercise 12.2 one can write

$$\mathrm{D}^\alpha(f \circ a) = \sum_{j=0}^{|\alpha|} (f^{(j)} \circ a)\, p_{\alpha,j}$$

where $p_{\alpha,j} = 0$ if $jm < |\alpha|$ and $p_{\alpha,j}$ is a homogeneous polynomial of degree $jm - |\alpha|$ and there is a constant $c_{\alpha,j} \ge 0$ such that

$$|p_{\alpha,j}(t)| \le c_{\alpha,j}\,|t|^{jm-|\alpha|} \qquad (t \in \mathbb{R}^d)$$

if $mj \ge |\alpha|$. With the ellipticity (12.4) this yields the estimate

$$
\begin{aligned}
|t|^{|\alpha|}\,|\mathrm{D}^\alpha(f\circ a)(t)| &\le \sum_{j=0}^{|\alpha|} |t|^{|\alpha|}\,\bigl|f^{(j)}(a(t))\bigr|\,|p_{\alpha,j}(t)| \\
&\le \sum_{j=0}^{|\alpha|} c_{\alpha,j}\,|t|^{jm}\,\bigl|f^{(j)}(a(t))\bigr| \le \sum_{j=0}^{|\alpha|} \frac{c_{\alpha,j}}{c^j}\,|a(t)|^j\,\bigl|f^{(j)}(a(t))\bigr| \lesssim \|f\|_{\infty,\mathrm{S}_\omega},
\end{aligned}
$$

where in the last step we have applied Lemma 12.5. □

We can now put together all the pieces. Fix $1<p<\infty$ and the polynomial a as before and define, for $\omega \in (\omega_a, \boldsymbol{\pi})$,

$$
\Phi_a : \mathrm{H}^\infty(\mathrm{S}_\omega) \to \mathcal{L}(\mathrm{L}^p(\mathbb{R}^d)), \qquad \Phi_a(f) = T_{f\circ a}.
$$

The mapping is well-defined and a bounded algebra representation by Lemma 12.6 and the Mikhlin multiplier Theorem 12.2.

The Operator

In the situation from before, for $\lambda \in \mathbb{C}\setminus\overline{\mathrm{S}_\omega}$ the function $r_\lambda := (\lambda - \mathbf{z})^{-1}$ is in $\mathrm{H}^\infty(\mathrm{S}_\omega)$ and hence

$$
R_\lambda := \Phi_a(r_\lambda), \qquad \lambda \in \mathbb{C}\setminus\overline{\mathrm{S}_\omega}
$$

is a pseudo-resolvent on $\mathrm{L}^p(\mathbb{R}^d)$.

Theorem 12.7. *The so-constructed pseudo-resolvent $(R_\lambda)_\lambda$ is the resolvent of a sectorial operator A_p of angle $\omega_{\mathrm{se}}(A_p) \le \omega_a$ with dense domain and range in $\mathrm{L}^p(\mathbb{R}^d)$. The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is a core for A_p and*

$$
A_p f = \sum_{|\alpha|=m} a_\alpha \mathrm{D}^\alpha f \qquad (f \in \mathcal{S}(\mathbb{R}^d)).
$$

The calculus Φ_a coincides with the sectorial calculus for A_p on $\mathrm{H}^\infty(\mathrm{S}_\omega)$.

Proof. Fix $\alpha \in (\omega, \boldsymbol{\pi})$. Then (cf. Remark 9.3)

$$
\sup_{\lambda\in\mathbb{C}\setminus\mathrm{S}_\alpha} \|\lambda r_\lambda\|_{\infty,\mathrm{S}_\omega} < \infty.
$$

The boundedness of the calculus Φ_a hence yields

$$
\sup_{\lambda\in\mathbb{C}\setminus\mathrm{S}_\alpha} \|\lambda R_\lambda\| < \infty.
$$

Next, observe that $n(n+a)^{-1} \to \mathbf{1}$ pointwise on $\mathbb{R}^d$ as $n\to\infty$. Since the associated Fourier multiplier operators are uniformly bounded, part g) of Theorem A.56 yields

$$(-n)R_{-n} = T_{n(n+a)^{-1}} \to \mathrm{I} \quad \text{weakly} \tag{12.8}$$

as $n \to \infty$. It follows that the pseudo-resolvent $(R_\lambda)_\lambda$ consists of injective operators, and hence is the resolvent of a unique closed operator A_p on $\mathrm{L}^p(\mathbb{R}^d)$ (Remark A.14). By what we have shown earlier, A_p is sectorial of angle at most ω. As ω can be chosen as close to ω_a as one likes, we find that $\omega_{\mathrm{se}}(A_p) \le \omega_a$.

The approximation (12.8) already shows that $\mathrm{dom}(A_p)$ is weakly dense, hence strongly dense. (We could also have invoked Theorem 9.2.) With a similar argument as before one can show that

$$n^{-1}(n^{-1} + A_p)^{-1} \to 0 \quad \text{weakly}$$

as $n \to \infty$. This implies that A_p has dense range.

Now let $f \in \mathcal{S}(\mathbb{R}^d)$, $\lambda \in \mathbb{C} \setminus \overline{\mathrm{S}_\omega}$ and define $g := (\lambda - A)f$, where A is the differential operator defined in (12.6). Then $g \in \mathcal{S}(\mathbb{R}^d)$ as well and

$$R(\lambda, A_p)g = \Phi_a(r_\lambda)g = \mathcal{F}^{-1}((\lambda - a)^{-1}\widehat{g}) = \mathcal{F}^{-1}((\lambda - a)^{-1}(\lambda - a)\widehat{f}) = f$$

by (12.7). Hence, $f \in \mathrm{dom}(A_p)$ and $A_p f = Af$ as claimed. Since $\mathcal{S}(\mathbb{R}^d)$ is dense in $\mathrm{L}^p(\mathbb{R}^d)$ and invariant under the application of the resolvent of A_p, it must be a core for A_p.

Finally, let us show that for $f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$ one has $\Phi_a(f) = f(A_p)$ where the latter is defined in the sectorial calculus for A_p. By general theory, it suffices to consider $f \in \mathcal{E}(\mathrm{S}_\omega)$ here. By Lemma 9.4 we have

$$f \circ a = \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\delta} \frac{f(z)}{z - a} \, \mathrm{d}z \tag{12.9}$$

(with an arbitrary $\delta \in (\omega_a, \omega)$), where initially this identity is to be understood pointwise on $\mathbb{R}^d$. However, if under the integral sign we pass to the associated Fourier multiplier operators the right hand side of the identity becomes

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\delta} f(z) R(z, A_p) \, \mathrm{d}z,$$

and this integral is convergent with respect to the operator norm. As the correspondence of symbols $m \in \mathcal{M}_p(\mathbb{R}^d)$ and operators T_m on $\mathrm{L}^p(\mathbb{R}^d)$ is isometric, we conclude that (12.9) can actually be interpreted as an identity in $\mathcal{M}_p(\mathbb{R}^d)$. This yields

$$\Phi_a(f) = T_{f \circ a} = \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\delta} f(z) R(z, A_p) \, \mathrm{d}z = f(A_p)$$

as desired. □

Supplementary Remarks 12.8. One can show that, actually, $\omega_{\mathrm{se}}(A_p) = \omega_a$ and that $\mathrm{dom}(A_p) = \mathrm{H}^m_p(\mathbb{R}^d)$, the Bessel potential space of order m.

Our theory so far only deals with homogeneous polynomials. One can deal with more general polynomials of the form $p = a+b$, where a is a homogeneous elliptic polynomial of order $m \in \mathbb{N}$ and b is any polynomial of degree $m-1$. On the operator level, one obtains as before the operator A_p associated with a and an operator B defined as the closure of the operator $x \mapsto \mathcal{F}^{-1}(b \cdot \widehat{x})$ on the Schwartz space. One can view B as a lower order perturbation of A_p and prove that for some $\lambda \geq 0$ the operator $\lambda + A_p + B$ is sectorial of angle ω_a and has a bounded $\mathrm{H}^\infty(\mathrm{S}_\omega)$-calculus for each $\omega \in (\omega_a, \pi)$. See, e.g., [2, Section 5.5] or [3, Section 13].

Exercises

12.1. Let X be a Banach space and recall the definition

$$\mathrm{E}(\mathbb{R}^d; X) := \{f \in \mathrm{L}^1(\mathbb{R}^d; X) \mid \widehat{f} \in \mathrm{L}^1(\mathbb{R}^d; X)\}.$$

Prove the following statements for $f \in \mathrm{L}^1(\mathbb{R}^d; X)$, $\mu \in \mathrm{M}(\mathbb{R}^d)$, $m \in \mathrm{L}^\infty(\mathbb{R}^d)$ and $1 \leq p < \infty$:

a) $\mathcal{F}^{-1}(\widehat{\mu} \cdot f) = \mu * (\mathcal{F}^{-1} f)$.

b) If $f \in \mathrm{E}(\mathbb{R}^d; X)$ then $\mu * f \in \mathrm{E}(\mathbb{R}^d; X)$ and
$$T_m(\mu * f) = \mu * (T_m f). \tag{12.10}$$

c) If $m \in \mathcal{M}^X_p(\mathbb{R}^d)$ then (12.10) holds for all $f \in \mathrm{L}^p(\mathbb{R}^d; X)$.

[Hint for a): Identity (5.11) and Exercise 5.10 plus a density argument.]

12.2. Let $a : \mathbb{R}^d \to \mathbb{C}$ be a homogeneous polynomial of degree $m \in \mathbb{N}$. Prove the following assertions:

a) There is $C \geq 0$ such that $|a(t)| \leq C\, |t|^m$ for all $t \in \mathbb{R}^d$.

b) For each $j = 1, \dots, d$ the function $\partial_j a$ is a homogeneous polynomial of degree $m - 1$.

c) For each $n \in \mathbb{N}_0$ and each multi-index $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = n$ there are polynomials $p_{\alpha,j}$, $j = \lceil n/m \rceil, \dots, n$ such that each $p_{\alpha,j}$ is homogeneous of degree $jm - n \geq 0$ and
$$D^\alpha(f \circ a) = \sum_{j=\lceil n/m \rceil}^{n} (f^{(j)} \circ a)\, p_{\alpha,j} \quad \text{on } a^{-1}(U)$$
wherever $U \subseteq \mathbb{C}$ is open and $f \in \mathrm{C}^n(U)$.

[Hint for c): Use b) and employ induction on $n \in \mathbb{N}_0$.]

12.3. Let $1 < p < \infty$ and A_p be the operator on $\mathrm{L}^p(\mathbb{R})$ associated (by the results of Section 12.2) with the elliptic homogeneous polynomial $a = \mathrm{i}\mathbf{t}$. Show that $-A_p$ is the generator of the right shift semigroup $(\tau_s)_{s\geq 0}$.

12.4. Let $1 < p < \infty$ and let $-\mathrm{i}A$ be the generator of the right shift group $(\tau_s)_{s\in\mathbb{R}}$ on $\mathrm{L}^p(\mathbb{R})$. Let $\omega > 0$ and $f \in \mathrm{Hol}(\mathrm{St}_\omega)$. The following refers to the canonical extension of the Fourier–Stieltjes calculus for A.

a) Show that if $x \in \mathcal{S}(\mathbb{R})$ then $x \in \mathrm{dom}(f(A))$ and $f(A)x \in \mathrm{L}^1(\mathbb{R})$ with

$$\widehat{f(A)x} = f \cdot \widehat{x} \quad \text{on } \mathbb{R}.$$

b) Show that $f(A) \in \mathcal{L}(\mathrm{L}^p(\mathbb{R}))$ if and only if $m := f|_{\mathbb{R}} \in \mathcal{M}_p(\mathbb{R})$, in which case $f(A) = T_m$.

c) Show that $(\tau_s)_{s\in\mathbb{R}}$ is the group of imaginary powers of some sectorial operator B with $\omega_{\mathrm{se}}(B) = 0$.

[Hint: Use the anchor element $e := (\mathrm{i} - \mathbf{z})^{-2}$ and observe that $e(A)$ is an isomorphism on $\mathcal{S}(\mathbb{R})$. For b) observe Remark A.55. For c) use b) and the Mikhlin multiplier theorem to show that e^A is sectorial of angle 0. The rest follows from Remark 10.6.]

Remark: In the case $p \neq 2$ one can show that the operator A in Exercise 12.4 does not have a bounded H^∞-calculus on any horizontal strip. So the operator $B = \mathrm{e}^A$ is sectorial and has BIP, but does not have a bounded H^∞-calculus on any sector.

Supplementary Exercises

12.5. Let $-\mathrm{i}A$ be the generator of the right shift group $(\tau_s)_{s\in\mathbb{R}}$ on $X = \mathrm{L}^1(\mathbb{R})$. Let $\omega > 0$ and $f \in \mathrm{Hol}(\mathrm{St}_\omega)$. The following refers to the canonical extension of the Fourier–Stieltjes calculus for A.

a) Show that $\mathrm{C}^2_{\mathrm{c}}(\mathbb{R}) \subseteq \mathrm{dom}(f(A))$.

b) Show that if $x \in \mathrm{dom}(f(A))$ then

$$\widehat{f(A)x} = f \cdot \widehat{x} \quad \text{on } \mathbb{R}.$$

c) Suppose that $f(A) \in \mathcal{L}(\mathrm{L}^1(\mathbb{R}))$. Show that $f|_{\mathbb{R}} \in \mathrm{FS}(\mathbb{R})$.

[Hint: a) The function $(\mathrm{i} - \mathbf{z})^{-2}$ is an anchor element for f. b) Example 6.2. c) Theorem 6.23.]

12.6. a) **(Wiener's Lemma)** Let $m = \widehat{\mu}$ for some $\mu \in \mathrm{M}(\mathbb{R})$. Show that both Cesàro-limits

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T m(s)\,\mathrm{d}s \qquad \text{and} \qquad \lim_{T\to\infty} \frac{1}{T} \int_{-T}^0 m(s)\,\mathrm{d}s$$

exist and are equal to $\mu\{0\}$. Conclude that if both limits $\lim_{s\to\infty} m(s)$ and $\lim_{s\to-\infty} m(s)$ exist, these limits are the same.

b) Show that the function $\frac{1}{1+e^x}$ on $\mathbb{R}$ is not contained in $\mathrm{FS}(\mathbb{R})$.

c) Prove that the shift group $(\tau_s)_{s\in\mathbb{R}}$ on $\mathrm{L}^1(\mathbb{R})$ is not the group of imaginary powers of a sectorial operator.

[Hint for c): Use Exercise 12.5.c) and part b) of the present exercise to show that e^A is not a sectorial operator. The rest then follows from Remark 10.6.]

References

[1] L. Grafakos. *Classical Fourier analysis*. Second. Vol. 249. Graduate Texts in Mathematics. Springer, New York, 2008, pp. xvi+489.

[2] M. Haase. *The Functional Calculus for Sectorial Operators*. Vol. 169. Operator Theory: Advances and Applications. Basel: Birkhäuser Verlag, 2006, pp. xiv+392.

[3] P. C. Kunstmann and L. Weis. "Maximal L_p-Regularity for Parabolic Equations, Fourier Multiplier Theorems and H^∞-functional Calculus". In: *Functional Analytic Methods for Evolution Equations (Levico Terme 2001)*. Vol. 1855. Lecture Notes in Math. Berlin: Springer, 2004, pp. 65–312.

Chapter 13
The Dore–Venni Theorem

13.1 A Transference Principle

Let $-\mathrm{i}A$ be the generator of a C_0-group $U = (U_s)_{s\in\mathbb{R}}$ on a Banach space X and $1 \le p < \infty$. For each $N > 0$ we shall construct a **partial dilation** of U on the interval $[-N, N]$ to the shift group $(\tau_s)_{s\in\mathbb{R}}$ on $\mathrm{L}^p(\mathbb{R}; X)$. By this we mean a pair (J, P) of bounded operators

$$J : X \to \mathrm{L}^p(\mathbb{R}; X) \quad \text{and} \quad P : \mathrm{L}^p(\mathbb{R}; X) \to X$$

such that the diagram

$$\begin{tikzcd} \mathrm{L}^p(\mathbb{R};X) \arrow[r, "\tau_s"] & \mathrm{L}^p(\mathbb{R};X) \arrow[d, "P"] \\ X \arrow[u, "J"] \arrow[r, "U_s"] & X \end{tikzcd}$$

is commutative for all $s \in [-N, N]$. The idea for this dilation comes from the observation that the left-hand side of the identity

$$U_s x = U_t U_{s-t} x$$

does not depend on $t \in \mathbb{R}$ and hence one has

$$U_s x = \frac{1}{2K}\int_{-K}^{K} U_t U_{s-t} x \,\mathrm{d}t$$

for each $K > 0$ and $s \in \mathbb{R}$. Defining

$$P : \mathrm{L}^p(\mathbb{R}; X) \to X, \qquad Pf := \frac{1}{2K}\int_{-K}^{K} U_t f(t)\,\mathrm{d}t$$

and $Jx := U_{-\mathbf{t}}x$ one formally obtains

$$U_s = P\,\tau_s\,J \qquad (s \in \mathbb{R}).$$

However, for $x \neq 0$ the (reflected) group orbit Jx is not contained in $\mathrm{L}^p(\mathbb{R}; X)$. To remedy this, we restrict the range of t to the interval $[-N, N]$ and define

$$J : X \to \mathrm{L}^p(\mathbb{R}; X), \qquad Jx := \mathbf{1}_{[\,|\mathbf{t}|\le K+N\,]} U_{-\mathbf{t}}x.$$

With these choices, the pair (J, P) does what we have promised.

As a consequence, for each $\mu \in \mathrm{M}[-N, N]$ we obtain the commutative diagram

$$\begin{tikzcd} \mathrm{L}^p(\mathbb{R}; X) \arrow[r, "\tau_\mu"] & \mathrm{L}^p(\mathbb{R}; X) \arrow[d, "P"] \\ X \arrow[u, "J"] \arrow[r, "U_\mu"] & X \end{tikzcd}$$

and hence the estimate

$$\|U_\mu\| \le \|J\|\|P\|\,\|\tau_\mu\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R};X))},$$

which with $f := \widehat{\mu}$ can be equivalently written as

$$\|f(A)\| \le \|J\|\|P\|\,\|f\|_{\mathcal{M}_p^X}.$$

This inequality allows to “transfer” an estimate for the Fourier multiplier norm of f to an estimate for the norm of $f(A)$, hence the name **transference principle**.

Theorem 13.1. *Let $-\mathrm{i}A$ be the generator of a C_0-group $(U_s)_{s\in\mathbb{R}}$ on a Banach space X. Define*

$$w(t) := \sup_{0\le|s|\le t} \|U_s\| \qquad (t \ge 0).$$

Then the following assertions hold:

a) *For each $1 \le p < \infty$ and $N > 0$ there is a constant $C = C(p, N, w)$ such that*

$$\big\|f(A)\big\| \le C\,\|f\|_{\mathcal{M}_p^X}$$

for all $f = \widehat{\mu}$, $\mu \in \mathrm{M}(\mathbb{R})$, $\mathrm{supp}(\mu) \subseteq [-N, N]$.

b) *If U is a bounded group, then with $M := \sup_{s\in\mathbb{R}} \|U_s\|$,*

$$\big\|f(A)\big\| \le M^2\|f\|_{\mathcal{M}_p^X}$$

for all $f = \widehat{\mu}$, $\mu \in \mathrm{M}(\mathbb{R})$ and $1 \le p < \infty$.

Proof. a) For the partial dilation (J, P) constructed above we easily find $\|J\| \le w(K + N) \cdot \big(2(K+N)\big)^{\frac{1}{p}}$ and $\|P\| \le w(K) \cdot (2K)^{\frac{-1}{p}}$. This yields the claim of a) with

$$C(p,N,w) := \inf_{K>0} w(K+N)\, w(K) \left(1 + \frac{N}{K}\right)^{\frac{1}{p}}. \tag{13.1}$$

b) If $\|U_s\| \le M$ for all $s \in \mathbb{R}$ then for the constant from a) we obtain

$$C \le \inf_{K>0} M^2 \left(1 + \frac{N}{K}\right)^{\frac{1}{p}} = M^2.$$

Hence, in this case the dependence on $N > 0$ has vanished and since $\mathrm{M}_c(\mathbb{R})$ is dense in $\mathrm{M}(\mathbb{R})$, the claim of b) follows. □

Remark 13.2. Part b) of Theorem 13.1 is the special case $G = \mathbb{R}$ of a result by Berkson–Gillespie-Muhly [3] for general locally compact amenable groups G. That theorem, in turn, is the vector-valued version of its scalar analogue due to Coifman and Weiss from [6]. Our proof is the original one, suitably adapted.

Lemma 13.3. *Let $1 \le p < \infty$ and let the Banach space $X \neq \{0\}$ be a closed subspace of a space $\mathrm{L}^p(\Omega)$ for some measure space Ω. Then*

$$\|\widehat{\mu}\|_{\mathcal{M}_p} = \|\widehat{\mu}\|_{\mathcal{M}_p^X}.$$

for each $\mu \in \mathrm{M}(\mathbb{R}^d)$.

Proof. The implication $\|\widehat{\mu}\|_{\mathcal{M}_p} \le \|\widehat{\mu}\|_{\mathcal{M}_p^X}$ is Theorem A.56.b) and holds for any Banach space X. For the converse, note that one has an isometric isomorphism

$$\iota : \mathrm{L}^p(\mathbb{R}^d; \mathrm{L}^p(\Omega)) \to \mathrm{L}^p(\Omega; \mathrm{L}^p(\mathbb{R}^d))$$

that commutes with the translation group and hence with the convolution operator τ_μ. By this it is meant that

$$\iota \tau_\mu f = \tau_\mu \circ (\iota f) \qquad \big(f \in \mathrm{L}^p(\mathbb{R}^d; \mathrm{L}^p(\Omega))\big).$$

Since ι is isometric, it easily follows that

$$\|\tau_\mu f\|_{\mathrm{L}^p(\mathbb{R}^d;X)} \le \|\widehat{\mu}\|_{\mathcal{M}_p} \|f\|_{\mathrm{L}^p(\mathbb{R}^d;X)}$$

for each $f \in \mathrm{L}^p(\mathbb{R}^d; X)$, i.e., the claim. □

Remark 13.4. With more effort one can even show $\mathcal{M}_p^X(\mathbb{R}^d) = \mathcal{M}_p(\mathbb{R}^d)$ isometrically if X is a closed subspace of an L^p-space, but we shall not need this in the following.

13.2 The Hilbert Transform and UMD Spaces

One of the simplest non-trivial candidates for an $\mathrm{L}^p(\mathbb{R}; X)$-Fourier multiplier is the function

$$h := -\mathrm{i}\,\mathrm{sgn}\,\mathbf{t}.$$

Since $h' = 0$ on $\mathbb{R} \setminus \{0\}$, h is a Mikhlin function on $\mathbb{R} \setminus \{0\}$, and hence, by the Mikhlin multiplier theorem, $\mathcal{H} := T_h$ is a bounded operator on $\mathrm{L}^p(\mathbb{R})$ for $1 < p < \infty$. It is called the **Hilbert transform**[1].

A Banach space X is called an $\mathcal{HT}_p$ **space** if $h \in \mathcal{M}_p^X(\mathbb{R})$. In this case,

$$C_{\mathcal{HT}_p}(X) := \|h\|_{\mathcal{M}_p^X}$$

is called the $\mathcal{HT}_p$**-constant** of X. Recall that, unless $X = \{0\}$, any L^1-Fourier multiplier is the Fourier transform of a measure and hence continuous. So $X \neq \{0\}$ can be an $\mathcal{HT}_p$ space only if $1 < p < \infty$. Define

$$h_{\varepsilon,T} := \mathcal{F}\Big(\frac{\mathbf{1}_{[\,\varepsilon\leq|\mathbf{s}|\leq T\,]}}{\pi\,\mathbf{s}}\Big) = \frac{2}{\pi \mathrm{i}}\int_\varepsilon^T \frac{\sin(s\mathbf{t})}{s}\,\mathrm{d}s \qquad (0 < \varepsilon \leq T < \infty).$$

The following is an important characterization of the $\mathcal{HT}_p$-property. (We use the convention that a Fourier multiplier norm $\|m\|_{\mathcal{M}_p^X}$ equals $+\infty$ if $m \notin \mathcal{M}_p^X(\mathbb{R})$.)

Theorem 13.5. *There is a constant $C \geq 1$ such that for every Banach space X and every $1 < p < \infty$*

$$\|h\|_{\mathcal{M}_p^X} \leq \sup_{0<\varepsilon\leq 1} \|h_{\varepsilon,1}\|_{\mathcal{M}_p^X} = \sup_{0<\varepsilon\leq T<\infty} \|h_{\varepsilon,T}\|_{\mathcal{M}_p^X} \leq C\,\|h\|_{\mathcal{M}_p^X}.$$

In particular, the following assertions are equivalent:

(i) *X is an $\mathcal{HT}_p$ space;*

(ii) *$\sup_{0<\varepsilon\leq 1} \|h_{\varepsilon,1}\|_{\mathcal{M}_p^X} < \infty$;*

(iii) *$\sup_{0<\varepsilon\leq T<\infty} \|h_{\varepsilon,T}\|_{\mathcal{M}_p^X} < \infty$.*

Proof. Trivially, the inequality $\sup_{0<\varepsilon\leq 1} \|h_{\varepsilon,1}\|_{\mathcal{M}_p^X} \leq \sup_{0<\varepsilon\leq T<\infty} \|h_{\varepsilon,T}\|_{\mathcal{M}_p^X}$ holds. For the converse inequality, note that

$$h_{\varepsilon,T} = \frac{2}{\pi \mathrm{i}}\int_\varepsilon^T \frac{\sin(\mathbf{t}s)}{s}\,\mathrm{d}s = \frac{2}{\pi \mathrm{i}}\int_{\varepsilon/T}^1 \frac{\sin(\mathbf{t}Ts)}{s}\,\mathrm{d}s = h_{\frac{\varepsilon}{T},1}(T\mathbf{t}).$$

Hence, $\|h_{\varepsilon,T}\|_{\mathcal{M}_p^X} = \|h_{\frac{\varepsilon}{T},1}\|_{\mathcal{M}_p^X}$ by Theorem A.56.h) (with A being multiplication by T).

[1] Actually, the boundedness of the Hilbert transform is one of the "first" results in the theory of singular integrals and multipliers, and does not need the full force of the Mikhlin multiplier theorem. See [9, Thm.4.1.7].

Define $m_n := h_{\frac{1}{n},n}$. Then, by the well-known Dirichlet integral (see (A.25) in Appendix A.11)

$$\int_0^\infty \frac{\sin s}{s}\,\mathrm{d}s = \frac{\pi}{2},$$

$m_n \to h$ pointwise on $\mathbb{R} \setminus \{0\}$. Hence $h \in \mathcal{M}_p^X(\mathbb{R})$ and

$$\pi \|h\|_{\mathcal{M}_p^X} \le \sup_n \|m_n\|_{\mathcal{M}_p^X}$$

by Theorem A.56.g).

For the final inequality we write

$$\frac{\pi \mathrm{i}}{2} h_{\varepsilon,T} = \int_\varepsilon^T \frac{\sin(\mathbf{t}s)}{s}\,\mathrm{d}s = \operatorname{sgn}(\mathbf{t}) \int_\varepsilon^T \frac{\sin(|\mathbf{t}|\,s)}{s}\,\mathrm{d}s = \operatorname{sgn}(\mathbf{t})\bigl(f(\varepsilon\mathbf{t}) - f(T\mathbf{t})\bigr)$$

where

$$f(t) := \int_{|t|}^\infty \frac{\sin s}{s}\,\mathrm{d}s \qquad (t \in \mathbb{R}).$$

By Exercise 13.3, $f \in \mathcal{F}(\mathrm{L}^1(\mathbb{R})) \subseteq \mathrm{FS}(\mathbb{R})$. It follows that the functions $f(\varepsilon\mathbf{t}) - f(T\mathbf{t})$, $0 < \varepsilon \le T < \infty$, are uniformly bounded in $\mathcal{M}_p^X(\mathbb{R})$, by $2\|f\|_{\mathrm{FS}}$. Since $\mathcal{M}_p^X(\mathbb{R})$ is a Banach algebra (Theorem 12.1, Theorem A.56.f)), we obtain

$$\sup_{0<\varepsilon\le T<\infty} \|h_{\varepsilon,T}\|_{\mathcal{M}_p^X} \le \bigl({}^4\!/\!_\pi \|f\|_{\mathrm{FS}}\bigr)\, \|h\|_{\mathcal{M}_p^X}$$

as claimed. □

Corollary 13.6. *Let* $1 < p < \infty$.

a) *Each Hilbert space is an* $\mathcal{HT}_2$ *space.*

b) *Each closed subspace of an* $\mathcal{HT}_p$ *space is an* $\mathcal{HT}_p$ *space.*

c) $\mathrm{L}^p(\Omega)$ *is an* $\mathcal{HT}_p$ *space for any measure space* Ω. *More generally, if* X *is an* $\mathcal{HT}_p$ *space, then so is* $\mathrm{L}^p(\Omega; X)$.

Proof. The first part of c) follows from Lemma 13.3. For the second part one needs a vector-valued version of Lemma 13.3, but we skip the proof. □

Remark 13.7. It follows from a result of Benedek, Calderón and Panzone from [2] that if a Banach space X is an $\mathcal{HT}_p$ space for one $p \in (1, \infty)$ then it is an $\mathcal{HT}_p$ space for *all* $p \in (1, \infty)$. Hence, one can drop the reference to p and call them simply $\mathcal{HT}$ **spaces**.

By results of Burkholder [5] and Bourgain [4] the class of $\mathcal{HT}$ spaces coincides with the class of the so-called **UMD spaces**. These are defined via the requirement that certain vector-valued **m**artingale **d**ifferences are **u**nconditional in L^p. We do not need this description and hence refer to [12] for a thorough treatment. However, we shall adopt the name "UMD spaces" in the following.

One can show that UMD spaces are reflexive, see [12, Thm.4.3.3].

13.3 Singular Integrals for Groups and Monniaux's Theorem

We now combine the transference principle and Theorem 13.5.

Theorem 13.8. *Let $-\mathrm{i}A$ be the generator of a C_0-group $U = (U_s)_{s\in\mathbb{R}}$ on a UMD space X. Then the following assertions hold:*

a) *The principle value integral*

$$\mathcal{H}_1^U x := \frac{1}{\pi}\text{p.v.}\int_{-1}^{1} \frac{U_s x}{s}\,\mathrm{d}s := \lim_{\varepsilon\searrow 0}\frac{1}{\pi}\int_{\varepsilon\le|s|\le 1}\frac{U_s x}{s}\,\mathrm{d}s \tag{13.2}$$

converges for all $x \in X$.

b) *If U is bounded and A is injective, the principle value integral*

$$\mathcal{H}^U x := \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty} \frac{U_s x}{s}\,\mathrm{d}s := \lim_{\varepsilon\downarrow 0, T\uparrow\infty}\frac{1}{\pi}\int_{\varepsilon\le|s|\le T}\frac{U_s x}{s}\,\mathrm{d}s \tag{13.3}$$

converges for all $x \in X$.

Proof. a) Note that, for any $x \in X$ and $0 < \varepsilon \le 1$,

$$\int_{\varepsilon\le|s|\le 1}\frac{U_s x}{s}\,\mathrm{d}s = \int_{\varepsilon\le|s|\le 1}\frac{U_s x - x}{s}\,\mathrm{d}s.$$

This shows that the principle value integral (13.2) exists for all $x \in \mathrm{dom}(A)$. Since $\mathrm{dom}(A)$ is dense, it suffices to show that the family of operators

$$S_\varepsilon := \int_{\varepsilon\le|s|\le 1}\frac{U_s}{s}\,\mathrm{d}s \qquad (0 < \varepsilon \le 1)$$

is uniformly bounded. Since X is an $\mathcal{HT}_p$ space for any $1 < p < \infty$, Theorem 13.5 together with the transference principle (Theorem 13.1) yields the claim.

b) As before, the principal value integral (13.2) converges as $\varepsilon \searrow 0$ for $x \in \mathrm{dom}(A)$. If $x = (-\mathrm{i}A)y \in \mathrm{ran}(A)$, however, integration by parts yields

$$\begin{aligned}\int_{1\le|s|\le T}\frac{U_s x}{s}\,\mathrm{d}s &= \int_{1\le|s|\le T}\frac{U_s(-\mathrm{i}Ay)}{s}\,\mathrm{d}s\\ &= \frac{U_T y + U_{-T} y}{T} - (U_1 y + U_{-1} y) + \int_{1\le|s|\le T}\frac{U_s y}{s^2}\,\mathrm{d}s\end{aligned}$$

and this converges as $T \nearrow \infty$. Hence, the principal value integral (13.3) converges for $x \in \mathrm{dom}(A) \cap \mathrm{ran}(A)$.

By Remark 13.7, X is reflexive. Since $-\mathrm{i}A$ is a densely defined injective sectorial operator on X, $\mathrm{dom}(A) \cap \mathrm{ran}(A)$ is dense in X. Hence, as before it suffices to show that the family of operators

$$S_{\varepsilon,T} := \int_{\varepsilon\le|s|\le T} \frac{U_s}{s}\,\mathrm{d}s \qquad (0<\varepsilon\le T<\infty)$$

is uniformly bounded. Again, Theorem 13.5 combined with the transference principle (Theorem 13.1) yields the claim. □

Remark 13.9. The proof of Theorem 13.8 actually yields more than what is stated in the theorem. Indeed, a re-examination of the proof and employing (13.1) yields

$$\|\mathcal{H}_1^U\|_{\mathcal{L}(X)} \le C\,2^{\frac{1}{p}} \big(\sup_{|s|\le 2}\|U_s\|\big)^2\, C_{\mathcal{HT}_p}(X)$$

for the (universal) constant C from Theorem 13.5. A similar remark is valid for the norm of $\mathcal{H}^U$ in case of a bounded group.

Monniaux's Theorem

Theorem 13.8 can be reformulated in functional calculus terms. Namely, for scalars $z\in\mathbb{C}$ we have

$$h_{0,1}(z) := \frac{1}{\pi}\mathrm{p.v.}\!\int_{-1}^{1} \frac{\mathrm{e}^{-\mathrm{i}sz}}{s}\,\mathrm{d}s = \frac{2}{\pi\mathrm{i}}\int_0^1 \frac{\sin zs}{s}\,\mathrm{d}s.$$

By Exercise 13.4, $h_{0,1}\in\mathrm{H}^\infty(\mathrm{St}_\omega)$ for each $\omega>0$ satisfying

$$\lim_{|\operatorname{Im} z|\le\omega,\operatorname{Re} z\to\pm\infty} h_{0,1}(z) = \mp\mathrm{i}.$$

Moreover,

$$h_{0,1}(z) = \frac{2}{\pi\mathrm{i}}\int_0^z \frac{\sin s}{s}\,\mathrm{d}s,$$

where the right hand side is to be understood as a complex line integral over the straight line segment from 0 to z. In other words, $h_{0,1}$ is the unique primitive of the function ${}^{2\sin z}/_{\pi\mathrm{i}z}$ which vanishes at $z=0$.

So, part a) of Theorem 13.8 essentially says that $h_{0,1}(A)\in\mathcal{L}(X)$ whenever $-\mathrm{i}A$ generates a C_0-group on a UMD space.

Theorem 13.10 (Monniaux [14]). *Let $-\mathrm{i}A$ be the generator of a C_0-group $(U_s)_{s\in\mathbb{R}}$ on a UMD space X with $\theta(U)<\pi$. Then e^A is a sectorial operator. In particular, A is the logarithm of a sectorial operator.*

Proof. By Remark 10.6, the second assertion follows from the first. The first, in turn, amounts to proving that

$$\Big(\frac{t}{t+\mathrm{e}^{\mathbf{z}}}\Big)(A), \qquad t>0$$

is a bounded family of bounded operators. To achieve this, we make use of the representation

$$\frac{t}{t+\mathrm{e}^z} = \frac{1}{2} + \frac{1}{2\mathrm{i}}\,\mathrm{p.v.}\!\int_{\mathbb{R}} t^{\mathrm{i}s}\mathrm{e}^{-\mathrm{i}sz}\,\frac{\mathrm{d}s}{\sinh(\pi s)} \tag{13.4}$$

which holds for all $z \in \mathrm{St}_\pi$ and $t > 0$. (Actually, by replacing z by $z + \log t$ we see that it suffices to establish the formula for $t = 1$. A proof is left as Exercise 13.9.)

Observe that the integral (13.4) is singular only at 0 since $|\mathrm{Im}\, z| < \pi$. Taking (13.4) for granted, we can write

$$\begin{aligned}(2\mathrm{i})\frac{t}{t+\mathrm{e}^z} &= \mathrm{i} + \int_{|s|\ge 1} t^{\mathrm{i}s}\mathrm{e}^{-\mathrm{i}sz}\,\frac{\mathrm{d}s}{\sinh(\pi s)} + \int_{-1}^{1} t^{\mathrm{i}s}\mathrm{e}^{-\mathrm{i}sz}\Big(\frac{1}{\sinh(\pi s)} - \frac{1}{\pi s}\Big)\,\mathrm{d}s \\ &\quad + \frac{1}{\pi}\,\mathrm{p.v.}\!\int_{-1}^{1} \frac{t^{\mathrm{i}s}\mathrm{e}^{-\mathrm{i}sz}}{s}\,\mathrm{d}s \\ &= \big(\widehat{\mu} + h_{0,1}\big)(z - \log t)\end{aligned}$$

for some $\mu \in \mathrm{M}_\omega(\mathbb{R})$. By Theorem 13.8, inserting A we find that $t(t+\mathrm{e}^A)^{-1}$ is a bounded operator. Since for $t > 0$ all the groups $t^{\mathbf{i}\mathbf{s}}U_{\mathbf{s}}$ have the same growth behaviour, it follows from Remark 13.9 that $\sup_{t>0} \|t(t+\mathrm{e}^A)^{-1}\| < \infty$. □

13.4 The Maximal Regularity Problem

Let A be a densely defined sectorial operator of angle $\omega_{\mathrm{se}}(A) < \pi/2$ on a Banach space X. (For example, A could be a strongly elliptic operator on $X = \mathrm{L}^p(\mathbb{R}^d)$ as treated in Chapter 12.) As such, $-A$ generates a bounded holomorphic C_0-semigroup $T = (T_t)_{t\ge 0}$ on X (Chapter 9).

Given an initial value $x \in X$, the trajectory $u(t) := T_t x$ is a so-called "mild" solution to the homogeneous Cauchy problem

$$\begin{cases} u'(t) + Au(t) = 0 \quad (t > 0), \\ u(0) = x. \end{cases}$$

For example, if $A = -\Delta$ on $\mathrm{L}^p(\mathbb{R}^d)$ then $(T_t)_{t\ge 0}$ is the heat semigroup and u solves the homogeneous parabolic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}u = \Delta u.$$

Using the semigroup T one can also construct "mild" solutions to the (finite time) *inhomogeneous* Cauchy problem

$$\begin{cases} u'(t) + Au(t) = f(t) \quad (0 < t < 1), \\ u(0) = x, \end{cases} \tag{13.5}$$

where $f \in \mathrm{L}^1((0,1);X)$, namely

$$u(t) = T_t x + \int_0^t T_{t-s} f(s)\,\mathrm{d}s \qquad (0 \le t \le 1).$$

In the following we shall restrict ourselves to the case $x = 0$, which amounts to

$$u(t) = (Sf)(t) := \int_0^t T_{t-s} f(s)\,\mathrm{d}s \qquad (0 \le t \le 1). \tag{13.6}$$

(See Exercise 13.5 for a proof that u is a "mild" solution of (13.5) with $x = 0$.)

Given $p \in (1,\infty)$ one says that A has **maximal** L^p**-regularity** if the solution $u = Sf$ given by (13.6) satisfies

$$f \in \mathrm{L}^p((0,1);X) \quad \Rightarrow \quad u',\ Au \in \mathrm{L}^p((0,1);X). \tag{13.7}$$

Here, $Au \in \mathrm{L}^p((0,1);X)$ means that $u \in \mathrm{L}^p((0,1);\mathrm{dom}(A))$ and $u' \in \mathrm{L}^p((0,1);X)$ means that there is $v \in \mathrm{L}^p((0,1);X)$ with

$$u(t) = \int_0^t v(s)\,\mathrm{d}s \qquad (0 \le t \le 1).$$

The terminology stems from the fact that both summands on the left-hand side of the equation

$$u' + Au = f$$

should have the maximal "amount of regularity" that one can reasonably expect, given that the right-hand side f is in L^p.

The **maximal regularity problem** consists in deciding whether a given operator has maximal L^p-regularity or not. One can show that if A has maximal L^p-regularity for one $1 < p < \infty$, then this is true for *all* such p. As a result, one often drops the reference to p and only speaks of "maximal regularity".

Remark 13.11. One may wonder why we have confined ourselves to generators of holomorphic semigroups. Indeed, all the notions and definitions so far are meaningful for general C_0-semigroups. However, Dore has shown in [7] that the holomorphy of the semigroup is necessary for maximal L^p-regularity.

Operator-Theoretic Reformulation

In the following we briefly sketch how the maximal regularity problem can be reformulated in purely operator-theoretic terms. To this aim we fix $p \in (1,\infty)$ and pass to the new Banach space

$$\mathcal{X}_p := \mathrm{L}^p((0,1);X).$$

On $\mathcal{X}_p$ we consider the operator $\mathcal{A}$ given by

$$\operatorname{dom}(\mathcal{A}) := \mathrm{L}^p((0,1);\operatorname{dom}(A)), \qquad (\mathcal{A}u)(t) := A(u(t)) \qquad (t \in (0,1)).$$

It is easy to see that $\mathcal{A}$ inherits from A many of its properties (Exercise 13.6). For example, $\mathcal{A}$ is a densely defined sectorial operator of angle $\omega_{\mathrm{se}}(\mathcal{A}) = \omega_{\mathrm{se}}(A)$.

We also consider the operator $\mathcal{B} := V^{-1}$, where V is the Volterra operator on $\mathcal{X}$ defined by

$$(Vu)(t) := \int_0^t u(s)\,\mathrm{d}s \qquad (0 \le t \le 1).$$

As in the scalar case (Exercise 6.6) one can prove that $-\mathcal{B}$ is the generator of the right shift semigroup $(\tau_s)_{s\ge 0}$. As such, $\mathcal{B}$ is a densely defined and invertible sectorial operator of angle $\pi/2$. Its domain is

$$\operatorname{dom}(\mathcal{B}) = \operatorname{ran}(V) = \{u \in \mathrm{W}^{1,\mathrm{p}}((0,1);X) \mid u(0) = 0\},$$

but we shall not need the second identity.

Using the operators $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{X}$, one can reformulate the maximal regularity property of A as follows.

Lemma 13.12. *Let A be a sectorial operator of angle $\omega_{\mathrm{se}}(A) < \pi/2$ on a Banach space X, and let the operators $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{X}_p = \mathrm{L}^p((0,1);X)$ be defined as above. Then the following assertions are equivalent:*

(i) *A has maximal L^p-regularity;*

(ii) *There is a constant $K \ge 0$ such that*

$$\|\mathcal{A}u\|_{\mathcal{X}_p} + \|\mathcal{B}u\|_{\mathcal{X}_p} \le K\|\mathcal{A}u + \mathcal{B}u\|_{\mathcal{X}_p}$$

for all $u \in \operatorname{dom}(\mathcal{A}) \cap \operatorname{dom}(\mathcal{B})$;

(iii) *The operator $\mathcal{A} + \mathcal{B}$ defined on $\operatorname{dom}(\mathcal{A}) \cap \operatorname{dom}(\mathcal{B})$ is closed.*

Proof. By definition of $\mathcal{A}$ and $\mathcal{B}$, (i) is equivalent to the assertion: *For each $f \in \mathcal{X}_p$ one has $Sf \in \operatorname{dom}(\mathcal{A}) \cap \operatorname{dom}(\mathcal{B})$.* Recall that, since $\mathcal{A}$ and $\mathcal{B}$ are closed operators, $\operatorname{dom}(\mathcal{A}) \cap \operatorname{dom}(\mathcal{B})$ is a Banach space with respect to the norm

$$|\!|\!|\, u \,|\!|\!| := \|u\|_{\mathcal{X}_p} + \|\mathcal{A}u\|_{\mathcal{X}_p} + \|\mathcal{B}u\|_{\mathcal{X}_p}.$$

(ii) $\Rightarrow$ (i): It follows from (ii) that

$$|\!|\!|\, u \,|\!|\!| \le \|u\|_{\mathcal{X}_p} + K\|\mathcal{A}u + \mathcal{B}u\|_{\mathcal{X}_p} \le (K+1)(\|\mathcal{A}u + \mathcal{B}u\|_{\mathcal{X}_p} + \|u\|_{\mathcal{X}_p})$$

for all $u \in \operatorname{dom}(\mathcal{A}) \cap \operatorname{dom}(\mathcal{B})$. Now, given $f \in \mathrm{C}([0,1];\operatorname{dom}(A))$, let $u := Sf$ be defined by (13.6). Then $u \in \mathrm{C}^1([0,1];X) \cap \mathrm{C}([0,1];\operatorname{dom}(A))$ and $Au + u' = f$

(Exercise 13.5). Since $u(0) = 0$, we have even $u \in \mathrm{dom}(\mathcal{A}) \cap \mathrm{dom}(\mathcal{B})$, and therefore we can write $f = \mathcal{A}u + \mathcal{B}u$. That implies

$$||| \, Sf \, ||| \le K'(\|f\|_{\mathcal{X}_p} + \|Sf\|_{\mathcal{X}_p}) \lesssim \|f\|_{\mathcal{X}_p}.$$

Since the space $\mathrm{C}([0,1]; \mathrm{dom}(A))$ is dense in $\mathcal{X}_p$, we conclude that S maps $\mathcal{X}_p$ into $\mathrm{dom}(\mathcal{A}) \cap \mathrm{dom}(\mathcal{B})$, which is equivalent to (i).

(i) $\Rightarrow$ (ii): (We only give a sketch here.) Suppose that (i) holds, i.e., S maps $\mathcal{X}_p$ into $\mathrm{dom}(\mathcal{A}) \cap \mathrm{dom}(\mathcal{B})$. By an application of the closed graph theorem, there is a constant $K \ge 0$ such that

$$||| \, Sf \, ||| \le K\|f\|_{\mathcal{X}_p}$$

for all $f \in \mathcal{X}_p$. Now let $u \in \mathrm{dom}(\mathcal{A}) \cap \mathrm{dom}(\mathcal{B})$ and define $f := \mathcal{A}u + \mathcal{B}u$. By the uniqueness of the mild solutions of the inhomogeneous Cauchy problem (13.5) (see [10, Sec.9.3.1] or [1, Prop.3.1.16]), $u = Sf$, and hence

$$\|\mathcal{A}u\|_{\mathcal{X}_p} + \|\mathcal{B}u\|_{\mathcal{X}_p} \le K\|f\|_{\mathcal{X}_p} = K\|\mathcal{A}u + \mathcal{B}u\|_{\mathcal{X}_p}$$

as claimed.

The proof of the equivalence (i),(ii) $\Longleftrightarrow$ (iii) is Exercise 13.7. □

Lemma 13.12 is the key equivalence for many classical results on maximal regularity. In particular, it is fundamental for the following result from [8].

Theorem 13.13 (Dore–Venni). *Let A be an injective sectorial operator on a UMD Banach space X. Suppose that A has BIP with $\theta_A < \pi/2$. Then A has maximal L^p-regularity for all $p \in (1, \infty)$.*

Here, we denote by θ_A the group type of the group $(A^{\mathrm{i}s})_{s \in \mathbb{R}}$, i.e.,

$$\theta_A := \inf\{\omega \ge 0 \mid \sup_{s \in \mathbb{R}} \mathrm{e}^{-\omega|s|}\|A^{\mathrm{i}s}\| < \infty\}.$$

Recall from Remark 13.7 that each UMD space is reflexive, and hence A is densely defined and has dense range by Theorem 9.2.

Remark 13.14. The Dore–Venni theorem was, at the time, a landmark result in the study of maximal regularity. It was superseded, in a certain sense, by results of Kalton and Weis from [13] and by Weis' characterization of the maximal regularity property from [15]. However, these papers use even more involved notions and techniques, whereas the Dore–Venni theorem is now in our reach. This is the reason why we present it here.

13.5 The Dore–Venni Theorem

The proof of Theorem 13.13 aims at verifying condition (ii) from Lemma 13.12. It rests on a more abstract result (Theorem 13.15 below) and a deep result from harmonic analysis on UMD spaces. Let us start with collecting some important properties of the involved operators $\mathcal{A}$ and $\mathcal{B}$ and the space $\mathcal{X}_p$.

First of all, the operators $\mathcal{A}$ and $\mathcal{B}$ are **resolvent commuting**, by which it is meant that

$$R(\lambda, \mathcal{A})R(\mu, \mathcal{B}) = R(\mu, \mathcal{B})R(\lambda, \mathcal{A})$$

for one/all $(\lambda, \mu) \in \rho(\mathcal{A}) \times \rho(\mathcal{B})$, cf. Corollary A.19. Next, by Exercise 13.6, $\mathcal{A}$ has pretty much the same functional calculus properties as A does. In particular, $\mathcal{A}$ has BIP and $\theta_{\mathcal{A}} < \pi/2$.

On the other hand, $\mathcal{B}$ is an invertible and densely defined sectorial operator of angle $\pi/2$. Its functional calculus properties depend on how good the space X is. Now, it turns out that if X is a UMD space, then $\mathcal{B}$ has BIP and $\theta_{\mathcal{B}} = \pi/2$. This is actually a deep result in vector-valued harmonic analysis which we cannot prove here in detail. (We have tried to give more insight into the matter in the supplementary Section 13.6 below. See in particular Corollary 13.19.)

Finally, by Corollary 13.6 and Remark 13.7, $\mathcal{X}_p$ is a UMD space since X is one. So, we have shown that $\mathcal{A}$ and $\mathcal{B}$ satisfy all the hypotheses of the following "abstract Dore–Venni theorem", which implies (ii) of Lemma 13.12. This concludes the proof of Theorem 13.13.

Theorem 13.15 (Dore–Venni). *Let A and B be two resolvent commuting sectorial operators with dense domain and range on a UMD Banach space X. Suppose that both A and B have BIP with $\theta_A + \theta_B < \pi$. Then there is a constant $K \geq 0$ such that*

$$\|Ax\| + \|Bx\| \leq K\|Ax + Bx\| \qquad (x \in \mathrm{dom}(A) \cap \mathrm{dom}(B)). \tag{13.8}$$

Proof. Let us abbreviate $A_l := \log A$ and $B_l := -\log B$. Then $-\mathrm{i}A_l$ and $\mathrm{i}B_l$ generate the C_0-groups $(A^{-\mathrm{i}s})_{s\in\mathbb{R}}$ and $(B^{\mathrm{i}s})_{s\in\mathbb{R}}$, respectively. Since A and B are resolvent commuting, one has $f(A)g(B) = g(B)f(A)$ for all elementary functions f, g such that $f(A)$ and $g(B)$ are defined. Consequently, the groups of imaginary powers of A and B commute. We thus obtain a new C_0-group U by defining

$$U_s := A^{-\mathrm{i}s}B^{\mathrm{i}s} \qquad (s \in \mathbb{R}).$$

Its generator is denoted by $-\mathrm{i}C_l$.

Clearly, $\theta(U) \leq \theta_A + \theta_B < \pi$, hence by Monniaux's theorem (Theorem 13.10) the operator $C := \mathrm{e}^{C_l}$ is sectorial. In particular, $1 + C$ is invertible, which is why there is $K' \geq 0$ such that

$$\|x\| \le K'\|(1+C)x\| \qquad (x \in \operatorname{dom}(C)). \tag{13.9}$$

Suppose that we can establish the inclusion

$$AB^{-1} \subseteq C. \tag{13.10}$$

Then, given $x \in \operatorname{dom}(A) \cap \operatorname{dom}(B)$ one has $Bx \in \operatorname{dom}(AB^{-1})$, and hence $Bx \in \operatorname{dom}(C)$ with $CBx = Ax$; so (13.9) yields

$$\|Bx\| \le K'\|(1+C)Bx\| = K'\|Ax + Bx\|.$$

It follows that

$$\|Ax\| + \|Bx\| \le \|Ax + Bx\| + 2\|Bx\| \le (1 + 2K')\|Ax + Bx\|,$$

and this is (13.8) with $K = 2K' + 1$.

Therefore, it remains to prove (13.10) . To this end, consider the unbounded 2-parameter group

$$V(s) := A^{-\mathrm{i}s_1} B^{\mathrm{i}s_2} \qquad (s = (s_1, s_2) \in \mathbb{R}^2)$$

and the associated Fourier–Stieltjes calculus given by

$$\Psi_V(f) := \int_{\mathbb{R}^2} V(s)\,\mu(\mathrm{d}s),$$

where $f = \widehat{\mu}$ and $\mu \in \mathrm{M}(\mathbb{R}^2)$ is such that

$$\int_{\mathbb{R}^2} \|V(s)\|\ |\mu|\,(\mathrm{d}s) < \infty.$$

Note that $f = f(\mathbf{z}_1, \mathbf{z}_2)$ is a function of two variables. The calculi for A_l and B_l are incorporated into this calculus via the formulae

$$f(A_l) = \Psi_V(f(\mathbf{z}_1)) = \Psi_V(f \otimes \mathbf{1}) \quad \text{and} \quad g(B_l) = \Psi_V(g(\mathbf{z}_2) = \Psi_V(\mathbf{1} \otimes g).$$

But the calculus for C_l is incorporated as well, namely via

$$f(C_l) = \Psi_V(f(\mathbf{z}_1 + \mathbf{z}_2))$$

(Exercise 13.8). It follows that

$$AB^{-1} = \mathrm{e}^{A_l}\mathrm{e}^{B_l} = \Psi_V(\mathrm{e}^{\mathbf{z}_1})\Psi_V(\mathrm{e}^{\mathbf{z}_2}) \subseteq \Psi_V(\mathrm{e}^{\mathbf{z}_1}\mathrm{e}^{\mathbf{z}_2}) = \Psi_V(\mathrm{e}^{\mathbf{z}_1+\mathbf{z}_2}) = \mathrm{e}^{C_l} = C$$

by general functional calculus rules. This concludes the proof. $\square$

13.6 Supplement: The Derivative as a Sectorial Operator

Fix a Banach space X and $1 < p < \infty$ and let $-A$ be the generator of the right shift group on $\mathrm{L}^p(\mathbb{R}; X)$. We alternatively write $A = \frac{\mathrm{d}}{\mathrm{d}s}$ because—as in Exercise 6.5 for scalar functions—one can show that A is the closure in $\mathrm{L}^p(\mathbb{R}; X)$ of the operator $\frac{\mathrm{d}}{\mathrm{d}s}$ defined on $\mathrm{C}_{\mathrm{c}}^\infty(\mathbb{R}; X)$. As the right shift group is bounded, A is sectorial of angle $\pi/2$. By Exercise 13.10, A has dense range.

Lemma 13.16. *Let $1 < p < \infty$ and $A = \frac{\mathrm{d}}{\mathrm{d}s}$ on $\mathrm{L}^p(\mathbb{R}; X)$, X some Banach space. Then for $\omega \in (\pi/2, \pi)$ and $f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$ the following statements are equivalent:*

(i) $f(A) \in \mathcal{L}(\mathrm{L}^p(\mathbb{R}; X))$;

(ii) $f(\mathbf{i}\mathbf{t}) \in \mathcal{M}_p^X(\mathbb{R})$.

In this case,

$$f(A) = T_{f(\mathbf{i}\mathbf{t})} \tag{13.11}$$

is the Fourier multiplier with symbol $f(\mathbf{i}\mathbf{t})$.

Proof. The identity (13.11) is clear if $f = \mathcal{L}\mu$ for some $\mu \in \mathrm{M}(\mathbb{R}_+)$, by (6.7) and the definition of the Hille–Phillips calculus. Suppose that $f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$ such that $f(\mathbf{i}\mathbf{t}) \in \mathcal{M}_p^X(\mathbb{R})$ and let $e = \mathbf{z}(1+\mathbf{z})^{-2}$ be the usual anchor element. Then $e, ef \in \mathcal{E}(\mathrm{S}_\omega)$. By Remark 9.8 and what we have just seen,

$$(ef)(A) = T_{(ef)(\mathbf{i}\mathbf{t})} = T_{e(\mathbf{i}\mathbf{t})} T_{f(\mathbf{i}\mathbf{t})} = e(A) T_{f(\mathbf{i}\mathbf{t})}.$$

(We have used Theorem 12.1.) It follows that $f(A) = T_{f(\mathbf{i}\mathbf{t})}$ and hence that $f(A)$ is a bounded operator.

The proof of the remaining implication is skipped since we shall not use it in the following. $\square$

We shall need the following one-dimensional version of the Mikhlin multiplier theorem for X-valued functions, due to Zimmermann [16].

Theorem 13.17 (Mikhlin, vector-valued). *Let X be a UMD space. Then for each $p \in (1, \infty)$ the space $\mathrm{Mi}(\mathbb{R} \setminus \{0\})$ of Mikhlin functions embeds continuously into $\mathcal{M}_p^X(\mathbb{R})$. In other words: for each $1 < p < \infty$ there is a constant C_p such that each Mikhlin function $m \in \mathrm{Mi}(\mathbb{R}^d \setminus \{0\})$ is an $\mathrm{L}^p(X)$-multiplier with*

$$\|m\|_{\mathcal{M}_p^X} \le C_p \|m\|_{\mathrm{Mi}}.$$

As in the scalar case, we take this theorem for granted. Its proof, or rather a proof of a much more general result, can be found in [12, Section 5.3.c].

Theorem 13.18. *Let, as before, $1 < p < \infty$ and $A = \frac{\mathrm{d}}{\mathrm{d}s}$ on $\mathrm{L}^p(\mathbb{R}; X)$, X some Banach space. Then the following assertions are equivalent:*

(i) *X is a UMD space;*

(ii) *A has BIP;*

(iii) *A has a bounded $\mathrm{H}^\infty(\mathrm{S}_\omega)$-calculus for one/all $\omega \in (\pi/2, \pi)$.*

In this case, there is a constant C such that

$$\|A^{\mathrm{i}s}\| \le C(1+|s|)\mathrm{e}^{\frac{\pi}{2}|s|} \qquad (s \in \mathbb{R}). \tag{13.12}$$

Proof. (i) ⇒ (iii): Let $\omega \in (\pi/2, \pi)$ and $f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$. Then by Lemma 12.5, $f(\mathrm{i}\mathbf{t})$ is a Mikhlin function on $\mathbb{R} \setminus \{0\}$, and hence by Theorem 13.17, $f(\mathrm{i}\mathbf{t}) \in \mathcal{M}_p^X(\mathbb{R})$. By Lemma 13.16, $f(A) \in \mathcal{L}(X)$. Moreover, by going through the arguments again we see that

$$\|f(A)\| = \|f(\mathrm{i}\mathbf{t})\|_{\mathcal{M}_p^X} \lesssim \|f(\mathrm{i}\mathbf{t})\|_{\mathrm{Mi}} \lesssim \|f\|_{\infty,\mathrm{S}_\omega}.$$

Fix $s \in \mathbb{R}$ and specialize $f = \mathbf{z}^{\mathrm{i}s}$. Then $f(\mathrm{i}\mathbf{t}) = \mathrm{e}^{\mathrm{i}s\log(\mathrm{i}\mathbf{t})}$. A simple estimation yields $\|f(\mathrm{i}\mathbf{t})\|_\infty = \mathrm{e}^{\frac{\pi}{2}|s|}$ and

$$\|\mathbf{t}f'(\mathrm{i}\mathbf{t})\|_\infty = |s|\,\mathrm{e}^{\frac{\pi}{2}|s|}.$$

This establishes the inequality (13.12).

(iii) ⇒ (ii): This follows from Theorem 10.7 since A has dense domain and range.

(ii) ⇒ (i): By (the unproven implication in) Lemma 13.16, the hypothesis (ii) implies that $(\mathrm{i}\mathbf{t})^{\mathrm{i}s} \in \mathcal{M}_p^X(\mathbb{R})$ for all $s \in \mathbb{R}$. Since $\mathcal{M}_p^X(\mathbb{R})$ is reflection invariant and an algebra (Theorem A.56, parts h) and f)), also the function

$$\mathrm{e}^{\pi \operatorname{sgn}\mathbf{t}} = (-\mathrm{i}\mathbf{t})^{-\mathrm{i}s}(\mathrm{i}\mathbf{t})^{\mathrm{i}s}$$

is in $\mathcal{M}_p^X(\mathbb{R})$. But then also

$$-\mathrm{i}\operatorname{sgn}\mathbf{t} = -\mathrm{i}\frac{\mathrm{e}^{\pi\operatorname{sgn}\mathbf{t}} - \mathrm{e}^{-\pi\operatorname{sgn}\mathbf{t}}}{\mathrm{e}^{\pi} - \mathrm{e}^{-\pi}}$$

is in $\mathcal{M}_p^X(\mathbb{R})$, which means that X is an $\mathcal{HT}_p$ space, viz. a UMD space. □

The results for the derivative on the line can be transferred to an interval.

Corollary 13.19. *Let $1 < p < \infty$ and $A_1 = \frac{\mathrm{d}}{\mathrm{d}s}$ on $\mathrm{L}^p((0,1);X)$, where X is a UMD space. Then A_1 has a bounded $\mathrm{H}^\infty(\mathrm{S}_\omega)$-calculus for each $\omega \in (\pi/2, \pi)$. Moreover, there is a constant $C \ge 0$ such that*

$$\|A_1^{\mathrm{i}s}\| \le C(1+|s|)\mathrm{e}^{\frac{\pi}{2}|s|} \qquad (s \in \mathbb{R}). \tag{13.13}$$

Proof. Let $A = \frac{\mathrm{d}}{\mathrm{d}s}$ on $\mathrm{L}^p(\mathbb{R};X)$. Fix $\omega \in (\pi/2, \pi)$ and $f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$. Then, by Theorem 13.18, $f(A)$ is a bounded operator.

We claim that $Y := \mathrm{L}^p(\mathbb{R}_+; X)$, considered as a closed subspace of $\mathrm{L}^p(\mathbb{R}; X)$, is invariant under $f(A)$. This is clear for $f = \mathcal{L}\mu$ for some $\mu \in \mathrm{M}(\mathbb{R}_+)$ since Y is invariant under τ_s for each $s \geq 0$. In particular (Remark 9.8), Y is invariant under $e(A)$ for each $e \in \mathcal{E}(\mathrm{S}_\omega)$. But as in the proof of Theorem 11.1 one can find a sequence $(e_n)_n$ in $\mathcal{E}(\mathrm{S}_\omega)$ which approximates f in such a way that $e_n(A) \to f(A)$ strongly. Hence, Y is invariant under $f(A)$ as claimed.

Now we let $P : \mathrm{L}^p(\mathbb{R}; X) \to \mathrm{L}^p((0,1); X)$ be the restriction operator, defined by $Px := x|_{(0,1)}$ and $J : \mathrm{L}^p((0,1); X) \to \mathrm{L}^p(\mathbb{R}; X)$ the operator defined by

$$Jx = \begin{cases} x & \text{on } (0,1), \\ 0 & \text{on } \mathbb{R} \setminus (0,1). \end{cases}$$

Then $\tau_s P = P\tau_s$ on Y. It follows that $e(A_1)P = Pe(A)$ on Y for each $e \in \mathrm{LS}(\mathbb{C}_+)$, in particular for each $e \in \mathcal{E}(\mathrm{S}_\omega)$. Taking into account that Y is invariant under $f(A)$ we obtain

$$(ef)(A_1)P = P(ef)(A) = Pe(A)f(A) = e(A_1)Pf(A) \qquad \text{on } Y.$$

Multiplying from the right with J then yields

$$(ef)(A_1) = e(A_1)Pf(A)J.$$

But e can be any anchor element for f, so it follows that $f(A_1)$ is bounded and $f(A_1) = Pf(A)J$. Now all claims follow from Theorem 13.18. □

Exercises

13.1. A function $m : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ is called **positively homogeneous** of **order** $\gamma \in \mathbb{R}$ if

$$m(\lambda \mathbf{x}) = \lambda^\gamma m(\mathbf{x})$$

for all $\lambda > 0$. Prove the following assertions:

a) Let $k \in \mathbb{N}$, let $m \in \mathrm{C}^k(\mathbb{R}^d \setminus \{0\})$ be positively homogeneous of order $\gamma \in \mathbb{R}$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$. Then $\mathrm{D}^\alpha m$ is positively homogeneous of order $\gamma - |\alpha|$.

b) Let $m \in \mathrm{C}^{k_d}(\mathbb{R}^d \setminus \{0\})$ be positively homogeneous of order $\gamma = 0$. Then $m \in \mathrm{Mi}(\mathbb{R}^d \setminus \{0\})$.

13.2 (Carlson–Beurling Inequality). Let $f \in \mathrm{C}_0(\mathbb{R})$ be differentiable on $\mathbb{R} \setminus \{0\}$ and such that $f, f' \in \mathrm{L}^2(\mathbb{R})$. Show that $\mathcal{F}f \in \mathrm{L}^1(\mathbb{R})$ and

$$\|\mathcal{F}f\|_1 \leq 2\pi \sqrt{\|f\|_2 \, \|f'\|_2}.$$

Conclude that f is the Fourier transform of an L^1-function.

[Hint: Show first that $\mathcal{F}f = (\mathrm{i}\mathbf{t})^{-1}\mathcal{F}f'$ on $\mathbb{R} \setminus \{0\}$. Then take $a, b > 0$ and apply Cauchy–Schwarz and Plancherel to the integral

$$\|\mathcal{F}f\|_1 = \int_{\mathbb{R}} \frac{\sqrt{a^2 + b^2t^2}}{\sqrt{a^2 + b^2t^2}} \, |\mathcal{F}f(t)| \, \mathrm{d}t.$$

Finally, optimize with respect to a and b (cf. [11, Thm.E.5])].

13.3. Show that the function

$$f(t) := \int_{|t|}^{\infty} \frac{\sin s}{s} \, \mathrm{d}s \qquad (t \in \mathbb{R})$$

is the Fourier transform of an L^1-function.
[Hint: Use integration by parts to show that $f(t) = O(|t|^{-1})$ as $|t| \to \infty$. Then employ Exercise 13.2.]

13.4. For $z \in \mathbb{C}$ let

$$h_{0,1}(z) := \frac{1}{\pi} \mathrm{p.v.}\!\int_{-1}^{1} \frac{\mathrm{e}^{-\mathrm{i}sz}}{s} \, \mathrm{d}s.$$

Show that

$$h_{0,1}(z) = \frac{2}{\pi \mathrm{i}} \int_0^1 \frac{\sin zs}{s} \, \mathrm{d}s = \frac{2}{\pi \mathrm{i}} \int_0^z \frac{\sin s}{s} \, \mathrm{d}s,$$

where the right hand side is to be understood as a complex line integral over the straight line segment from 0 to z. Furthermore, show that $h_{0,1} \in \mathrm{H}^\infty(\mathrm{St}_\omega)$ for each $\omega > 0$ with

$$\lim_{|\mathrm{Im}\, z| \le \omega, \mathrm{Re}\, z \to \pm\infty} h_{0,1}(z) = \mp \mathrm{i}.$$

13.5. Let $-A$ be the generator of a bounded C_0-semigroup $(T_t)_{t \ge 0}$ on a Banach space X. For $f \in \mathrm{L}^1((0,1); X)$ define

$$u(t) := (Sf)(t) := \int_0^t T_{t-s} f(s) \, \mathrm{d}s \qquad (0 \le t \le 1).$$

Prove the following assertions:

a) S is a well-defined bounded operator $S : \mathrm{L}^1((0,1); X) \to \mathrm{C}([0,1]; X)$.

b) For each $t \in [0,1]$:

$$\int_0^t u(r) \, \mathrm{d}r \in \mathrm{dom}(A) \quad \text{and} \quad -A \int_0^t u(r) \, \mathrm{d}r = u(t) - \int_0^t f(s) \, \mathrm{d}s.$$

(This means by definition that u is a **mild solution** to (13.5) with $x = 0$.)

c) If $f \in \mathrm{C}([0,1]; \mathrm{dom}(A))$ then $Sf \in \mathrm{C}([0,1]; \mathrm{dom}(A)) \cap \mathrm{C}^1([0,1]; X)$ with $u' + Au = f$.

d) Let $1 < p < \infty$ and suppose that there is a constant $K \ge 0$ such that

$$\|ASf\|_{\mathrm{L}^p((0,1);X)} \le K\|f\|_{\mathrm{L}^p((0,1);X)} \qquad \big(f \in \mathrm{C}([0,1];\mathrm{dom}(A))\big).$$

Then A has maximal L^p-regularity.

13.6. Let A be a closed operator on a Banach space X, let Ω be a measure space and let, for $1 \le p < \infty$, $\mathcal{X}_p := \mathrm{L}^p(\Omega; X)$. Define $\mathcal{A}$ on $\mathcal{X}_p$ by

$$\mathrm{dom}(\mathcal{A}) := \mathrm{L}^p(\Omega; \mathrm{dom}(A)), \qquad \mathcal{A}u := A \circ u.$$

(Here, $\mathrm{dom}(A)$ has to be viewed as a Banach space with respect to the graph norm.) Prove the following assertions:

a) If A is densely defined/injective, then so is $\mathcal{A}$.

b) If $\mathrm{ran}(A)$ is dense in X, then $\mathrm{ran}(\mathcal{A})$ is dense in $\mathcal{X}_p$.

c) $\rho(A) \subseteq \rho(\mathcal{A})$ with

$$R(\lambda, \mathcal{A})u = R(\lambda, A) \circ u \qquad (u \in \mathcal{X}_p)$$

and $\|R(\lambda, \mathcal{A})\| \le \|R(\lambda, A)\|$.

d) If A is sectorial then so is $\mathcal{A}$, with $\omega_{\mathrm{se}}(\mathcal{A}) \le \omega_{\mathrm{se}}(A)$.

e) Suppose that A is sectorial, $\omega \in (\omega_{\mathrm{se}}(A), \pi)$, and $f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$ is such that $f(A)$ is defined and bounded. Then also $f(\mathcal{A})$ is defined and bounded and

$$f(\mathcal{A})u = f(A) \circ u \qquad (u \in \mathcal{X}_p).$$

In particular $\|f(\mathcal{A})\| \le \|f(A)\|$.

f) If A is sectorial and has BIP or a bounded $\mathrm{H}^\infty(\mathrm{S}_\omega)$ calculus, then so does $\mathcal{A}$.

13.7. Let A be a densely defined sectorial operator of angle $\omega_{\mathrm{se}}(A) < \pi/2$ and let $1 < p < \infty$. Let the space $\mathcal{X}_p$ and the operators $\mathcal{A}$ and $\mathcal{B}$ be defined as in Section 13.4. Show that the following assertions are equivalent:

(i) A has maximal L^p-regularity;

(ii) The operator $\mathcal{A} + \mathcal{B}$ with domain $\mathrm{dom}(\mathcal{A}) \cap \mathrm{dom}(\mathcal{B})$ is closed.

Show further that, in this case, $\mathcal{A} + \mathcal{B}$ is invertible.
[Hint: Use Lemma 13.12 and that $\mathcal{B}$ is invertible; in order to show that $\mathcal{A}+\mathcal{B}$ has dense range, look into the proof of Lemma 13.12.]

13.8 (Fourier–Stieltjes Calculus for d-Parameter Groups). Let X be a Banach space and let $U : \mathbb{R}^d \to \mathcal{L}(X)$ be any strongly continuous d-parameter group on X. Define

$$\omega : \mathbb{R}^d \to (0, \infty), \qquad \omega(s) := \|U(s)\| \quad (s \in \mathbb{R}^d),$$

and let $\mathrm{M}(\mathbb{R}^d, \omega)$ consist of all $\mu \in \mathrm{M}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \omega(s)\, |\mu|\,(\mathrm{d}s) < \infty.$$

Prove the following assertions:

a) $\mathrm{M}(\mathbb{R}^d, \omega)$ is a unital convolution subalgebra of $\mathrm{M}(\mathbb{R}^d)$.

b) For $\mu \in \mathrm{M}(\mathbb{R}^d, \omega)$ let

$$U_\mu := \int_{\mathbb{R}^d} U_s \, \mathrm{d}s.$$

Then the mapping

$$\mathrm{M}(\mathbb{R}^d, \omega) \to \mathcal{L}(X), \quad \mu \mapsto U_\mu$$

is a unital homomorphism.

Define $\mathrm{FS}(\mathbb{R}^d, \omega) := \{\widehat{\mu} \mid \mu \in \mathrm{M}(\mathbb{R}^d, \omega)\}$. The mapping

$$\Psi_U : \mathrm{FS}(\mathbb{R}^d, \omega) \to \mathcal{L}(X), \qquad \Psi_U(f) := U_\mu \qquad (f = \widehat{\mu},\ \mu \in \mathrm{M}(\mathbb{R}^d, \omega))$$

is the **Fourier–Stieltjes calculus** for U. Its canonical extension is also denoted by Ψ_U.

c) Define for $j \in \{1, \dots, d\}$ the 1-parameter group U^j by

$$U^j_{s_j} := U(s_j \mathrm{e}_j) \qquad (s_j \in \mathbb{R}),$$

where e_j is the j-th canonical basis vector of $\mathbb{R}^d$. Let $-\mathrm{i}A_j$ be the generator of U^j. Show that

$$f(A_j) = \Psi_U(f(\mathbf{z}_j))$$

whenever the left-hand side is defined in the (canonically extended) Fourier–Stieltjes calculus for A_j.

d) Let $-\mathrm{i}C$ be the generator of the 1-parameter group V defined by

$$V(s) := U(s\mathrm{e}_1 + \cdots + s\mathrm{e}_d) \qquad (s \in \mathbb{R}).$$

Show that

$$f(C) = \Psi_U(f(\mathbf{z}_1 + \cdots + \mathbf{z}_d))$$

whenever the left-hand side is defined in the (canonically extended) Fourier–Stieltjes calculus for C.

Supplementary Exercises

13.9. This exercise is to establish the formula

$$\frac{1}{1+\mathrm{e}^z} = \frac{1}{2} + \frac{1}{2\mathrm{i}} \mathrm{p.v.}\!\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}sz} \frac{1}{\sinh(\pi s)}\, \mathrm{d}s \qquad (13.14)$$

used in the proof of Theorem 13.10. We abbreviate $f := (1+\mathrm{e}^{\mathbf{z}})^{-1}$. Then $f' = -\mathrm{e}^{\mathbf{z}}(1+\mathrm{e}^{\mathbf{z}})^{-2}$.

a) For $s \in \mathbb{R}$ let

$$J_s := \mathcal{F}^{-1}(f')(s) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}sz}\frac{-\mathrm{e}^{z}}{(1+\mathrm{e}^{z})^2}\,\mathrm{d}z.$$

Shift the contour of integration to $\mathbb{R} + 2\pi\mathrm{i}$ and show that

$$(1-\mathrm{e}^{-2\pi s})J_s = R_s,$$

where R_s is the residue of the function $g_s := \mathrm{i}\mathrm{e}^{\mathrm{i}s\mathbf{z}}\frac{-\mathrm{e}^{\mathbf{z}}}{(1+\mathrm{e}^{\mathbf{z}})^2}$ at $z = \pi\mathrm{i}$.

b) Show that $R_s = -s\mathrm{e}^{-\pi s}$, e.g. by passing to $g_s(\mathbf{z}+\pi\mathrm{i})$ and using a power series argument.

c) Conclude that

$$f' = \frac{1}{2}\mathcal{F}\Big(\frac{-\mathbf{s}}{\sinh(\pi\mathbf{s})}\Big).$$

d) Define the function u by

$$u(z) = \frac{1}{2\mathrm{i}}\,\mathrm{p.v.}\!\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}sz}\frac{1}{\sinh(\pi s)}\,\mathrm{d}s \qquad (z \in \mathbb{R}).$$

Show, e.g. by splitting the integral as in the proof of Theorem 13.10, that u is well defined, $u(0) = 0$ and $u' = f'$.

e) Prove the validity of the representation (13.14).

Remark: One can prove (13.14) more directly. Can you imagine, how?

13.10. Let X be a Banach space and let $A = \frac{\mathrm{d}}{\mathrm{d}s}$ on $\mathrm{L}^p(\mathbb{R};X)$ for $1 < p < \infty$. Show that A has dense range.
[Hint: Let $D := \{\varphi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}) : \int_{\mathbb{R}} f = 0\}$ and show that $D \otimes X$ is dense in $\mathrm{L}^p(\mathbb{R};X)$ and contained in $\mathrm{ran}(A)$.]

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Appendix

A.1 Notational and Terminological Conventions

In this appendix we list certain of our conventions regarding notation and terminology.

Point and Number Sets. The set of natural numbers is $\mathbb{N} := \{1, 2, 3, \dots\}$ and $Z_+ := \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ We write $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \ge 0\}$ for the positive real line and, a little inconsistently, $\mathbb{C}_+ := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ for the *open* right half-plane. The open unit disc is $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ and $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle.

Functions. As is usual in functional analysis, we normally write f for a function and $f(x)$ for its value at the argument x. We use $\mathbf{z}$ for the function $z \mapsto z$ (where the domain $D \subseteq \mathbb{C}$ is usually understood). Accordingly, we use $\mathbf{z}_j : \mathbb{C}^d \to \mathbb{C}$ to denote the projection onto the j-th component. Similar conventions apply to other domains like $\mathbb{R}$ or $\mathbb{R}_+$, where the coordinates are often denoted by $\mathbf{s}$ or $\mathbf{t}$, and $\mathbf{s}_j$ and $\mathbf{t}_j$ for the projections $\mathbb{R}^d \to \mathbb{R}$ onto the j-th component.

Metric Spaces. Open and closed balls of radius $r \ge 0$ round a point $x \in \Omega$ in a metric space (Ω, d) are generically written as

$$\mathrm{B}(x, r) := \{y \in \Omega \mid d(x, y) < r\} \quad \text{and} \quad \mathrm{B}[x, r] := \{y \in \Omega \mid d(x, y) \le r\}.$$

Functional Analysis and Operator Theory. The generic notation for Banach spaces is X, Y, Z. The dual space of a Banach space X is denoted by X', and the canonical duality is

$$\langle x, x' \rangle \qquad (x \in X,\ x' \in X').$$

The set of bounded linear operators $T : X \to Y$ is $\mathcal{L}(X; Y)$, with $\mathcal{L}(X) := \mathcal{L}(X; X)$. The dual or Banach space adjoint of an operator $T \in \mathcal{L}(X; Y)$ is

$T' \in \mathcal{L}(Y'; X')$. An operator $T \in \mathcal{L}(X;Y)$ is called a **contraction** if $\|T\| \le 1$ and **isometric** or an **isometry** if $\|Tx\| = \|x\|$ for all $x \in X$. Each isometry is an injective contraction with closed range. A surjective isometry is called an **isometric isomorphism**.

Hilbert spaces are generically denoted by H, K. The **inner product** of a (pre-)Hilbert space H is usually written as

$$(\, x \,|\, y \,) \qquad (x, y \in H).$$

The **Hilbert space adjoint** of an operator $T \in \mathcal{L}(H;K)$ is denoted by T^*. An operator $T \in \mathcal{L}(H;K)$ is isometric if and only if $T^*T = \mathrm{I}_H$, i.e., if

$$(\, Tx \,|\, Ty \,) = (\, x \,|\, y \,) \quad \text{for all } x,\, y \in H.$$

An isometric isomorphism $T : H \to K$ is called **unitary**. Equivalently, $T \in \mathcal{L}(H;K)$ is invertible and $T^{-1} = T^*$.

Function Spaces. For any set Ω we denote by $\ell^\infty(\Omega)$ the space of all **bounded functions**, endowed with the supremum norm

$$\|f\|_\infty := \|f\|_{\infty,\Omega} := \sup_{z \in \Omega} |f(z)| \,.$$

For a topological space Ω, the space of **continuous functions** is $\mathrm{C}(\Omega)$, whereas $\mathrm{C_b}(\Omega)$ is the space of all **bounded and continuous** functions. If Ω is metric, we write $\mathrm{UC_b}(\Omega)$ for the space of bounded and **uniformly continuous** functions. If Ω is locally compact, we have $\mathrm{C_c}(\Omega)$, the space of **continuous functions with compact support** and its sup-norm closure $\mathrm{C}_0(\Omega)$, the space of continuous functions **vanishing at infinity**. Note that the **support** of a function f is

$$\operatorname{supp}(f) := \overline{[\, f \neq 0 \,]}.$$

All these function spaces have analogues for vector-valued functions. So, for example, we write $\ell^\infty(\Omega; X)$ for the bounded X-valued functions and $\mathrm{C_b}(\Omega; X)$ for the bounded and continuous functions.

For an open subset $O \subseteq \mathbb{C}$ we denote by $\mathrm{Hol}(O)$ the set of all scalar-valued holomorphic functions defined on O. According to our general convention, $\mathrm{Hol}(O; X)$ denotes the space of X-valued holomorphic functions on O. (See Appendix A.3 for more on such functions.)

The space of **bounded holomorphic** functions on O is $\mathrm{H}^\infty(O) := \ell^\infty(O) \cap \mathrm{Hol}(O)$. It is a unital Banach algebra (see below) with respect to the norm $\|f\|_{\mathrm{H}^\infty} := \|f\|_\infty = \sup_{z \in O} |f(z)|$.

If (Ω, Σ) is a **measurable space** (i.e., Ω is a set and Σ is a σ-algebra on Ω), then

$$\mathcal{M}(\Omega, \Sigma) := \{ f : \Omega \to \mathbb{C} \mid f \text{ is measurable} \}$$

is the space of all measurable functions. The space of **bounded and measurable functions** is

$$\mathcal{M}_{\mathrm{b}}(\Omega, \Sigma) := \{f \in \mathcal{M}(\Omega, \Sigma) \mid f \text{ bounded}\}.$$

If the σ-algebra Σ is understood, we simply write $\mathcal{M}(\Omega)$ and $\mathcal{M}_{\mathrm{b}}(\Omega)$.

BP-Convergence. Let Ω be a set. We say that a sequence of bounded functions $(f_n)_n$ on Ω converges **boundedly and pointwise** (in short: bp-converges) to a function f on Ω if

$$\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty \quad \text{and} \quad f_n(x) \to f(x) \quad \text{for all } x \in \Omega.$$

Measure Theory. We expect the reader to be familiar with the basics of measure theory and measure-theoretic integration as it is treated in many books. We stick to common notation and terminology apart from a few peculiarities that shall now be explained briefly.

Our standard notation for a measure space is $\mathrm{\Omega} = (\Omega, \Sigma, \mu)$. Note the difference between $\mathrm{\Omega}$ (denoting the whole measure space) and Ω (denoting the underlying set). This convention allows us to denote the corresponding L^p-spaces by $\mathrm{L}^p(\mathrm{\Omega})$.

We use the symbol

$$\bigsqcup_{n=1}^{\infty} A_n$$

to denote the union of a sequence $(A_n)_n$ of **pairwise disjoint** sets.

The **characteristic function** (sometimes also called indicator function) of a set A is denoted by $\mathbf{1}_A$.

The **integral** of $f \in \mathrm{L}^1(\mathrm{\Omega})$ is denoted by

$$\int_{\mathrm{\Omega}} f := \int_{\Omega} f \,\mathrm{d}\mu.$$

Generically, for "the set of points in Ω where …happens" we use square brackets $[\dots]$ where other books may use curly brackets. E.g., if $f : \Omega \to X$ is a function into any set X and $A \subseteq X$, then

$$[\, f \in A\,] := \{\omega \in \Omega \mid f(\omega) \in A\}.$$

Analogously, if $g : \Omega \to X$ is another function, $[\, f = g\,] := \{\omega \in \Omega \mid f(\omega) = g(\omega)\}$.

A **null set** in $\mathrm{\Omega}$ is any subset of Ω that is contained in a measurable subset of zero measure. So a null set need not be measurable with respect to Σ, but would be with respect to the μ-completion of Σ.

Two functions $f, g : \Omega \to X$ are equal **almost everywhere** if $[\, f \neq g\,]$ is a null set.

A function $f : \Omega \to \mathbb{C}$ is **essentially measurable** if f coincides almost everywhere with a measurable function. Equivalently, for each Borel set $B \subseteq \mathbb{C}$ the set $[\, f \in B\,]$ is contained in the μ-completion of Σ. The **essential range** of such a function is

$$\operatorname{essran}(f) = \{z \in \mathbb{C} \mid \mu[\,|f - z| < \varepsilon\,] > 0 \text{ for all } \varepsilon > 0\}.$$

The set of all essentially measurable functions (modulo equality almost everywhere) is denoted by $\mathrm{L}^0(\Omega)$.

If $\Omega = (\Omega, \Sigma, \mu)$ is a measure space and $f : \Omega \to \Omega'$ is a mapping then we denote by $f_*\mu$ the **image measure** of μ under f, defined on (sub-σ-algebras of) the σ-algebra $f_*(\Sigma) := \{A \subseteq \Omega' \mid f^{-1}(A) \in \Sigma\}$.

A **monolith** in a measure space Ω is any measurable set of infinite measure whose only measurable subsets either have infinite measure as well or are null sets. A measure space is called **semi-finite** if it does not have monoliths. More formally, Ω is semi-finite if it has the following property:

$$\forall\, A \in \Sigma : \mu(A) > 0 \;\; \Rightarrow \;\; \exists\, B \in \Sigma : B \subseteq A \;\wedge\; 0 < \mu(B) < \infty.$$

Monoliths, or rather their characteristic functions, play a role for L^∞ but none in L^p whenever $p < \infty$. In the presence of a monolith, L^∞ cannot be identified with the dual of L^1, since multiplication with the characteristic function of a monolith is the zero operator on each L^p for $p < \infty$.

Complex Measures. If (Ω, Σ) is a measurable space, we denote by $\mathrm{M}(\Omega, \Sigma)$ the space of all **complex measures** on it, endowed with the total variation norm. If Σ is understood, then we simply write $\mathrm{M}(\Omega)$. For a complex measure $\mu \in \mathrm{M}(\Omega, \Sigma)$, its **total variation** measure is $|\mu|$.

If Ω is a locally compact or a metric space, then $\Sigma = \mathrm{Bo}(\Omega)$, the **Borel** algebra, is the canonical choice. In this case, $\mathrm{M}(\Omega)$ denotes the set of complex **regular** Borel measures on Ω.

The **support** of a positive Borel measure μ on a topological space Ω is the set

$$\operatorname{supp}(\mu) := \{x \in \Omega \mid \mu(U) > 0 \text{ for each open neighborhood } U \text{ of } x\}.$$

For more on complex measures see Appendix A.7.

Algebras. In this text, by an **algebra** we mean an *associative* algebra over the complex field $\mathbb{C}$. An algebra A is **commutative** if $ab = ba$ for all $a, b \in A$. An algebra is **unital** if it contains a unit element (usually denoted by e). A **$*$-algebra** or algebra with **involution** is an algebra A with an antilinear self-map $(a \mapsto a^*) : A \to A$ satisfying

$$(ab)^* = b^*a^* \quad \text{and} \quad (a^*)^* = a \qquad (a, b \in A).$$

(If A is unital, it then follows that $\mathrm{e}^* = \mathrm{e}$.) A norm $\|\cdot\|$ on an algebra is called an **algebra norm** if it is **submultiplicative**, i.e., if

$$\|a{\cdot}b\| \le \|a\|\,\|b\| \quad (a,\, b \in A).$$

An algebra endowed with an algebra norm is called a **normed algebra**. If, in addition, the algebra is complete with respect to the induced metric, then it is called a **Banach algebra**. A **C^*-algebra** is a Banach algebra A with involution such that

$$\|x^* x\| = \|x\|^2 \quad (x \in A).$$

(It then follows that $\|x^*\| = \|x\|$ for all $x \in A$.)

An **ideal** in an algebra A is a subspace I of A such that ax, $xa \in I$ for all $a \in A$ and $x \in I$. If A is a $*$-algebra, then an ideal I of A is a $*$-ideal if $x \in I$ implies $x^* \in I$.

Groups and Semigroups. A **semigroup** is a set S together with an associative operation $S \times S \to S$. A **unit element** in a semigroup S is an element e such that

$$\mathrm{e}\, s = s\, \mathrm{e} = s \quad \text{for all } s \in S.$$

Semigroups with unit are sometimes called *monoids*, but we shall avoid this terminology and call them just semigroups again. (The reason is that one can always adjoin a unit element.) A semigroup homomorphism between semigroups S and T is a mapping $f : S \to T$ such that

$$f(st) = f(s)f(t) \qquad (s,\, t \in S)$$

and $f(\mathrm{e}_S) = \mathrm{e}_T$ if e_S and e_T are units in S and T, respectively.

An inverse of an element s in a semigroup with unit is an element $t \in S$ such that $st = ts = \mathrm{e}$. A group is a semigroup with unit in which each element is invertible. The set $S^\times$ of invertible elements of a semigroup S is a group.

A.2 Calculus in Banach Spaces

In this appendix we review basic differential and integral calculus on intervals for vector-valued functions.

Weak Integrability

Let $\Omega = (\varOmega, \varSigma, \mu)$ be any measure space and X a Banach space with dual space X'. A function $f : \varOmega \to X$ is (essentially) **weakly measurable** if $x' \circ f$ is (essentially) measurable for each $x' \in X'$. It is **weakly integrable** if $x' \circ f \in \mathrm{L}^1(\Omega)$ for each $x' \in X'$. Note that $x' \circ f = \langle f(\cdot), x' \rangle$.

If f is weakly integrable, then one can consider the following linear mapping:

$$X' \to \mathrm{L}^1(\Omega), \qquad x' \mapsto x' \circ f.$$

An application of the closed graph theorem shows: this mapping is bounded, i.e., there is $c \geq 0$ such that

$$\|x' \circ f\|_{\mathrm{L}^1(\Omega)} = \int_\Omega |x' \circ f| \leq c \|x'\|_{X'} \qquad (x' \in X').$$

Hence we can define an element $\int_\Omega f$ of X'' by

$$\langle \int_\Omega f \, , x' \rangle := \int_\Omega \langle f(\cdot), x' \rangle \qquad (x' \in X').$$

This element of X'' is called the **(weak) integral** of f. Obviously

$$\Big\| \int_\Omega f \Big\| \leq \sup_{\|x'\| \leq 1} \int_\Omega |x' \circ f| \, .$$

If $T : X \to Y$ is a bounded linear mapping and $f : \Omega \to X$ is weakly integrable, then $T \circ f : \Omega \to Y$ is weakly integrable and

$$\int_\Omega T \circ f = T'' \int_\Omega f. \tag{A.1}$$

(Note that $y' \circ T \circ f = (T'y') \circ f$ for each $y' \in Y'$.)

Recall from elementary functional analysis that X can be regarded via the canonical embedding $\iota_X : X \to X''$ as a closed subspace of X''. Usually, ee shall perform this identification tacitly and do not distinguish notationally between elements $x \in X$ and their images $\iota_X(x) \in X''$.

Under this identification it is reasonable to ask whether the weak integral $\int_\Omega f$ of a weakly integrable function $f : \Omega \to X$ is a member of X rather than just of X''. This will not hold in general, but under additional hypotheses, e.g. Bochner integrability (Appendix A.6). However, *if* it is the case for a given f as above, then it is also true for $T \circ f$, where $T : X \to Y$ is any bounded linear mapping, and one has

$$\int_\Omega T \circ f = T \int_\Omega f. \tag{A.2}$$

This follows from (A.1) and the fact that $T'' \circ \iota_X = \iota_Y \circ T$.

Weak* Integrability

Let, again, $\Omega = (\varOmega, \varSigma, \mu)$ be any measure space and X a Banach space with dual space X'. A function $f : \varOmega \to X'$ is (essentially) **weakly* measurable** if $\langle x, f(\cdot)\rangle$ is (essentially) measurable for each $x \in X$. It is **weakly* integrable** if $\langle x, f(\cdot)\rangle \in \mathrm{L}^1(\Omega)$ for each $x \in X$.

If f is weakly* integrable, then one can consider the following linear mapping:

$$X \to \mathrm{L}^1(\Omega), \qquad x \mapsto \langle x, f(\cdot)\rangle .$$

As before, an application of the closed graph theorem shows that this mapping is bounded, i.e., there is $c \geq 0$ such that

$$\| \langle x, f(\cdot)\rangle \|_{\mathrm{L}^1(\Omega)} = \int_\Omega |\langle x, f(\cdot)\rangle| \leq c\|x\|_X \qquad (x \in X).$$

Hence we can define an element $\int_\Omega f$ of X' by

$$\langle x\,, \int_\Omega f\,\rangle := \int_\Omega \langle x, f(\cdot)\rangle \qquad (x \in X).$$

This element of X' is called the **(weak*) integral** of f. Obviously

$$\Big\|\int_\Omega f\Big\| \leq \sup_{\|x\|\leq 1} \int_\Omega |\langle x, f(\cdot)\rangle| \,.$$

If $T : X \to Y$ is a bounded linear mapping and $f : \varOmega \to Y'$ is weakly* integrable, then $T' \circ f : \varOmega \to X'$ is weakly integrable and

$$\int_\Omega T' \circ f = T' \int_\Omega f. \tag{A.3}$$

Clearly if $f : \varOmega \to X'$ is even weakly integrable, then it is weakly* integrable, and the integrals coincide.

Regulated and Continuous Functions

Let $(\varOmega, d)$ be any metric space and, as before, X a Banach space. We write $\mathrm{C}(\varOmega; X)$ for the space of continuous functions $f : \varOmega \to X$. For each $f \in \mathrm{C}(\varOmega; X)$ the function $\|f\|_X : \varOmega \to \mathbb{R}$ defined by

$$\|f\|_X(t) := \|f(t)\|_X \qquad (t \in \varOmega)$$

is continuous.

A basic result of functional analysis states that the space $\ell^\infty(\varOmega; X)$ of bounded and the space $\mathrm{C_b}(\varOmega; X)$ of bounded and continuous functions are

Banach spaces with respect to the norm

$$\|f\|_\infty := \sup_{t\in\Omega} \|f(t)\|_X.$$

Now let $\Omega = [a,b]$ be a compact subinterval of $\mathbb{R}$. An X-valued **step function** is a function $f : [a,b] \to X$ such that there is a partition

$$a = t_0 < t_1 < \cdots < t_n = b$$

such that f is constant on each interval (t_{j-1}, t_j), $j = 1, \ldots, n$. The space of step functions is $\mathrm{Step}([a,b];X)$. Its closure in $\ell^\infty([a,b];X)$ is

$$\mathrm{Reg}([a,b];X),$$

the space of **regulated functions**.

Lemma A.1. *The space of continuous functions* $\mathrm{C}([a,b];X)$ *is a closed subspace of* $\mathrm{Reg}([a,b];X)$.

Proof. Let $f \in \mathrm{C}([a,b];X)$. Since $[a,b]$ is compact, f is uniformly continuous. Now the proof proceeds as for scalar functions. $\square$

If $f \in \mathrm{Step}([a,b];X)$ then $\|f\|_X$ is a scalar step function and $x'\circ f$ is a scalar step function for each $x' \in X'$. Hence f is weakly integrable. It is a simple exercise to show that

$$\int_a^b \mathbf{1}_{(c,d)}(t)x \,\mathrm{d}t = (d-c)x \in X.$$

It follows (how?) that if f is regulated then $\|f\|_X$ is regulated and $x'\circ f$ is regulated for each $x' \in X'$.

Theorem A.2. *Let* $f \in \mathrm{Reg}([a,b];X)$ *be a regulated function. Then* $\|f\|_X$ *is regulated and* f *is weakly integrable, and the integral satisfies*

$$\int_a^b f(t)\,\mathrm{d}t \in X$$

and

$$\Big\|\int_a^b f(t)\,\mathrm{d}t\Big\| \le \int_a^b \|f(t)\|_X \,\mathrm{d}t \le (b-a)\|f\|_\infty. \tag{A.4}$$

Proof. Note that $|x'\circ f| \le \|x'\|\|f\|_X$ and therefore

$$\|x'\circ f\|_\infty \le \|x'\|\|f\|_\infty$$

for each $f \in \ell^\infty([a,b];X)$. Approximating f by a sequence of step functions we see (how precisely?) that $\|f\|_X$ is regulated and $x'\circ f$ is regulated for each $x' \in X'$. Consequently, f is weakly integrable and one has

$$\int_a^b |\langle f(t), x' \rangle| \, \mathrm{d}t \leq \|x'\| \int_a^b \|f(t)\|_X \, \mathrm{d}t \leq (b-a)\|x'\| \|f\|_\infty$$

for each $x' \in X'$. This implies (A.4), which tells that the integral

$$\mathrm{Reg}([a,b]; X) \to X'', \qquad f \mapsto \int_a^b f(t) \, \mathrm{d}t$$

is a bounded linear mapping. Since this mapping maps the dense subspace of step functions into X (viewed as a closed linear subspace of X''), it follows that $\int_a^b f(t) \, \mathrm{d}t \in X$ for each $f \in \mathrm{Reg}([a,b]; X)$. $\square$

Note that if $f : [a,b] \to X$ is regulated (a step function, continuous) and $T : X \to Y$ is a bounded linear mapping into another Banach space Y, then $Tf := T \circ f$ is regulated (a step function, continuous), and

$$T \int_a^b f(t) \, \mathrm{d}t = \int_a^b T f(t) \, \mathrm{d}t.$$

Improper Integrals. Suppose that $f : (0, \infty) \to X$ is a function that is regulated on each finite interval and satisfies $\int_0^\infty \|f(t)\|_X \mathrm{d}t < \infty$. Then

$$\int_0^\infty f(t) \, \mathrm{d}t := \lim_{T \to \infty} \int_0^T f(t) \, \mathrm{d}t$$

exists in X, by Cauchy's criterion.

Strongly Continuous Mappings. Let Ω be any topological space and X, Y Banach spaces. A mapping $F : \Omega \to \mathcal{L}(X; Y)$ is called **strongly continuous**, if for each $x \in E$ the function

$$\Omega \to Y, \qquad t \mapsto F(t)x$$

is continuous. Analogously, a function $F : [a,b] \to \mathcal{L}(X; Y)$ is called **strongly regulated** if $F(\cdot)x$ is regulated for each $x \in X$. For a strongly regulated function F its integral $\int_a^b F(t) \, \mathrm{d}t \in \mathcal{L}(X; Y)$ is defined by

$$\big(\int_a^b F(t) \, \mathrm{d}t\big)x := \int_a^b F(t)x \, \mathrm{d}t \qquad (x \in X).$$

The Fundamental Theorem of Calculus

A function $u : [a,b] \to X$ is **differentiable** at $t_0 \in [a,b]$ if

$$\lim_{t \to t_0} \frac{1}{t - t_0} \big(u(t) - u(t_0)\big)$$

exists in X. In that case this limit is denoted by $u'(t_0)$ or by

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t_0).$$

As for scalar functions, if u is differentiable at each point of $[a,b]$, the function u' is called the **derivative** of u.

Theorem A.3 (Fundamental Theorem). *Let $u \in \mathrm{C}([a,b];X)$. Then the function*

$$Ju(t) := \int_a^t u(s)\,\mathrm{d}s$$

is differentiable and $(Ju)' = u$.
On the other hand, let u be differentiable and $u' \in \mathrm{Reg}([a,b];X)$. Then

$$\int_a^b u'(s)\,\mathrm{d}s = u(b) - u(a).$$

Proof. The first part is proved as in the scalar case. The second can be reduced to the scalar case by applying elements of X'. $\square$

Corollary A.4. *Let $u : [a,b] \to X$ be differentiable and $u' = 0$. Then u is constant.*

Let us push the calculus a little further.

Lemma A.5 (Product rule). *Let $f : [a,b] \to X$ differentiable and let $T : [a,b] \to \mathcal{L}(X;Y)$ be strongly continuous such that $T(\cdot)f(t)$ is differentiable for each $t \in [a,b]$. Then the product function $Tf : [a,b] \to Y$ is differentiable, with*

$$(Tf)'(t) = T(t)f'(t) + [T(\cdot)f(t)]'(t).$$

Proof. Since T is strongly continuous, it is uniformly norm bounded. Fix $t \in [a,b]$ and $s \in \mathbb{R}$ such that $s + t \in [a,b]$. Then

$$\begin{aligned}
&\frac{1}{s}\Big(T(t+s)f(t+s) - T(t)f(t)\Big) \\
&\quad = T(t+s)\frac{f(t+s)-f(t)}{s} + \frac{1}{s}\Big(T(t+s)f(t) - T(t)f(t)\Big).
\end{aligned}$$

Since T is uniformly bounded and strongly continuous, the first summand converges to $T(t)f'(t)$ as $s \to 0$. The second summand is clear. $\square$

Note that if $T(\cdot)$ is actually differentiable with respect to the operator norm, then the product rule takes the common form

$$(Tf)' = Tf' + T'f.$$

From the product rule one derives immediately an integration-by-parts formula:

$$\int_a^b T(s)f'(s)\,\mathrm{d}s = T(t)f(t)\Big|_{t=a}^{t=b} - \int_a^b [T(\cdot)f(t)]'(t)\,\mathrm{d}t.$$

A.3 Vector-valued Holomorphic Functions

Let $D \subseteq \mathbb{C}$ be a nonempty open set and X a Banach space. A function

$$f : D \to X$$

is called **holomorphic** if the derivative

$$f'(z) := \lim_{w\to z} \frac{f(w) - f(z)}{w - z}$$

exists as a $\|\cdot\|_X$-limit in X for each $z \in D$.

If f is holomorphic and $T : X \to Y$ is a bounded operator, then Tf is holomorphic and

$$(Tf)'(z) = Tf'(z) \qquad (z \in D)$$

as an easy computation shows. Also, it follows easily that a holomorphic function is continuous.

In particular, if f is holomorphic, then it is **weakly holomorphic**, i.e., for each $x' \in X'$ the mapping $x' \circ f$ is holomorphic, and one has

$$(x' \circ f)'(z) = \langle f'(z), x'\rangle \qquad (z \in D).$$

(Do not confuse the different dashes here!) Since elements of X' separate the points of X, one can transfer results from scalar function theory to vector-valued function theory. Maybe the most important one is Cauchy's theorem.

Theorem A.6. *Let Γ be a path (or cycle) in D such that for all scalar-valued holomorphic functions f on D the formula*

$$\int_\Gamma f(z)\,\mathrm{d}z = 0$$

holds. Then it holds as well for all vector-valued holomorphic functions f.

Note that, since a holomorphic function is continuous, the integral in Theorem A.6 is defined in the sense of Appendix A.2. (Our paths are continuous and piecewise continuously differentiable.)

Proof. Let $f : D \to X$ be holomorphic and $x' \in X'$. Then

$$\Big\langle \int_\Gamma f(z)\,\mathrm{d}z, x' \Big\rangle = \int_\Gamma \langle f(z), x'\rangle\;\mathrm{d}z = 0$$

by hypothesis. Since X' separates the points, the claim is proved. □

Analogously, a Cauchy integral formula

$$f(a) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{f(z)}{z-a}\,\mathrm{d}z$$

holds for vector-valued holomorphic functions if it holds for scalar-valued ones.

As a consequence, one obtains in the same way as for scalar functions that a vector-valued holomorphic function f is infinitely differentiable and, locally around each point $a \in D$, given by a power series

$$f(z) = \sum_{n=0}^{\infty} v_n (z-a)^n,$$

where $v_n = \frac{1}{n!} f^{(n)}(a) \in X$ for all $n \in \mathbb{N}_0$.

The next theorem is often helpful to decide whether a vector-valued function is holomorphic. We call a subset $N \subseteq X'$ of the dual X' of a Banach space X **norming** if by

$$|||\, x \,||| := \sup_{x' \in N} |\langle x, x' \rangle| \qquad (x \in X)$$

an equivalent norm is defined on X. (Then, necessarily, N is bounded (why?) and separates the points of X.)

Theorem A.7. *Let X be a Banach space, let $N \subseteq X'$ be a norming subset of X', and let $f : D \to X$ be locally bounded. Then the following assertions are equivalent:*

(i) *f is holomorphic.*

(ii) *$x' \circ f$ is holomorphic for each $x' \in N$.*

Proof. The implication (i) ⇒ (ii) is clear. In order to prove the converse, fix $a \in D$ and $r > 0$ such that $\mathrm{B}[a, r] \subseteq D$, and consider the functions

$$g(z) := \frac{f(z) - f(a)}{z - a} \quad \text{and} \quad h(z, w) = g(z) - g(w)$$

for $w, z \in D \setminus \{a\}$. It suffices to show that

$$\lim_{(z,w) \to (a,a)} h(z, w) = 0.$$

(Then one can apply Cauchy's criterion to conclude that $\lim_{z \to a} \frac{f(z)-f(a)}{z-a}$ exists.)

We write $f_{x'} = x' \circ f$ for $x' \in N$. By hypothesis, the function $f_{x'}$ is holomorphic on D. Hence, by Cauchy's theorem, for $0 < |z-a|\,, |w-a| < r$:

$$
\begin{aligned}
\langle h(z,w), x' \rangle &= g_{x'}(z) - g_{x'}(w) \\
&= \frac{1}{2\pi \mathrm{i}(z-a)} \int_{|u-a|=r} f_{x'}(u) \left(\frac{1}{u-z} - \frac{1}{u-a} \right) \mathrm{d}u \\
&\qquad - \frac{1}{2\pi \mathrm{i}(w-a)} \int_{|u-a|=r} f_{x'}(u) \left(\frac{1}{u-w} - \frac{1}{u-a} \right) \mathrm{d}u \\
&= \frac{1}{2\pi \mathrm{i}} \int_{|u-a|=r} f_{x'}(u) \frac{1}{(u-z)(u-a)} \,\mathrm{d}u \\
&\qquad - \frac{1}{2\pi \mathrm{i}} \int_{|u-a|=r} f_{x'}(u) \frac{1}{(u-w)(u-a)} \,\mathrm{d}u \\
&= \frac{1}{2\pi \mathrm{i}} \int_{|u-a|=r} f_{x'}(u) \frac{z-w}{(u-z)(u-a)(u-w)} \,\mathrm{d}u.
\end{aligned}
$$

By taking the modulus and the supremum over $x' \in N$ we find that

$$
|||\, h(z,w) \,||| \le M \frac{|z-w|}{2r\pi} \int_{|u-a|=r} \frac{|\mathrm{d}u|}{|u-z|\,|u-w|},
$$

where $M := \sup_{|u-a|=r} |||\, f(u) \,||| < \infty$, by hypothesis. Hence,

$$
|||\, h(z,w) \,||| = O(|z-w|)
$$

as $(z,w) \to (a,a)$. □

Sometimes the set N is large enough even to ensure the local boundedness of f, like in the following corollary.

Corollary A.8. *Let X, Y be Banach spaces, $D \subseteq \mathbb{C}$ a nonempty open set. Then for a mapping $f : D \to \mathcal{L}(X;Y)$ the following assertions are equivalent:*

(i) *The function f is holomorphic.*

(ii) *The function f is strongly holomorphic, i.e., for each $x \in X$ the function $f(\cdot)x : D \to Y$ is holomorphic.*

(iii) *The function f is weakly holomorphic, i.e., for each $x \in X$ and $y' \in Y'$ the function $f_{x,y'} := \langle f(\cdot)x, y' \rangle$ is holomorphic.*

Proof. Since for $x \in X$ and $y' \in Y'$ the mappings

$$
\delta_x : \mathcal{L}(X;Y) \to Y, \quad \delta_x(T) = Tx
$$

and

$$
\delta_{y'} : Y \to \mathbb{C}, \qquad \delta_{y'}(y) = \langle y, y' \rangle
$$

are linear and bounded, the implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are clear.

Now suppose (iii) and let $K \subseteq D$ be a compact set. For each pair $(x, y') \in X \times Y'$ the function $\langle f(\cdot)x, y' \rangle$ is holomorphic, hence continuous, hence bounded on K. By the uniform boundedness principle(!), it follows that f is bounded on K. So f is locally bounded, and one can apply Theorem A.7 with

$$N := \{T \mapsto \langle Tx, y' \rangle \mid \|x\| \leq 1, \|y'\| \leq 1\}.$$

(Note that N is strongly norming in the sense that

$$\sup_{\varphi \in N} |\varphi(T)| = \|T\|$$

for each $T \in \mathcal{L}(X; Y)$.) □

Remark A.9. Arendt and Nikolski have shown that Theorem A.7 holds under the even weaker assumption that N is just a point-separating subset of X', see [1, Theorem A.7].

Next, we turn to a remarkable consequence.

Theorem A.10 (Vitali). *Let X be a Banach space, $D \subseteq \mathbb{C}$ open and connected, and $f_n : D \to X$ holomorphic ($n \in \mathbb{N}$). Suppose that $(f_n)_n$ is locally uniformly bounded and the set*

$$C := \{z \in D \mid \lim_{n\to\infty} f_n(z) \text{ exists}\}$$

of convergence points is non-discrete. Then

$$f(z) := \lim_{n\to\infty} f_n(z)$$

exists for all $z \in D$, f is holomorphic, and the convergence is locally uniform in all derivatives.

Proof. Define $F : D \to \ell^\infty(\mathbb{N}; X)$ by $F(z) = (f_n(z))_n$. Then F is locally bounded. Write $\pi_n : \ell^\infty(\mathbb{N}; X) \to X$ for the projection onto the n-th coordinate. Then the functionals

$$x' \circ \pi_n \qquad (x' \in X',\, n \in \mathbb{N})$$

are norming. Moreover, for each $x' \in X'$ and $n \in \mathbb{N}$ the function $x' \circ \pi_n \circ F = x' \circ f_n$ is holomorphic. By Theorem A.7 it follows that F is holomorphic.

The space $\mathrm{c}(\mathbb{N}; X)$ of convergent sequences in X is a closed subspace of $\ell^\infty(\mathbb{N}; X)$, so we can consider the factor space

$$Y := \ell^\infty(\mathbb{N}; X)/\mathrm{c}(\mathbb{N}; X)$$

and the induced holomorphic mapping $F^\sim : D \to Y$. By definition, $F^\sim$ vanishes on C, which by hypothesis is a non-discrete set. Hence, by the identity theorem of complex function theory, $F^\sim$ must vanish identically. So $F(z) \in \mathrm{c}(\mathbb{N}; X)$ for all $z \in D$. The remaining assertions are standard consequences. □

A.4 Unbounded Operators

In this appendix we treat the basic theory of unbounded operators on Banach spaces.

Closed Linear Relations and Operators

Let X and Y be Banach spaces. A **linear relation** between X and Y is a linear subspace

$$A \subseteq X \oplus Y.$$

We call the subspace

$$\mathrm{dom}(A) := \{x \in X \mid \exists\, y \in Y : (x,y) \in A\}$$

of X the **domain** and the subspace

$$\mathrm{ran}(A) := \{y \in Y \mid \exists\, x \in X : (x,y) \in A\}$$

of Y the **range** of the relation A. The space

$$\ker(A) := \{x \in X \mid (x,0) \in A\}$$

is called its **kernel**. With

$$A^{-1} := \{(y,x) \mid (x,y) \in A\}$$

we denote the **inverse relation**. (Hence, $\mathrm{dom}(A) = \mathrm{ran}(A^{-1})$ and $\mathrm{ran}(A) = \mathrm{dom}(A^{-1})$.)

A linear relation A is called a (linear) **operator** if the space

$$A[0] := \ker(A^{-1}) = \{y \in Y \mid (0,y) \in A\} \subseteq Y$$

is trivial. Because of linearity, this is the case if and only if A is a functional relation, i.e.: for each $x \in \mathrm{dom}(A)$ there is exactly one $y \in Y$ such that $(x,y) \in A$. This element y is then usually denoted by Ax, and A is the graph of a linear mapping

$$A : X \supseteq \mathrm{dom}(A) \to Y.$$

Set theory purists would claim that there is anyway no difference between a mapping and its graph. So we are justified to not distinguish them notationally.

A linear relation/operator A is called **densely defined** if $\mathrm{dom}(A)$ is dense in X, **fully defined** if $\mathrm{dom}(A) = X$, and **closed**, if A is closed as a subspace of $X \oplus Y$. Since flipping the entries is a topological isomorphism $X \oplus Y \cong Y \oplus X$, A is closed if and only if A^{-1} is closed.

If A is an operator, the **graph norm** on $\mathrm{dom}(A)$ is given by

$$\|x\|_A := \|x\|_X + \|Ax\|_Y \qquad (x \in \mathrm{dom}(A)).$$

It is a norm on $\mathrm{dom}(A)$ such that both the inclusion mapping $\mathrm{dom}(A) \to X$ and the operator $A : \mathrm{dom}(A) \to Y$ are bounded for the graph norm.

Lemma A.11. *Let X and Y be Banach spaces and let $A : X \supseteq \mathrm{dom}(A) \to Y$ be a linear operator.*

a) *The following assertions are equivalent:*
   (i) *A is closed.*
   (ii) *Whenever $\mathrm{dom}(A) \ni x_n \to x \in X$ and $Ax_n \to y \in Y$ then $x \in \mathrm{dom}(A)$ and $Ax = y$.*
   (iii) *The space $(\mathrm{dom}(A), \|\cdot\|_A)$ is a Banach space.*

b) *Each two of the following three assertions together imply the third one:*
   (i) *A is continuous (for the norm of X), i.e., there is $c \geq 0$ such that $\|Ax\| \leq c\|x\|$ for all $x \in \mathrm{dom}(A)$.*
   (ii) *A is closed.*
   (iii) *$\mathrm{dom}(A)$ is a closed subspace of X.*

Proof. This is left as Exercise 2.1. □

In accordance with the usual terminology, we call a linear operator **bounded** if $A \in \mathcal{L}(X;Y)$. By the closed graph theorem, this is equivalent to $\mathrm{dom}(A) = X$ and A being closed.

A linear relation $A \subseteq X \oplus Y$ is called **invertible** if $A^{-1} \in \mathcal{L}(Y;X)$. (Invertible relations are necessarily closed and have trivial kernel.)

Linear operators are sometimes called **unbounded** operators, even if the possibility of having a bounded operator is not excluded. It would be better to speak of **possibly unbounded operators**, but this terminology is rarely used. Because of this ambiguity, we prefer to call them just "operators".

We write

$$A \subseteq B$$

for two linear relations when this inclusion holds as subspaces of $X \oplus Y$. If both are operators, this just means that $\mathrm{dom}(A) \subseteq \mathrm{dom}(B)$ and $Ax = Bx$ for all $x \in \mathrm{dom}(A)$.

A linear operator A is called **closable** if there is a closed operator B such that $A \subseteq B$. Alternatively, A is closable if the closure $\overline{A}$ within $X \oplus Y$ is again an operator. Closability of A can be characterized by means of sequences through the requirement

$$\operatorname{dom}(A) \ni x_n \to 0 \in X, \quad Ax_n \to y \in Y \quad \Longrightarrow \quad y = 0.$$

If A is closed then a subspace $V \subseteq \operatorname{dom}(A)$ is called a **core** for A if A is the closure of its restriction to V or, in other words, if V is dense in $\operatorname{dom}(A)$ with respect to the graph norm.

The Algebra of Linear Relations and Operators

Let us turn to the algebra of linear relations/operator. The **scalar multiple** λA of a relation A with a scalar $\lambda \in \mathbb{K}$ is defined by

$$\lambda A := \{(x, \lambda y) \mid (x, y) \in A\}.$$

If A is an operator, this just means that

$$\operatorname{dom}(\lambda A) = \operatorname{dom}(A), \qquad (\lambda A)x := \lambda(Ax).$$

Given two relation A, B between X and Y, their **sum** $A + B$ is defined by

$$A + B := \{(x, y + z) \mid (x, y) \in A, \, (x, z) \in B\}.$$

If both are operators, this just means that

$$\operatorname{dom}(A + B) := \operatorname{dom}(A) \cap \operatorname{dom}(B), \qquad (A + B)x := Ax + Bx.$$

Addition of linear relation is associative

$$(A + B) + C = A + (B + C)$$

as a moment's thought reveals.

If A is a linear relation in $X \oplus Y$ and B is a linear relation in $Y \oplus Z$ then their **product** is the (linear!) relation BA in $X \oplus Z$ given by

$$BA := \{(x, z) \mid \exists\, y \in Y : (x, y) \in Y, \, (y, z) \in B\}.$$

If both A and B are operators, this just means that

$$\operatorname{dom}(BA) = \{x \in \operatorname{dom}(A) \mid Ax \in \operatorname{dom}(B)\}, \quad (BA)x := B(Ax).$$

Multiplication of linear relations is associative:

$$(CB)A = C(BA)$$

as is easily seen. Moreover, one has the inversion rule

$$(BA)^{-1} = A^{-1}B^{-1}.$$

Unfortunately, due to the domains of the operators, the algebra of unbounded operators is usually not as simple as for bounded operators. For example, one has

$$0 \cdot A = 0|_{\mathrm{dom}(A)} \subseteq 0$$

but in general *not* $0 \cdot A = 0$, due to the different domains. The following theorem lists a few important properties.

Theorem A.12. *Let A, B, C linear operators. Then the following assertions hold:*

a) $(A + B)C = AC + BC$.

b) $C(A + B) \supseteq CA + CB$, *with equality, e.g., if* $\mathrm{ran}(A) \subseteq \mathrm{dom}(C)$.

c) *If A is closed and B is bounded, then AB is closed.*

d) *If B is closed and A is invertible, then AB is closed.*

Proof. This is left as Exercise 2.2. □

Spectral Theory

We assume the reader to be familiar with the the usual spectral theory for bounded operators (definition of resolvent and spectrum, formula for the spectral radius) as can be found in many introductory texts on functional analysis. Here we want to extend these notions to closed operators, and even to linear relations.

Let X be a Banach space over $\mathbb{K}$, and let $A \subseteq X \oplus X$ be a linear relation. For each $\lambda \in \mathbb{K}$, the relation

$$\lambda - A := \lambda \mathrm{I} - A$$

has the same domain as A and is closed if and only if A is. (Check that!)

We call $\lambda \in \mathbb{K}$ a **spectral point** of A if $\lambda - A$ is *not invertible*. The set

$$\sigma(A) := \{\lambda \in \mathbb{K} \mid \lambda - A \text{ is not invertible}\}$$

is called the **spectrum** of A, its complement

$$\rho(A) := \mathbb{K} \setminus \sigma(A) = \{\lambda \in \mathbb{K} \mid (\lambda - A)^{-1} \in \mathcal{L}(X)\}$$

the **resolvent set** of A. The mapping

$$R(\cdot, A) : \rho(A) \to \mathcal{L}(X), \qquad R(\lambda, A) := (\lambda - A)^{-1}$$

is called the **resolvent** of A. Note that

$$A = \lambda - R(\lambda, A)^{-1} \tag{A.5}$$

for each $\lambda \in \rho(A)$. It follows that a linear relation with non-empty resolvent set must be closed and is uniquely determined by any of its resolvent operators. In particular, A is an operator if and only if one/all resolvent operators $R(\lambda, A)$ are injective.

In order to state the main result, we need another notion. A mapping $R : \Omega \to \mathcal{L}(X)$ (where $\Omega \subseteq \mathbb{K}$ is any non-empty subset) is called a **pseudo-resolvent** if it satisfies the **resolvent identity**

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu) \qquad (\lambda,\ \mu \in \Omega).$$

Note that if $R(\cdot)$ is a pseudo-resolvent, then by interchanging λ and μ it follows that

$$R(\lambda)R(\mu) = R(\mu)R(\lambda) \qquad (\lambda,\ \mu \in \Omega).$$

The connection of resolvents and pseudo-resolvents is given by the next result.

Theorem A.13. *Let A be a linear relation with $\rho(A) \neq \emptyset$. Then its resolvent $R(\cdot, A) : \rho(A) \to \mathcal{L}(X)$ satisfies the resolvent identity, i.e., is a pseudo-resolvent.*

Conversely, let $R : \Omega \to \mathcal{L}(X)$ be any pseudo-resolvent. Then there is a unique linear relation A such that $R(\lambda) = (\lambda - A)^{-1}$ for one (equivalently: all) $\lambda \in \Omega$.

Proof. Let A be a linear relation. Then (A.5) can be written equivalently as ($\lambda \in \rho(A)$ and $x, y \in X$):

$$(x, y) \in A \quad \iff \quad R(\lambda, A)(\lambda x - y) = x. \tag{A.6}$$

Let $z \in X$ and $\lambda, \mu \in \rho(A)$. Let $x := R(\lambda, A)z$ and $y := \lambda x - z$, so that $(x, y) \in A$ and $z = \lambda x - y$. Now (A.6) as it stands and with λ replaced by μ yields

$$\begin{aligned} R(\lambda, A)z = x &= R(\mu, A)(\mu x - y) = R(\mu, A)((\mu - \lambda)x + z) \\ &= (\mu - \lambda)R(\mu, A)x + R(\mu, A)z \\ &= (\mu - \lambda)R(\mu, A)R(\lambda, A)z + R(\mu, A)z. \end{aligned}$$

Conversely, suppose that $R : \Omega \to \mathcal{L}(X)$ is a pseudo-resolvent. According to (A.6) it suffices to show that the relation A defined by

$$(x, y) \in A \quad \overset{\text{def.}}{\iff} \quad R(\lambda)(\lambda x - y) = x$$

is independent of $\lambda \in \Omega$. To this aim take $\lambda,\ \mu \in \Omega$ and $x,\ y \in X$ such that $R(\lambda)(\lambda x - y) = x$. Then, with $z := \lambda x - y$,

$$\begin{aligned} R(\mu)(\mu x - y) &= R(\mu)((\mu-\lambda)x + z) \\ &= (\mu-\lambda)R(\mu)R(\lambda)z + R(\mu)z = R(\lambda)z = x, \end{aligned}$$

and this suffices (by symmetry). □

Theorem A.13 can be rephrased as follows: Each pseudo-resolvent is the restriction of a resolvent of a linear relation.

Remark A.14. Let $R : \Omega \to \mathcal{L}(X)$ be a pseudo-resolvent and A the unique linear relation such that $(\lambda - A)^{-1} = R(\lambda)$ for $\lambda \in \Omega$. Then for each $\lambda \in \Omega$

$$\ker(R(\lambda)) = \{y \in X \mid (0, y) \in A\}.$$

It follows that A is an operator (and not just a linear relation) if and only if $R(\lambda)$ is injective for one/all $\lambda \in \Omega$.

Corollary A.15. *Let A be a linear relation on X, $\mu \in \rho(A)$ and $\lambda \in \mathbb{K}$ such that*

$$R(\lambda) := \sum_{n=0}^{\infty} (\mu - \lambda)^n R(\mu, A)^{n+1}$$

is convergent. Then $\lambda \in \rho(A)$ and $R(\lambda) = R(\lambda, A)$.

Proof. By an elementary calculation, $R(\lambda) - R(\mu, A) = (\mu - \lambda) R(\lambda) R(\mu, A)$. Theorem A.13 with $\Omega := \{\mu, \lambda\}$ yields the claim. □

Corollary A.16. *Let A be a linear relation. Then the resolvent set $\rho(A)$ is an open subset of $\mathbb{K}$ and the resolvent $R(\cdot, A)$ is an analytic mapping. At each point $\mu \in \rho(A)$ it has the power series expansion*

$$R(\lambda, A) = \sum_{k=0}^{\infty} (\lambda - \mu)^k (-1)^k R(\mu, A)^{k+1}$$

which is valid at least for $|\lambda - \mu| < \|R(\mu, A)\|^{-1}$. In particular,

$$\|R(\mu, A)\| \geq \frac{1}{\operatorname{dist}(\mu, \sigma(A))}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} R(\lambda, A) = -R(\lambda, A)^2 \qquad \textit{on } \rho(A).$$

Proof. For $|\lambda - \mu| < \|R(\mu, A)\|^{-1}$ the series

$$R(\lambda) := \sum_{n=0}^{\infty} (\mu - \lambda)^n R(\mu, A)^{n+1}$$

converges. By Corollary A.15 it follows that $\lambda \in \rho(A)$ and $R(\lambda, A) := R(\lambda)$. The remaining statements follow readily as in the theory of scalar-valued holomorphic mappings. □

If $\mathbb{K} = \mathbb{C}$, the resolvent of a linear relation is an operator-valued holomorphic mapping. A short introduction to such mappings is given in Appendix A.3.

(Approximate) Eigenvalues

From now on we confine to operators for convenience. A scalar $\lambda \in \mathbb{K}$ is called an **eigenvalue** of an operator A if

$$\ker(\lambda - A) = \{x \in \mathrm{dom}(A) \mid Ax = \lambda x\} \neq \{0\}.$$

Each vector $0 \neq x \in \ker(\lambda - A)$ is called a corresponding **eigenvector**. The set

$$\sigma_{\mathrm{p}}(A) := \{\lambda \in \mathbb{K} \mid \lambda \text{ eigenvalue of } A\}$$

is called the **point spectrum** of A. Clearly, $\sigma_{\mathrm{p}}(A) \subseteq \sigma(A)$.

Lemma A.17. *For a closed operator A and a scalar $\lambda \in \mathbb{K}$ the following assertions are equivalent:*

(i) *There is $c \geq 0$ such that*

$$\|x\| \leq c\|(\lambda - A)x\| \quad \textit{for all } x \in \mathrm{dom}(A).$$

(ii) *$\lambda - A$ is injective and has closed range.*

Moreover, (i) *and* (i) *are not satisfied if and only if there is a sequence $(x_n)_n$ of $\|\cdot\|_X$-unit vectors in* $\mathrm{dom}(A)$ *such that $\|\lambda x_n - Ax_n\| \to 0$.*

Proof. The equivalence of (i) and (ii) is proved as for bounded operators. The rest is easy. □

A scalar $\lambda \in \mathbb{K}$ such that the equivalent conditions (i) and (ii) do *not* hold, is called an **approximate eigenvalue** of A, and a sequence $(x_n)_n$ as in the last part, is called an **approximate eigenvector** of A. The set

$$\sigma_{\mathrm{a}}(A) = \{\lambda \in \mathbb{C} \mid \lambda \text{ approximate eigenvalue of } A\}$$

is called the **approximate point spectrum** of A. Clearly, $\sigma_{\mathrm{a}}(A) \subseteq \sigma(A)$.

Lemma A.18. *The topological boundary $\partial\sigma(A)$ of the spectrum of an operator A consists of approximate eigenvalues:*

$$\partial\sigma(A) \subseteq \sigma_{\mathrm{a}}(A).$$

Proof. Let $\lambda \in \partial\sigma(A)$ and let $\lambda_n \in \rho(A)$ such that $\lambda_n \to \lambda$. Then, by Theorem A.16, $\|R(\lambda_n, A)\| \to \infty$. Hence, we can find vectors $y_n \in X$ such that $\|y_n\| \leq 1$ and $\|R(\lambda_n, A)y_n\| \to \infty$. Define

$$x_n := \frac{R(\lambda_n, A)y_n}{\|R(\lambda_n, A)y_n\|} \in X.$$

Then $\|x_n\| = 1$ and

$$(\lambda - A)x_n = (\lambda - \lambda_n)x_n + \frac{y_n}{\|R(\lambda_n, A)y_n\|} \to 0. \qquad \square$$

What Does It Mean To Commute?

Two bounded operators A and B on a Banach space commute if $AB = BA$ holds. It would be straighforward to extend this notion of commutation also to general operators, or even to linear relations. However, from such a definition practically nothing useful follows. In particular, for general operators A and B and resolvent points $\lambda \in \rho(A)$ and $\mu \in \rho(B)$ one would expect

$$A \text{ and } B \text{ commute} \quad \Rightarrow \quad R(\lambda, A) \text{ and } R(\mu, B) \text{ commute},$$

but such an implication is sometimes false if the left-hand side just means that $AB = BA$ as linear relations. So we need to look for something better.

Let, as always, X and Y be Banach spaces and let A and B linear relations on X and Y, respectively. A bounded operator $T \in \mathcal{L}(X; Y)$ is said to **intertwine** A with B if

$$TA \subseteq BT. \tag{A.7}$$

Equivalently(!): T intertwines A with B if for all x, $y \in X$ the implication

$$(x, y) \in A \quad \Rightarrow \quad (Tx, Ty) \in B$$

holds. If T intertwines A with B then it also intertwines A^{-1} with B^{-1} and $\lambda - A$ with $\lambda - B$, for each $\lambda \in \mathbb{K}$. (Proof as exercise.)

Note that (A.7) becomes the identity

$$TA = BT$$

if A is fully defined and B is an operator. It follows that if $\lambda \in \rho(A) \cap \rho(B)$ then

$$TA \subseteq BT \quad \Longleftrightarrow \quad T\,R(\lambda, A) = R(\lambda, B)\,T.$$

Corollary A.19. *Let A be a linear relation on a Banach space X such that $\rho(A) \neq \emptyset$ and let $T \in \mathcal{L}(X)$. Then the following statements are equivalent:*

(i) *T intertwines A with itself, i.e., $TA \subseteq AT$.*

(ii) *$TR(\lambda, A) = R(\lambda, A)T$ for one/all $\lambda \in \rho(A)$.*

(iii) *$R(\mu, T)A \subseteq AR(\mu, T)$ for one/all $\mu \in \rho(T)$.*

(iv) *$R(\mu, T)R(\lambda, A) = R(\lambda, A)R(\mu, T)$ for one/all $(\lambda, \mu) \in \rho(A) \times \rho(T)$.*

Cf. also Exercise 2.3.

Polynomials of Operators

Let A be an operator on a Banach space X. Then the powers A^j of A for $j \in \mathbb{N}_0$ are defined recursively by

$$A^0 := \mathrm{I}, \qquad A^j = A^{j-1}A \qquad (j \in \mathbb{N}).$$

It follows from the associativity of operator multiplication that

$$A^k A^m = A^{k+m} \qquad (k, m \in \mathbb{N}_0).$$

A simple induction yields the equivalence

$$x \in \mathrm{dom}(A^n) \quad \iff \quad x \in \mathrm{dom}(A),\ Ax \in \mathrm{dom}(A), \dots, A^{n-1}x \in \mathrm{dom}(A)$$

for each $n \in \mathbb{N}$ and $x \in X$. Moreover, one has the chain of inclusions

$$\mathrm{dom}(A) \supseteq \mathrm{dom}(A^2) \supseteq \mathrm{dom}(A^3) \supseteq \dots .$$

For a polynomial $p(z) = \sum_{j=0}^n a_j z^j \in \mathbb{C}[z]$ with $a_n \neq 0$ we define

$$p(A) := \sum_{j=0}^n a_j A^j$$

so that $\mathrm{dom}(p(A)) = \mathrm{dom}(A^n)$. It follows that

$$p(A) + q(A) = (p+q)(A)$$

for polynomials $p, q \in \mathbb{C}[z]$ as long as the leading coefficients of p and q do not sum up to zero.

Theorem A.20. *Let A be an operator on a Banach space X and let $p, q \in \mathbb{C}[z]$ be polynomials. Then the following assertions hold.*

a) *If $p \neq 0$ then $p(A)q(A) = (pq)(A)$.*

b) *If $\rho(A) \neq \emptyset$ then $p(A)$ is a closed operator.*

Proof. a) We may suppose that $q \neq 0$ as well. By a) and b) of Theorem A.12 it is easy to see that

$$p(A)q(A) \supseteq (pq)(A).$$

So the result follows if we can show that

$$x \in \mathrm{dom}(A^{m+n}) \quad \iff \quad x \in \mathrm{dom}(A^n),\ q(A)x \in \mathrm{dom}(A^m)$$

whenever $x \in X$, $m \in \mathbb{N}_0$ and $\deg(q) = n \in \mathbb{N}_0$. This is clearly true if $n = 0$ or $m = 0$. Also, the implication $\Rightarrow$ is easy. For the converse we induct over $n + m$. So suppose that $\deg(q) = n \geq 1$ and $x \in \mathrm{dom}(A^n)$ such that $q(A)x \in \mathrm{dom}(A^m)$ with $m \geq 1$. Then $q(A)x \in \mathrm{dom}(A^{m-1})$, hence by induction applied to $(n, m-1)$ we know that $x \in \mathrm{dom}(A^{n+m-1})$. But $n \geq 1$, so $x \in \mathrm{dom}(A^m)$. Now write $q(z) = r(z)z + a_0$ with $\deg(r) = n - 1$. Then $r(A)Ax = r(A)Ax + a_0x - a_0x = q(A)x - a_0x \in \mathrm{dom}(A^m)$. So by induction applied to $(m, n-1)$ we conclude that $Ax \in \mathrm{dom}(A^{n-1+m})$, which yields $x \in \mathrm{dom}(A^{n+m})$ as desired.

b) Fix $\lambda \in \rho(A)$. We prove the statement by induction on $n = \deg(p) \in \mathbb{N}_0$, the case $n = 0$ being trivial. Write $p(z) = (\lambda - z)q(z) - \mu$ for some scalar $\mu \in \mathbb{C}$ and a polynomial q with $\deg(q) = n - 1$. Suppose that $x_n \in \mathrm{dom}(A^n)$ with $x_n \to x$ and $p(A)x_n \to y$. Then

$$q(A)x_n = R(\lambda, A)p(A)x_n + \mu R(\lambda, A)x_n \to R(\lambda, A)(y + \mu x).$$

By induction, $x \in \mathrm{dom}(A^{n-1})$ and $q(A)x = R(\lambda, A)(y + \mu x) \in \mathrm{dom}(A)$. So $x \in \mathrm{dom}(A^n)$ and

$$p(A)x = (\lambda - A)q(A)x - \mu x = (\lambda - A)R(\lambda, A)(y + \mu x) - \mu x = y$$

as claimed. □

A.5 Operators on Hilbert Space

In this appendix, an "operator" is always meant to be a possibly unbounded linear operator in the sense of Appendix A.4.

The Hilbert Space Adjoint

Let H and K be Hilbert spaces. The **adjoint** relation A^* of an operator $A : H \supseteq \mathrm{dom}(A) \to K$ is given by

$$(u, v) \in A^* \quad \overset{\text{def.}}{\Longleftrightarrow} \quad (\, x \,|\, v \,)_H = (\, Ax \,|\, u \,)_K \quad \text{for all } x \in \mathrm{dom}(A).$$

The following lemma summarizes its properties. The proof is quite straightfoward.

Lemma A.21. *Let A, B be operators between Hilbert spaces H and K and $\lambda \in \mathbb{K}$. Then the following assertions hold:*

a) *A^* is closed. It is an operator if and only if* $\mathrm{dom}(A)$ *is dense, and in this case*

$$(Ax \,|\, u) = (x \,|\, A^*u) \qquad (x \in \operatorname{dom}(A),\, u \in \operatorname{dom}(A^*)).$$

b) $\ker(A^*) = \operatorname{ran}(A)^\perp$.

c) *A^* is injective if and only if A has dense range.*

d) *If A is injective, then $(A^{-1})^* = (A^*)^{-1}$.*

e) $(\lambda - A)^* = \overline{\lambda} - A^*$.

f) *If A is bounded then so is A^*, and one has $\|A\| = \|A^*\|$.*

g) *If $A \subseteq B$ then $B^* \subseteq A^*$.*

h) *$A^* + B^* \subseteq (A+B)^*$ with equality if $B \in \mathcal{L}(H)$.*

i) *$A^*B^* \subseteq (BA)^*$ with equality if $B \in \mathcal{L}(H)$.*

Proof. As exercise. □

Recall that even if A is an operator, its adjoint need not be an operator as well. Nevertheless, the following corollary yields information about the spectrum of the linear relation A^*.

Corollary A.22 (Spectral mapping theorem for the adjoint). *Let A be a closed operator on H. Then*

$$\sigma(A^*) = \{\overline{\lambda} \mid \lambda \in \sigma(A)\}$$

and $R(\overline{\lambda}, A^*) = R(\lambda, A)^*$ for $\lambda \in \rho(A)$.

Proof. Suppose that $\lambda \in \rho(A)$. Then $(\lambda - A)$ is invertible, and

$$\mathcal{L}(H) \ni R(\lambda, A)^* = [(\lambda - A)^{-1}]^* = [(\lambda - A)^*]^{-1} = (\overline{\lambda} - A^*)^{-1}$$

by d), e) and f) of Lemma A.21. Hence, $\overline{\lambda} \in \rho(A^*)$ and $R(\overline{\lambda}, A^*) = R(\lambda, A)^*$. Conversely, suppose that $\overline{\lambda} - A^* = (\lambda - A)^*$ is invertible. We need to show that $\lambda - A$ is invertible. Abbreviate $B := \lambda - A$. Since B^* is invertible, B has dense range by Lemma A.21.c. Let $x \in \operatorname{dom}(B)$ and $u := (B^*)^{-1}x$. Then $(u, x) \in B^*$ and hence

$$\|x\|^2 = (x \,|\, x) = |(Bx \,|\, u)| \le \left\|(B^*)^{-1}\right\| \|Bx\| \|x\|.$$

It follows that $\|x\| \le \left\|(B^*)^{-1}\right\| \|Bx\|$ for all $x \in \operatorname{dom}(B)$, hence B is injective and has closed range. Since B also has dense range, B is invertible. □

One can rephrase the definition of A^* in terms of orthogonality. On $H \oplus H$ we consider the canonical inner product

$$(\,(u, v) \,|\, (x, y)\,) := (u \,|\, x) + (v \,|\, y)$$

which amounts to

$$\|(x,y)\|^2 = \|x\|^2 + \|y\|^2.$$

Let $J : H \oplus H \to H \oplus H$ be the (unitary!) operator $J(x,y) := (-y,x)$. Then $J^2 = -\mathrm{I}$ and

$$A^* = [J(A)]^\perp = J(A^\perp)$$

as subsets of $H \oplus H$ (the last equation holds since J is unitary). It follows that

$$A^{**} = JA^{*\perp} = J[J(A)]^{\perp\perp} = J\overline{J(A)} = J^2\overline{A} = \overline{A}.$$

In particular, $A^{**} = A$ if and only if A is closed. Since A is an operator, it follows that *the adjoint of a closed and densely defined operator is densely defined.*

The Numerical Range

For an operator A on a Hilbert space H its **numerical range** is defined as

$$\mathrm{W}(A) := \{(\, Ax \,|\, x\,) \ \mid x \in \mathrm{dom}(A),\ \|x\| = 1\}.$$

By the Toeplitz–Hausdorff theorem, the numerical range is always a convex subset of $\mathbb{K}$. Let us collect some useful facts about the numerical range.

Theorem A.23. *Let A be an operator on a Hilbert space H. Then the following assertions hold:*

a) $\sigma_{\mathrm{p}}(A) \subseteq \mathrm{W}(A)$ *and* $\sigma_{\mathrm{a}}(A) \subseteq \overline{\mathrm{W}(A)}$.

b) *If* $\lambda \in \rho(A) \setminus \overline{\mathrm{W}(A)}$, *then*

$$\|R(\lambda, A)\| \le \frac{1}{\mathrm{dist}(\lambda, \mathrm{W}(A))}.$$

c) *If* $U \subseteq \mathbb{K} \setminus \overline{\mathrm{W}(A)}$ *is connected and* $U \cap \rho(A) \neq \emptyset$, *then* $U \subseteq \rho(A)$.

d) *If* $A \in \mathcal{L}(H)$ *then* $\sigma(A) \subseteq \overline{\mathrm{W}(A)}$.

Proof. a) If $\lambda \in \mathbb{K}$ is an eigenvalue of A then there is $x \in \mathrm{dom}(A)$ with $\|x\| = 1$ such that $Ax = \lambda x$. Hence, $\langle Ax, x\rangle = \langle \lambda x, x\rangle = \lambda$. If λ is merely an approximate eigenvalue, then there is a sequence $(x_n)_n$ of unit vectors in $\mathrm{dom}(A)$ with $(\lambda - A)x_n \to 0$. Hence,

$$\lambda - (\, Ax_n \,|\, x_n\,) = (\,(\lambda - A)x_n \,|\, x_n\,) \to 0,$$

so $\lambda \in \overline{\mathrm{W}(A)}$.

b) Let $\delta := \mathrm{dist}(\lambda, \mathrm{W}(A))$. Then, for each $x \in \mathrm{dom}(A)$,

$$\delta\|x\|^2 \le \big|\lambda\|x\|^2 - (\, Ax \,|\, x\,)\big| = |(\,(\lambda - A)x \,|\, x\,)| \le \|(\lambda - A)x\|\|x\|.$$

Replacing x by $R(\lambda, A)x$ we find $\|R(\lambda, A)x\| \le \delta^{-1}\|x\|$ as claimed.

c) Since $\rho(A)$ is open in $\mathbb{K}$, $U \cap \rho(A)$ is open in U. If $(\lambda_n)_n$ is a sequence in $U \cap \rho(A)$ with $\lambda_n \to \lambda \in U$, then by b) $\sup_n \|R(\lambda_n, A)\| < \infty$. Hence, $\lambda \notin \sigma(A)$. Consequenty, $U \cap \rho(A)$ is closed in U. Since U is connected, the claim is proved.

d) Let $A \in \mathcal{L}(H)$ and $\lambda \in \mathbb{K} \setminus \overline{\mathrm{W}(A)}$. Then, as seen above $\delta\|x\| \le \|(\lambda - A)x\|$ for all $x \in H$, so $\lambda - A$ is injective and has closed range. Let $y \perp \operatorname{ran}(\lambda - A)$. Then $A^* y = \overline{\lambda} y$, so that

$$(Ay \,|\, y) = (y \,|\, A^* y) = \big(y \,\big|\, \overline{\lambda} y \big) = \lambda \|y\|^2.$$

Since $\lambda \notin \mathrm{W}(A)$, $y = 0$. It follows that $\lambda - A$ has dense range and is therefore invertible. □

Self-Adjoint Operators

An operator A on a Hilbert space H is **symmetric** if

$$(Ax \,|\, y) = (x \,|\, Ay) \qquad \text{for all } x, y \in \operatorname{dom}(A)$$

and **self-adjoint** if $A^* = A$. A look at the definition of A^* teaches that a densely defined operator is symmetric if and only if $A \subseteq A^*$. Since A^* is a closed operator, it follows that *a densely defined symmetric operator is closable and its closure is again symmetric*. A closable operator A with self-adjoint closure is called **essentially self-adjoint**.

Note that, by Theorem A.23, if A is symmetric then $\lambda - A$ is injective whenever $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Remark A.24. It is a standard result about sesquilinear forms that in the case $\mathbb{K} = \mathbb{C}$ one has

$$A \text{ is symmetric} \quad \Longleftrightarrow \quad \mathrm{W}(A) \subseteq \mathbb{R}.$$

Lemma A.25. *Let A be a symmetric operator. If $\lambda - A$ is surjective and $\overline{\lambda} - A$ has dense range for some $\lambda \in \mathbb{C}$, then A is self-adjoint and $\lambda, \overline{\lambda} \in \rho(A)$.*

Proof. We first show that A is densely defined. Let $y \perp \operatorname{dom}(A)$. By hypothesis there is $x \in \operatorname{dom}(A)$ such that $(\lambda - A)x = y$. Then for all $u \in \operatorname{dom}(A)$

$$\big(x \,\big|\, (\overline{\lambda} - A)u \big) = ((\lambda - A)x \,|\, u) = (y \,|\, u) = 0.$$

It follows that $x \perp \operatorname{ran}(\overline{\lambda} - A)$ and hence $x = 0$. Therefore, $y = 0$.

As A is densely defined, its adjoint A^* is an operator and, by symmetry, $A \subseteq A^*$. Hence,

$$\lambda - A \subseteq \lambda - A^* = (\overline{\lambda} - A)^*.$$

As $\overline{\lambda} - A$ has dense range, its adjoint is injective. But $\lambda - A$ is surjective, so $\lambda - A = \lambda - A^*$, from which it follows that $A = A^*$. Moreover, it follows that

$\lambda - A$ is bijective, so $\lambda \in \rho(A)$. If $\mathbb{K} = \mathbb{R}$, we are done. Else, we first note that $\overline{\lambda} - A = (\lambda - A)^*$ is injective, and

$$(\overline{\lambda} - A)^{-1} = \big((\lambda - A)^*\big)^{-1} = R(\lambda, A)^* \in \mathcal{L}(H) \qquad \square$$

The following is an immediate corollary.

Corollary A.26. *A symmetric operator A with $\rho(A) \cap \mathbb{R} \neq \emptyset$ is self-adjoint.*

For symmetric operators on a complex Hilbert space we have the following important characterization.

Theorem A.27. *For an operator A on a complex Hilbert space H, the following assertions are equivalent:*

(i) *A is essentially self-adjoint.*

(ii) *A is symmetric and densely defined, and for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$ both operators $\lambda - A$, $\overline{\lambda} - A$ have dense range.*

(iii) *A is closable, $\sigma(\overline{A}) \subseteq \mathbb{R}$ and*

$$\|R(\lambda, \overline{A})\| \leq \frac{1}{|\operatorname{Im} \lambda|} \qquad (\lambda \in \mathbb{C} \setminus \mathbb{R}).$$

Proof. All three conditions state or imply the closability of A, so without loss of generality we may suppose that A is closed.

(i) $\Rightarrow$ (ii): Only the assertions about the ranges are to be proved. But since $A = A^*$ it follows that $\mathrm{W}(A^*) = \mathrm{W}(A) \subseteq \mathbb{R}$. Hence, $\lambda - A^*$ and $\overline{\lambda} - A^*$ are both injective, so $\overline{\lambda} - A$ and $\lambda - A$ both have dense range.

(ii) $\Rightarrow$ (iii): As A is symmetric, $\mathrm{W}(A) \subseteq \mathbb{R}$. As A is closed, $\sigma_{\mathrm{a}}(A) \subseteq \overline{\mathrm{W}(A)} \subseteq \mathbb{R}$. It follows from Theorem A.23.a that $\lambda - A$ and $\overline{\lambda} - A$ are injective and have closed ranges. Since, by hypothesis, these ranges are dense, $\lambda, \overline{\lambda} \in \rho(A)$. Then from Theorem A.23.c we conclude that $\mathbb{C} \setminus \mathbb{R} \subseteq \rho(A)$ and

$$\|R(\lambda, A)\| \leq \frac{1}{\operatorname{dist}(\lambda, \mathrm{W}(A))} \leq \frac{1}{\operatorname{dist}(\lambda, \mathbb{R})} = \frac{1}{|\operatorname{Im} \lambda|}$$

for *all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

(iii) $\Rightarrow$ (i): Define $B := \mathrm{i}A$. Then $\mathbb{R} \setminus \{0\} \subseteq \rho(B)$ and $\|\lambda R(\lambda, B)\| \leq 1$ for all $0 \neq \lambda \in \mathbb{R}$. Take $x \in \operatorname{dom}(A) = \operatorname{dom}(B)$. Then

$$\|\lambda x\|^2 \leq \|(\lambda - B)x\|^2 \leq \|\lambda x\|^2 + \|Bx\|^2 - 2\lambda \operatorname{Re}\,(\,x \,|\, Bx\,)$$

It follows that $0 = \operatorname{Re}\,(\,x \,|\, Bx\,) = \operatorname{Im}\,(\,Ax \,|\, x\,)$, i.e., $\mathrm{W}(A) \subseteq \mathbb{R}$. Hence A is symmetric (Remark A.24). By Lemma A.25, A is self-adjoint. $\square$

From now on we want to allow for $\mathbb{K} = \mathbb{R}$ as well as for $\mathbb{K} = \mathbb{C}$.

Lemma A.28. *Let $A \in \mathcal{L}(H)$ be a bounded self-adjoint operator on a Hilbert space H. Let $\lambda_{\min}$ and $\lambda_{\max}$ the smallest and the largest spectral value of A. Then the following assertions hold:*

a) $\lambda_{\max} = \|A\|$ *or* $\lambda_{\min} = -\|A\|$.

b) $\overline{\mathrm{W}(A)} = [\lambda_{\min}, \lambda_{\max}]$.

Proof. a) We may suppose that $c := \|A\| \neq 0$. Then $\sigma(A) \subseteq [-c, c]$. By the definition of the norm there is a sequence $(x_n)_n$ in H such that $\|x\|_n = 1$ and $\|Ax_n\| \to c$. Then, by virtue of the identity $\|A^*A\| = \|A\|^2 = c^2$,

$$\begin{aligned}\|A^*Ax_n - c^2x_n\| &= \|A^*Ax_n\|^2 + -2c^2 \operatorname{Re}(\, A^*Ax_n \,|\, x_n \,) + c^4\|x_n\|^2 \\ &\leq 2c^4 - 2c^2\|Ax_n\|^2 \to 0.\end{aligned}$$

Hence, c^2 is an approximate eigenvalue of $A^*A = A^2$, i.e.,

$$c^2 - A^2 = -(c - A)(-c - A)$$

is not invertible. But this means that at least one of the operators $c - A$ and $-c - A$ is not invertible.

b) By Theorem A.23 , $\sigma(A) \subseteq \overline{\mathrm{W}(A)}$. Since the latter set is convex and closed, $[\lambda_{\min}, \lambda_{\max}] \subseteq \overline{\mathrm{W}(A)}$. For proving the converse we shift the operator and consider $B := A - \alpha$, where $\alpha := \frac{1}{2}(\lambda_{\min} + \lambda_{\max})$. By a) applied to B, we have

$$\lambda_{\min} - \alpha = -\|B\| \quad \text{and} \quad \lambda_{\max} - \alpha = \|B\|.$$

Hence,

$$\mathrm{W}(A) - \alpha = \mathrm{W}(B) \subseteq [-\|B\|, \|B\|] = [\lambda_{\min}, \lambda_{\max}] - \alpha. \qquad \square$$

A symmetric operator A on a Hilbert space H is called **positive** (symbolically: $A \geq 0$) if $(\, Ax \,|\, x \,) \geq 0$ for all $x \in \mathrm{dom}(A)$.

Theorem A.29. *For a self-adjoint operator A on a Hilbert space H the following assertions are equivalent:*

(i) *A is positive, i.e.,* $\mathrm{W}(A) \subseteq [0, \infty)$.

(ii) $\sigma(A) \subseteq [0, \infty)$.

In this case, $\|\lambda(\lambda + A)^{-1}\| \leq 1$ *for all* $\lambda > 0$.

Proof. (i) $\Rightarrow$ (ii): Let $\lambda > 0$. Then $\lambda + A$ is injective and has closed range. But since $A = A^*$ and $\lambda \in \mathbb{R}$, it follows that $\lambda + A$ also has dense range, so is invertible.

(ii) $\Rightarrow$ (i): Again, let $\lambda > 0$. Then $\lambda + A$ is invertible. Denote $B := (\lambda + A)^{-1}$. Then B is a bounded self-adjoint operator with spectrum $\sigma(B) \subseteq [0, \frac{1}{\lambda}]$. By Lemma A.28 it follows that $\|B\| \leq \frac{1}{\lambda}$ and $\mathrm{W}(B) \subseteq [0, \frac{1}{\lambda}]$. But this implies that

$$\lambda + (Ax \mid x) = ((\lambda + A)x \mid x) = (y \mid By) \geq 0$$

whenever $x \in \operatorname{dom}(A)$ with $\|x\| = 1$ and $y := (\lambda + A)x$. As $\lambda > 0$ was arbitrary, $(Ax \mid x) \geq 0$ follows. □

A.6 The Bochner Integral

Let $\Omega = (\Omega, \Sigma, \mu)$ be any measure space and X a Banach space with dual space X'. We built on the notions of weak measurability and integrability developed in Section A.2.

The Pettis Measurability Theorem

For a scalar function $f : \Omega \to \mathbb{C}$ and an element $x \in X$ one writes $f \otimes x$, or fx, for the function

$$f \otimes x : \Omega \to X, \qquad (f \otimes x)(t) := f(t)x.$$

An X-valued **simple function** is a linear combination of functions $\mathbf{1}_A \otimes x$, for $x \in X$ and $A \in \Sigma$. That is,

$$\operatorname{Simp}(\Omega; X) = \operatorname{span}\{\mathbf{1}_A \otimes x \mid A \in \Sigma,\ x \in X\}$$

In other words, one can write

$$f = \sum_{j=1}^{n} \mathbf{1}_{A_j} x_j \tag{A.8}$$

where the A_j are measurable sets and $x_j \in X$. Equivalently, f has finitely many values and for each $x \in f(\Omega)$ the set $[\, f = x \,]$ is in Σ. One can hence find a representation (A.8) where the sets A_j are pairwise disjoint. In this case the function $\|f\|_X$, defined as $\|f\|_X(t) := \|f(t)\|_X$, is a simple function, namely

$$\|f\|_X = \sum_{j=1}^{n} \mathbf{1}_{A_j} \|x_j\|.$$

In particular, $\|f\|_X$ is measurable. Obviously, $\operatorname{Simp}(\Omega; X)$ is a vector space, and invariant under pointwise multiplication with scalar simple functions.

A mapping $f : \Omega \to X$ is called **μ-measurable** if there is a sequence of simple functions $f_n \in \operatorname{Simp}(\Omega; X)$ such that $f_n \to f$ μ-almost everywhere (in the norm of X). If $X = \mathbb{C}$, a μ-measurable function is just an essentially measurable function. Obviously, the set of μ-measurable functions is a vector space, and invariant under multiplication with scalar-valued (essentially) measurable functions. Moreover, if f is μ-measurable then so is $\|f\|_X$.

Lemma A.30. *If $f : \Omega \to X$ is μ-measurable, there is a sequence f_n of X-valued simple functions such that $f_n \to f$ μ-almost everywhere and $\|f_n\|_X \le 2\|f\|_X$ μ-almost everywhere for each $n \in \mathbb{N}$.*

Proof. Since f is μ-measurable, there is a sequence of simple functions $(g_n)_n$ such that $g_n \to f$ almost everywhere. Then $f_n := \mathbf{1}_{[\,\|g_n\|_X \le 2\|f\|_X\,]} g_n$ does the job. (Details as exercise.) □

If f is a simple function with representation (A.8) then for each $x' \in X'$ the function

$$x' \circ f = \sum_{j=1}^{n} \langle x', x_j \rangle \, \mathbf{1}_{A_j}$$

is simple. Since measurability is preserved under limits of sequences, it is clear that a μ-measurable function is essentially weakly measurable. Moreover, it is **essentially separably valued**, by which we mean that there is a null set $N \subseteq \Omega$ such that $f(\Omega \setminus N)$ is contained in a separable subspace X_s of X. The following famous theorem states the converse.

Theorem A.31 (Pettis' Measurability Theorem). *Let $\Omega = (\Omega, \Sigma, \mu)$ be a measure space, X a Banach space and $f : \Omega \to X$ weakly measurable and essentially separably-valued. Then it is μ-measurable.*

Proof. By changing f on a measurable null set, we may suppose that f has values in a separable subspace X_s of X. The Hahn–Banach theorem yields a countable set $M \subseteq \mathrm{B}_{X'}[0,1]$ of the unit ball of X' which is norming for X_s, i.e. such that

$$\|x\|_X = \sup_{x' \in M} |\langle x, x' \rangle| \qquad \text{for all } x \in X_s.$$

Hence the mapping $\|f(\cdot) - x\|$ is measurable for any $x \in X_s$ (being the pointwise supremum of a countable set of measurable functions). Now let $(x_n)_n$ be a sequence, dense in X_s. For fixed $m \in \mathbb{N}$ and $1 \le j \le m$ we let

$$A_{m,j} := \big[\, \|f(\cdot) - x_j\| = \min_{1 \le k \le m} \|f(\cdot) - x_k\| \,\big]$$

and $(B_{m,j})_j$ the "disjointification" of the $A_{m,j}$, i.e.,

$$B_{m,1} := A_{m,1}, \quad B_{m,2} = A_{m,2} \setminus B_{m,1}, \quad \ldots \quad B_{m,m} := A_{m,m} \setminus \bigcup_{j=1}^{m-1} B_{m,j}.$$

Then the $B_{m,j}$ form a finite measurable partition of Ω. Define the step function f_m by

$$f_m := \sum_{j=1}^{m} \mathbf{1}_{B_{m,j}} x_j.$$

Note that for each $t \in \Omega$ one has

$$\|f_m(t) - f(t)\|_X = \min_{1 \le k \le m} \|f(t) - x_k\| \searrow 0$$

as $m \to \infty$ (check that!) since the set $\{x_n \mid n \in \mathbb{N}\}$ is dense in X_s. □

Corollary A.32. *If f_n is μ-measurable and $f_n \to f$ μ-almost everywhere, then f is again μ-measurable.*

Bochner Integrability

A μ-measurable function $f : \Omega \to X$ is called **(Bochner) integrable** if the function $\|f\|_X$ is integrable, i.e.,

$$\int_\Omega \|f(t)\|_X \, \mu(\mathrm{d}t) < \infty.$$

The space of Bochner integrable functions (modulo almost everywhere null-functions) is denoted by

$$\mathrm{L}^1(\Omega; X).$$

This is a vector space and

$$\|f\|_{\mathrm{L}^1} := \int_\Omega \|f\|_X$$

is a norm on it.

Theorem A.33 (Dominated Convergence). *Let $(f_n)_n$ be a sequence in $\mathrm{L}^1(\Omega; X)$ such that $f_n \to f$ μ-almost everywhere and there is a function $0 \le g \in \mathrm{L}^1(\Omega)$ such that $\|f_n\|_X \le g$ almost everywhere for all $n \in \mathbb{N}$. Then $f \in \mathrm{L}^1(\Omega; X)$ and $\|f_n - f\|_{\mathrm{L}^1} \to 0$.*

Proof. By Corollary A.32, f is μ-measurable. Moreover, $\|f_n\|_X \to \|f\|_X$ a.e., hence $\|f\|_X \le g$ a.e.. It follows that $f \in \mathrm{L}^1(\Omega; X)$ and and $\|f - f_n\|_X \le \|f\|_X + \|f_n\|_X \le 2g$ a.e. for each $n \in \mathbb{N}$. So by the scalar dominated convergence theorem, $\|f - f_n\|_{\mathrm{L}^1} \to 0$. □

The previous result implies the following, with a proof exactly as in the scalar case.

Corollary A.34. *The space $\mathrm{L}^1(\Omega; X)$ is complete. If $(f_n)_n$ is a sequence in $\mathrm{L}^1(\Omega, X)$ that converges in L^1-norm to $f \in \mathrm{L}^1(\Omega; X)$, then there is a subsequence $(f_{n_k})_k$ and a function $0 \le g \in \mathrm{L}^1(\Omega)$ such that $f_{n_k} \to f$ a.e. and $\|f_{n_k}\|_X \le g$ a.e. for all $k \in \mathbb{N}$.*

Another consequence of the dominated convergence theorem is that the space of integrable simple functions is dense in $\mathrm{L}^1(\Omega)$.

Corollary A.35. *The space of integrable simple functions is dense in* $\mathrm{L}^1(\Omega; X)$. *More precisely: A function* $f : \Omega \to X$ *is in* $\mathrm{L}^1(\Omega; X)$ *if and only if there is an* L^1*-Cauchy sequence* $(f_n)_n$ *of integrable simple functions such that* $f_n \to f$ μ*-a.e.*.

(In some texts, for example in [3], the definition of $\mathrm{L}^1(\Omega; X)$ is based on the last characterization.)

The Bochner Integral

Let $f \in \mathrm{L}^1(\Omega; X)$, i.e., a Bochner integrable function. Since $|x' \circ f| \leq \|x'\| \|f\|_X$, f is weakly integrable with

$$\Big\| \int_\Omega f \Big\| \leq \int_\Omega \|f(\cdot)\|_X = \|f\|_1.$$

The norm on the left-hand side is by defnition the norm of X''. However, we shall show that actually $\int_\Omega f \in X$.

This is pretty clear for an integrable simple function. Namely, let

$$f = \sum_{j=1}^n \mathbf{1}_{A_j} x_j$$

be an integrable simple function and suppose without loss of generality that the A_j are pairwise disjoint. Then

$$\|f\|_X = \sum_{j=1}^n \|x_j\| \mathbf{1}_{A_j}$$

and since this is integrable, either $\mu(A_j) < \infty$ or $x_j = 0$ for each j. By leaving out zeros from the sum we may further suppose that $\mu(A_j) < \infty$ for all j. A simple computation then yields

$$\int_\Omega f = \sum_{j=1}^n \mu(A_j) x_j \in X.$$

So, since the space of integrable simple functions is dense in $\mathrm{L}^1(\Omega; X)$ and X is (identified with) a closed subspace of X'', it follows that

$$\int_\Omega f \in X \quad \text{for all } f \in \mathrm{L}^1(\Omega; X).$$

The integral of a Bochner integrable function is called its **Bochner integral**.

Lemma A.36. *Let Y be a closed subspace of X and $f \in \mathrm{L}^1(\Omega; X)$ is such that $f \in Y$ μ-almost everywhere. Then*

$$\int_\Omega f \in Y.$$

Proof. For $x' \in X'$ vanishing on Y we obtain

$$\langle \int_\Omega f\,, x' \rangle := \int_\Omega \langle f(\cdot), x' \rangle = 0$$

since $x' \circ f = 0$ μ-almost everywhere. The claim follows from the Hahn–Banach theorem. □

(A closer analysis would show that actually $f \in \mathrm{L}^1(\Omega; Y)$ in Lemma A.36.)

Theorem A.37 (Hille). *Let $A : X \supseteq \mathrm{dom}(A) \to Y$ be a closed operator and $f \in \mathrm{L}^1(\Omega; X)$ such that $f \in \mathrm{dom}(A)$ almost everywhere and $Af := A \circ f \in \mathrm{L}^1(\Omega; Y)$. Then*

$$\int_\Omega f \in \mathrm{dom}(A) \quad \textit{and} \quad A \int_\Omega f = \int_\Omega Af.$$

Proof. By hypothesis, the mapping $(f, Af) : \Omega \to X \oplus Y$ is well defined and Bochner integrable. Since it has (almost everywhere) values in the closed subspace graph(A), its Bochner integral also lies in there (Lemma A.36). Applying the (bounded) projections onto the summands X and Y yields the claim, by virtue of (A.2). □

The Bochner-Lebesgue Spaces

Let $1 \le p < \infty$. A μ-measurable function $f : \Omega \to X$ is called **Bochner p-integrable** if

$$\|f\|_{\mathrm{L}^p}^p := \int_\Omega \|f\|_X^p < \infty.$$

The space of Bochner p-integrable functions (modulo almost everywhere null-functions) is denoted by

$$\mathrm{L}^p(\Omega; X).$$

It is a (useful) exercise to prove the following theorem:

Theorem A.38. *Let Ω be a measure space, X a Banach space, and $1 \le p < \infty$. Then the following assertions hold:*

a) *The set $\mathrm{L}^p(\Omega; X)$ is a space, and $f \mapsto \|f\|_{\mathrm{L}^p}$ is a norm on it.*

b) *A dominated convergence theorem similar to Theorem A.33 holds for $\mathrm{L}^p(\Omega; X)$.*

c) *The space of p-integrable simple functions (which coincides with the space of integrable simple functions) is dense in* $\mathrm{L}^p(\Omega; X)$. *In particular, the space*

$$\mathrm{L}^p(\Omega) \otimes X := \operatorname{span}\{f \otimes x \mid f \in \mathrm{L}^p(\Omega),\, x \in X\}$$

is dense in $\mathrm{L}^p(\Omega; X)$.

d) *The space* $\mathrm{L}^p(\Omega; X)$ *is complete, and each convergent sequence in it has a dominated subsequence that converges almost everywhere to the limit.*

Finally we consider the case $p = \infty$. A μ-measurable function f is called essentially bounded, if there is $c \geq 0$ such that $\|f\|_X \leq c$ almost everywhere, i.e. if $\|f\|_X \in \mathrm{L}^\infty(\Omega)$. We let

$$\mathrm{L}^\infty(\Omega; X)$$

be the space (!) of all essentially bounded μ-measurable functions $f : \Omega \to X$ (modulo almost everywhere null-functions) endowed with the norm(!)

$$\|f\|_{\mathrm{L}^\infty} = \|\|f\|_X\|_{\mathrm{L}^\infty} = \inf\{c \geq 0 \mid \mu[\,\|f\|_X \geq c\,] = 0\}.$$

Convergence in this norm is uniform convergence outside some null set. The space $\mathrm{L}^\infty(\Omega; X)$ is a Banach space. Unless $\dim(X) < \infty$ the space of simple functions is *not* dense in $\mathrm{L}^\infty(\Omega; X)$.

Bochner Spaces over Intervals. Let $J = (a, b) \subseteq \mathbb{R}$ be an interval. (We allow $a = -\infty$ and/or $b = +\infty$.) Then we abbreviate

$$\mathrm{L}^p(a, b; X) := \mathrm{L}^p((a, b), \lambda; X),$$

where λ is the Lebesgue measure.

If $f : J \to X$ is piecewise continuous then it is μ-measurable. This is easily seen by Pettis' measurability theorem. (Note that a continuous image of a separable metric space is separable.)

Simultaneous Approximation

In certain situations it is helpful to know that a given function can be approximated by step functions in L^p-norm for more than one p.

Theorem A.39. *Let* Ω *a measure space,* X *a Banach space and* $1 \leq p < r < q \leq \infty$. *Then the following assertions hold:*

a) $\mathrm{L}^p(\Omega; X) \cap \mathrm{L}^q(\Omega; X) \subseteq \mathrm{L}^r(\Omega; X)$ *and*

$$\|f\|_r \leq \|f\|_p + \|f\|_q \qquad (f \in \mathrm{L}^p \cap \mathrm{L}^q).$$

b) *For each* $f \in \mathrm{L}^1(\Omega; X) \cap \mathrm{L}^\infty(\Omega; X)$ *and each* $\varepsilon > 0$ *there is a sequence of integrable simple functions* $(f_n)_n$ *with* $\|f_n - f\|_1 \to 0$ *and* $\sup_n \|f_n\|_\infty \le \|f\|_\infty + \varepsilon$.

c) *The space of integrable simple functions is dense in the space* $\mathrm{L}^p(\Omega; X) \cap \mathrm{L}^r(\Omega; X)$ *with respect to the norm* $\|f\|_p + \|f\|_r$.

Proof. a) For scalar functions, this is a well-known conseqence of Hölder's inequality. The statement for vector-valued functions follows readily.

b) Let $(g_n)_n$ be a dominated sequence of integrable step functions such that $g_n \to f$ almost everywhere. Then $f_n := g_n \mathbf{1}_{[\,\|g_n(\cdot)\| \le \|f\|_\infty + \varepsilon\,]}$ does the job.

c) Let $f \in \mathrm{L}^p(\Omega; X) \cap \mathrm{L}^r(\Omega; X)$. Then $f_n := f \mathbf{1}_{[\,\|f(\cdot)\| \le n\,]} \to f$ almost everywhere and dominated by $\|f(\cdot)\|$. Hence $f_n \to f$ both in L^p- and in L^r-norm. Observe that each $f_n \in \mathrm{L}^1 \cap \mathrm{L}^\infty$. Now apply b). □

A.7 Complex Measures

A **complex measure** on a measurable space (Ω, Σ) is a mapping $\mu : \Sigma \to \mathbb{C}$ which is σ-additive and satisfies $\mu(\emptyset) = 0$. If the range of μ is contained in $\mathbb{R}$, μ is called a **signed measure**. The sets of complex and signed measures on (Ω, Σ) are denoted by $\mathrm{M}(\Omega, \Sigma)$ and $\mathrm{M}(\Omega, \Sigma; \mathbb{R})$, respectively. They are vector spaces with the natural operations.

The **conjugate** of a complex measure $\mu \in \mathrm{M}(\Omega, \Sigma)$ is the complex measure $\overline{\mu}$ defined by

$$\overline{\mu}(B) := \overline{\mu(B)} \qquad (B \in \Sigma).$$

The **real part** and **imaginary part** of μ are then given by

$$\operatorname{Re} \mu := \frac{1}{2}(\mu + \overline{\mu}), \qquad \text{and} \qquad \operatorname{Im} \mu := \frac{1}{2\mathrm{i}}(\mu - \overline{\mu}).$$

Clearly $\operatorname{Re} \mu$ and $\operatorname{Im} \mu$ are signed measures, and $\mu = \operatorname{Re} \mu + \mathrm{i} \operatorname{Im} \mu$.

A complex measure $\mu \in \mathrm{M}(\Omega, \Sigma)$ is **positive**, written $\mu \ge 0$, if $\mu(B) \ge 0$ for all $B \in \Sigma$. It is then a positive finite measure in the sense of elementary measure theory. The set of positive finite measures is denoted by $\mathrm{M}_+(\Omega, \Sigma)$. The signed measures are ordered by the partial ordering given by

$$\mu \le \nu \quad \overset{\text{def.}}{\iff} \quad \nu - \mu \ge 0.$$

This turns $\mathrm{M}(\Omega, \Sigma; \mathbb{R})$ into a (real) ordered vector space.

The Total Variation

The **total variation** or **modulus** $|\mu|$ of a complex measure $\mu \in \mathrm{M}(\Omega, \Sigma)$ is defined by

$$|\mu|(B) := \sup\Big\{\sum_{n=1}^{\infty} |\mu(B_n)| \;\mid\; (B_n)_{n\in\mathbb{N}} \in \Sigma^{\mathbb{N}},\ B = \bigsqcup_{n\in\mathbb{N}} B_n\Big\}$$

for $B \in \Sigma$. Then $|\mu|$ is a positive finite measure, see [4, Thm. 6.2]. It is characterized by the property

$$\nu \in \mathrm{M}(\Omega, \Sigma),\ \forall\, B \in \Sigma : |\mu(B)| \le \nu(B) \quad\Longrightarrow\quad |\mu| \le \nu.$$

Consequently,

$$|\mu| = \sup_{c\in\mathbb{T}} \operatorname{Re}(c\mu) = \sup_{t\in\mathbb{Q}} \operatorname{Re}(\mathrm{e}^{\mathrm{i}\pi t}\mu).$$

From this, the following statements about the total variation are easy to prove (μ, $\nu \in \mathrm{M}(\Omega, \Sigma)$, $c \in \mathbb{C}$):

a) $|\mu + \nu| \le |\mu| + |\nu|$.

b) $|c\mu| = |c|\,|\mu|$.

c) $|\operatorname{Re}\mu|\,,\ |\operatorname{Im}\mu| \le |\mu|$.

The definition

$$\|\mu\|_{\mathrm{M}} := |\mu|(\Omega) \qquad (\mu \in \mathrm{M}(\Omega, \Sigma))$$

turns $\mathrm{M}(\Omega, \Sigma)$ into a Banach space. (This is a simple exercise.)

If μ is a signed measure, then $\mu^+ := \frac{1}{2}(|\mu| + \mu)$ and $\mu^- := \frac{1}{2}(|\mu| - \mu)$ are positive measures satisfying

$$\mu^+ - \mu^- = \mu, \quad \mu^+ + \mu^- = |\mu|\,.$$

It follows that each complex measure μ is a linear combination of four positive measures μ_j satisfying $\mu_j \le |\mu|$.

Integration with respect to a Complex Measure

Let $\mu \in \mathrm{M}(\Omega, \Sigma)$. For a step function $f = \sum_{j=1}^{n} x_j \mathbf{1}_{A_j} \in \mathrm{Step}(\Omega, \Sigma, |\mu|)$ one defines

$$\int_\Omega f \,\mathrm{d}\mu := \sum_{j=1}^{n} x_j \mu(A_j)$$

as usual, and shows (using finite additivity) that this does not depend on the representation of f. Moreover, one obtains

$$\Bigl|\int_\Omega f\,\mathrm{d}\mu\Bigr| \le \int_\Omega |f|\;\mathrm{d}|\mu| = \|f\|_{\mathrm{L}^1(\Omega,\Sigma,|\mu|)}, \tag{A.9}$$

whence the integral has a continuous linear extension to $\mathrm{L}^1(\Omega, \Sigma, |\mu|)$. By continuity, (A.9) remains true for all $f \in \mathrm{L}^1(\Omega, \Sigma, |\mu|)$.

Alternatively, one can write μ as a linear combination of four positive measures μ_j as

$$\mu = (\mu_1 - \mu_2) + \mathrm{i}(\mu_3 - \mu_4) \tag{A.10}$$

where each $\mu_j \le |\mu|$. By the latter inequality, $\mathrm{L}^1(\Omega, \Sigma, |\mu|) \subseteq \mathrm{L}^1(\Omega, \Sigma, \mu_j)$ canonically. Hence, for $f \in \mathrm{L}^1(\Omega, \Sigma, |\mu|)$ one can let

$$\int_\Omega f\,\mathrm{d}\mu := \int_\Omega f\,\mathrm{d}\mu_1 - \int_\Omega f\,\mathrm{d}\mu_2 + \mathrm{i}\Bigl(\int_\Omega f\,\mathrm{d}\mu_3 - \int_\Omega f\,\mathrm{d}\mu_4\Bigr).$$

Of course, one has to make sure that this definition is independent of the representation (A.10).

By using either way of computing the integral, one finds that

$$\int_\Omega f\,\mathrm{d}\overline{\mu} = \overline{\int_\Omega \overline{f}\,\mathrm{d}\mu} \quad \text{for all } f \in \mathrm{L}^1(\Omega, \Sigma, |\mu|).$$

The definition of weak or weak*-integrals from Appendix A.2 can be easily extended to complex measures. Suppose that $\mu \in \mathrm{M}(\Omega, \Sigma)$ is a complex measure, X is a Banach space and $f : \Omega \to X$ is weakly integrable with respect to $|\mu|$. Then for each $x' \in X'$ one has

$$\Bigl|\int_\Omega \langle f(\cdot), x'\rangle\,\mathrm{d}\mu\Bigr| \le \int_\Omega |\langle f(\cdot), x'\rangle|\,\mathrm{d}|\mu| \le c\|x'\| \tag{A.11}$$

for some $c \ge 0$ independent of x'. Hence, as for a positive measure, one can define the integral $\int_\Omega f\,\mathrm{d}\mu$ as the unique element of X'' satisfying

$$\langle \int_\Omega f\,\mathrm{d}\mu\,, x'\rangle := \int_\Omega \langle f(\cdot), x'\rangle\;\mathrm{d}\mu \qquad (x' \in X'). \tag{A.12}$$

In general one may have $\int_\Omega f\,\mathrm{d}\mu \notin X$. However, if $f \in \mathrm{L}^1(\Omega, \Sigma, |\mu|\,; X)$ (the Bochner space), then $\int_\Omega f\,\mathrm{d}\mu \in X$. This is most easily seen by decomposing μ as in (A.10) and noting that one has

$$\mathrm{L}^1(\Omega, \Sigma, |\mu|\,; X) \subseteq \mathrm{L}^1(\Omega, \Sigma, \mu_j; X)$$

canonically for each j.

If f is weakly integrable and $\|f(\cdot)\|_X$ is integrable with respect to $|\mu|$, then

$$\Bigl\|\int_\Omega f\,\mathrm{d}\mu\Bigr\| \le \int_\Omega \|f(\cdot)\|_X \mathrm{d}|\mu|. \tag{A.13}$$

This follows from (A.12) and (A.11) by taking the supremum over $x' \in X'$ with $\|x'\| \le 1$.

The situation for weak* integrals is analogous.

The Radon–Nikodym Theorem

Let $\Omega = (\Omega, \Sigma, \mu)$ be a measure space. For $f \in \mathrm{L}^1(\Omega)$ a complex measure $f\mu$ is defined by

$$(f\mu)(B) := \int_B f \,\mathrm{d}\mu \qquad (B \in \Sigma).$$

The mapping

$$\mathrm{L}^1(\Omega) \to \mathrm{M}(\Omega, \Sigma), \qquad f \mapsto f\mu$$

is an *isometric embedding* satisfying

$$|f\mu| = |f|\,\mu.$$

Integration with respect to $f\mu$ is simple: one has

$$g \in \mathrm{L}^1(\Omega, \Sigma, |f\mu|) \quad \Longleftrightarrow \quad fg \in \mathrm{L}^1(\Omega)$$

and, in this case,

$$\int_\Omega g \,\mathrm{d}f\mu = \int_\Omega fg \,\mathrm{d}\mu.$$

The following famous result yields a characterization of those complex measures that are of the form $f\mu$ for some $f \in \mathrm{L}^1(\Omega)$.

Theorem A.40 (Radon–Nikodym). *Let $\Omega = (\Omega, \Sigma, \mu)$ be σ-finite measure space. A complex measure $\nu \in \mathrm{M}(\Omega, \Sigma)$ is of the form $\nu = f\mu$ for some $f \in \mathrm{L}^1(\Omega)$ if and only if it satisfies*

$$\forall\, B \in \Sigma : \mu(B) = 0 \quad \Longrightarrow \quad \nu(B) = 0. \tag{A.14}$$

The property (A.14) is called the **absolute continuity** of ν with respect to μ. A proof of Theorem A.40 can be found in [4, p. 6.10].

Locally Compact Spaces

A topological space Ω is called **locally compact** if each point in Ω has a compact neighborhood. It can be shown that if Ω in addition is Hausdorff, then each point even has a neighborhood *base* consisting of compact sets.

Locally compact Hausdorff spaces provide a rich supply of continuous functions on it, a fact that makes them particularly suitable objects for functional analysis. For example, **Urysohn's lemma** tells that to each compact subset K and open subset O of a locally compact Hausdorff space Ω there is $f \in \mathrm{C_c}(\Omega)$ such that $0 \le f \le 1$, $f = 1$ on K and $\operatorname{supp}(f) \subseteq O$. (Here, $\operatorname{supp}(f) := \overline{[\, f \neq 0\,]}$ is the **support** of f and $\mathrm{C_c}(\Omega)$ is the space of continuous functions with compact support.) Urysohn's lemma is easy to prove if Ω is metric. A proof in the general case can be found in [4, Sec. 2.12].

Obviously, $\mathrm{C_c}$-functions are bounded. The supremum-norm closure of $\mathrm{C_c}(\Omega)$ is $\mathrm{C_0}(\Omega)$, the space of continuous functions that **vanish at infinity**. A continuous function f on Ω is contained in $\mathrm{C_0}(\Omega)$ if and only if for each $\varepsilon > 0$ there is a compact set $K \subseteq \Omega$ such that $|f| \le \varepsilon$ on $\Omega \setminus K$. (This follows from Urysohn's lemma, see [4, Sec. 3.17].)

The Riesz–Markov–Kakutani Theorem

Let Ω be a locally compact Hausdorff space. We take, canonically, $\Sigma = \mathrm{Bo}(\Omega)$ the Borel σ-algebra and suppress explicit reference to Σ henceforth.

A finite positive Borel measure is **regular** if for each Borel set A

$$\begin{aligned} \mu(A) &= \inf\{\mu(O) \mid O \supseteq A,\, O \text{ open}\} \\ &= \sup\{\mu(K) \mid K \subseteq A,\, K \text{ compact}\}. \end{aligned}$$

A consequence of regularity is (by Urysohn's lemma) that the space $\mathrm{C_c}(\Omega)$ of continuous functions of compact support is dense in $\mathrm{L}^p(\Omega, \mu)$ for $1 \le p < \infty$, see [4, p. 3.14].

A complex Borel measure μ on Ω is called **regular** if $|\mu|$ is regular. It is [4, Exe. 6.3] to show that the space

$$\mathrm{M}(\Omega) := \{\mu \in \mathrm{M}(\Omega, \mathrm{Bo}(\Omega)) \mid \mu \text{ is regular}\}$$

of regular complex Borel measures is a closed subspace of all complex Borel measures. If Ω is separable and metrizable, each complex Borel measure is regular.

Theorem A.41 (Riesz–Markov–Kakutani (RMK)). *The mapping*

$$\mathrm{M}(\Omega) \to \mathrm{C_0}(\Omega)', \qquad \mu \mapsto (f \mapsto \int_\Omega f \,\mathrm{d}\mu)$$

is a positivity-preserving isometric isomorphism.

A proof can be found in [4, Thm. 6.19] or [3, Thm. IX.4.2].

A.8 Accretive Operators on Hilbert Spaces

This appendix is a continuation of Appendix A.5.

An operator A on a Hilbert space H is called **accretive** if its numerical range $\mathrm{W}(A)$ is contained in $[\,\mathrm{Re}\,\mathbf{z} \ge 0\,]$, i.e., if

$$\mathrm{Re}\,(\,Ax \,|\, x\,) \ge 0 \qquad \text{for all } x \in \mathrm{dom}(A).$$

An operator A is called **dissipative** if $-A$ is accretive. Note that an operator A is symmetric if and only if $\pm\mathrm{i}A$ both are accretive. And a self-adjoint operator is accretive if and only if it is positive.

Here are a couple of equivalent characterizations of accretivity.

Lemma A.42. *Let A be an operator on a Hilbert space H and let $\mu > 0$. Then the following assertions are equivalent:*

(i) *A is accretive;*

(ii) $\|(A+\mu)x\| \ge \|(A-\mu)x\|$ *for all* $x \in \mathrm{dom}(A)$*;*

(iii) $\|(A+\lambda)x\| \ge (\mathrm{Re}\,\lambda)\|x\|$ *for all* $x \in \mathrm{dom}(A)$, $\mathrm{Re}\,\lambda \ge 0$*;*

(iv) $\|(A+\lambda)x\| \ge \lambda\|x\|$ *for all* $x \in \mathrm{dom}(A)$, $\lambda \ge 0$.

Proof. (i) $\Leftrightarrow$ (ii) follows from $\|(A+\mu)x\|^2 - \|(A-\mu)x\|^2 = 4\,\mathrm{Re}\,(\,Ax\,|\,x\,)$ for all $x \in \mathrm{dom}(A)$.

(i) $\Leftrightarrow$ (iv): For $\lambda > 0$ we have $\|(A+\lambda)x\|^2 - \lambda^2\|x\|^2 = \|Ax\|^2 + 2\lambda\,\mathrm{Re}\,(\,Ax\,|\,x\,)$ for all $x \in \mathrm{dom}(A)$. This shows that (i) $\Rightarrow$ (iv). Dividing by λ and letting $\lambda \to \infty$ yields the converse implication.

(iii) $\Rightarrow$ (iv): This is obvious.

(i) $\Rightarrow$ (iii): Suppose that (i) holds and let $\mathrm{Re}\,\lambda \ge 0$. Define $\alpha := \mathrm{Im}\,\lambda$. Then (i) holds with A replaced by $A + \mathrm{i}\alpha$. Since we have already established the implication (i) $\Rightarrow$ (iv),

$$\|(A+\lambda)x\| = \|((A+\mathrm{i}\alpha) + \mathrm{Re}\,\lambda)x\| \ge (\mathrm{Re}\,\lambda)\|x\|$$

for all $x \in \mathrm{dom}(A)$, which is (iii). $\square$

An operator A is called **m-accretive** if it is accretive and closed and $\mathrm{ran}(1 + A)$ is dense in H.

Theorem A.43. *Let A be an operator on H, $\alpha \in \mathbb{R}$ and $\lambda > 0$. The following assertions are equivalent:*

(i) *A is m-accretive;*

(ii) *$A + \mathrm{i}\alpha$ is m-accretive;*

(iii) *$A + \varepsilon$ is m-accretive for all $\varepsilon > 0$;*

(iv) $-\lambda \in \rho(A)$ *and* $\|(A-\lambda)(A+\lambda)^{-1}\| \le 1$*;*

(v) $\{\operatorname{Re}\mathbf{z} < 0\} \subseteq \rho(A)$ *and*

$$\|R(\lambda, A)\| \le \frac{1}{|\operatorname{Re}\lambda|} \qquad (\operatorname{Re}\lambda < 0);$$

(vi) $(-\infty, 0) \subseteq \rho(A)$ *and* $\sup_{t>0} \|t(t+A)^{-1}\| \le 1$*;*

(vii) *A is closed and densely defined, and A^* is m-accretive.*

Proof. Let B be a closed, accretive operator on H. By a) and c) of Theorem A.23, if $\operatorname{ran}(B+\mu)$ is dense for some μ with $\operatorname{Re}\mu > 0$, then $[\operatorname{Re}\mathbf{z} < 0] \subseteq \rho(B)$. (This consideration will be used several times in the following.) It follows that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

The equivalence (i) $\Leftrightarrow$ (iv) holds by (ii) of Lemma A.42. Similarly, (i)$\Leftrightarrow$(v) and (i)$\Leftrightarrow$(vi) hold by (iii) and (iv) of Lemma A.42, respectively.

(vi) $\Rightarrow$ (vii): By (vi), A is sectorial. As H is reflexive, $\operatorname{dom}(A)$ is dense (Theorem 9.2.c)). Taking the adjoint yields that (vi) also holds for A^* in place of A. Hence, by what was already shown, A^* is m-accretive.

(vii) $\Rightarrow$ (i): By the already established implication (i)$\Rightarrow$(vii) we conclude that $A = \overline{A} = A^{**}$ is m-accretive. □

The operator $(A-1)(A+1)^{-1}$ is called the **Cayley transform** of A.

Theorem A.44 (Lumer–Phillips). *An operator A on a Hilbert space H is m-accretive if and only if $-A$ generates a strongly continuous contraction semigroup.*

Proof. If A is m-accretive, parts (vi) and (vii) of Theorem A.43 show that the Hille–Yosida theorem (Theorem 8.17) is applicable with $\omega = 0$ and $M = 1$ to the operator $-A$. Conversely, suppose that $-A$ generates the C_0-semigroup T such that $\|T(t)\| \le 1$ for all $t \ge 0$. Then, by the "easy" part of the Hille–Yosida theorem, (vi) of Theorem A.43 holds. □

Theorem A.45. (Stone) *An operator $-\mathrm{i}A$ on a Hilbert space H generates a C_0-group of unitary operators if and only if A is self-adjoint.*

Proof. Suppose that A is self-adjoint. Then $\pm\mathrm{i}A$ are both m-accretive and hence, by the Lumer–Phillips theorem, generate contraction semigroups. By Theorem 6.8, $-\mathrm{i}A$ generates a unitary group. This proof also works in the converse direction. □

A.9 The Fourier Transform for Vector-Valued Functions

In Chapter 5 the Fourier transform of bounded measures and integrable functions on $\mathbb{R}^d$ was introduced. In this appendix we back up those findings with some results about the **vector-valued Fourier transform** defined by

$$\widehat{f}(t) := (\mathcal{F}f)(t) := \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}t\cdot s} f(s)\,\mathrm{d}s \qquad (t \in \mathbb{R}^d,\ f \in \mathrm{L}^1(\mathbb{R}^d; X)),$$

where X is any Banach space. Obviously,

$$\mathcal{F}(f \otimes x) = \widehat{f} \otimes x$$

where $f \in \mathrm{L}^1(\mathbb{R}^d)$ is a scalar function and $x \in X$.

Since $\mathrm{L}^1(\mathbb{R}^d) \otimes X$ is dense in $\mathrm{L}^1(\mathbb{R}^d; X)$ (Theorem A.38), results from the scalar case extend by approximation to the vector-valued case. For example, the **Riemann–Lebesgue lemma** tells that the Fourier transform is a linear contraction

$$\mathcal{F} : \mathrm{L}^1(\mathbb{R}^d; X) \to \mathrm{C}_0(\mathbb{R}^d; X).$$

Recall from Example 5.2.6) that for $1 \le p < \infty$ the space $\mathrm{L}^p(\mathbb{R}^d; X)$ is shift invariant and the regular representation $(\tau_t)_{t\in\mathbb{R}^d}$ is strongly continuous and contractive thereon. In particular, if $\mu \in \mathrm{M}(\mathbb{R}^d)$ and $f \in \mathrm{L}^p(\mathbb{R}^d; X)$ one can form the convolution $\mu * f = \int_{\mathbb{R}^d} \tau_t f\, \mu(\mathrm{d}t)$ and obtains the estimate $\|\mu * f\|_p \le \|\mu\|_{\mathrm{M}} \|f\|_p$.

The identity

$$\mathcal{F}(\mu * f) = \widehat{\mu} \cdot \widehat{f}, \tag{A.15}$$

established in Theorem 5.7 for $\mu \in \mathrm{M}(\mathbb{R}^d)$ and $f \in \mathrm{L}^1(\mathbb{R}^d)$ continues to hold for $f \in \mathrm{L}^1(\mathbb{R}^d; X)$. Analogously, the identity

$$\mathcal{F}(\psi \widehat{f}) = \widehat{\psi} * (\mathcal{S}f)$$

from (5.11) extends to the case $f \in \mathrm{L}^1(\mathbb{R}^d; X)$ and $\psi \in \mathrm{L}^1(\mathbb{R}^d)$. Finally, if $(\varphi_n)_n$ is an approximation of the identity in $\mathrm{L}^1(\mathbb{R}^d)$, then

$$\varphi_n * f \to f \quad \text{in } \mathrm{L}^p(\mathbb{R}^d; X)$$

for all $f \in \mathrm{L}^p(\mathbb{R}^d; X)$, $1 \le p < \infty$.

Lemma A.46. *Let $0 \ne \psi \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{C}_0(\mathbb{R}^d)$ such that $\widehat{\psi} \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{C}_0(\mathbb{R}^d)$ and $\psi(0) = 1$ and $\psi_n = \psi(\mathbf{t}/n)$ for $n \in \mathbb{N}$. Then the following statements hold:*

a) *$\psi_n \to \mathbf{1}$ boundedly and uniformly on compacts.*

b) *$((2\pi)^{-d}\widehat{\psi_n})_n$ is a Dirac sequence.*

c) *$\widehat{\psi_n} * f \to (2\pi)^d f$ in L^p-norm for each $f \in \mathrm{L}^p(\mathbb{R}^d; X)$, $1 \le p < \infty$.*

Proof. a) is clear and c) follows from b). For b) note that $\widehat{\psi_n} = n^d \widehat{\psi}(n\mathbf{s})$. So, by Exercise 6.9, $(c^{-1}\widehat{\psi_n})_n$ is a Dirac sequence, where

$$c = \int_{\mathbb{R}^d} \widehat{\psi}.$$

For the concrete choice $\psi := \exp(-\sum_{j=1}^d |\mathbf{x}_j|)$ one can compute the Fourier transform explicitly by elementary methods and finds $c = (2\pi)^d$. It will be a consequence of the Fourier inversion theorem below (Theorem A.47) that c is actually independent of the choice of ψ. □

Theorem A.47 (Fourier inversion). *If $f \in \mathrm{L}^1(\mathbb{R}^d; X)$ is such that $\widehat{f} \in \mathrm{L}^1(\mathbb{R}^d; X)$, then f has a representative in $\mathrm{C}_0(\mathbb{R}^d; X)$ given by*

$$f(t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(s) \mathrm{e}^{\mathrm{i}s\cdot t} \mathrm{d}s$$

for all $t \in \mathbb{R}^d$. In other words: $f = (2\pi)^{-d} \mathcal{S}\mathcal{F}\widehat{f}$ (almost everywhere).

Proof. Take any function $\psi \in \mathrm{C}_0(\mathbb{R}^d) \cap \mathrm{L}^1(\mathbb{R}^d)$ such that $\widehat{\psi} \in \mathrm{L}^1(\mathbb{R}^d)$ and the sequence $(\psi_n)_n$ satisfies a)–c) of Lemma A.46. Then

$$(2\pi)^d \mathcal{S}f = \lim_{n\to\infty} \widehat{\psi_n} * \mathcal{S}f = \lim_{n\to\infty} \mathcal{F}(\psi_n \widehat{f}) \tag{A.16}$$

as a limit in $\mathrm{L}^1(\mathbb{R}^d; X)$. On the other hand, $\psi_n \widehat{f} \to \widehat{f}$ in $\mathrm{L}^1(\mathbb{R}^d; X)$ as well, hence $\mathcal{F}(\psi_n \widehat{f}) \to \mathcal{F}^2 f$ uniformly. The claim follows. □

Corollary A.48. *The space*

$$\mathrm{E}(\mathbb{R}^d; X) := \{f \mid f, \widehat{f} \in \mathrm{L}^1(\mathbb{R}^d; X) \cap \mathrm{C}_0(\mathbb{R}^d; X)\}$$

is shift and reflection invariant. It is dense in $\mathrm{L}^p(\mathbb{R}^d; X)$ for all $1 \le p < \infty$. The Fourier transform restricts to an isomorphism of $\mathrm{E}(\mathbb{R}^d; X)$ with inverse $\mathcal{F}^{-1} = (2\pi)^{-d}\mathcal{S}\mathcal{F}$.

Proof. The first and last assertions follow from the inversion theorem since $\mathcal{S}\mathcal{F} = \mathcal{F}\mathcal{S}$ and $\mathcal{F}(\tau_t f) = \mathrm{e}^{-\mathrm{i}t\mathbf{s}}\widehat{f}$. For the middle statement let $f \in \mathrm{L}^1(\mathbb{R}^d; X) \cap \mathrm{C}_0(\mathbb{R}^d; X)$ and let $(\psi_n)_n$ be a sequence as in Lemma A.46. Then $f_n := \widehat{\psi_n} * f \to (2\pi)^d f$ in $\mathrm{L}^p(\mathbb{R}^d; X)$. Moreover, $f_n \in \mathrm{L}^1(\mathbb{R}^d; X)$ and

$$\widehat{f_n} = (\mathcal{F}^2 \psi_n) \cdot \widehat{f} = (2\pi)^d (\mathcal{S}\psi_n)\, \widehat{f} \in \mathrm{L}^1(\mathbb{R}^d)$$

for each $n \in \mathbb{N}$. The claim follows. □

We head for Plancherel's theorem. The following is an auxiliary result.

Lemma A.49. *Let $f \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^2(\mathbb{R}^d)$. Then $\widehat{f} \in \mathrm{L}^2(\mathbb{R}^d)$ and*

$$\|\widehat{f}\|_2^2 = (2\pi)^d \|f\|_2^2. \tag{A.17}$$

Proof. Define $h := f * \overline{\mathcal{S}f}$. Then $\widehat{h} = \widehat{f} \cdot \mathcal{F}(\overline{\mathcal{S}f}) = |\widehat{f}|^2$ and $h \in \mathrm{C}_0(\mathbb{R}^d)$. (By the Cauchy–Schwarz inequality and since $\mathrm{C}_\mathrm{c}(\mathbb{R}^d)$ is dense in $\mathrm{L}^2(\mathbb{R}^d)$, convolution maps $\mathrm{L}^2 \times \mathrm{L}^2$ boundedly into $\mathrm{C}_0(\mathbb{R}^d)$.) Let $(\psi_n)_n$ be a sequence of functions as in Lemma A.46. Then

$$\int_{\mathbb{R}^d} |\widehat{f}|^2 \psi_n = \int_{\mathbb{R}^d} \widehat{h}\, \psi_n = \int_{\mathbb{R}^d} h\, \widehat{\psi_n}.$$

By b) of Lemma A.46, passing to the limit yields

$$\|\widehat{f}\|_2^2 = \int_{\mathbb{R}^d} |\widehat{f}|^2 = (2\pi)^d h(0) = (2\pi)^d \|f\|_2^2$$

as claimed. □

Identity (A.17) is called **Plancherel's identity**.

Theorem A.50 (Plancherel). *Let H be a Hilbert space. Then the Fourier transform maps $\mathrm{L}^1(\mathbb{R}^d; H) \cap \mathrm{L}^2(\mathbb{R}^d; H)$ into $\mathrm{L}^2(\mathbb{R}^d; H)$ and extends to a bounded operator (again denoted by $\mathcal{F}$) on $\mathrm{L}^2(\mathbb{R}^d; H)$ in such a way that $\mathcal{F}^2 = (2\pi)^d \mathcal{S}$ and the operator*

$$(2\pi)^{-\frac{d}{2}} \mathcal{F} : \mathrm{L}^2(\mathbb{R}^d; H) \to \mathrm{L}^2(\mathbb{R}^d; H)$$

is unitary. Moreover, for each $\mu \in \mathrm{M}(\mathbb{R}^d)$ and $f \in \mathrm{L}^2(\mathbb{R}^d; H)$

$$\mathcal{F}(\mu * f) = \widehat{\mu} \cdot \widehat{f}. \tag{A.18}$$

Proof. Let $f : \mathbb{R}^d \to H$ be an integrable simple function. Then there is a finite orthonormal system $e_1, \dots, e_n$ in H and $f_1, \dots, f_n \in \mathrm{L}^2(\mathbb{R}^d) \cap \mathrm{L}^1(\mathbb{R}^d)$ such that $f = \sum_{j=1}^n f_j \otimes e_j$. Hence, by Lemma A.49,

$$\begin{aligned} \|\mathcal{F} f\|^2_{\mathrm{L}^2(\mathbb{R}^d; H)} &= \int_{\mathbb{R}^d} \|f(s)\|_H^2 \,\mathrm{d}s = \int_{\mathbb{R}^d} \sum_{j=1}^n |\mathcal{F} f_j(s)|^2 \,\mathrm{d}s = \sum_{j=1}^n \|\mathcal{F} f_j\|_2^2 \\ &= \sum_{j=1}^n (2\pi)^d \|f_j\|_2^2 = \cdots = (2\pi)^d \|f\|^2_{\mathrm{L}^2(\mathbb{R}^d; H)}. \end{aligned}$$

Since the space of integrable simple functions is dense in $\mathrm{L}^2(\mathbb{R}^d; H)$, $\mathcal{F}$ has a unique bounded extension to $\mathrm{L}^2(\mathbb{R}^d; H)$. Since the integrable simple functions are even dense in $\mathrm{L}^1(\mathbb{R}^d; H) \cap \mathrm{L}^2(\mathbb{R}^d; H)$ (Theorem A.39), this bounded extension coincides with $\mathcal{F}$ on that space. Hence, it is allowed to denote it again by $\mathcal{F}$.

Plancherel's identity (A.17) has been shown already for integrable simple functions f, so it holds for all $f \in \mathrm{L}^2(\mathbb{R}^d; H)$ by approximation. Similarly, the identity (A.18) holds for $f \in \mathrm{L}^1(\mathbb{R}^d)$, hence for integrable simple functions, and hence for all $f \in \mathrm{L}^2(\mathbb{R}^d; H)$.

By Theorem A.47, the identity $\mathcal{F}^2 f = (2\pi)^d \mathcal{S} f$ holds for functions $f \in \mathrm{E}(\mathbb{R}^d; H)$. By Corollary A.48 these functions are dense in $\mathrm{L}^2(\mathbb{R}^d; H)$ so by approximation, the identity is true for all $f \in \mathrm{L}^2(\mathbb{R}^d; H)$. It follows, in particular, that $(2\pi)^{-d/2}\mathcal{F}$ is a surjective isometry, i.e., a unitary operator. $\square$

The Fourier Transform on the Schwartz Space

The **Schwartz space** of X-valued functions on $\mathbb{R}^d$ is defined by

$$\mathcal{S}(\mathbb{R}^d; X) := \left\{ f \in \mathrm{C}^\infty(\mathbb{R}^d; X) \;\middle|\; \mathbf{t}^\alpha \, \mathrm{D}^\beta f \in \mathrm{C_b}(\mathbb{R}^d; X) \text{ for all } \alpha, \beta \in \mathbb{N}_0^d \right\}.$$

Here we apply the usual notational conventions of multivariable differential calculus: the elements $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ are called **multi-indices** and their modulus is defined by

$$|\alpha| = \alpha_1 + \cdots + \alpha_d \qquad (\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d).$$

The **monomial** associated with such a multi-index is the function

$$\mathbf{t}^\alpha := \prod_{j=1}^d \mathbf{t}_j^{\alpha_j}$$

where $\mathbf{t}_j$ is the projection onto the j-th coordinate; and the associated **partial derivative operator** is

$$\mathrm{D}^\alpha := \prod_{j=1}^d \mathrm{D}_j^{\alpha_j},$$

where $\mathrm{D}_j = \partial/\partial t_j$ is the partial derivative operator in the t_j-direction.

The Schwartz space contains the space

$$\mathrm{C_c^\infty}(\mathbb{R}^d; X) := \{ f \in \mathrm{C}^\infty(\mathbb{R}^d; X) \mid \operatorname{supp}(f) \text{ is compact} \}$$

of X-valued **test functions** and is dense in $\mathrm{L}^p(\mathbb{R}^d; X)$ for $1 \le p < \infty$ and in $\mathrm{C}_0(\mathbb{R}^d; X)$.

A function $m \in \mathrm{C}^\infty(\mathbb{R}^d)$ is called of **tempered growth** if for each multi-index $\alpha \in \mathbb{N}_0^d$ there is $n \in \mathbb{N}$ such that

$$(1 + |\mathbf{t}|)^{-n} \, \mathrm{D}^\alpha f \quad \text{is bounded.}$$

Each polynomial is of tempered growth. If $f, g \in \mathrm{C}^\infty(\mathbb{R}^d)$ are of tempered growth, then so are $\overline{f}$, $f + g$, fg, $\mathrm{e}^{\mathrm{i}\,\mathrm{Re}\, f}$. (This is easy to see.)

Theorem A.51. *For a Schwartz function $f \in \mathcal{S}(\mathbb{R}^d; X)$ the following statements hold:*

a) $\mathrm{D}^\alpha f \in \mathcal{S}(\mathbb{R}^d; X)$ *for all* $\alpha \in \mathbb{N}_0^d$.

b) $f \circ A \in \mathcal{S}(\mathbb{R}^{d'}; X)$ *for each linear mapping* $A : \mathbb{R}^{d'} \to \mathbb{R}^d$, $d' \in \mathbb{N}$.

c) $g \cdot f \in \mathcal{S}(\mathbb{R}^d; X)$ *for all* $g \in \mathrm{C}^\infty(\mathbb{R}^d)$ *of tempered growth.*

d) $\mathcal{F}f \in \mathcal{S}(\mathbb{R}^d; X)$ *and*

$$\mathcal{F}(\mathrm{D}^\alpha f) = (\mathrm{i}\mathbf{t})^\alpha \widehat{f} \quad \textit{and} \quad \mathrm{D}^\alpha \widehat{f} = \mathcal{F}((-\mathrm{i}\mathbf{t})^\alpha f) \tag{A.19}$$

for each $\alpha \in \mathbb{N}_0^d$.

e) $\mu * f \in \mathcal{S}(\mathbb{R}^d; X)$ *for all* $\mu \in \mathrm{M}(\mathbb{R}^d)$ *such that* $\widehat{\mu} \in \mathrm{C}^\infty(\mathbb{R}^d)$ *is of tempered growth. In particular,* $\tau_s f \in \mathcal{S}(\mathbb{R}^d)$ *for all* $s \in \mathbb{R}^d$.

Proof. a) is clear from the definition of the Schwartz space. b) follows since each $\mathrm{D}^\alpha(f \circ A)$ is a linear combination of functions $(\mathrm{D}^\beta f) \circ A$. c) holds since $\mathrm{D}^\alpha(gf)$ is, by Leibniz' rule, a linear combination of functions $(\mathrm{D}^\beta g)(\mathrm{D}^\gamma f)$.

d) Fix $j \in \{1, \dots, d\}$ and let $g_j := (-\mathrm{i}\mathbf{t}_j)f$. Then $g_j \in \mathcal{S}(\mathbb{R}^d; X)$ by b). In particular, g_j is integrable. For $0 \neq h \in \mathbb{R}$ and $s \in \mathbb{R}^d$ we can write

$$\frac{1}{h}\big(\widehat{f}(s + h\mathrm{e}_j) - \widehat{f}(s)\big) = \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-\mathrm{i}ht_j} - 1}{-\mathrm{i}t_j h} \mathrm{e}^{-\mathrm{i}st} g_j(t)\,\mathrm{d}t.$$

Since the function $\frac{\mathrm{e}^{-\mathrm{i}x}-1}{-\mathrm{i}x}$ is uniformly bounded in $x \in \mathbb{R}$, one can apply the dominated convergence theorem and let $h \to 0$ under the integral sign. This yields $\mathrm{D}_j \widehat{f} = \mathcal{F}(g_j)$. Applying this result inductively we find that $\widehat{f} \in \mathrm{C}^\infty(\mathbb{R}^d; X)$ and the second formula in (A.19).

The first formula is also proved inductively, where in each step one performs an integration by parts. Combining both formulae yields $\widehat{f} \in \mathcal{S}(\mathbb{R}^d; X)$.

d) Let $\mu \in \mathrm{M}(\mathbb{R}^d)$ such that $\widehat{\mu} \in \mathrm{C}^\infty(\mathbb{R}^d)$ is of tempered growth. Then $\mathcal{F}(\mu * f) = \widehat{\mu}\widehat{f}$ is a Schwartz function. By the Fourier inversion theorem,

$$\mu * f = (2\pi)^{-d} \mathcal{S}\mathcal{F}(\widehat{\mu}\widehat{f}),$$

and this is a Schwartz function by what already has been proved. □

Corollary A.52. *The Fourier transform restricts to a linear isomorphism on* $\mathcal{S}(\mathbb{R}^d; X)$.

Corollary A.53. *The space* $\mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d; X)$ *is dense in the space* $\mathrm{E}(\mathbb{R}^d; X)$ *with respect to the norm*

$$\|f\|_\mathrm{E} = \|f\|_1 + \|\widehat{f}\|_1.$$

Proof. Let $f \in \mathrm{E}(\mathbb{R}^d; X)$ and let $\psi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d)$ be constant to $\mathbf{1}$ in a neighborhood of 0. Let $\varphi := \mathcal{F}^{-1}\psi$, which is a Schwartz function, and $\varphi_n := n^d \varphi(n\mathbf{t})$ for $n \in \mathbb{N}$. Then $(\varphi_n)_n$ is a Dirac sequence and hence $f_n := f * \varphi_n \to f$ in $\mathrm{L}^1(\mathbb{R}^d; X)$. Furthermore, $f_n \in \mathrm{C}^\infty(\mathbb{R}^d)$ with $\mathrm{D}^\alpha f_n = f * \mathrm{D}^\alpha \varphi_n$ for each $\alpha \in \mathbb{N}_0^d$.

Note that $\psi_n := \widehat{\varphi_n} = \psi(\mathbf{s}/n) \to \mathbf{1}$ boundedly and locally uniformly. We claim that $\psi_n f_n \to f$ in the norm of $\mathrm{E}(\mathbb{R}^d; X)$. Obviously

$$\psi_n f_n = \psi_n (f_n - f) + \psi_n f \to f \quad \text{in} \quad \mathrm{L}^1(\mathbb{R}^d; X).$$

For the Fourier transform we obtain

$$\mathcal{F}(\psi_n f_n) = \mathcal{F}(\widehat{\varphi_n} f_n) = \widehat{f_n} * (\mathcal{S}\varphi_n) = (\widehat{f}\psi_n) * (\mathcal{S}\varphi_n) \to \widehat{f}$$

in $\mathrm{L}^1(\mathbb{R}^d; X)$ as well. So our claim is established, and since $\psi_n f_n \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d; X)$, the proof is complete. □

Corollary A.54. *Let $U \subseteq \mathbb{R}^d$ be open and $t_0 \in U$. Then there is a function $f \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\widehat{f}(t_0) = 1$, $0 \le \widehat{f} \le 1$, and $\operatorname{supp}(\widehat{f}) \subseteq U$.*

Proof. Let $\psi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d)$ with $0 \le \psi \le 1$, $\psi(t_0) = 1$, and $\operatorname{supp}(\psi) \subseteq U$. Then $f := \mathcal{F}^{-1}\psi$ does the job. □

A.10 Fourier Multiplier Operators

Let X be a Banach space. Recall from Corollary A.48 that the space

$$\mathrm{E}(\mathbb{R}^d; X) := \{f \mid f, \widehat{f} \in \mathrm{L}^1(\mathbb{R}^d; X) \cap \mathrm{C}_0(\mathbb{R}^d; X)\}$$

is dense in $\mathrm{L}^p(\mathbb{R}^d; X)$ for all $1 \le p < \infty$. For a function $m \in \mathrm{L}^\infty(\mathbb{R}^d)$ we define the operator

$$T_m : \mathrm{E}(\mathbb{R}^d; X) \to \mathrm{C}_0(\mathbb{R}^d; X), \qquad T_m f := \mathcal{F}^{-1}(m \cdot \widehat{f}).$$

Here, $\mathcal{F}^{-1} = (2\pi)^{-d}\mathcal{S}\mathcal{F}$ is the inverse Fourier transform. The operator T_m is called a **Fourier multiplier operator** with **symbol** m. It is easy to see that T_m is uniquely determined by m, i.e., the mapping

$$m \mapsto T_m$$

is injective.

Now fix $1 \le p < \infty$. Then $m \in \mathrm{L}^\infty(\mathbb{R}^d)$ is called an $\mathrm{L}^p(X)$**-multiplier** if the associated Fourier multiplier operator T_m extends to a bounded operator on $\mathrm{L}^p(\mathbb{R}^d; X)$. By density of $\mathrm{E}(\mathbb{R}^d; X)$, this extension—again denoted by T_m—is unique. We let

$$\mathcal{M}_p^X(\mathbb{R}^d) := \{m \in \mathrm{L}^\infty(\mathbb{R}^d) \mid m \text{ is an } \mathrm{L}^p(X)\text{-multiplier}\}$$

and endow it with the norm

$$\|m\|_{\mathcal{M}_p^X} := \|T_m\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R}^d;X))}. \tag{A.20}$$

If $X = \mathbb{C}$ we abbreviate $\mathcal{M}_p(\mathbb{R}^d) := \mathcal{M}_p^{\mathbb{C}}(\mathbb{R}^d)$.

Remark A.55. It follows from Corollary A.53 that $m \in \mathcal{M}_p^X(\mathbb{R}^d)$ with $\|m\|_{\mathcal{M}_p^X} \le C$ if and only if

$$\|T_m\varphi\|_p \le C\|\varphi\|_p \quad \text{for all } \varphi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d; X).$$

The following result lists the most important properties of the spaces $\mathcal{M}_p^X(\mathbb{R}^d)$.

Theorem A.56. *Let $X \neq \{0\}$ be a Banach space and $1 \le p < \infty$. Then the following assertions hold:*

a) *If $\mu \in \mathrm{M}(\mathbb{R}^d)$ then $\widehat{\mu} \in \mathcal{M}_p^X(\mathbb{R}^d)$ and $T_{\widehat{\mu}} = \tau_\mu$, i.e., convolution with μ. In particular,*
$$\|\widehat{\mu}\|_{\mathcal{M}_p^X} \le \|\mu\|_\mathrm{M}.$$
b) *$\mathcal{M}_p^X(\mathbb{R}^d) \subseteq \mathcal{M}_p(\mathbb{R}^d)$ contractively.*
c) *If $1 < p < \infty$ then $\mathcal{M}_p(\mathbb{R}^d) = \mathcal{M}_{p'}(\mathbb{R}^d)$ isometrically ($1/p + 1/p' = 1$).*
d) *$\mathcal{M}_1^X(\mathbb{R}^d) = \mathcal{M}_1(\mathbb{R}^d) = \mathrm{FS}(\mathbb{R}^d)$ isometrically.*
e) *If X is a Hilbert space, then $\mathcal{M}_2^X(\mathbb{R}^d) = \mathrm{L}^\infty(\mathbb{R}^d)$ isometrically.*
f) *The space $\mathcal{M}_p^X(\mathbb{R}^d)$ is a subalgebra of $\mathrm{L}^\infty(\mathbb{R}^d)$ and a Banach algebra with respect to the norm* (A.20)*. The inclusion $\mathcal{M}_p^X(\mathbb{R}^d) \subseteq \mathrm{L}^\infty(\mathbb{R}^d)$ is contractive:*
$$\|m\|_{\mathrm{L}^\infty} \le \|m\|_{\mathcal{M}_p} \qquad (m \in \mathcal{M}_p^X(\mathbb{R}^d)).$$
Moreover, the mapping
$$\mathcal{M}_p^X(\mathbb{R}^d) \to \mathcal{L}(\mathrm{L}^p(\mathbb{R}^d; X)), \qquad m \mapsto T_m$$
is an isometric and unital algebra homomorphism onto a closed unital subalgebra of $\mathcal{L}(\mathrm{L}^p(\mathbb{R}^d; X))$.
g) *If $(m_n)_n$ is a bounded sequence in $\mathcal{M}_p^X(\mathbb{R}^d)$ which converges pointwise almost everywhere to a function $m \in \mathrm{L}^\infty(\mathbb{R}^d)$, then $m \in \mathcal{M}_p^X(\mathbb{R}^d)$ and*
$$\|m\|_{\mathcal{M}_p^X} \le \liminf_{n\to\infty} \|m_n\|_{\mathcal{M}_p^X}.$$
Moreover, if $1 < p < \infty$ then
$$\int_{\mathbb{R}^d} \langle T_{m_n} f, g\rangle_{X,X'} \to \int_{\mathbb{R}^d} \langle T_m f, g\rangle_{X,X'}$$
for all $f \in \mathrm{L}^p(\mathbb{R}^d; X)$ and $g \in \mathrm{L}^{p'}(\mathbb{R}^d; X')$.
h) *If $m \in \mathcal{M}_p^X(\mathbb{R}^d)$ then for $A \in \mathrm{GL}(\mathbb{R}^d)$ one has $m \circ A \in \mathcal{M}_p^X(\mathbb{R}^d)$ with $\|m \circ A\|_{\mathcal{M}_p^X} = \|m\|_{\mathcal{M}_p^X}$.*

i) *If $m \in \mathcal{M}_p^X(\mathbb{R}^d)$ then for $s \in \mathbb{R}^d$ one has $\tau_s m \in \mathcal{M}_p^X(\mathbb{R}^d)$ with $\|\tau_s\|_{\mathcal{M}_p^X} = \|m\|_{\mathcal{M}_p^X}$. Moreover, the mapping*

$$\mathbb{R}^d \to \mathcal{L}(\mathrm{L}^p(\mathbb{R}^d; X)), \qquad s \mapsto T_{\tau_s m}$$

is strongly continuous, and for each $\mu \in \mathrm{M}(\mathbb{R}^d)$ one has $\mu * m \in \mathcal{M}_p^X(\mathbb{R}^d)$ with

$$\|T_{\mu * m}\|_{\mathcal{M}_p^X} \le \|\mu\|_{\mathrm{M}} \|m\|_{\mathcal{M}_p^X}.$$

Proof. a) Let $f \in \mathrm{E}(\mathbb{R}^d; X)$ and $\mu \in \mathrm{M}(\mathbb{R}^d)$. Then

$$\mathcal{F}(\mu * f) = \widehat{\mu} \cdot \widehat{f}.$$

Hence, by the Fourier inversion theorem (Theorem A.47) and the definition of T_m, $\tau_\mu f = \mu * f = T_m f$. It follows that

$$\|T_m f\|_p = \|\tau_\mu f\|_p \le \|\mu\|_{\mathrm{M}} \|f\|_p$$

and hence the claim.

b) Let $m \in \mathcal{M}_p^X(\mathbb{R}^d)$, $f \in \mathrm{E}(\mathbb{R}^d)$ and $x \in X$ with $\|x\| = 1$. Then

$$T_m^X(f \otimes x) = T_m^{\mathbb{C}} f \otimes x$$

where for clarity we have written T_m^X and $T_m^{\mathbb{C}}$ to distinguish the operators on X-valued and on scalar functions. Taking p-norms yields

$$\|T_m^{\mathbb{C}} f\|_p = \|T_m^X(f \otimes x)\|_p \le \|m\|_{\mathcal{M}_p^X} \|f \otimes x\|_p = \|m\|_{\mathcal{M}_p^X} \|f\|_p.$$

The claim follows.

c) Let $f,\, g \in \mathrm{E}(\mathbb{R}^d)$. Then, since $\mathcal{F}^{-1} = (2\pi)^{-d}\mathcal{S}\mathcal{F} = (2\pi)^{-d}\mathcal{F}\mathcal{S}$,

$$\begin{aligned}\langle T_m f, g\rangle &= (2\pi)^{-1}\big\langle \mathcal{S}\mathcal{F}(m\widehat{f}), g\big\rangle = (2\pi)^{-1}\big\langle m\widehat{f}, \widehat{\mathcal{S}g}\big\rangle \\ &= (2\pi)^{-1}\big\langle f, \mathcal{F}(m\widehat{\mathcal{S}g})\big\rangle = \big\langle \mathcal{S}f, T_m(\mathcal{S}g)\big\rangle,\end{aligned}$$

where the pointed brackets indicate the usual $\mathrm{L}^p - \mathrm{L}^{p'}$-duality. Since $\mathrm{E}(\mathbb{R}^d)$ is reflection-invariant and dense in $\mathrm{L}^p(\mathbb{R}^d)$ and in $\mathrm{L}^{p'}(\mathbb{R}^d)$, the claim follows.

d) Let $\mu \in \mathrm{M}(\mathbb{R}^d)$. Then, by Lemma 6.9,

$$\|\mu\|_{\mathrm{M}} = \|\tau_\mu\|_{\mathcal{L}(\mathrm{L}^1(\mathbb{R}^d))} \le \|\tau_\mu\|_{\mathcal{L}(\mathrm{L}^1(\mathbb{R}^d;X))} \le \|\mu\|_{\mathrm{M}}.$$

Hence, by a),

$$\|\widehat{\mu}\|_{\mathcal{M}_1^X} = \|\tau_\mu\|_{\mathcal{L}(\mathrm{L}^1(\mathbb{R}^d;X))} = \|\mu\|_{\mathrm{M}} = \|\widehat{\mu}\|_{\mathrm{FS}(\mathbb{R}^d)}.$$

So the inclusion $\mathrm{M}(\mathbb{R}^d) \subseteq \mathcal{M}_1^X(\mathbb{R}^d)$ is isometric. Next, fix $m \in \mathcal{M}_1^X(\mathbb{R}^d)$. By b), $m \in \mathcal{M}_1(\mathbb{R}^d)$ and hence T_m is a bounded operator on $\mathrm{L}^1(\mathbb{R}^d)$ that commutes with all translations. By Theorem 6.23, there is $\mu \in \mathrm{M}(\mathbb{R}^d)$ such that $T_m = \tau_\mu = T_{\widehat{\mu}}$, from which it follows that $m = \widehat{\mu}$.

e) Suppose that $X = H$ is a Hilbert space and $m \in \mathrm{L}^\infty(\mathbb{R}^d)$. Then

$$T_m = \mathcal{F}^{-1} M_m \mathcal{F} \tag{A.21}$$

on $\mathrm{E}(\mathbb{R}^d; H)$, where M_m is the multiplication operator with m. Since, by Plancherel's theorem, the Fourier transform extends to an isomorphism on $\mathrm{L}^2(\mathbb{R}^d; H)$, T_m is bounded and (A.21) holds on the whole $\mathrm{L}^2(\mathbb{R}^d; H)$. Since the operator $(2\pi)^{-d/2}\mathcal{F}$ is unitary, $\|T_m\| = \|M_m\| = \|m\|_{\mathrm{L}^\infty}$. (Here we use Theorem 2.1.d).)

f) We first show that $\mathcal{M}_p^X(\mathbb{R}^d)$ is an algebra and the mapping

$$\mathcal{M}_p^X(\mathbb{R}^d) \to \mathcal{L}(\mathrm{L}^p(\mathbb{R}^d; X)), \qquad m \mapsto T_m$$

is a homomorphism. The only non-trivial issue is the multiplicativity. Let $f \in \mathrm{E}(\mathbb{R}^d; X)$ and $m_1, m_2 \in \mathcal{M}_p^X(\mathbb{R}^d)$. Let, furthermore, $(\psi_n)_n$ be a sequence in $\mathrm{E}(\mathbb{R}^d)$ as in Lemma A.46. Then $\widehat{\psi_n} * (m_1 \widehat{f}) \in \mathrm{L}^1(\mathbb{R}^d; X)$ and

$$\mathcal{F}^{-1}(\widehat{\psi_n} * m_1 \widehat{f}) = (2\pi)^d \psi_n(T_{m_1} f),$$

which is in $\mathrm{L}^1(\mathbb{R}^d; X)$ as well. Hence, $\mathcal{F}^{-1}(\widehat{\psi_n} * (m_1 \widehat{f})) \in \mathrm{E}(\mathbb{R}^d; X))$ and we can apply T_{m_2} to obtain

$$(2\pi)^d\, T_{m_2}\big(\psi_n \cdot (T_{m_1} f)\big) = \mathcal{F}^{-1}\big(m_2 \cdot (\widehat{\psi_n} * m_1 \widehat{f})\big).$$

Since $T_{m_1} f \in \mathrm{L}^p(\mathbb{R}^d; X)$ and T_{m_2} is L^p-bounded,

$$(2\pi)^d\, T_{m_2}\big(\psi_n \cdot (T_{m_1} f)\big) \to (2\pi)^d\, T_{m_2} T_{m_1} f$$

in $\mathrm{L}^p(\mathbb{R}^d; X)$. On the other hand,

$$\mathcal{F}^{-1}\big(m_2 \cdot (\widehat{\psi_n} * m_1 \widehat{f})\big) \to (2\pi)^d\, \mathcal{F}^{-1}\big(m_2 \cdot (m_1 \widehat{f})\big) = (2\pi)^d\, T_{m_2 m_1} f$$

in $\mathrm{C}_0(\mathbb{R}^d; X)$. It follows that $T_{m_2 m_1} f = T_{m_2} T_{m_1} f \in \mathrm{L}^p(\mathbb{R}^d; X)$ and hence the claim.

Next, we show that $\|m\|_{\mathrm{L}^\infty} \le \|m\|_{\mathcal{M}_p^X}$ for $m \in \mathcal{M}_p^X(\mathbb{R}^d)$. This is clear for $p = 1$ from d). For $1 < p < \infty$ we argue as follows. By b) it suffices to consider $X = \mathbb{C}$. By the Riesz–Thorin interpolation theorem [5, Thm.2.1.] and assertion c),

$$\|T_m\|_{\mathcal{L}(\mathrm{L}^2)} \le \|T_m\|_{\mathcal{L}(\mathrm{L}^p)}^\theta\, \|T_m\|_{\mathcal{L}(\mathrm{L}^{p'})}^{1-\theta} = \|T_m\|_{\mathcal{L}(\mathrm{L}^p)}$$

for some suitable $\theta \in [0, 1]$. Hence, an application of e) establishes the claim.

Finally, we prove that $\mathcal{M}_p^X(\mathbb{R}^d)$ is complete. Let $(m_n)_n$ be a Cauchy sequence in $\mathcal{M}_p^X(\mathbb{R}^d)$. Then $(T_{m_n})_n$ is a Cauchy sequence in $\mathcal{L}(\mathrm{L}^p(\mathbb{R}^d;X))$ and hence has a limit $T \in \mathcal{L}(\mathrm{L}^p(\mathbb{R}^d;X))$. On the other hand, by what we already have proved, $(m_n)_n$ is a Cauchy sequence in $\mathrm{L}^\infty(\mathbb{R}^d)$ and hence has a limit $m \in \mathrm{L}^\infty(\mathbb{R}^d)$. Therefore, given $f \in \mathrm{E}(\mathbb{R}^d;X)$ the sequence

$$T_{m_n} f = \mathcal{F}^{-1}(m_n \widehat{f})$$

converges in $\mathrm{L}^p(\mathbb{R}^d;X)$ to Tf and in $\mathrm{C}_0(\mathbb{R}^d;X)$ to $T_m f$. The rest is straightforward.

g) By what has already been proved, $\sup_{n\in\mathbb{N}} \|m_n\|_{\mathrm{L}^\infty} < \infty$. For each $f \in \mathrm{E}(\mathbb{R}^d;X)$ one hence has

$$T_{m_n} f = \mathcal{F}^{-1}(m_n \widehat{f}) \to T_m f$$

in $\mathrm{C}_0(\mathbb{R}^d;X)$. On the other hand, $\sup_n \|T_{m_n} f\|_p < \infty$ and hence Fatou's theorem yields that $T_m f \in \mathrm{L}^p(\mathbb{R}^d;X)$ and

$$\|T_m f\|_p \le \liminf_{n\to\infty} \|T_{m_n} f\|_p \le \big(\liminf_{n\to\infty} \|m_n\|_{\mathcal{M}_p^X}\big)\, \|f\|_p.$$

This establishes the first claim. For the second, suppose in addition that $1 < p < \infty$, $f \in \mathrm{E}(\mathbb{R}^d;X)$ and $g \in \mathrm{L}^1(\mathbb{R}^d;X')$. Then

$$\int_{\mathbb{R}^d} \langle T_{m_n} f, g\rangle_{X,X'} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}st} m_n(t) \big\langle \widehat{f}(t), g(s)\big\rangle_{X,X'} \,\mathrm{d}t\,\mathrm{d}s,$$

which converges to

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}st} m(t) \big\langle \widehat{f}(t), g(s)\big\rangle_{X,X'} \,\mathrm{d}t\,\mathrm{d}s = \int_{\mathbb{R}^d} \langle T_m f, g\rangle_{X,X'}$$

by Lebesgue's theorem. So the claim follows by approximation.

h) This follows (after some computation) from the formulae ($B \in \mathrm{GL}(\mathbb{R}^d)$)

$$\widehat{f \circ B^{-1}} = |\det B|\, (\widehat{f} \circ B^t)$$

and

$$\|f \circ B^{-1}\|_p = |\det B|^{\frac{1}{p}}\, \|f\|_p,$$

which can be established by a change of variables.

i) Let $f \in \mathrm{E}(\mathbb{R}^d;X)$ and $s \in \mathbb{R}^d$. Then by an application of 5.7.a) one obtains

$$T_{\tau_s m} f = \mathrm{e}^{-\mathrm{i}s\cdot} T_m\big(\mathrm{e}^{-\mathrm{i}s\cdot} f\big).$$

This yields $\|\tau_s m\|_{\mathcal{M}_p^X} = \|m\|_{\mathcal{M}_p^X}$ and the strong continuity of the mapping $s \mapsto T_{\tau_s m}$. The rest is straightforward. □

Remark A.57. If X is a reflexive Banach space and $1 < p < \infty$, then $\mathrm{L}^{p'}(\mathbb{R}^d; X') \cong \mathrm{L}^p(\mathbb{R}^d; X)'$ via the canonical pairing

$$\langle f, g \rangle = \int_{\mathbb{R}^d} \langle f(t), g(t) \rangle_{X,X'} \, \mathrm{d}t,$$

see [2, Cor.1.3.22]. So, in this case, the second part of Theorem A.56.g) just means that $T_{m_n} \to T_m$ in the weak operator topology on $\mathrm{L}^p(\mathbb{R}^d; X)$.

A.11 Some Analytical Identities

In this appendix we provide proofs for some analytical identities.

The Gaussian Kernel

One of the most useful identities in mathematics is

$$\int_{\mathbb{R}} \mathrm{e}^{-x^2} \, \mathrm{d}x = \sqrt{\pi}. \tag{A.22}$$

We shall provide a proof which only uses facts from elementary real analysis. To this end, consider the functions

$$f(t) := \Big(\int_0^t \mathrm{e}^{-x^2} \, \mathrm{d}x \Big)^2 \quad \text{and} \quad g(t) := \int_0^1 \frac{\mathrm{e}^{-t^2(1+x^2)}}{1+x^2} \, \mathrm{d}x \qquad (t \ge 0).$$

Then F and G are differentiable on $\mathbb{R}_+$ and

$$\begin{aligned} f'(t) &= 2 \int_0^t \mathrm{e}^{-x^2} \, \mathrm{d}x \, \mathrm{e}^{-t^2} = \int_0^t 2\mathrm{e}^{-(t^2+x^2)} \, \mathrm{d}x = \int_0^1 2t\mathrm{e}^{-t^2(1+x^2)} \, \mathrm{d}x \\ &= -g'(t), \end{aligned}$$

where we have used that we can differentiate g under the integral sign. It follows that $f + g$ is constant on $\mathbb{R}_+$, and inserting 0 yields

$$f(t) + g(t) = \int_0^1 \frac{\mathrm{d}x}{1+x^2} = \arctan 1 = \frac{\pi}{4}.$$

Estimating G with the triangle inequality for integrals yields

$$|g(t)| \le \mathrm{e}^{-t^2} \to 0 \quad \text{as} \quad t \to \infty.$$

Since the square root function is continuous, it follows that

$$\int_0^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x = \lim_{t\to\infty}\sqrt{f(t)} = \frac{\sqrt{\pi}}{2},$$

and hence (A.22).

The Fourier Transform of the Gaussian Kernel

Here we shall establish the formula

$$\frac{1}{\sqrt{4\pi}}\int_{\mathbb{R}} \mathrm{e}^{-s^2/4}\mathrm{e}^{-\mathrm{i}st}\,\mathrm{d}s = \mathrm{e}^{-t^2} \qquad (t \in \mathbb{R}). \tag{A.23}$$

Since the function $\mathrm{e}^{-\mathbf{s}^2/4}$ is even and for each $t \in \mathbb{R}$ the function $\sin(t\mathbf{s})$ is odd, it suffices to prove

$$f(t) := \frac{1}{\sqrt{4\pi}}\int_{\mathbb{R}} \mathrm{e}^{-s^2/4}\cos(ts)\,\mathrm{d}s = \mathrm{e}^{-t^2}.$$

Now, differentiating under the integral and performing integration by parts yields

$$f'(t) = \frac{2}{\sqrt{4\pi}}\int_{\mathbb{R}} \tfrac{-s}{2}\mathrm{e}^{-s^2/4}\sin(ts)\,\mathrm{d}s = \frac{-2}{\sqrt{4\pi}}\int_{\mathbb{R}} \mathrm{e}^{-s^2/4}t\,\cos(ts)\,\mathrm{d}s = -2t\cdot f(t).$$

Multiplying by e^{t^2} we obtain

$$0 = f'(t)\mathrm{e}^{t^2} + 2tf(t)\mathrm{e}^{t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\big[f(t)\mathrm{e}^{t^2}\big].$$

Hence $f(t) = c\mathrm{e}^{-t^2}$ for some constant c. Inserting $t = 0$ yields

$$c = f(0) = \frac{1}{\sqrt{4\pi}}\int_{\mathbb{R}} \mathrm{e}^{-s^2/4}\,\mathrm{d}s = 1$$

as desired.

The Inverse Laplace Transform of $\mathbf{e}^{-\sqrt{\mathbf{z}}}$

We shall establish the formula

$$\int_0^\infty \left(\frac{t\mathrm{e}^{-t^2/4s}}{2\sqrt{\pi}s^{3/2}}\right)\mathrm{e}^{-zs}\,\mathrm{d}s = \mathrm{e}^{-t\sqrt{z}} \qquad (t > 0,\ \mathrm{Re}\,z > 0). \tag{A.24}$$

To this end, we first prove a lemma.

Lemma A.58. *Let $\alpha > 0$. Then*

$$\int_0^\infty e^{-(\frac{\alpha}{t}-t)^2}\,dt = \int_0^\infty \frac{\alpha}{t^2} e^{-(\frac{\alpha}{t}-t)^2}\,dt = \frac{\sqrt{\pi}}{2}.$$

Proof. The first identity follows from a change of variables $t = \alpha/s$ and then renaming s by t. Adding the first and the second term and changing the variable to $s = t - \frac{\alpha}{t}$ yields

$$\int_0^\infty \Big(1+\frac{\alpha}{t^2}\Big) e^{-(\frac{\alpha}{t}-t)^2}\,dt = \int_{-\infty}^\infty e^{-s^2}\,ds = \sqrt{\pi}.$$

This yields the claim. □

We can now establish (A.24). By analyticity of the functions it suffices to prove the identity only for $z > 0$. Replacing z by z^2/t^2 and changing the variable in the integral from s to s^2t^2, the claim is equivalent with

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{s^2} e^{-1/4s^2} e^{-z^2s^2}\,ds = e^{-z}$$

for $z > 0$. Now

$$\begin{aligned} e^z \int_0^\infty \frac{1}{s^2} e^{-1/4s^2} e^{-z^2s^2}\,ds &= \int_0^\infty \frac{1}{s^2} e^{-(\frac{1}{2s}-zs)^2} ds \\ &= 2\int_0^\infty \frac{z}{2s^2} e^{-(\frac{z}{2s}-s)^2} ds = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi} \end{aligned}$$

by Lemma A.58.

The Dirichlet integral

Next, we shall provide a proof for the so-called Dirichlet integral

$$\int_0^\infty \frac{\sin s}{s}\,ds = \frac{\pi}{2}. \tag{A.25}$$

There is a well-known approach to this formula via complex analysis by integrating the function $\frac{e^{i\mathbf{z}}-1}{\mathbf{z}}$ over semi-circles with radius tending to infinity, and then taking the imaginary part.

A real-variable approach works as follows. Integration by parts easily yields

$$\lim_{a,b\to\infty} \int_a^b \frac{\sin s}{s}\,ds = 0.$$

Hence, by Cauchy's criterion, the improper integral

$$c := \lim_{T\to\infty} \int_0^T \frac{\sin s}{s}\,ds$$

exists. By the Laplace transform version of Abel's theorem (with an analogous proof) it follows that

$$c = \lim_{x \searrow 0} \int_0^\infty \mathrm{e}^{-sx} \frac{\sin s}{s} \, \mathrm{d}s.$$

(Note that the function $\frac{\sin \mathbf{s}}{\mathbf{s}}$ is not absolutely integrable, so the dominated convergence theorem is not applicable here.)

Now let

$$f(x) := \int_0^\infty \mathrm{e}^{-sx} \frac{\sin s}{s} \, \mathrm{d}s \qquad (x > 0).$$

Then, by a standard argument,

$$f'(x) = -\int_0^\infty \mathrm{e}^{-sx} \sin x \, \mathrm{d}s = \mathrm{Im} \int_0^\infty \mathrm{e}^{-s(x+\mathrm{i})} \, \mathrm{d}s = \frac{-1}{1+x^2}.$$

Since $\lim_{x\to\infty} f(x) = 0$, it follows that

$$f(x) = \frac{\pi}{2} - \arctan x$$

and hence $c = \lim_{x\searrow 0} f(x) = \frac{\pi}{2}$.

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