LECTURES ON GEOMETRY OF SUPERSYMMETRY

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ABSTRACT. $\boxed{\text{To be polished.}}$ This is note for my course on supersymmetry in the fall 2017 at Tsinghua university. We discuss supersymmetric gauge theories in various dimensions, their geometric structures and dualities.

CONTENTS

We will be working with the symmetric monoidal category of $\mathbb{Z}_2$-graded algebras over a field k of characteristic 0. Let A be a k-algebra with $\mathbb{Z}_2$-graded decomposition

$$A = A^0 \oplus A^1.$$

We will write $|a_i| = i, a_i \in A^i$, for the grading. The monoidal structure is given by the graded tensor product $\hat{\otimes}_k$ defined as follows.

Definition 0.1. Let A, B be two $\mathbb{Z}_2$-graded algebras over k. We define the graded tensor product $A\hat{\otimes}_k B$ as the $\mathbb{Z}_2$-graded algebra whose underlying $\mathbb{Z}_2$-graded vector space is $A \otimes_k B$, with multiplication defined by

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{|b_1||a_2|}(a_1 a_2) \otimes (b_1 b_2).$$

1. Clifford algebra

Definition 1.1. Let V be a vector space over the field k, and Q is a quadratic form on V. We always assume q be non-degenerate. Let $T(V) = \bigoplus\limits_{n\geq 0} V^{\otimes n}$ be the tensor algebra. The **Clifford algebra** $Cl(V, Q)$ is defined to be the quotient of $T(V)$ by the relation

$$x^2 = -Q(x), \quad x \in V.$$

The tensor product $\otimes$ on $T(V)$ induces a product on $Cl(V,Q)$ denoted by $\cdot$.

Equivalently, $Cl(V,Q)$ is the associative algebra freely generated by V and the relation

$$a \cdot b + b \cdot a = -2\langle a, b\rangle \quad \forall a, b \in V,$$

where $\langle -, -\rangle$ is the inner product on V associated to q.

Let $T(V) = T^{even}(V) \oplus T^{odd}(V)$ be the decomposition into even and odd number of tensors of V in $T(V)$. It equips $Cl(V,Q)$ with the structure of $\mathbb{Z}_2$-graded algebra by

$$Cl(V,Q) = Cl^0(V,q) \oplus Cl^1(V,Q)$$

where $Cl^0(V,Q)$ and $Cl^1(V,Q)$ are the images of $T^{even}(V)$ and $T^{odd}(V)$ respectively.

Let $T^{\leq p}(V) = \bigoplus_{n \leq p} V^{\otimes n}$. Let $\mathcal{F}^p$ be its image in $Cl(V,Q)$. Then the filtration

$$\mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset Cl(V,Q)$$

equips $Cl(V,Q)$ with the structure of filtered algebra. Let $Gr_{\mathcal{F}}(Cl(V,Q))$ be the associated graded algebra.

Lemma 1.2. *There is a canonical isomorphism of algebras*

$$Gr_{\mathcal{F}}(Cl(V,Q)) \cong \wedge^* V$$

where $\wedge^* V$ is the exterior algebra.

In particular, there is an explicit isomorphism of vector spaces

$$\rho : \wedge^* V \to Cl(V,Q), \quad v_1 \wedge \cdots \wedge v_p \to \sum_{\sigma \in S_p} (-1)^\sigma v_{\sigma(1)} \cdots v_{\sigma(p)}.$$

Lemma 1.3. *Suppose $V = V_1 \oplus V_2$ and $Q = Q_1 + Q_2$, where Q_i is a quadratic form on V_i. Then there is a canonical isomorphism of $\mathbb{Z}_2$-graded k-algebras*

$$Cl(V,Q) \cong Cl(V_1,Q_1) \hat{\otimes}_k Cl(V_2,Q_2).$$

Example 1.4. Let V be a real vector space, and $Q = x_1^2 + \cdots x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$. The associated Clifford algebra will be denoted by $Cl_{p,q}$. Moreover $Cl_p \equiv Cl_{p,0}$ for simplicity. We have

$$Cl_{p,q} \cong Cl_{p,0} \hat{\otimes}_{\mathbb{R}} Cl_{0,q}.$$

Example 1.5. Let V be a complex vector space of dimension n, then all non-degenerate quadratic forms are equivalent. The associated Clifford algebra will be denoted by $\mathbb{C}l_n$ and we have

$$\mathbb{C}l_n \cong \mathbb{C}l_1^{\hat{\otimes}_{\mathbb{C}}^n}.$$

Lemma 1.6. *Let $k(n)$ denote the $n \times n$ matrix algebra with entries in k. The we have algebra isomorphisms*

$$Cl_{1,0} \cong \mathbb{C}, \quad Cl_{0,1} \cong \mathbb{R} \oplus \mathbb{R}, \quad Cl_{2,0} \cong \mathbb{H}, \quad Cl_{1,1} \cong Cl_{0,2} \cong \mathbb{R}(2)$$

$$\mathbb{C}l_1 \cong \mathbb{C} \oplus \mathbb{C}, \quad \mathbb{C}l_2 \cong \mathbb{C}(2).$$

Here $\mathbb{H}$ are quaternions.

Proof. We prove $Cl_{1,1} \cong Cl_{0,2} \cong \mathbb{R}(2)$. As a vector space

$$Cl_{1,1} = \mathbb{R} \oplus \mathbb{R}x \oplus \mathbb{R}y \oplus \mathbb{R}xy$$

with multiplication structure by $x^2 = 1, y^2 = -1, xy = -yx$. It is identified with $\mathbb{R}(2)$ by

$$x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad xy = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Similarly,

$$Cl_{0,2} = \mathbb{R} \oplus \mathbb{R}x \oplus \mathbb{R}y \oplus \mathbb{R}xy$$

with multiplication structure by $x^2 = 1, y^2 = 1, xy = -yx$. It is identified with $\mathbb{R}(2)$ by

$$x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad xy = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

□

The Clifford algebras are easily classified with the help of the following proposition.

Proposition 1.7. *We have the following algebra isomorphisms*

(1) $Cl_{p+2,q} \cong Cl_{2,0} \otimes Cl_{q,p}, \quad Cl_{p,q+2} \cong Cl_{0,2} \otimes Cl_{q,p}, \quad Cl_{p+1,q+1} \cong Cl_{1,1} \otimes Cl_{p,q}$.
(2) *Bott periodicity (real case):* $Cl_{p+4,q} \cong Cl_{p,q+4}, \quad Cl(p+8,q) \cong Cl(p,q)(16)$.
(3) *Bott periodicity (complex case):* $\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}l_2 \cong \mathbb{C}l_n(2)$.
(4) $Cl_{p,q} \cong Cl^0_{p+1,q}$.

Proof. (1)The isomorphism

$$Cl_{2,0} \otimes Cl_{q,p} \to Cl_{p+2,q}$$

is realized in generators

$$e_i \to e_i, \tilde{e}_\alpha \to e_{12}\tilde{e}_\alpha, \quad 1 \le i \le 2, 1 \le \alpha \le p+q, \quad e_{12} = e_1 e_2.$$

(4) The isomorphism

$$Cl_{p,q} \cong Cl^0_{p+1,q}$$

is realized in generators

$$e_i \to e_1 e_{i+1}, \quad 1 \le i \le p+q.$$

□

Combining Lemma 1.6 and Proposition 1.7, we find the following table

n	1	2	3	4	5	6	7	8
$Cl_{n,0}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
$Cl_{0,n}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
$Cl_{n-1,1}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
$Cl_{1,n-1}$	$\mathbb{C}$	$\mathbb{R}(2)$	$\mathbb{R}(2) \oplus \mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4) \oplus \mathbb{H}(4)$	$\mathbb{H}(8)$
$\mathbb{C}l_n$	$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$	$\mathbb{C}(4)$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8) \oplus \mathbb{C}(8)$	$\mathbb{C}(16)$

TABLE 1. Clifford algebras

2. Spin group

Definition 2.1. We define the following operations on Clifford algebras

(1) **Reflection automorphism**: let $x = v_1 \cdots v_m$, then $\hat{x} = (-)^m v_1 \cdots v_m$.
(2) **Transpose anti-automorphism**: let $x = v_1 \cdots v_m$, then $x^t = v_m \cdots v_1$.
(3) **Conjugate anti-automorphism**: $x^* = \hat{x}^t$.

Definition 2.2. Let $Cl^\times(V,Q)$ be the multiplicative group of units in $Cl(V,Q)$. We define the twisted conjugation action of $Cl^\times(V,Q)$ on the Clifford algebra

$$\hat{Ad} : Cl^\times(V,Q) \to Gl(Cl(V,Q))$$

by

$$\hat{Ad}_x(y) = \hat{x} y x^{-1}.$$

Example 2.3. Let $x, y \in V$, then

$$\hat{Ad}_x(y) = y - 2\frac{\langle x,y\rangle}{\langle x,x\rangle} x$$

is the reflection along the hyperplane orthogonal to x.

Definition 2.4. The **Clifford group** $\Gamma(V,Q)$ is defined by

$$\Gamma(V,Q) = \{x \in Cl^\times(V,q) | \hat{Ad}_x : V \to V \text{ preserves } V\}.$$

Lemma 2.5. *The Clifford action $\Gamma(V,Q) : V \to V$ preserves the quadratic form.*

Proof. In fact, $\forall v \in V$,

$$\left\langle \hat{Ad}_x v, \hat{Ad}_x v \right\rangle = \hat{x} v x^{-1} \widehat{\hat{x} v x^{-1}} = \langle v, v\rangle .$$

$\square$

Corollary 2.6. *There is an exact sequence of groups*

$$0 \to k^\times \to \Gamma(V,Q) \xrightarrow{\hat{Ad}} O(V,Q) \to 0.$$

Here $O(V,Q)$ is the orthogonal group of q-preserving linear automorphisms of V.

Proof. $\hat{Ad}$ is defined by the previous lemma. Let $x \in \Gamma(V,Q) \cap \ker(\hat{Ad})$. Then

$$vx = \hat{x} v, \quad \forall v \in V.$$

Using the filtration $\mathcal{F}^\bullet$ on $Cl(V,Q)$, it is enough to show that given $x \in \wedge^p V$, $p > 0$, if

$$\iota_{\langle v,-\rangle} x = 0 \in \wedge^{p-1} V \text{ for any } \ v \in V,$$

then $x = 0$. Here $\langle v, -\rangle$ is viewed as an element of V^* and ι is the natural contraction. But this is obvious.

On the other hand, Cartan-Dieudonné theorem states that every element of $O(V,Q)$ is a composition of at most $\dim_k V$ reflections ($char(k) \neq 2$), hence a Clifford action. $\square$

Definition 2.7. The **spin norm** $N : \Gamma(V,Q) \to k^\times$ is a group homomorphism defined by

$$N(x) = x\hat{x}^t.$$

The reason that $N(x) \in k^\times$ comes from the observation that $x\hat{x}^t \in \Gamma(V,Q) \cap \ker(\hat{Ad})$. In fact,

$$\hat{Ad}_{x\hat{x}^t} v = \hat{x}\left(x^t v (\hat{x}^t)^{-1}\right) x^{-1} = \hat{x}\left(x^t v (\hat{x}^t)^{-1}\right)^t x^{-1} = v.$$

Definition 2.8. The **Pin group** and **Spin group** are defined by

$$\begin{aligned}\mathrm{Pin}(V,Q) &= \{v_1\cdots v_r \in Cl^{\times}(V,Q) | v_i \in V, Q(v_i) = \pm 1\}.\\ \mathrm{Spin}(V,Q) &= \mathrm{Pin}(V,Q)\cap Cl^0(V,Q).\end{aligned}$$

In the case when k is a spin field (i.e. $k^{\times} = (k^{\times})^2 \cup -(k^{\times})^2$), we still have a surjection

$$\hat{Ad} : \mathrm{Pin}(V,Q) \to O(V,Q)$$

and the spin norm is

$$N : \mathrm{Pin}(V,Q) \to \pm 1.$$

Proposition 2.9. *In the real case $k = \mathbb{R}$, we have exact sequences*

$$\begin{aligned}0 \to \mathbb{Z}_2 \to \mathrm{Spin}(p,q) \to SO(p,q) \to 1\\ 0 \to \mathbb{Z}_2 \to \mathrm{Pin}(p,q) \to O(p,q) \to 1.\end{aligned}$$

Here $SO(p,q) = \{A \in O(p,q) | \det A = 1\}$.

Proof. Let $x \in \mathrm{Pin}(V,Q)\cap \mathbb{R}^{\times}$, then

$$N(x) = x^2 = \pm 1 \Longrightarrow x = \pm 1.$$

This implies the second exact sequence. The first exact sequence follows from the fact that each reflection has determinant -1. □

Proposition 2.10. $\mathrm{Spin}(p,q) \cong \mathrm{Spin}(q,p)$.

Proof. We consider the map of $Cl_{p,q} \otimes_{\mathbb{R}} \mathbb{C}$ on generators

$$e_i \to \sqrt{-1}e_i.$$

This is well-defined on $Cl^0_{p,q}$ since it contains even number of products of $\sqrt{-1}$'s. □

Example 2.11. We can identify some spin groups in low dimensions

$$\begin{aligned}&\mathrm{Spin}(2) \cong U(1), && \mathrm{Spin}(3) \cong SU(2)\\ &\mathrm{Spin}(4) = SU(2)\times SU(2), && \mathrm{Spin}(3,1) \cong SL^*(2,\mathbb{C})\\ &\mathrm{Spin}(6) \cong SU(4), && \mathrm{Spin}(5,1) \cong SL^*(2,\mathbb{H}).\end{aligned}$$

Here $SL^*(2,k) = \{A \in GL(2,k) | \det g = \pm 1\}$. This can be seen as follows.

- $\mathrm{Spin}(2)$. In this case we have $Cl_2 \cong \mathbb{H} = \mathbb{R}\oplus\mathbb{R}\,\mathrm{i}\oplus\mathbb{R}\,\mathrm{j}\oplus\mathbb{R}\,\mathrm{k}$.

  $$\mathrm{Spin}(2) = \{a + b\,\mathrm{k}\,|a^2+b^2 = 1, \quad a,b \in \mathbb{R}\}.$$

  Given $\phi = e^{\mathrm{k}\,\theta} \in \mathrm{Spin}(2)$, $z = x\,\mathrm{i} + y\,\mathrm{j} \in \mathbb{R}^2$, we have $\hat{Ad}_{\phi} : z \to e^{2\,\mathrm{k}\,\theta}z$.
- $\mathrm{Spin}(3)$. Let e_1, e_2, e_3 be orthonormal basis of $\mathbb{R}^3$. $Cl^0_3 \cong Cl_2 \cong \mathbb{H}$. Explicitly,

  $$\mathbb{H} \to Cl^0_3, \quad \mathrm{i} \to e_2e_3, \quad \mathrm{j} \to e_3e_1, \quad \mathrm{k} \to e_1e_2.$$

  Observe that the spin norm on Cl^0_3 can be identified with the norm on $\mathbb{H}$. It follows that

  $$\mathrm{Spin}(3) \cong \{a \in \mathbb{H} | |a|^2 = 1\}.$$

  Consider a two-dim representation of $\mathbb{H}$ by

  $$\mathrm{i} = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}, \mathrm{j} = \begin{pmatrix} 0 & -\sqrt{-1}\\ -\sqrt{-1} & 0\end{pmatrix}, \mathrm{k} = \begin{pmatrix} \sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix}$$

It identifies

$$\mathbb{H} \cong \{A \in M_2(\mathbb{C}) | \bar{A} = \gamma A \gamma^{-1}\}, \quad \gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The norm of $\mathbb{H}$ can be identified with the determinant

$$t^2 + x^2 + y^2 + z^2 = \det(t + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

This gives another isomorphism

$$SU(2) \cong \{a \in \mathbb{H} | |a|^2 = 1\}.$$

It follows that $\mathrm{Spin}_3 \cong SU(2)$. If we identify $\mathbb{R}^3$ as the imaginary part of $\mathbb{H}$, then the homomorphism $\mathrm{Spin}(3) \to SO(3)$ is realized by

$$\hat{A}d_a : v \to ava^\dagger, \quad a \in \mathrm{Spin}(3) \subset H, \quad v \in \mathrm{Im}(\mathbb{H}).$$

- $\mathrm{Spin}(4)$. $SU(2)$ acts on $\mathbb{H} \cong \mathbb{R}^4$ from both sides, which gives the following map

$$SU(2)_L \times SU(2)_R \to GL(4, \mathbb{R}), \quad A_L \times A_R : q \to A_L q A_R^{-1}.$$

Since the norm is given by the derminant, it actually maps to $SO(4)$. It follows that $Spin_4$ can be identified with two copies of $SU(2)$

$$Spin(4) \cong SU(2) \times SU(2).$$

- $\mathrm{Spin}(3,1)$. We consider $\mathbb{R}^{3,1}$ with metric $\eta = \mathrm{diag}(-1,1,1,1)$. We define the Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $x = \{x^0, x^1, x^2, x^3\}$ be coordinates on $\mathbb{R}^{3,1}$. The Pauli representation

$$x \to A(x) = \sum_{i=0}^{3} x^i \sigma_i$$

identifies $\mathbb{R}^{3,1}$ with the space of 2×2 Hermitian matrices. It is easy to see that

$$\det A(x) = -\sum_{\mu,\nu=0}^{3} \eta_{\mu,\nu} x^\mu x^\nu.$$

Let $SL^*(2,\mathbb{C}) = \{A \in GL(2,\mathbb{C}) | \det A = \pm 1\}$. Then the following $\mathbb{Z}_2$-covering

$$\begin{array}{rcl} \pi : SL^*(2,\mathbb{C}) & \to & SO(3,1) \\ N & \to & \{A(x) \to \det(N) N A(x) N^\dagger\} \end{array}$$

identifies $\mathrm{Spin}(3,1)$ with $SL^*(2,\mathbb{C})$. The two cases $\det A = \pm 1$ correspond to matrix M in $SO(3,1)$ with $M_{00} > 0$ or $M_{00} < 0$. Sometimes $\mathrm{Spin}(3,1)$ just refers to the universal cover $SL(2,\mathbb{C})$ of the connected component $SO^+(3,1)$ of $SO(3,1)$ containing identity.

- $\mathrm{Spin}(6)$. Since $Cl_6^0 \cong Cl_5 \cong \mathbb{C}(4)$, we get a map $\mathrm{Spin}(6) \to SU(4)$ which turns out to be an isomorphism. This can be explicitly realized as follows.

  Let $V = \mathbb{C}^4$ with basis $\{e_1, e_2, e_3, e_4\}$. Let h be the standard hermitian metric such that $h(e_i, e_j) = \delta_{ij}$. It induces a $SU(4)$-equivariant $\mathbb{C}$-conjugate linear isomorphism

$$\alpha : V \to V^*, \quad \alpha(v) \to h(v, -).$$

Let $\omega = -e_1 \wedge e_2 \wedge e_3 \wedge d_4 \in \wedge^4 V$. Let us consider the composition

$$J : \wedge^2 V \xrightarrow{-\wedge^2 \alpha} \wedge^2 V^* \xrightarrow{\lrcorner\omega} \wedge^2 V.$$

Then J is $SU(4)$-equivariant, $\mathbb{C}$-conjugate linear, and $J^2 = 1$. In particular, J defines a real structure on $\wedge^2 V$, whose real points can be identified with $\mathbb{R}^6$. The induced hermitian metric on $\wedge^2 V$ gives a real metric on $\mathbb{R}^6$, which is preserved by $SU(4)$. This gives a double cover

$$SU(4) \to SO(6)$$

and identifies $\mathrm{Spin}(6) \cong SU(4)$.

- $\mathrm{Spin}(5,1)$. Recall the reprsentation $\mathbb{H} \hookrightarrow M_2(\mathbb{C})$ above. This identifies

$$SL(2,\mathbb{H}) \cong \left(A \in SL(4,\mathbb{C}) | \bar{A} = \Gamma A \Gamma^{-1}\right), \quad \Gamma = \begin{pmatrix} \gamma & \\ & \gamma \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Consider the $SL(2,\mathbb{H})$-equivariant $\mathbb{C}$-conjugate linear map

$$J : \wedge^2\mathbb{C}^4 \to \wedge^2\mathbb{C}^4, \quad J(a \wedge b) = \Gamma^{-1}\bar{a} \wedge \Gamma^{-1}\bar{b}.$$

Let $\langle -, - \rangle$ be the $SL(4,\mathbb{C})$-invariant pairing on $\wedge^2\mathbb{C}^4$

$$\langle -, - \rangle : \wedge^2\mathbb{C}^4 \otimes \wedge^2\mathbb{C}^4 \to \wedge^4\mathbb{C}^4 \cong \mathbb{C}.$$

We identify $\mathbb{R}^6$ with real points of $\wedge^2\mathbb{C}^4$ with respect to J. In terms of standard basis $\{e_i\}_{i=1}^4$ of $\mathbb{C}^4$,

$$\mathbb{R}^6 = \mathrm{Span}_{\mathbb{R}}\left\{e_1 \wedge e_2, e_1 \wedge e_3 + e_2 \wedge e_4, \mathrm{i}(e_1 \wedge e_3 - e_2 \wedge e_4), e_1 \wedge e_4 - e_2 \wedge e_3, \mathrm{i}(e_1 \wedge e_4 + e_2 \wedge e_3), e_3 \wedge e_4\right\}.$$

It is easy to check that $\langle -, - \rangle$ gives $\mathbb{R}^6$ an inner product with signature $(-1,1,1,1,1,1)$. $SL(2,\mathbb{H})$ acts on $\mathbb{R}^6$ and preserves $\langle -, - \rangle$, leading to

$$SL(2,\mathbb{H}) \to SO(5,1).$$

3. Spinor

Definition 3.1. Let A be an algebra or group over k. Let $k \subset K$. Then a K-representation of A is a k-linear homomorphism

$$A \to \mathrm{Hom}_K(W, W)$$

for a K-vector space W. Equivalently, W is a representation of $A \otimes_k K$.

In this section $k = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. We are interested in k-representations of $\mathrm{Spin}(p,q)$.

3.1. Real spin representation.

Let $e_1, \cdots, e_{p+q}$ be orthonormal basis of $\mathbb{R}^{n=p+q}$. Define volume forms

$$\omega = e_1 \cdots e_{p+q}, \quad \omega_{\mathbb{C}} = i^{[\frac{n+1}{2}]+q}\omega.$$

ω is a central element of $Cl(V,q)$ if n is odd. We have

$$\omega_{\mathbb{C}}^2 = 1, \quad \omega^2 = \begin{cases} (-1)^q & n \equiv 0,3 \mod 4 \\ (-1)^{q+1} & n \equiv 1,2 \mod 4 \end{cases}$$

Definition 3.2. For $n \equiv 3 \mod 4, q$ even, or $n \equiv 1 \mod 4, q$ odd, we define the **chirality decomposition**

$$Cl_{p,q} = Cl_{p,q}^+ \oplus Cl_{p,q}^-, \quad Cl_{p,q}^\pm = \frac{1 \pm \omega}{2} Cl_{p,q}.$$

Similarly, for n odd in the complex case, we have chirality decomposition

$$\mathbb{C}l_n = \mathbb{C}l_n^+ \oplus \mathbb{C}l_n^-, \quad \mathbb{C}l_n^\pm = \frac{1 \pm \omega_{\mathbb{C}}}{2}\mathbb{C}l_n.$$

Example 3.3. Table 1 illustrates chirality decompositions in Euclidean and Minkowsky spaces.

Proposition 3.4. *If $n \equiv 3 \mod 4, q$ even, or $n \equiv 1 \mod 4, q$ odd, then $Cl_{p,q}$ has two inequivalent irreducible real representations. Otherwise $Cl_{p,q}$ has an unique irreducible real representation.*

Proof. This follows from the fact that $K(n)$ are simple $\mathbb{R}$-algebras for $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$. □

Proposition 3.5. *Let $n \equiv 3 \mod 4, q$ even, or $n \equiv 1 \mod 4, q$ odd. Let W be the unique irreducible representation of $Cl_{p+1,q}$, $\omega_{p+1,q}$ be the corresponding volume form. Then*

(1) *$\omega^2_{p+1,q} = 1$, which induces a decomposition $W = W^+ \oplus W^-$ as $Cl^0_{p+1,q}$-modules.*
(2) *Under $Cl_{p,q} \cong Cl^0_{p+1,q}$, $W^\pm$ are the two inequivalent irreducible real representations of $Cl_{p,q}$.*

Proof. (1) is obvious. (2) is proved by comparing the volume forms. □

Lemma 3.6. *Let q even, $n \equiv 3 \mod 4$, or q odd, $n \equiv 1 \mod 4$. Let $W^\pm$ be the two irreducible representations of $Cl_{p,q}$. Let*

$$\Delta^\pm : \mathrm{Spin}(p,q) \subset Cl^0_{p,q} \subset Cl_{p,q} \to GL(W^\pm, \mathbb{R})$$

be the induced real representations. Then $\Delta^\pm$ are equivalent real representations of $\mathrm{Spin}(p,q)$.

Proof. The reflection automorphism switches $Cl^+_{p,q} \leftrightarrow Cl^-_{p,q}$ since $\hat{\omega} = -\omega$. It follows that

$$Cl^0_{p,q} = \{x \oplus \hat{x} | x \in Cl^+_{p,q}\}.$$

The lemma follows immediately. □

Definition 3.7. We define the real spinor representation $S = S_{p,q}$ of $\mathrm{Spin}(p,q)$ as the induced representation

$$\Delta_{p,q} : \mathrm{Spin}(p,q) \to GL(S, \mathbb{R})$$

from an irreducible representation S of $Cl_{p,q}$ under $\mathrm{Spin}(p,q) \subset Cl^0_{p,q} \subset Cl_{p,q}$.

Lemma 3.6 implies that this definition is well-defined for any (p,q). By construction, $\Delta_{p,q}$ does not come from a representation of $SO(p,q)$. Proposition 3.4 implies that $S_{p,q}$ is reducible when $n \equiv 0 \mod 4, q$ even or $n \equiv 2 \mod 4, q$ odd.

Example 3.8. Euclidean space.

n	1	2	3	4	5	6	7	8
$Cl_{n,0}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
$S_{n,0}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}_\pm$	$\mathbb{H}^2$	$\mathbb{C}^4$	$\mathbb{R}^8$	$\mathbb{R}^8_\pm$	$\mathbb{R}^{16}$
Irreducible $\mathbb{R}$-spinors	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}_\pm$	$\mathbb{H}^2$	$\mathbb{C}^4$	$\mathbb{R}^8$	$\mathbb{R}^8_\pm$

Using $Cl_{n-1,0} = Cl^0_{n,0}$, we find

- $n \equiv 3,5,6,7 \mod 8$. S is irreducible, quaternion for $n = 3,5$, complex for $n = 6$, real for $n = 7$.
- $n \equiv 1 \mod 8$. S is a direct sum of two equivalent irreducible $\mathbb{R}$-representations.
- $n \equiv 2 \mod 8$. S is a direct sum of two equivalent irreducible $\mathbb{C}$-representations.
- $n \equiv 4 \mod 8$. S is a direct sum of two inequivalent irreducible $\mathbb{H}$-representations.
- $n \equiv 8 \mod 8$. S is a direct sum of two inequivalent irreducible $\mathbb{R}$-representations.

Example 3.9. Minkowski space.

n	1	2	3	4	5	6	7	8
$Cl_{n-1,1}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
$S_{n-1,1}$	$\mathbb{R}_\pm$	$\mathbb{R}^2$	$\mathbb{C}^2$	$\mathbb{H}^2$	$\mathbb{H}^2_\pm$	$\mathbb{H}^4$	$\mathbb{C}^8$	$\mathbb{R}^{16}$
Irreducible $\mathbb{R}$-spinors	$\mathbb{R}$	$\mathbb{R}_\pm$	$\mathbb{R}^2$	$\mathbb{C}^2$	$\mathbb{H}^2$	$\mathbb{H}^2_\pm$	$\mathbb{H}^4$	$\mathbb{C}^8$

- $n \equiv 1,5,7,8 \mod 8$. S is irreducible, quaternion for $n=5,7$, complex for $n=8$, real for $n=1$.
- $n=3 \mod 8$. S is a direct sum of two equivalent irreducible $\mathbb{R}$-representations.
- $n=4 \mod 8$. S is a direct sum of two equivalent irreducible $\mathbb{C}$-representations.
- $n=2 \mod 8$. S is a direct sum of two inequivalent irreducible $\mathbb{R}$-representations.
- $n=6 \mod 8$. S is a direct sum of two inequivalent irreducible $\mathbb{H}$-representations.

3.2. **Complex spin representation.** Consider $V_\mathbb{C}=\mathbb{C}^n$, and $\omega_\mathbb{C}$ be the volume form as above, $\omega_\mathbb{C}^2=1$. We have the chirality decomposition

$$\mathbb{C}l_n=\mathbb{C}l_n^+\oplus\mathbb{C}l_n^-,\quad n \text{ odd}.$$

The following proposition is the complex analogue of the previous discussion.

Proposition 3.10. *For n even, $\mathbb{C}l_n$ has a unique irreducible representation W. Moreover,*

(1) *W is decomposed $W=W^+\oplus W^-$ as $\mathbb{C}l^0_{n-1}$-modules.*
(2) *Under $\mathbb{C}l_{n-1}\cong\mathbb{C}l_n^0$, $W^\pm$ are the two inequivalent irreducible representations of $\mathbb{C}l_{n-1}$.*
(3) *$W^\pm$ are equivalent* $\mathrm{Spin}_\mathbb{C}(n-1)$ *representations under* $\mathrm{Spin}_\mathbb{C}(n-1)\subset\mathbb{C}l^0_{n-1}\subset\mathbb{C}l_{n+1}$.

Definition 3.11. Let $p+q=n$. We define the complex spinor representation $\mathbb{S}=\mathbb{S}_n$ of $\mathrm{Spin}_\mathbb{C}(n)$

$$\Delta_n^\mathbb{C}:\mathrm{Spin}_\mathbb{C}(n)\to GL(\mathbb{S},\mathbb{C})$$

to be the induced one from an irreducible representation $\mathbb{S}$ of $\mathbb{C}l_n$ under $\mathrm{Spin}_\mathbb{C}(n)\subset\mathbb{C}l_n^0\subset\mathbb{C}l_n$.

Example 3.12. In the complex case, we have the following table

n	2m	2m+1
$\mathbb{C}l_n$	$\mathbb{C}(2^m)$	$\mathbb{C}(2^m)\oplus\mathbb{C}(2^m)$
$\mathbb{S}_n$	$\mathbb{C}^{2^m}$	$\mathbb{C}^{2^m}$
Irreducible $\mathbb{C}$-spinors	$\mathbb{C}_\pm^{2^{m-1}}$	$\mathbb{C}^{2^m}$

We have the following concrete realization of the above complex represenations.

- $\boxed{n=2m}$. $V=\mathbb{R}^{2m}=\mathbb{C}^m, V_\mathbb{C}=V\otimes_\mathbb{R}\mathbb{C}=V_\mathbb{C}^{1,0}\oplus V_\mathbb{C}^{0,1}$. Let us represent

$$V=\mathrm{Span}_\mathbb{R}\{dx^i,dy^i\}_{1\le i\le m},\quad V_\mathbb{C}^{1,0}=\mathrm{Span}_\mathbb{C}\{dz^i\}_{1\le i\le m},\quad V_\mathbb{C}^{0,1}=\mathrm{Span}_\mathbb{C}\{d\bar z^i\}_{1\le i\le m}.$$

Then the Clifford action $Cl(V)\to\mathrm{End}_\mathbb{C}(\wedge^*V_\mathbb{C}^{1,0})$ has a geometric realization

$$dx^i\to dz^i-\iota_{z^i},\quad dy^i\to\frac{1}{\sqrt{-1}}(dz^i+\iota_{z^i}).$$

The two irreducible $\mathbb{C}$-spinors $\mathbb{S}_\pm$ are given by

$$\boxed{\mathbb{S}_+=\wedge^{even}V_\mathbb{C}^{1,0},\quad \mathbb{S}_-=\wedge^{odd}V_\mathbb{C}^{1,0},\quad \mathbb{S}_n\cong\wedge^*V_\mathbb{C}^{1,0}.}$$

Similarly there is a Clifford acton $Cl(V)\to\mathrm{End}_\mathbb{C}(\wedge^*V_\mathbb{C}^{0,1})$ realized by

$$dx^i\to d\bar z^i-\iota_{\bar z^i},\quad dy^i\to-\frac{1}{\sqrt{-1}}(d\bar z^i+\iota_{\bar z^i}).$$

$\wedge^*V_\mathbb{C}^{1,0}$ and $\wedge^*V_\mathbb{C}^{0,1}$ are isomorphic Cl_{2m}-modules under the complex conjugation.

Proposition 3.13. *Let $\mathbb{S}_n$ be the complex spinor representation. Then we have isomorphic* $\mathrm{Spin}(n)$*-modules*

$$\mathbb{S}_n \otimes_{\mathbb{C}} \mathbb{S}_n \cong \begin{cases} \wedge^* \mathbb{C}^n, & \text{for } n \text{ even.} \\ \wedge^{even}\mathbb{C}^n \cong \wedge^{odd}\mathbb{C}^n, & \text{for } n \text{ odd.} \end{cases}$$

Proof. Assume $n = 2m$. By dimension reason, $\mathbb{S}_{2m} \otimes_{\mathbb{C}} \mathbb{S}_{2m}$ is the irreducible $\mathbb{C}l_{2m} \otimes_{\mathbb{C}} \mathbb{C}l_{2m} \cong \mathbb{C}l_{4m}$-module. On the other hand, $\mathbb{C}l_{2m}$ is the irreducible $\mathbb{C}l_{2m} \otimes_{\mathbb{C}} \mathbb{C}l_{2m}$-module by

$$\Phi : \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}l_n \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}l_n), \quad \Phi_{x,y}(u) = xuy^t.$$

It follows that we have equivalent $\mathbb{C}l_{2m} \otimes_{\mathbb{C}} \mathbb{C}l_{2m}$-modules

$$\mathbb{S}_{2m} \otimes_{\mathbb{C}} \mathbb{S}_{2m} \cong \mathbb{C}l_{2m}.$$

Restricting to $\mathrm{Spin}(2m)$-modules, and observing that for $x \in \mathrm{Spin}(2m)$, $x^t = x^{-1}, x = \hat{x}$, we find

$$\mathbb{S}_{2m} \otimes_{\mathbb{C}} \mathbb{S}_{2m} \cong \wedge^* \mathbb{C}^{2m} \quad \text{as } \mathrm{Spin}(2m)\text{-modules.}$$

Assume $n = 2m + 1$. $\mathbb{C}l_{2m+1} = \mathbb{C}l^+_{2m+1} \oplus \mathbb{C}l^-_{2m+1}$, with diagonal embedding

$$\mathbb{C}l^0_{2m+1} = \{x \oplus \hat{x} | x \in \mathbb{C}l^+_{2m+1}\}.$$

Let $\mathbb{S}_{2m+1}$ denote the irreducible representation of $\mathbb{C}l^+_{2m+1} \cong \mathbb{C}l^0_{2m+1}$. Similar argument as above shows

$$\mathbb{S}_{2m+1} \otimes_{\mathbb{C}} \mathbb{S}_{2m+1} \cong \mathbb{C}l^0_{2m+1} = \wedge^{even}\mathbb{C}^{2m+1}.$$

□

3.3. **Spinors in physics.**

Definition 3.14. Let V be a $\mathbb{C}$-representation of a real group G.

- V is said to be of **real type** if there exists a G-equivariant real structure $J : V \to V$ (i.e. J is complex conjugate linear and $J^2 = 1$). The real points $\mathrm{Re}(V) = \{v \in V | J(v) = v\}$ is a $\mathbb{R}$-representation of G.
- V is said to be of **quaternionic type** if there exists a G-equivariant quaternionic structure $J : V \to V$ (i.e. J is complex conjugate linear and $J^2 = -1$). We can define symplectic real points on even copies of V as follows: let

$$\Phi_J = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} : V^{2m} \to V^{2m},$$

which defines a G-equivariant real structure on V^{2m}. Then $\mathrm{Re}_{\Phi_J}(V^{2m})$ defines a real G-representation.

Various spinors in physics terminology have the following interpretation.

(1) **Dirac spinor**. $\mathbb{S}_{n=p+q}$ gives a $\mathbb{C}$-representation of $\mathrm{Spin}(p,q)$ under an isomorphism

$$\mathrm{Spin}(p,q) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Spin}_{\mathbb{C}}(n).$$

This representation is called a Dirac spinor.

(2) **Weyl spinor**. When $n = 2m$ is even, $\mathbb{S}_n$ is decomposed into two irreducible $\mathbb{C}$-representations

$$\mathbb{S}_n = \mathbb{S}^+_n \oplus \mathbb{S}^-_n.$$

Each $\mathbb{S}^\pm_n$ is called a Weyl spinor.

(3) **Majorana spinor**. If the $\mathbb{C}$-representation $\mathbb{S}_{n=p+q}$ of $\mathrm{Spin}(p,q)$ is of real type, then its real points M_n is called a Majorana spinor.

(4) **Symplectic-Majorana spinor**. If the $\mathbb{C}$-representation $\mathbb{S}_{n=p+q}$ of $\mathrm{Spin}(p,q)$ is of quaternionic type, we can impose the symplectic-Majorana reality condition. The symplectic real points of even copies of $\mathbb{S}_n$ is called a symplectic-Majorana spinor.

(5) **Majorana-Weyl spinor**. When $n = 2m = p + q$ is even and the weyl spinors $\mathbb{S}_n^{\pm}$ are of real type, then the real points are called Majorana-Weyl spinors.

(6) **Symplectic-Majorana-Weyl spinor**. When $n = 2m = p + q$ is even and the weyl spinors $\mathbb{S}_n^{\pm}$ are of quaternionic type, we can impose the symplectic-Majorana-Weyl reality condition. The symplectic real points of even copies of $\mathbb{S}_n^{\pm}$ is called a Symplectic-Majorana-Weyl spinor.

Example 3.15 (Euclidean space)**.** The real and complex representations for Euclidean spaces are summarized as follows.

n	1	2	3	4	5	6	7	8
$Cl_{n,0}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$
$S_{n,0}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}_{\pm}$	$\mathbb{H}^2$	$\mathbb{C}^4$	$\mathbb{R}^8$	$\mathbb{R}^8_{\pm}$	$\mathbb{R}^{16}$
Irred $\mathbb{R}$-spinor	$\mathbb{R}$ (M)	$\mathbb{C}$ (W)	$\mathbb{H}$ (SM)	$\mathbb{H}_{\pm}$ (SMW)	$\mathbb{H}^2$ (SM)	$\mathbb{C}^4$ (W)	$\mathbb{R}^8$ (M)	$\mathbb{R}^8_{\pm}$ (MW)
$\mathbb{C}l_n$	$\mathbb{C}\oplus\mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2)\oplus\mathbb{C}(2)$	$\mathbb{C}(4)$	$\mathbb{C}(4)\oplus\mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8)\oplus\mathbb{C}(8)$	$\mathbb{C}(16)$
$\mathbb{S}_n$	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}^4$	$\mathbb{C}^4$	$\mathbb{C}^8$	$\mathbb{C}^8$	$\mathbb{C}^{16}$
Irred $\mathbb{C}$-spinors	$\mathbb{C}$	$\mathbb{C}_{\pm}$	$\mathbb{C}^2$	$\mathbb{C}^2_{\pm}$	$\mathbb{C}^4$	$\mathbb{C}^4_{\pm}$	$\mathbb{C}^8$	$\mathbb{C}^8_{\pm}$

(1) $n \equiv 8 \mod 8$. The two chiral irreducible $\mathbb{R}$-spinors are **Majorana-Weyl** (MW) spinors.
(2) $n \equiv 4 \mod 8$. The two chiral irreducible $\mathbb{R}$-spinors are **Symplectic-Majorana-Weyl** (SMW) spinor.
(3) $n \equiv 2,6 \mod 8$. The irreducible chiral $\mathbb{C}$-spinors are **Weyl** spinors. They give rise to equivalent $\mathbb{R}$-spinors which are complex conjugate of each other.
(4) $n \equiv 1,7 \mod 8$. The irreducible $\mathbb{R}$-spinors are **Majorana** spinors.
(5) $n \equiv 3,5 \mod 8$. The irreducible $\mathbb{R}$-spinors are **Symplectic-Majorana** spinors.

Example 3.16 (Minkowski space)**.** The real and complex representations for Minkowski spaces are summarized as follows.

n	1	2	3	4	5	6	7	8
$Cl_{n-1,1}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
$S_{n-1,1}$	$\mathbb{R}_{\pm}$	$\mathbb{R}^2$	$\mathbb{C}^2$	$\mathbb{H}^2$	$\mathbb{H}^2_{\pm}$	$\mathbb{H}^4$	$\mathbb{C}^8$	$\mathbb{R}^{16}$
Irred $\mathbb{R}$-spinor	$\mathbb{R}$ (M)	$\mathbb{R}_{\pm}$ (MW)	$\mathbb{R}^2$ (M)	$\mathbb{C}^2$ (W)	$\mathbb{H}^2$ (SM)	$\mathbb{H}^2_{\pm}$ (SMW)	$\mathbb{H}^4$ (SM)	$\mathbb{C}^8$ (W)
$\mathbb{C}l_n$	$\mathbb{C}\oplus\mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2)\oplus\mathbb{C}(2)$	$\mathbb{C}(4)$	$\mathbb{C}(4)\oplus\mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8)\oplus\mathbb{C}(8)$	$\mathbb{C}(16)$
$\mathbb{S}_n$	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}^4$	$\mathbb{C}^4$	$\mathbb{C}^8$	$\mathbb{C}^8$	$\mathbb{C}^{16}$
Irred $\mathbb{C}$-spinors	$\mathbb{C}$	$\mathbb{C}_{\pm}$	$\mathbb{C}^2$	$\mathbb{C}^2_{\pm}$	$\mathbb{C}^4$	$\mathbb{C}^4_{\pm}$	$\mathbb{C}^8$	$\mathbb{C}^8_{\pm}$

(1) $n \equiv 2 \mod 8$. The two chiral irreducible $\mathbb{R}$-spinors are **Majorana-Weyl** (MW) spinors.
(2) $n \equiv 6 \mod 8$. The two chiral irreducible $\mathbb{R}$-spinors are **Symplectic-Majorana-Weyl** (SMW) spinor.
(3) $n \equiv 4,8 \mod 8$. The irreducible chiral $\mathbb{C}$-spinors are **Weyl** spinors. They give rise to equivalent $\mathbb{R}$-spinors which are complex conjugate of each other.
(4) $n \equiv 1,3 \mod 8$. The irreducible $\mathbb{R}$-spinors are **Majorana** spinors.
(5) $n \equiv 5,7 \mod 8$. The irreducible $\mathbb{R}$-spinors are **Symplectic-Majorana** spinors.

3.4. **Unitarity.**

Definition 3.17. Let V be a vector space over k [1]. A k-hermitian form h is a $\mathbb{R}$-bilinear pairing

$$h(-,-): V \otimes_{\mathbb{R}} V \to k$$

such that for any $v_1, v_2 \in V, \lambda \in k$

- $h(v_1, v_2\lambda) = h(v_1, v_2)\lambda$.
- $h(v_1, v_2) = \overline{h(v_2, v_1)}$.

h is called positive-definite if $h(v,v) > 0$ for any nonzero $v \in V$.

Remark 3.18. We have the alternate description of hermitian forms

(1) $\mathbb{R}$-hermitian form is the same as an inner product.
(2) $\mathbb{C}$-hermitian form is the same as a I-invariant sympletic pairing $\omega : \wedge^2_{\mathbb{R}} V \to \mathbb{R}$. Here I ($I^2 = -1$) defines the complex structure on V. Then

$$h(v_1, v_2) = \omega(v_1, Iv_2) + \mathrm{i}\,\omega(v_1, v_2).$$

(3) $\mathbb{H}$-hermitian form is the same as a sympletic pairing $\omega_{\mathbb{C}} : \wedge^2_{\mathbb{C}} V \to \mathbb{C}$ such that

$$\omega_{\mathbb{C}}(v_1 J, v_2 J) = \overline{\omega_{\mathbb{C}}(v_1, v_2)}.$$

Here $J : V \to V$ ($J^2 = -1$) is the complex conjugate linear operator defining the quaternionic structure on V. Then

$$h(v_1, v_2) = \overline{\omega_{\mathbb{C}}(v_1, v_2 J)} + \mathrm{j}\,\omega_{\mathbb{C}}(v_1, v_2).$$

Note that $\omega_{\mathbb{C}}(\cdot\,\mathrm{j}, \cdot)$ defines a $\mathbb{C}$-hermitian form on V.

Lemma 3.19. *Let G be a finite group or compact Lie group. Let W be a k-representation of G. Then W carries a G-invariant positive definite k-hermitian form.*

Proof. Let h be any positive definite k-hermitian form. Averaging h over G gives a desired hermitian form. □

3.4.1. *Euclidean space.* We consider the Euclidean space $\mathbb{R}^n$ and unitarity of spinors.

Proposition 3.20. *Let W be a k-representation of Cl_n. Then there exists a positive definite k-hermitian form h on W that is invariant under Clifford multiplication by unit vectors $e \in \mathbb{R}^n$, i.e.,*

$$h(e \cdot s_1, e \cdot s_2) = h(s_1, s_2), \quad \forall e \in \mathbb{R}^n, |e|^2 = 1, \quad s_i \in W.$$

In particular, h leads to group homomorphism

$$\mathrm{Spin}(n) \to \begin{cases} SO(W) & k = \mathbb{R} \\ SU(W) & k = \mathbb{C} \\ Sp(W) & k = \mathbb{H}. \end{cases}$$

Proof. Consider the finite group with presentation

$$\Gamma_n = \left\langle e_1, \cdots, e_n, -1 \middle| e_i^2 = -1, (-1)^2 = 1, e_i e_j = (-1)e_j e_i, (-1)e_i = e_i(-1) \right\rangle.$$

Then W carries a representation of Γ_n such that (-1) acts as $-\mathrm{Id}$. Then h is given by a Γ_n-invariant positive definite k-hermitian form. □

[1] When $k = \mathbb{H}$, V is a right $\mathbb{H}$-module

Remark 3.21. For later applications, we collect formulae for real dimensions

$$\begin{cases}\dim_{\mathbb{R}} SO(n) = \frac{1}{2}n(n-1)\\ \dim_{\mathbb{R}} SU(n) = n^2-1\\ \dim_{\mathbb{R}} Sp(n) = n(2n+1).\end{cases}$$

Example 3.22. The irreducible real spinor of Euclidean $\mathbb{R}^6$ is $\mathbb{C}^4$. This leads to an isomorphism

$$\mathrm{Spin}(6) \to SU(4).$$

This is explicitly realized in Example 2.11.

3.4.2. *Minkowski space.* Now we consider the Minkowski space. Let $\{e_i\}_{i=1,\cdots,n}$ be the orthonormal generators of $Cl_{n,0}$. Then the generators $\tilde{e}_0, \tilde{e}_1, \cdots, \tilde{e}_{n-1}$ of $Cl_{n-1,1}$ can be realized inside $\mathbb{C}l_n$ via

$$\tilde{e}_1 = e_1, \cdots, \tilde{e}_{n-1} = e_{n-1}, \tilde{e}_0 = \sqrt{-1}e_n.$$

Let $\mathbb{S}$ be a $\mathbb{C}$-representation of $\mathbb{C}l_n$, with a $\mathbb{C}$-hermitian form h by Proposition 3.20. Then

$$h\left(\tilde{e}_0 s_1, s_2\right) = h\left(s_1, \tilde{e}_0 s_2\right), \quad h\left(\tilde{e}_i s_1, s_2\right) = -h\left(s_1, \tilde{e}_i s_2\right), \quad 1 \le i \le n-1.$$

Let us denote

$$\langle s_1, s_2\rangle_0 = h\left(\tilde{e}_0 s_1, s_2\right).$$

Then $\langle s_1, s_2\rangle$ is a $\mathbb{C}$-hermitian form satisfying

$$\langle s_1, \tilde{e}_i s_2\rangle_0 = \langle \tilde{e}_i s_1, s_2\rangle_0, \quad 0 \le i \le n-1.$$

In particular, $\langle -,-\rangle_0$ is $\mathrm{Spin}(n-1,1)$-invariant but not positive definite.

Proposition 3.23. *Let $x \in \wedge^k \mathbb{R}^{n-1,1} \subset Cl_{n-1,1}$. Then*

$$\langle s_1, x\cdot s_2\rangle_0 = (-)^{k(k-1)/2}\overline{\langle s_2, x\cdot s_1\rangle_0}.$$

In particular, $\mathrm{i}^{k(k-1)/2}\langle s, xs\rangle_0$ *is real.*

Remark 3.24. In physics application, this proposition shows the reality of the following expression

$$\int \psi^\dagger \gamma^0 (i\gamma^\mu \partial_\mu - m)\psi.$$

3.5. **Charge conjugation.**

Definition 3.25. Let $\mathbb{S}$ be a k-representation of $\mathrm{Spin}(p,q)$. We define a **charge conjugation** on $\mathbb{S}$ to be a $\mathrm{Spin}(p,q)$-invariant non-degenerate bilinear form $C : \mathbb{S}\otimes_k \mathbb{S} \to k$.

We consider charge conjugation for complex spinors.

Definition 3.26. Let $Cl(V)$ be a Clifford algebra, W be a Clifford k-module. We define two Clifford structures ons its k-linear dual, denoted by $W^\vee, W^{\bar{\vee}}$ respectively, via

$$W^\vee : \quad (x\cdot\varphi)(w) = \varphi(x^t\cdot w), \quad \forall w \in W, x \in Cl(V).$$

and

$$W^{\bar{\vee}} : \quad (x\cdot\varphi)(w) = \varphi(\hat{x}^t\cdot w), \quad \forall w \in W, x \in Cl(V).$$

They induce identical spin representations. A **charge conjugation** C on W is a Clifford module isomorphism between W and $W^\vee$ or $W^{\bar{\vee}}$.

Equivalently, C can be viewed as a non-degenerate bilinear form $C : W \otimes_k W \to k$ such that

$$C(e \cdot s_1, s_2) = \eta C(s_1, e \cdot s_2) \quad \forall s_i \in W, e \in V.$$

Here η is either $+1$ or -1. We divide our discussion into cases when n is even or odd.

3.5.1. *n=2m.*

Lemma/Definition 3.27. *Let $\mathbb{S}_{2m}$ be the complex spin representation. Then there exists unique (up to rescaling) $\mathbb{C}l_{2m}$-module isomorphisms*

$$C_+ : \mathbb{S}_{2m} \to \mathbb{S}_{2m}^{\vee}, \quad C_- : \mathbb{S}_{2m} \to \mathbb{S}_{2m}^{\bar{\vee}}.$$

The corresponding charge conjugation is denoted by

$$C_{\pm} : \mathbb{S}_{2m} \otimes_{\mathbb{C}} \mathbb{S}_{2m} \to \mathbb{C}.$$

It satisfies the following symmetry properties

$$\boxed{C_{\pm}(\alpha, \beta) = (-)^{m(m\mp 1)/2} C_{\pm}(\beta, \alpha)}, \quad \alpha, \beta \in \mathbb{S}_{2m}.$$

Proof. The definition of $\mathbb{C}_{\pm}$ follows from the uniqueness of $\mathbb{C}l_{2m}$-representation.

To see the symmetry property, we use the presentation in Example 3.12. Let $V = \mathbb{R}^{2m} = \mathbb{C}^m, V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} = V_{\mathbb{C}}^{1,0} \oplus V_{\mathbb{C}}^{0,1}$. Let us represent $\mathbb{S}_{2m} = \wedge^* V_{\mathbb{C}}^{1,0}$. Then the pairing

$$C_{\pm} : \wedge^* V_{\mathbb{C}}^{1,0} \otimes \wedge^* V_{\mathbb{C}}^{1,0} \to \wedge^m V_{\mathbb{C}}^{1,0} \cong \mathbb{C}$$

is given by

$$C_+(\alpha, \beta) = (\alpha^t \wedge \beta)^{top}, \quad C_-(\alpha, \beta) = (\hat{\alpha}^t \wedge \beta)^{top}, \quad \alpha, \beta \in \wedge^* V_{\mathbb{C}}^{1,0}.$$

Here $(dz_{i_1} \cdots dz_{i_k})^t = dz_{i_k} \cdots dz_{i_1}$, $\left(\widehat{dz_{i_1} \cdots dz_{i_k}}\right)^t = (-1)^k dz_{i_k} \cdots dz_{i_1}$. The symmetry property follows. □

Definition 3.28. Let $V_{\mathbb{C}} = \mathbb{C}^{2m}$. We define the pairing

$$\Gamma_{\pm}^k : \mathbb{S}_{2m} \otimes_{\mathbb{C}} \mathbb{S}_{2m} \to \wedge^k V_{\mathbb{C}}^{\vee}$$

by the formula

$$\Gamma_{\pm}^k(s_1, s_2)(\alpha) = C_{\pm}(s_1, \alpha \cdot s_2)$$

where $\alpha \cdot s_2$ is the Clifford action.

Proposition 3.29. $\Gamma_{\pm}^k$ *has the following symmetry property:*

$$\boxed{\Gamma_{\pm}^k(s_1, s_2) = (-1)^{k(k\mp 1)/2 + m(m\mp 1)/2} \Gamma_{\pm}^k(s_2, s_1)}, \quad s_i \in \mathbb{S}_{2m}.$$

Proof.

$$\begin{aligned}
\Gamma_{\pm}^k(s_1, s_2)(\alpha) &= C_{\pm}(s_1, \alpha s_2) = (\pm)^k C_{\pm}(\alpha^t s_1, s_2) = (\pm)^k (-)^{k(k-1)/2} C_{\pm}(\alpha s_1, s_2) \\
&= (\pm)^k (-)^{k(k-1)/2} (-)^{m(m\mp 1)/2} C_{\pm}(s_2, \alpha s_1) = (-1)^{k(k\mp 1)/2 + m(m\mp 1)/2} \Gamma_{\pm}^k(s_2, s_1)(\alpha).
\end{aligned}$$

□

Remark 3.30. Let $\omega_{\mathbb{C}}$ be the complex volume form. Then $C_{\pm}$ are related by

$$C_+(\cdot, \cdot) = C_-(\omega_{\mathbb{C}} \cdot, \cdot).$$

Remark 3.31. Note that for $m+k$ is odd, the pairing $\Gamma^k_\pm$ is in fact on different chiral spinors

$$\Gamma^k_\pm : \mathbb{S}_+ \otimes \mathbb{S}_- \to \wedge^k V^\vee_{\mathbb{C}}.$$

On the other hand, when $m+k$ is even, then $\Gamma^k_\pm$ have the same symmetry property.

3.5.2. $n = 2m+1$. Let us first describe the charge conjugation. Let $\mathbb{S} = \mathbb{S}_{2m+1}$ be the irreducible representation of $\mathbb{C}l^+_{2m+1}$. Observe that the volume form has the property

$$\omega^t = (-1)^m \omega, \quad \hat{\omega}^t = -(-1)^m \omega.$$

This implies that

$$\mathbb{S}_{2m+1} \cong \begin{cases} \mathbb{S}^\vee_{2m+1} & m \text{ even} \\ \hat{\mathbb{S}}^\vee_{2m+1} & m \text{ odd} \end{cases}.$$

Definition 3.32. When $n = 2m+1$, there is only one charge conjugation (up to rescaling) by

$$C_+ : \mathbb{S}_{2m+1} \otimes \mathbb{S}_{2m+1} \to \mathbb{C}, \quad m \text{ even}$$
$$C_- : \mathbb{S}_{2m+1} \otimes \mathbb{S}_{2m+1} \to \mathbb{C}, \quad m \text{ odd}$$

We will just denote it by $C : \mathbb{S}_{2m+1} \otimes \mathbb{S}_{2m+1} \to \mathbb{C}$.

Lemma 3.33. *The charge conjugation C has the following symmetry property*

$$\boxed{C(\alpha,\beta) = (-)^{m(m+1)/2} C(\beta,\alpha)}.$$

Proof. Let us consider the embedding

$$j : \mathbb{C}l_{2m} \cong \mathbb{C}l^0_{2m+1} \subset \mathbb{C}l_{2m+1}.$$

It is easy to see that

$$j(\hat{x}^t) = j(x)^t = \overline{j(x)}^t.$$

Therefore the symmetry property of C on $\mathbb{S}_{2m+1}$ is the same as C_- on $\mathbb{S}_{2m}$ for any m. $\square$

Definition 3.34. Let $V_{\mathbb{C}} = \mathbb{C}^{2m+1}$. We can define the pairing

$$\Gamma^k : \mathbb{S}_{2m+1} \otimes \mathbb{S}_{2m+1} \to \wedge^k V^\vee_{\mathbb{C}}.$$

Similarly,

Proposition 3.35. Γ^k *has the following symmetry property:*

$$\boxed{\Gamma^k(s_1,s_2) = (-1)^{k(k-1)/2+mk+m(m+1)/2}\Gamma^k(s_2,s_1)}, \quad s_i \in \mathbb{S}_{2m+1}.$$

Note that $(-1)^{k(k-1)/2+mk+m(m+1)/2} = (-1)^{(m-k)(m-k+1)/2}$.

3.5.3. *Majorana spinor revisited.* Now we revisit the meaning of Majorana spinor for $\mathrm{Spin}(p,q)$. Let e_i be the Clifford generator of $Cl(p,q)$ such that

$$e_i^2 = \begin{cases} -1 & \text{if } 1 \le i \le p \\ +1 & \text{if } p+1 \le i \le n = p+q. \end{cases}$$

Let $\mathbb{S}_n$ be the complex spin representation. Let $(-,-)$ be a hermitian form on $\mathbb{S}_n$ such that

$$(e_i \cdot s_1, s_2) = \eta\,(s_1, e_i \cdot s_2)\,, \quad 1 \le i \le n, s_1, s_2 \in \mathbb{S}_n.$$

Here $\eta = \pm 1$ is a fixed sign (we can choose $\eta = (-1)^{q+1}$). In the Euclidean case, $(-,-)$ is the hermitian inner product with $\eta = -1$. In the Minkowski case, $(-,-) = \langle -,- \rangle_0$ defined in Section 3.4.2 with $\eta = 1$.

Let $C(-,-)$ be a charge conjugation. We define a complex conjugate linear map $* : \mathbb{S}_n \to \mathbb{S}_n, s \to s^*$

$$h(s_1, s_2) = C(s_1^*, s_2), \quad \forall s_i \in \mathbb{S}_n.$$

For any unit generator e_i and $s \in \mathbb{S}_n$,

$$(e_i \cdot s)^* = \pm e_i \cdot s^*.$$

where the sign $\pm$ depends on the signature and charge conjugation. In particular, $*$ is $\mathrm{Spin}(n)$-equivariant.

Majorana-type conditions for $\mathbb{S}_n$ appear precisely when $*^2 = \pm 1$. Precisely,

$$(*)^2 = \begin{cases} 1 & \text{Majorana} \\ -1 & \text{Symplectic-Majorana} \end{cases}.$$

- When $*^2 = 1$, $*$ defines a real structure. The Majorana spinors can be expressed by
$$s^* = s, \quad s \in \mathbb{S}_n.$$
- When $*^2 = -1$, $*$ defines a quaternionic structure. We need several spinors $\mathbb{S}_n^{\oplus N}$ to impose the symplectic-Majorana condition
$$s^* = \Omega s, \quad s \in \mathbb{S}_n^{\oplus N},$$
where Ω is a anti-symmetric $N \times N$-matrix with $\Omega\overline{\Omega} = -1$.

4. Poincaré group

The **Poincaré group** is the isometry group of $\mathbb{R}^{p,q}$. We work with its universal cover and denote by

$$\mathrm{Poin}(p,q) = \mathbb{R}^{p,q} \rtimes \mathrm{Spin}(p,q).$$

In physics, particles are organized into unitary representations of Poincaré group. There is an essential difference between Euclidean and Minkowski cases: $\mathrm{Spin}(d)$ is a compact simple Lie group while $\mathrm{Spin}(d-1,1)$ is a non-compact simple Lie group. It is known that every non-trivial irreducible unitary representation of a non-compact simple Lie group is infinite dimensional, while for compact Lie groups they are all finite dimensional. We will focus on the Minkowski space in this section.

4.1. **Poincaré algebra.** Let $\mathrm{poin}(d-1,1)$ be the Lie algebra of $\mathrm{Poin}(d-1,1)$, called the **Poincaré algebra**.

Let us choose linear coordinates x^μ of $\mathbb{R}^{d-1,1}$ with metric

$$\eta = -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^{d-1})^2.$$

A basis of $\mathrm{poin}(d-1,1)$ can be represented by

$$\mathbf{P}_\mu = -i\frac{\partial}{\partial x^\mu}, \quad \mathbf{M}_{\mu\nu} = -i\left(x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}\right)$$

satisfying the Poincaré algebra relations

$$\begin{aligned} \left[\mathbf{P}_\mu, \mathbf{P}_\nu\right] &= 0 \\ \left[\mathbf{M}_{\mu\nu}, \mathbf{P}_\rho\right] &= i\eta_{\mu\rho}\mathbf{P}_\nu - i\eta_{\nu\rho}\mathbf{P}_\mu \\ \left[\mathbf{M}_{\mu\nu}, \mathbf{M}_{\rho\sigma}\right] &= i\eta_{\mu\rho}\mathbf{M}_{\nu\sigma} - i\eta_{\nu\rho}\mathbf{M}_{\mu\sigma} - (\rho \leftrightarrow \sigma) \end{aligned}$$

There are two Casimir operators

$$C_1 = -\mathbf{P}^2 = -\mathbf{P}^\mu\mathbf{P}_\mu, \quad C_2 = -\frac{1}{2}\mathbf{P}^2\mathbf{M}_{\mu\nu}\mathbf{M}^{\mu\nu} + \mathbf{M}_{\mu\rho}\mathbf{P}^\rho\mathbf{M}^{\mu\sigma}\mathbf{P}_\sigma.$$

Remark 4.1. When $d = 4$, C_2 is the square of the Pauli-Lubanski vector

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathbf{M}^{\nu\rho}\mathbf{P}^\sigma.$$

W^μ commutes with $\mathbf{P}_\mu$, transfers as a vector under $\mathbf{M}_{\mu\nu}$, and with its own commutator relation

$$\left[W_\mu, W_\nu\right] = i\epsilon_{\mu\nu\rho\sigma}W^\rho\mathbf{P}^\sigma.$$

C_1, C_2 essentially classify unitary irreducible representations in four dimension. More Casimir operators are present in higher dimensions.

4.2. **Unitary representation.** We discuss Wigner's classification of nonnegative-energy irreducible unitary representations of $\mathrm{Poin}(d-1,1)$ in terms of induced representations of the little group.

The first Casimir operator

$$C_1 = -\mathbf{P}^\mu\mathbf{P}_\mu = m^2$$

has the physics interpretation of **mass** and $\mathbf{P}^0$ is the **energy**. We only consider non-negative energy representations, i.e., $\mathbf{P}^0 \geq 0$. The second Casimir operator C_2 is the spin operator. The representations will be characterized by the mass and spin.

Let H be a irreducible unitary representation of $\mathrm{Poin}(d-1,1)$. Since the translation subgroup is Abelian, we can decompose into common eigenvalues

$$\mathrm{H} = \bigoplus_{p\in\mathcal{O}} \mathrm{H}_p, \quad \mathbf{P}^\mu = p^\mu \text{ on } \mathrm{H}_p\,.$$

Here $\mathcal{O}$ is a $SO(d-1,1)$-orbit in $\mathbb{R}^{d-1,1}$. The eigenvalue $p = \{p^\mu\}$ is called the **momentum**. Let $\mathrm{Stab}_{\mathcal{O}} \subset \mathrm{Spin}(d-1,1)$ be the stablizer subgroup of the orbit $\mathcal{O}$. This is Wigner's little group. Then H is induced by a representation V of $\mathrm{Stab}_{\mathcal{O}}$

$$\mathrm{H} = \mathrm{Spin}(d-1,1) \otimes_{\mathrm{Stab}_{\mathcal{O}}} V$$

which carries a natural Poincaré group action. $\dim V$ is often called the **physics degree of freedom**.

- $m^2 > 0$. This case is called $\boxed{\textit{massive}}$. $\mathcal{O}$ is the orbit of $p = (m, 0, \cdots, 0)$. $\mathrm{Stab}_{\mathcal{O}} = \mathrm{Spin}(d-1)$.
- $m^2 = 0$. This case is called $\boxed{\textit{massless}}$.

- If $p \neq 0$, then $\mathcal{O}$ is the orbit of $(E, 0, \cdots, 0, E)$. $\mathrm{Stab}_{\mathcal{O}} = \mathrm{Poin}(d-2, 0)$, which can be seen by using the light cone coordinate
$$x^{\pm} = \frac{1}{\sqrt{2}}(x^{d-1} \pm x^0).$$
In this light cone frame, $\mathcal{O}$ is the orbit of $(p^-, 0, \cdots, 0)$. Then $\mathrm{Stab}_{\mathcal{O}}$ is generated by $\{\mathbf{M}_{mn}, \mathbf{M}_{m+}\}_{1 \leq m,n \leq d-2}$. The representation of $\mathrm{Poin}(d-2, 0)$ is further induced: let ξ^m be the eigenvalue of $\mathbf{M}_{m+}$.
    * $\xi \neq 0$. The little group is $\mathrm{Spin}(d-3)$. This case is called $\boxed{\textit{infinite spin}}$.
    * $\xi = 0$. The little group is $\mathrm{Spin}(d-2)$. This case is called $\boxed{\textit{helicity}}$.
- If $p = 0$, then $\mathcal{O}$ is the origin. $\mathrm{Stab}_{\mathcal{O}} = \mathrm{Spin}(d-1, 1)$. This case is called $\boxed{\textit{zero momentum}}$.

- $m^2 < 0$. This case is called $\boxed{\textit{tachyonic}}$. $\mathcal{O}$ is the orbit of $(0, \cdots, 0, \sqrt{-m^2})$. $\mathrm{Stab}_{\mathcal{O}} = \mathrm{Spin}(d-2, 1)$.

4.3. **Coleman-Mandula Theorem.** Let H be a unitary representation of $\mathrm{Poin}(d-1,1)$. The physics system is described by the **S-matrix**, which is a $\mathrm{Poin}(d-1,1)$-equivariant unitary operator
$$S : \mathrm{Sym}(\mathrm{H}) \to \mathrm{Sym}(\mathrm{H}).$$
H is $\mathbb{Z}_2$-graded and Sym is the graded symmetric product. By a symmetry of the S-matrix, we mean an operator $B : \mathrm{Sym}(H) \to \mathrm{Sym}(H)$ which is a derivation and commutes with S.

Under a suitable assumption in the massive case, Coleman-Mandula Theorem says that the Lie algebra of all even symmetries of the S-matrix ($d > 2$) of is a direct sum
$$\mathrm{poin}(d-1,1) \oplus I$$
I is called **internal symmetry**, which does not mix with Poincaré group. In the case when only massless representations exist, Poincaré algebra may be enlarged to conformal algebra.

However, if we allow odd symmetries, then there exists nontrivial extensions of Poincaré algebra. They are called super Poincaré algebras and classified by the Haag-Lopuszanski-Sohnius Theorem.

5. SUSY ALGEBRA

5.1. **Super Lie algebra.**

Definition 5.1. A **super Lie algebra** is a $\mathbb{Z}_2$-graded k-vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ ($\mathfrak{g}_0$ is called even and $\mathfrak{g}_1$ is called odd) together with a k-bilinear super Lie bracket $[-,-]$ satisfying the following properties:

- $[-,-]$ is even, i.e.,
$$[-,-] : \mathfrak{g}_0 \otimes \mathfrak{g}_0 \to \mathfrak{g}_0, \quad \mathfrak{g}_0 \otimes \mathfrak{g}_1 \to \mathfrak{g}_1, \quad \mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \mathfrak{g}_0.$$
- Super skew-symmetry:
$$[x, y] = -(-1)^{|x||y|}[y, x].$$
Here $|x| = i$ for $x \in \mathfrak{g}_i, i = 0, 1$.
- Super Jacobi identity:
$$[[x, y], z] = [x, [y, z]] - (-1)^{|x||y|}[y, [x, z]].$$

Example 5.2. Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded k-vector space. The space of linear operators $\mathrm{End}_k(V)$ is naturally a super Lie algebra:

- $\mathrm{End}_k(V) = \mathrm{End}_k(V)_0 \oplus \mathrm{End}_k(V)_1$ where
$$\mathrm{End}_k(V)_0 = \mathrm{Hom}_k(V_0, V_0) \oplus \mathrm{Hom}_k(V_1, V_1), \quad \mathrm{End}_k(V)_1 = \mathrm{Hom}_k(V_0, V_1) \oplus \mathrm{Hom}_k(V_1, V_0).$$
- Given two linear operators $A, B \in \mathrm{End}_k(V)$, we define the **super commutator**
$$[A, B] = A \circ B - (-1)^{|A||B|} B \circ A.$$

Notation 5.3. In this note, $[-,-]$ always denote super commutator for linear operators on $\mathbb{Z}_2$-graded space.

Definition 5.4. A **super Hilbert space** is a super $\mathbb{C}$-vector space $\mathrm{H} = \mathrm{H}_0 \oplus \mathrm{H}_1$ together with a hermitian inner product such that H_0 and H_1 are orthogonal. Let $\alpha : \mathrm{H} \to \mathrm{H}$ be a bounded linear operator. We define its **super adjoint** $\alpha^\dagger : \mathrm{H} \to \mathrm{H}$ by
$$\alpha^\dagger := \begin{cases} \alpha^* & \alpha \text{ is even} \\ -\sqrt{-1}\alpha^* & \alpha \text{ is odd} \end{cases}.$$
Here α^* is the usual adjoint. α will be called **super hermitian** (or simply just hermitian in this lecture) if
$$\alpha^\dagger = \alpha.$$

Definition 5.5. We define the space of **anti-hermitian** operators on a super Hilbert space H by
$$u(\mathrm{H}) := \{\alpha \in \mathrm{Hom}(\mathrm{H}, \mathrm{H}) | \alpha^\dagger = -\alpha\}.$$
A **unitary representation** of a super Lie algebra $\mathfrak{g}$ on H is a super Lie algebra morphism $\mathfrak{g} \to u(\mathrm{H})$.

Remark 5.6.

- Our definition of super adjoint implies that $u(\mathrm{H})$ forms a super Lie algebra. In fact, the following equation always holds
$$(\alpha\beta)^\dagger = (-1)^{|\alpha||\beta|}\beta^\dagger\alpha^\dagger, \quad [\alpha, \beta]^\dagger = -[\beta, \alpha]^\dagger.$$
- For an odd operator Q on H, the following expression is semi-positive definite
$$\sqrt{-1}(QQ^\dagger + Q^\dagger Q) = QQ^* + Q^*Q \geq 0.$$
- For an odd operator, $[Q, Q] = 2Q^2$ may not vanish.

5.2. **Super Poincaré algebra.** Let (V, Q) be a real vector space with quadratic from Q. Let S be a real representation of $\mathrm{Spin}(V, Q)$, together with a symmetric $\mathrm{Spin}(V, Q)$-equivariant pairing
$$\Gamma : S \otimes S \to V.$$
This defines a super Lie algebra
$$V \oplus S$$
where V is the even component and S is the odd component. The Lie-algebra structure given by
$$[v_1 \oplus s_1, v_2 \oplus s_2] = -2\Gamma(s_1, s_2), \quad v_i \in V, s_i \in S.$$
We denote by $\exp \mathcal{L}$ the corresponding super Lie group, which is in fact the super linear space
$$V \times \Pi S.$$
Here Π is the parity changing operator. ΠS means that the underlying vector space is S, but is purely odd (fermionic). $V \times \Pi S$ carries a natural $\mathrm{Spin}(V, Q)$-action with equivariant group law
$$(v_1, s_1) \cdot (v_2, s_2) = (v_1 + v_2 + \frac{1}{2}[s_1, s_2], s_1 + s_2).$$

Definition 5.7. We define the **super Poincaré group** (or SUSY group) by the semi-product

$$\mathrm{Poin}_S(V) = (V \times \Pi S) \rtimes \mathrm{Spin}(V).$$

Its Lie algebra

$$\mathrm{poin}_S(V) = V \oplus S \oplus so(V)$$

is called the **super Poincaré algebra** (or SUSY algebra).

We will be mainly interested in super Poincaré groups $\mathrm{Poin}_S(d-1,1)$ in Minkowski spaces $V = \mathbb{R}^{d-1,1}$. Recall the following minimal real spinors in $\mathbb{R}^{d-1,1}$

d	1	2	3	4	5	6	7	8	10
$Cl_{d-1,1}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(32)$
Irred $\mathbb{R}$-spinor	$\mathbb{R}$ (M)	$\mathbb{R}_\pm$ (MW)	$\mathbb{R}^2$ (M)	$\mathbb{C}^2$ (W)	$\mathbb{H}^2$ (SM)	$\mathbb{H}^2_\pm$ (SMW)	$\mathbb{H}^4$ (SM)	$\mathbb{C}^8$ (W)	$\mathbb{R}_\pm(16)(MW)$
$\dim_{\mathbb{R}}$(Irred)	1	1	2	4	8	8	16	16	16

Definition 5.8. We introduce the following notations

- We denote our super space and super Poincaré group by
$$\mathbb{R}^{d-1,1} \rtimes \Pi S = \begin{cases} M^{d|s} \\ M^{d|(s_+,s_-)} \end{cases} \quad \mathbb{R}^{d-1,1} \rtimes \Pi S = \begin{cases} \mathrm{Poin}(d|s) & \text{if } d \not\equiv 2,6 \mod 8 \\ \mathrm{Poin}(d|(s_+,s_-)) & \text{if } d \equiv 2,6 \mod 8. \end{cases}$$
Here in the first case $s = \dim_{\mathbb{R}} S$. In the second case, let $S = S_+ \oplus S_-$ be decomposed into copies of two chiral components, then $s_\pm = \dim_{\mathbb{R}} S_\pm$.
- A representation of (globla) $N = n$ (or $N = (n_+, n_-)$) supersymmetry is a representation of $\mathrm{Poin}(d|s)$ (or $\mathrm{Poin}(d|(s_+,s_-))$) where
$$s = N \dim_{\mathbb{R}}(\text{Irred}) \quad \text{or} \quad s_\pm = N_\pm \dim_{\mathbb{R}}(\text{Irred}).$$
The dimensions s or $s_\pm$ are called the number of super charges.

SUSY algebra can be extended in various ways. In general, let us consider the decomposition

$$\mathrm{Sym}^2(S) \cong V \oplus \mathbb{R}^m \oplus \bigoplus_i \wedge^{p_i} V.$$

- The first component $V = \mathbb{R}^{d-1,1}$ gives our super Poincaré algebra.
- The second component gives several copies of $\mathbb{R}$, which we can add into SUSY algebra as a central extension. Their values in a SUSY representation are ofter called **central charges**.
- We can also add part of the third component into SUSY algebra. They are not central extensions, and play the role of central charges when coupled with **"extended objects"** such as "D-branes".
- Outer automorphisms of the super Poincaré algebra that commutes with the Poincaré subalgebra are called **R-symmetries**.

We will be interested in unitary representations of SUSY algebra. This is similar to Wigner's classification. A new phenomenon arises when central charges are nonzero, leading to the notion of **BPS** state.

5.3. **Physics degree of freedom.**

Theorem 5.9. *Let the super Hilbert space* $\mathrm{H} = \mathrm{H}_0 \oplus \mathrm{H}_1$ *be a unitary representation of super Poincare algebra (without the zero momentum irreducible components), then* H_0 *and* H_1 *have the same physics degree of freedom.*

5.4. **d=2 and $\mathbb{R}^\times$.** $V=\mathbb{R}^{1,1}$. In this case

$$SO(1,1)=\left\{e^{\theta J}\,\middle|\,\theta\in\mathbb{R},J=\begin{pmatrix}0&1\\1&0\end{pmatrix}\right\}\cong\mathbb{R}_{>0},\quad \mathrm{Spin}(1,1)=\mathbb{R}^\times.$$

We have two Majorana-Weyl spinors $\mathbb{R}_\pm$ and $\mathbb{R}_\pm\cong\mathbb{R}_\mp^\vee$. Then

$$\mathrm{Sym}^2(R_\pm)\cong V_\pm,\quad \mathbb{R}_+\otimes\mathbb{R}_-\cong\mathbb{R},\quad V=V_+\oplus V_-.$$

Let $Q_\pm$ be a basis of $\mathbb{R}_\pm$, $\partial_\pm$ be a basis of $V_\pm$. Then $N=(N_+,N_-)$-SUSY algebra reads

$$\begin{aligned}\left[Q_+^a,Q_+^b\right]&=-2\delta^{ab}\partial_+,\\ \left[Q_-^{\tilde a},Q_-^{\tilde b}\right]&=-2\delta^{\tilde a\tilde b}\partial_-,\\ \left[Q_+^a,Q_-^{\tilde b}\right]&=2Z^{a\tilde b}\end{aligned}$$

where $1\le a,b\le N_+, 1\le\tilde a,\tilde b\le N_-$. $Z^{a\tilde b}$ are central charges.

For unitary representations, the reality condition reads

$$(Q_+^a)^\dagger=Q_+^a,\quad (Q_-^{\tilde a})^\dagger=Q_-^{\tilde a}.$$

For example, consider unitary representation of $N=(1,1)$ SUSY algebra. We have

$$\frac{1}{4}\left[Q_+\pm Q_-,Q_+\pm Q_-\right]=-\partial_t\pm Z.$$

Here $\partial_t=\frac{1}{2}(\partial_++\partial_-)$ is the time direction, whose eigenvalue is $\sqrt{-1}E$. E is the energy. Reality condition says that Z is pure imaginary and

$$\sqrt{-1}\left[Q_+\pm Q_-,Q_+\pm Q_-\right]=\sqrt{-1}\left[Q_+\pm Q_-,Q_+^\dagger\pm Q_-^\dagger\right]=\left[Q_+\pm Q_-,Q_+^*\pm Q_-^*\right]\ge 0.$$

This implies the BPS bound

$$E\ge|Z|.$$

Representations which saturate the BPS bound (when $Z\neq0$) are called BPS.

5.5. **d=3 and $SL(2,\mathbb{R})$.** $V=\mathbb{R}^{2,1}$, $\mathrm{Spin}(2,1)=SL(2,\mathbb{R})$. The irreducible Majorana spinor $S=\mathbb{R}^2$ is the fundamental representation of $SL(2,\mathbb{R})$.

V can be identified with symmetric 2-by-2 matrices which are spanned by

$$\sigma_0=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \sigma_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

Let e_0,e_1,e_2 be orthonormal basis of V. Then we have the following correspondence

$$v=\sum_{\mu=0}^2 x^\mu e_\mu\Longleftrightarrow A(v)=\sum_{\mu=0}^2 x^\mu\sigma_\mu$$
$$|v|^2=-\det(A(v)).$$

The $\mathrm{Spin}(2,1)$-action on V is realized by

$$N:A(x)\to(N^t)^{-1}A(x)N^{-1},\quad N\in SL(2,\mathbb{C}),\quad A^t(x)=A(x)\in V.$$

There is a natural isomorphism of spinors

$$\mathrm{Sym}^2_{\mathbb{R}}(S)\to V^\vee\cong V$$

given by the Spin-equivariant expression

$$s^t A(v)s \in \mathbb{R}, \quad s \in S, v \in V.$$

This leads to the minimal N=1 supersymmetry

$$\boxed{[Q_\alpha, Q_\beta] = -2\sigma^\mu_{\alpha\beta}\partial_\mu}, \quad 1 \le \alpha, \beta \le 2.$$

The reality condition reads

$$Q_\alpha^\dagger = Q_\alpha.$$

Explicitly,

$$\begin{aligned}
[Q_1, Q_1] &= -2\partial_0 - 2\partial_1 \\
[Q_1, Q_2] &= 2\partial_2 \\
[Q_2, Q_2] &= -2\partial_0 + 2\partial_1.
\end{aligned}$$

In particular, when ∂_2 is represented by a constant, this algebra is reduced to $d = 2, N = (1,1)$ supersymmetry. This is precisely the method of dimensional reduction.

$$\boxed{\text{d=3, N=1} \Longrightarrow \text{d=2, N=(1,1)}}.$$

5.6. **d=4 and** $SL(2,\mathbb{C})$**.** $V = \mathbb{R}^{3,1}, \mathrm{Spin}(3,1) = SL(2,\mathbb{C})$. The irreducible Weyl spinor $S = \mathbb{C}^2$ is the fundamental representation of $SL(2,\mathbb{C})$.

V can be identified with hermitian 2-by-2 matrices which are spanned by Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let e_0, e_1, e_2, e_3 be an orthonormal basis of V. Then we have the following correspondence

$$v = \sum_{\mu=0}^{3} x^\mu e_\mu \Longleftrightarrow A(v) = \sum_{\mu=0}^{3} x^\mu \sigma_\mu$$

$$|v|^2 = -\det(A(v)).$$

The $\mathrm{Spin}(3,1)$-action on V is realized by

$$N : A \to (N^\dagger)^{-1} A N^{-1}, \quad N \in SL(2,\mathbb{C}), A = A^\dagger.$$

Let $\bar{S}$ be the complex conjugate representation. S and $\bar{S}$ are equivalent $\mathbb{R}$-representations but inequivalent $\mathbb{C}$-representations of $\mathrm{Spin}(3,1)$. The direct sum

$$\mathbb{S} = S \oplus \bar{S}$$

forms the Dirac spinor. The Clifford multiplication e_μ on $\mathbb{S}$ is represented by gamma matrices

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \hat{\sigma}_\mu & 0 \end{pmatrix}, \quad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = -2\eta_{\mu\nu}.$$

Here

$$\hat{\sigma}_\mu = (\mathrm{i}\,\sigma_2)\bar{\sigma}_\mu(\mathrm{i}\,\sigma_2)^t, \quad \text{or equivalently} \quad \boxed{\hat{\sigma}_0 = \sigma_0, \hat{\sigma}_i = -\sigma_i}, \quad 1 \le i \le 3.$$

The volume form is represented on $\mathbb{S}$ by

$$\omega = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} \mathrm{i}\,\sigma_0 & 0 \\ 0 & -\mathrm{i}\,\sigma_0 \end{pmatrix}$$

which gives the complex structure on $\mathbb{S}$.

The Dirac spinor $\mathbb{S}$ has a quaternionic structure that is compatible with the Clifford acton. This is realized by the complex conjugate linear map

$$J : \mathbb{S} \to \mathbb{S}, \quad (x, \bar{y}) \to (\mathrm{i}\,\sigma_2 y, \mathrm{i}\,\sigma_2 \bar{x})$$

which satisfies the relation

$$J \circ e_\mu = e_\mu \circ J, \quad J^2 = -1.$$

The charge conjugations

$$\langle -, - \rangle_\pm : \mathbb{S} \otimes_{\mathbb{C}} \mathbb{S} \to \mathbb{C}$$

are given by

$$\langle s_1, s_2 \rangle_\pm = \begin{cases} \mathrm{i}\, s_1^t \sigma_2 s_2 & s_1, s_2 \in S \\ \pm\, \mathrm{i}\, s_1^t \sigma_2 s_2 & s_1, s_2 \in \bar{S} \\ 0 & s_1 \in S, s_2 \in \bar{S} \text{ or } s_1 \in \bar{S}, s_2 \in S. \end{cases}$$

It is easy to check directly that for $s_1, s_2 \in \mathbb{S}$

$$\langle s_1, s_2 \rangle_\pm = - \langle s_2, s_1 \rangle_\pm, \quad \langle s_1, e_\mu s_2 \rangle_\pm = \pm \langle e_\mu s_1, s_2 \rangle_\pm .$$

The reality properties are

$$\overline{\langle s_1, s_2 \rangle}_\pm = \pm \langle \bar{s}_1, \bar{s}_2 \rangle_\pm, \quad \langle J(s_1), J(s_2) \rangle_\pm = \pm \overline{\langle s_1, s_2 \rangle}_\pm .$$

By Remark 3.18, we have the $\mathbb{H}$-hermitian form on $\mathbb{S}$

$$h(s_1, s_2) = \overline{\langle s_1, J(s_2) \rangle_+} + \mathrm{j} \langle s_1, s_2 \rangle_+, \quad s_i \in \mathbb{S},$$

satisfying $h(e_\mu s_1, s_2) = h(s_1, e_\mu s_2)$ and

$$h(S, S), h(\bar{S}, \bar{S}) \in \mathrm{j}\,\mathbb{C}, \quad h(S, \bar{S}), h(\bar{S}, S) \in \mathbb{C}.$$

Lemma 5.10. *Let $\alpha \in \wedge^k_{\mathbb{R}} V$, $s_1, s_2 \in S$, we have*

$$\overline{h(s_1, \alpha s_2)} = (-1)^{k(k-1)/2} h(s_2, \alpha s_1).$$

Proof.

$$\overline{h(s_1, \alpha s_2)} = h(\alpha s_2, s_1) = h(s_2, \alpha^t s_1) = (-1)^{k(k-1)/2} h(s_2, \alpha s_1).$$

□

Proposition 5.11. *We have the following isomorphisms of* Spin*-representations*

$$\wedge^2_{\mathbb{R}} S \cong \mathbb{R} \oplus \wedge^3 V \oplus \wedge^4 V, \quad \mathrm{Sym}^2_{\mathbb{R}}(S) \cong V \oplus \wedge^2 V.$$

Proof. Consider morphisms of Spin-representations

$$h^{(k)} : S \otimes_{\mathbb{R}} S \to \wedge^k_{\mathbb{R}} V^\vee \otimes_{\mathbb{R}} \mathbb{H}$$

defined by the pairing

$$h^{(k)}(s_1, s_2)(\alpha) = h(s_1, \alpha s_2), \quad \alpha \in \wedge^k_{\mathbb{R}} V, s_i \in S.$$

By the previous lemma,

$$\overline{h^{(k)}(s_1,s_2)(\alpha)} = \begin{cases} h^{(k)}(s_2,s_1)(\alpha) \in \mathrm{j}\,\mathbb{C} & k=0,4 \\ h^{(k)}(s_2,s_1)(\alpha) \in \mathbb{C} & k=1 \\ -h^{(k)}(s_2,s_1)(\alpha) \in \mathrm{j}\,\mathbb{C} & k=2 \\ -h^{(k)}(s_2,s_1)(\alpha) \in \mathbb{C} & k=3 \end{cases}$$

It follows that we have morphisms

$$\wedge^2_{\mathbb{R}} S \xrightarrow{h^{(k)}} \wedge^0 V^\vee \oplus \wedge^3 V^\vee \oplus \wedge^4 V^\vee, \quad \mathrm{Sym}^2_{\mathbb{R}} S \xrightarrow{h^{(k)}} V^\vee \oplus \wedge^2 V^\vee$$

which can be shown to be isomorphisms. The proposition follows since $V \cong V^\vee$. $\square$

The main part of $N=1$ SUSY algebra in $d=4$ is obtained from the pairing

$$\mathrm{Sym}^2_{\mathbb{R}}(\mathcal{S}) \to V, \quad s \to \left\langle s, \hat{\sigma}_\mu J(s) \right\rangle_+ \eta^{\mu\nu} e_\nu = -(s^t \sigma^\mu \bar{s}) e_\mu. \quad \sigma^\mu = \eta^{\mu\nu}\sigma_\nu.$$

The SUSY generator is a Majorana spinor $\{Q_\alpha\}_{\alpha=1,2}$. $h^{(1)}$ gives the SUSY commutator relation

$$\boxed{\left[Q_\alpha, \bar{Q}_{\dot\beta}\right] = -2\sigma^\mu_{\alpha\dot\beta}\partial_\mu}.$$

For unitary representations, the reality condition reads

$$Q_\alpha^\dagger = \bar{Q}_{\dot\alpha}.$$

Explicitly, we find

$$\begin{aligned} [Q_1, \bar{Q}_1] &= -2(\partial_0 + \partial_3) \\ [Q_2, \bar{Q}_2] &= -2(\partial_0 - \partial_3) \\ [Q_1, \bar{Q}_2] &= -2(\partial_1 - \mathrm{i}\,\partial_2) \\ [Q_2, \bar{Q}_1] &= -2(\partial_1 + \mathrm{i}\,\partial_2). \end{aligned}$$

We observe that under dimensional reduction along the direction ∂_3, $\{\mathrm{Re}(Q_1), \mathrm{Re}(Q_2)\}$ and $\{\mathrm{Im}(Q_1), \mathrm{Im}(Q_2)\}$ give two sets of $N=1$ SUSY algebra in $d=3$. In other words, dimensional reduction leads to

$$\boxed{\text{d=4, N=1} \Longrightarrow \text{d=3, N=2}}.$$

5.7. **d=6 and** $SL(2,\mathbb{H})$**.** $V = \mathbb{R}^{5,1}, \mathrm{Spin}(5,1) = SL(2,\mathbb{H})$. There are two symplectic-Majorana-Weyl spinor $S_\pm \cong \mathbb{H}^2$:

$$S_+ = \begin{pmatrix} \mathbb{H} \\ \mathbb{H} \end{pmatrix}, \quad g : \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \to g \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad g \in SL(2,\mathbb{H}).$$

$$S_- = \begin{pmatrix} \mathbb{H} \\ \mathbb{H} \end{pmatrix}, \quad g : \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \to (g^\dagger)^{-1} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad g \in SL(2,\mathbb{H}).$$

We can identify V by 2-by-2 Hermitian quaternion matrices

$$V = \{A \in M_2(\mathbb{H}) | A = A^\dagger\} = \left\{ A(x) = \begin{pmatrix} x^0 + x^5 & x^1 + \mathrm{i}\,x^2 + \mathrm{j}\,x^3 + \mathrm{k}\,x^4 \\ x^1 - \mathrm{i}\,x^2 - \mathrm{j}\,x^3 - \mathrm{k}\,x^4 & x^0 - x^5 \end{pmatrix} \right\}_{x^i \in \mathbb{R}}.$$

where

$$\det A(x) = -|x|^2.$$

$SL(2,\mathbb{H})$ acts on V by

$$g : A \to (g^\dagger)^{-1} A g^{-1}, \quad A \in V, \quad g \in SL(2,\mathbb{H}).$$

We also deonte their $\mathbb{H}$-hermitian conjugate by

$$S_+^\dagger = \begin{pmatrix}\mathbb{H} & \mathbb{H}\end{pmatrix}, \quad g: \begin{pmatrix}q_1 & q_2\end{pmatrix} \to \begin{pmatrix}q_1 & q_2\end{pmatrix} g^\dagger, \quad g \in SL(2,\mathbb{H}).$$

$$S_-^\dagger = \begin{pmatrix}\mathbb{H} & \mathbb{H}\end{pmatrix}, \quad g: \begin{pmatrix}q_1 & q_2\end{pmatrix} \to \begin{pmatrix}q_1 & q_2\end{pmatrix} g^{-1}, \quad g \in SL(2,\mathbb{H}).$$

$S_\pm$ and $S_\pm^\dagger$ are $\mathbb{H}$-bimodules. Spin action on $S_\pm$ are right $\mathbb{H}$-linear while on $S_\pm^\dagger$ are left $\mathbb{H}$-linear.

Consider the Dirac spinor

$$\mathbb{S} = S_+ \oplus S_-.$$

We define the Spin-equivariant $\mathbb{H}$-hermitian form on $\mathbb{S}$

$$h(-,-): \mathbb{S} \otimes_{\mathbb{R}} \mathbb{S} \to \mathbb{H}$$

by

$$h(s_1, s_2) = s_1^\dagger s_2 = \overline{h(s_2, s_1)}, \quad s_1 \in S_-, s_2 \in S_+.$$

By Remark 3.18, this is equivalent to a Spin-equivariant symplectic pairing

$$\langle -,- \rangle_+ : \wedge^2_{\mathbb{C}} \mathbb{S} \to \mathbb{C}, \quad \text{such that} \quad \langle s_1\,\mathrm{j}, s_2\,\mathrm{j}\rangle_+ = \overline{\langle s_1, s_2\rangle}_+.$$

$\langle -,- \rangle_+$ is precisely the charge conjugation C_+. Explicitly,

$$\langle s_1, s_2\rangle_+ = \mathrm{Im}_{\mathrm{j}}\, s_1^\dagger s_2, \quad s_1 \in S_\pm, s_2 \in S_\mp,$$

where

$$\mathrm{Im}_{\mathrm{j}}(z_1 + \mathrm{j}\, z_2) = z_2, \quad z_i \in \mathbb{C}, \quad z_1 + \mathrm{j}\, z_2 \in \mathbb{H}.$$

The charge conjugation C_- is given by

$$\langle -,- \rangle_- = \langle \omega -, - \rangle_+.$$

The analogue of Pauli matrices are

$$\sigma_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \sigma_2 = \begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}, \sigma_3 = \begin{pmatrix}0 & \mathrm{j}\\ -\mathrm{j} & 0\end{pmatrix}, \sigma_4 = \begin{pmatrix}0 & \mathrm{k}\\ -\mathrm{k} & 0\end{pmatrix}, \sigma_5 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

They represent the Clifford multiplication

$$e_\mu : S_+ \to S_-.$$

Similarly we have $\hat{\sigma}_\mu$ representing the Clifford action

$$e_\mu : S_- \to S_+$$

by

$$\hat{\sigma}_\mu = \{\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3, -\sigma_4, -\sigma_5\}, \quad \hat{\sigma}_\mu = \mathrm{i}\,\sigma_2 \bar{\sigma}_\mu (\mathrm{i}\,\sigma_2)^{-1}.$$

They together represent the Clifford multiplication e_μ on $S_+ \oplus S_-$ by

$$\gamma_\mu = \begin{pmatrix}0 & \sigma_\mu\\ \hat{\sigma}_\mu & 0\end{pmatrix}, \quad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = -2\eta_{\mu\nu}.$$

Proposition 5.12. *We have isomorphisms of spin representations*

$$\mathrm{Sym}^2_{\mathbb{R}}(S_\pm) \cong V \oplus 3 \wedge^3_\pm V, \quad \wedge^2_{\mathbb{R}}(S_\pm) = 3V \oplus \wedge^3_\pm V, \quad S_+ \otimes_{\mathbb{R}} S_- = 2 \wedge^{even} V.$$

Here $\wedge^3_\pm V \subset \wedge^3_{\mathbb{R}} V$ *are* ± 1*-eigenvectors of the Hodge star operator* $\star : \wedge^3_{\mathbb{R}} V \to \wedge^3_{\mathbb{R}} V$.

Proof. The pairing

$$\mathrm{Sym}^2_{\mathbb{R}}(S_+) \to V^\vee \cong V$$

is realized by

$$h(s, vs) \in \mathbb{R}, \quad s \in S_+, v \in V.$$

Three copies of

$$\mathrm{Sym}^2_{\mathbb{R}}(S_+) \to \wedge^3_+ V^\vee \cong \wedge^3_+ V$$

are realized by

$$h(s, \alpha s) \in \mathrm{Im}(\mathbb{H}), \quad s \in S_+, \alpha \in \wedge^3_+ V.$$

To see the self-duality, we observe that Clifford multiplication by the volume form ω

$$\omega : \wedge^* V \to \wedge^{6-*} V$$

is the Hodge star. For $s \in S_+$, $\alpha_\pm \in \wedge^3_\pm V$,

$$h(s, \alpha_\pm s) = h(\omega s, \alpha_\pm s) = h(s, \omega\alpha_\pm s) = \pm h(s, \alpha_\pm s).$$

∎

$N = (1, 0)$ SUSY algebra in $d = 6$ is obtained by

$$\mathrm{Sym}^2_{\mathbb{R}}(S) \to V.$$

$$\boxed{\{\epsilon^\alpha Q_\alpha, \epsilon^\beta Q_\beta\} = -2\bar{\epsilon}^\alpha \sigma^\mu_{\alpha\beta} \epsilon^\beta \partial_\mu}, \quad \epsilon^1, \epsilon^2 \in \mathbb{H}.$$

Dimensional reduction leads to

$$\boxed{\text{d=6, N=(1,0)} \Longrightarrow \text{d=4, N=2}}.$$

5.8. **d=10.** $V = \mathbb{R}^{9,1}, Cl_{9,1} = \mathbb{R}(32)$. This case is related to the Octonions. We give a detailed discussion later in N=1 super Yang-Mills theory.

We have two Majorana-Weyl spinors $S_\pm = \mathbb{R}^{16}_\pm$, which are dual of each other by a natural pairing

$$\langle -, - \rangle : S_+ \otimes_{\mathbb{R}} S_- \to \mathbb{R}, \qquad \langle e_\mu -, - \rangle = \langle -, e_\mu - \rangle.$$

Here $e_0, e_1, \cdots, e_9$ is an orthonormal basis of V. The pairing

$$\begin{cases} \Gamma^k : S_\pm \otimes_{\mathbb{R}} S_\pm \to V^k & k \text{ odd} \\ \Gamma^k : S_+ \otimes_{\mathbb{R}} S_- \to V^k & k \text{ even} \end{cases}$$

leads to the isomorphisms of spin representations

$$S_\pm \otimes_{\mathbb{R}} S_\pm = V \oplus \wedge^3 V \oplus \wedge^5_\pm V$$
$$S_+ \otimes_{\mathbb{R}} S_- = \mathbb{R} \oplus \wedge^2 V \oplus \wedge^4 V$$

where $\wedge^5_\pm V$ are the ± 1-eigenvectors of the Hodge star $\star : \wedge^5 V \to \wedge^5 V$. The symmetry properties are

$$\mathrm{Sym}^2_{\mathbb{R}} S_\pm = V \oplus \wedge^5_\pm V, \quad \wedge^2_{\mathbb{R}} S_\pm = \wedge^3 V.$$

We can have chiral supersymmetry in $d = 10$ from either $\mathrm{Sym}^2 \mathcal{S}_\pm \to V$.

6. N=1 Super Yang-Mills

6.1. **Normed division algebra.**

Definition 6.1. A normed division algebra $\mathbb{K}$ is a (finite-dimensional, possibly nonassociative) $\mathbb{R}$-algebra equipped with a multiplicative unit and a norm $|\cdot| : \mathbb{K} \to \mathbb{R}$ satisfying

$$|a \cdot b| = |a||b|, \quad \forall a, b \in \mathbb{K}.$$

There are only four normed division algebras: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$. These algebras have real dimension 1, 2, 4, and 8. Each one has a conjugation $* : \mathbb{K} \to \mathbb{K}$ such that

$$(a^*)^* = a, \quad ab^* = b^* a^*, \quad |a|^2 = aa^* = a^* a, \quad \forall a.b \in \mathbb{K}.$$

The real part and imaginary part are defined by

$$\mathrm{Re}(a) = \frac{a + a^*}{2}, \quad \mathrm{Im}(a) = \frac{a - a^*}{2}.$$

The norm is polarized to an inner product on $\mathbb{K}$ by

$$(a, b) = \mathrm{Re}(ab^*) = \mathrm{Re}(a^* b).$$

Example 6.2. $\mathbb{O}$ can be represented by two copies of $\mathbb{H}$ with

$$(p, q) \cdot (r, s) = (pr - s^* q, sp + qr^*), \quad (p, q)^* = (p^*, -q).$$

$\mathbb{O}$ is neither commutative nor associative.

We define the associator

$$[-, -, -] : \mathbb{K} \times \mathbb{K} \times \mathbb{K} \to \mathbb{K}, \quad [a, b, c] = (ab)c - a(bc).$$

For normed division algebra, this is only nontrivial for $\mathbb{K} = \mathbb{O}$. In this case it is easy to see that

$$[-, -, -] : \wedge^3(\mathbb{O}/\mathbb{R}) \to \mathrm{Im}(\mathbb{O}).$$

The fact that the associator is completely anti-symmetric (such property is called **alternative**) implies that the subalgebra generated by any two elements of $\mathbb{O}$ is associative. In particular, this implies that

$$\mathrm{Re}((ab)c) = \mathrm{Re}(a(bc)) = \text{cyclic permutations}, \quad \forall a, b, c \in \mathbb{O}.$$

Given a normed division algebra $\mathbb{K}$, we consider the vector space

$$V = \{A \in M_2(\mathbb{K}) | A = A^\dagger\} = \left\{ A(x) = \begin{pmatrix} t + x & y \\ y^* & t - x \end{pmatrix} \right\}_{t, x \in \mathbb{R}, y \in \mathbb{K}}.$$

V is the Minkowski space whose inner product is identified with determinant

$$|v|^2 = -\det(v), \quad v \in V.$$

The spin group $\mathrm{Spin}(V)$ is naturally identified with $SL(2, \mathbb{K})$. Let us define

$$\tilde{v} = \begin{pmatrix} t - x & -y \\ -y^* & t + x \end{pmatrix}, \quad \text{for} \quad v = \begin{pmatrix} t + x & y \\ y^* & t - x \end{pmatrix}.$$

Then it is easy to see that

$$v\tilde{v} = \tilde{v}v = -|v|^2, \quad \forall v \in V.$$

The inner product is polarized as

$$(v_1, v_2) = -\frac{1}{2} \operatorname{Re} \operatorname{Tr}(\tilde{v}_1 v_2).$$

The Dirac spinor is given by

$$\mathbb{S} = S_+ \oplus S_-$$

where each $S_\pm \cong \mathbb{K}^2$ as a vector space. The Clifford multiplication is

$$\gamma : V \to \operatorname{End}(S_+ \oplus S_-), \quad \gamma(v) = \begin{pmatrix} 0 & \tilde{v} \\ v & 0 \end{pmatrix},$$

that is,

$$\gamma(v) = v : S_+ \to S_-, \quad \gamma(v) = \tilde{v} : S_- \to S_+.$$

The charge conjugation

$$\langle -, - \rangle_+ : \operatorname{Sym}^2_{\mathbb{R}}(\mathbb{S}) \to \mathbb{R}$$

is induced by

$$S_+ \otimes S_- \to \mathbb{R}, \quad \langle A, B \rangle_+ = \operatorname{Re}(A^\dagger B).$$

It is compatible with Clifford multiplication as a consequence of the property of associator

$$\langle \gamma(v) A, B \rangle_+ = \langle A, \gamma(v) B \rangle_+ .$$

It leads to the pairing for chiral SUSY algebra

$$\Gamma : S_\pm \otimes_{\mathbb{R}} S_\pm \to V, \quad (v, \Gamma(A, B)) = \langle A, \gamma(v) B \rangle_+ , \quad \forall v \in V, A, B \in S_\pm.$$

Explicitly, we can check that

$$-\Gamma(A, B) = \begin{cases} A\widetilde{B^\dagger + B}A^\dagger & A, B \in S_+ \\ AB^\dagger + BA^\dagger & A, B \in S_- \end{cases}$$

Proposition 6.3 (3-ψ's rule)**.** *For any $\psi \in S_\pm$,*

$$\boxed{\Gamma(\psi, \psi) \cdot \psi = 0}.$$

Here $\cdot$ is the Clifford multiplication. Equivalently, for any $A_1, A_2, A_3 \in S_\pm$,

$$\Gamma(A_1, A_2) \cdot A_3 + \Gamma(A_2, A_3) \cdot A_1 + \Gamma(A_3, A_1) \cdot A_3 = 0.$$

Proof. Assume $\psi \in S_+$, then

$$\Gamma(\psi, \psi) \cdot \psi = -2(\widetilde{\psi\psi^\dagger})\psi = 2(\psi\psi^\dagger - \operatorname{Tr} \psi\psi^\dagger)\psi = 2[\psi, \psi^\dagger, \psi] = 0.$$

$\square$

6.2. **N=1 Super Yang-Mills.** We consider a principle G-bundle over the Minkowski space V

$$P \to V.$$

The charge conjugation $\langle -, - \rangle_+$, Killing pairing on the Lie algebra $\mathfrak{g}$ of G, and the Minkowski metric on V define an inner product on $\Omega^*(V, \mathfrak{g})$ and a pairing between $\Omega^*(V, S_\pm \otimes \mathfrak{g})$, all denoted by $\langle -, - \rangle$. We will always identify $V \cong V^\vee$ when its meaning is clear from the context.

The field space of N=1 super Yang-Mills theory is given by the superspace

$$\mathcal{E} = \Omega^1(V, \mathfrak{g}) \oplus \Omega^0(V, S_+ \otimes \mathfrak{g}).$$

Here $A \in \Omega^1(V, \mathfrak{g})$ is even and represents a connection 1-form, while $\psi \in \Omega^0(V, S_+ \otimes \mathfrak{g})$ is odd and represents a fermion. We consider the following odd action of S_+ on $\mathcal{E}$ by

$$S_+ : \Omega^0(V, S_+ \otimes \mathfrak{g}) \to \Omega^1(V, \mathfrak{g}), \quad \epsilon : \psi \to \Gamma(\epsilon, \psi)$$

and

$$S_+ : \Omega^1(V, \mathfrak{g}) \to \Omega^0(V, S_+ \otimes \mathfrak{g}), \quad \epsilon : A \to \frac{1}{2} F_A \cdot \epsilon.$$

Here $F_A = dA + \frac{1}{2}[A, A]$ is the curvature 2-form, and $\cdot$ is the Clifford multiplication. In terms of the usual convention of variation,

$$\delta_\epsilon A = \Gamma(\epsilon, \psi), \quad \delta_\epsilon \psi = \frac{1}{2} F_A \cdot \epsilon.$$

Definition 6.4. Consider the following lagrangian density on $\mathcal{E}$

$$\mathcal{L}[A, \psi] = \frac{1}{4} \langle F_A, F_A \rangle + \frac{1}{2} \langle \psi, \not{D}_A \psi \rangle .$$

Here $\not{D}_A = \not{D} + A$ and $\not{D} = \gamma^\mu \partial_\mu$ is the Dirac operator. N=1 super Yang-Mills functional is defined by

$$S_{YM}^{N=1}[A, \psi] = \int_V \mathcal{L}[A, \psi].$$

Proposition 6.5. *$S_{YM}^{N=1}$ is invariant under the odd transformation*

$$\delta_\epsilon S_{YM}^{N=1} = 0, \quad \forall \epsilon \in S_+.$$

Proof.

$$\begin{aligned} \delta_\epsilon \langle F_A, F_A \rangle &= 2 \langle F_A, d_A \delta_\epsilon A \rangle = 2 \langle d_A^* F_A, \delta_\epsilon A \rangle + \text{divergence} \\ &= 2 \langle \epsilon, d_A^* F_A \cdot \psi \rangle + \text{divergence} = -2 \langle \psi, d_A^* F_A \cdot \epsilon \rangle + \text{divergence}. \end{aligned}$$

Here there is an extra minus sign in the last line since ψ, ϵ are odd.

$$\begin{aligned} \delta_\epsilon \langle \psi, \not{D}_A \psi \rangle &= 2 \langle \psi, \not{D}_A \delta_\epsilon \psi \rangle + \langle \psi, \delta_\epsilon A \cdot \psi \rangle + \text{divergence} \\ &= \langle \psi, \not{D}_A (F_A \cdot \epsilon) \rangle + \langle \psi, \Gamma(\epsilon, \psi) \cdot \psi \rangle + \text{divergence} \\ &= \langle \psi, \not{D}_A (F_A \cdot \epsilon) \rangle + \langle \epsilon, \Gamma(\psi, \psi) \cdot \psi \rangle + \text{divergence}. \end{aligned}$$

Observe that

$$\not{D}_A (F_A \cdot \epsilon) = (d_A F_A + d_A^* F_A) \cdot \epsilon = d_A^* F_A \cdot \epsilon.$$

It follows that

$$\delta_\epsilon \mathcal{L} = \frac{1}{2} \langle \epsilon, (\psi \cdot \psi) \cdot \psi \rangle + \text{divergence}.$$

By the 3-ψ's rule, the first term vanishes. The proposition follows. ∎

Let us examine the SUSY algebra relation. Let δ^G denote the gauge transformation

$$\delta_\lambda^G A = d_A \lambda, \quad \delta_\lambda^G \psi = [\lambda, \psi], \quad \lambda \in \Omega^0(V, \mathfrak{g}).$$

Proposition 6.6. *For any $\epsilon_1, \epsilon_2 \in S_+$,*

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]A = -\mathcal{L}_\lambda A - \delta^G_{\iota_\lambda A} A, \quad [\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi = -\mathcal{L}_\lambda \psi - \delta^G_{\iota_\lambda A}\psi + \not{D}_A \psi\text{-terms}$$

Here $\lambda = \Gamma(\epsilon_1, \epsilon_2)$.

Proof. Since ϵ_i are odd and chiral, for $\alpha_k \in \wedge^k V$,

$$\langle \epsilon_1, \alpha_k \cdot \epsilon_2 \rangle = \begin{cases} -\langle \epsilon_2, \alpha_k \cdot \epsilon_1 \rangle & k = 1 \\ \langle \epsilon_2, \alpha_k \cdot \epsilon_1 \rangle & k = 3 \end{cases}$$

It follows that

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]A = \frac{1}{2}\Gamma(\epsilon_2, (F_A \cdot \epsilon_1)) - (\epsilon_1 \leftrightarrow \epsilon_2) = -\iota_\lambda F_A = -\mathcal{L}_\lambda A - \delta^G_{\iota_\lambda A} A.$$

Here $\lambda = \Gamma(\epsilon_1, \epsilon_2)$ is viewed as a vector field. On the other hand,

$$\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi &= \frac{1}{2} d_A \Gamma(\epsilon_1, \psi) \cdot \epsilon_2 - (\epsilon_1 \leftrightarrow \epsilon_2) \\
&= \frac{1}{2}\not{D}_A (\Gamma(\epsilon_1, \psi) \cdot \epsilon_2) - \frac{1}{2} d_A^* \Gamma(\epsilon_1, \psi) \cdot \epsilon_2 - (\epsilon_1 \leftrightarrow \epsilon_2) \\
&= \frac{1}{2}\not{D}_A (\Gamma(\epsilon_1, \psi) \cdot \epsilon_2 - \Gamma(\epsilon_2, \psi) \cdot \epsilon_1) + \frac{1}{2} \left(\langle \epsilon_1, \not{D}_A \psi \rangle \epsilon_2 - \langle \epsilon_2, \not{D}_A \psi \rangle \epsilon_1 \right) \\
&\overset{3\psi}{=} \frac{1}{2}\not{D}_A \left(\Gamma(\epsilon_1, \epsilon_2) \cdot \psi \right) + \frac{1}{2} \left(\langle \epsilon_1, \not{D}_A \psi \rangle \epsilon_2 - \langle \epsilon_2, \not{D}_A \psi \rangle \epsilon_1 \right) \\
&= -\mathcal{L}_\lambda \psi - \delta^G_{\iota_\lambda A} \psi - \frac{1}{2}\Gamma(\epsilon_1, \epsilon_2) \cdot \not{D}_A \psi + \frac{1}{2} \left(\langle \epsilon_1, \not{D}_A \psi \rangle \epsilon_2 - \langle \epsilon_2, \not{D}_A \psi \rangle \epsilon_1 \right)
\end{aligned}$$

□

Theorem 6.7. *The solution space of equation of motions of $S^{N=1}_{YM}$ modulo gauge transformations*

$$Crit(S^{N=1}_{YM})/Gauge$$

carries a representation of N=1 SUSY algebra.

Proof. This is a direct consequence of the previous two lemmas and the gauge invariance of $S^{N=1}_{YM}$. □

Remark 6.8. Since SUSY algebra of our model is closed modulo equation of motion, we call it **on-shell supersymmetry**. In case when equation of motion is not needed, we call it **off-shell supersymmetry**.

6.3. **Berkovits construction.** Berkovits gives a remarkable formulation of 10d N=1 super Yang-Mills theory in terms of Chern-Simons type theory via pure spinor formalism. Let S_+ be a minimal chiral spinor in $d = 10$. Let us define the Berkovits algebra

$$\mathcal{B} = \Omega^0(V) \otimes \wedge(S_+) \otimes \mathrm{Sym}(S_+)/\Gamma$$

Here $\Gamma : \mathrm{Sym}^2(S_+) \to V$ gives 10 quadratic polynomials on S_+, and $\mathrm{Sym}(S_+)/\Gamma$ is the quotient by these quadratic relations. Let us use $\{\theta^\alpha\}_{1\le\alpha\le16}$ to represent odd coordinate of S_+, $\{u^\alpha\}_{1\le\alpha\le16}$ to represent even coordinate of S_+, and $\{x^\mu\}_{0\le\mu\le9}$ to represent coordinate of V, then

$$\mathcal{B} = \mathbb{R}\{x^\mu\}[\theta^\alpha, \lambda^\alpha] / \left\langle \lambda, e_\mu \lambda \right\rangle.$$

An element $\lambda \in S_+$ satisfying $\Gamma(\lambda, \lambda) = 0$ is also called a pure spinor in this case.

Let us define a differential

$$Q = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha} + \mathcal{L}_{\Gamma\langle\lambda,\theta\rangle}$$

where $\mathcal{L}$ is the Lie derivative. Then Berkovits shows [?B-superparticle] that Maurer-Carton elements of the dga $(\mathcal{B}, Q)$ can be identified with $\mathrm{Crit}(S_{YM}^{d=10,N=1})/\mathrm{Gauge}$. In particular, a Chern-Simons type action functional can be constructed to reformulate d=10 N=1 super Yang-Mills theory.

7. Supersymmetry in D=4

7.1. **SUSY representations.** Let S be the minimal Majorana spinor for $V = \mathbb{R}^{3,1}$, $\bar{S}$ its complex conjugate. We can also identify $S, \bar{S}$ with the two Weyl spinors S_+, S_-, and the Dirac spinor is given by

$$\mathbb{S} = S \oplus \bar{S}.$$

The Clifford multiplication is realized by

$$\gamma_\mu = \begin{pmatrix} 0 & \hat{\sigma}_\mu \\ \sigma_\mu & 0 \end{pmatrix}, \quad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = -2\eta_{\mu\nu}$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\sigma}_0 = \sigma_0, \sigma_i = -\sigma_i, i = 1,2,3.$$

There are isomorphisms of Spin-representations

$$\wedge^2_{\mathbb{R}} S \cong \mathbb{R} \oplus \wedge^3 V \oplus \wedge^4 V, \quad \mathrm{Sym}^2_{\mathbb{R}}(S) \cong V \oplus \wedge^2 V.$$

The simplest N-SUSY with scalar central charges is

$$\begin{aligned} (Q^A_\alpha)^\dagger &= \bar{Q}^A_{\dot\alpha} \quad \text{(Reality)} \\ \left[Q^A_\alpha, \bar{Q}^B_{\dot\beta}\right] &= -2\delta^{AB}\sigma^\mu_{\alpha\dot\beta}\partial_\mu \\ \left[Q^A_\alpha, Q^B_\beta\right] &= -2\,\mathrm{i}\,\epsilon_{\alpha\beta}Z^{AB} \\ \left[\bar{Q}^A_{\dot\alpha}, \bar{Q}^B_{\dot\beta}\right] &= -2\,\mathrm{i}\,\epsilon_{\dot\alpha\dot\beta}\bar{Z}^{AB} \end{aligned}$$

where $1 \leq A, B \leq N, 1 \leq \alpha, \dot\beta \leq 2, \epsilon_{12} = -\epsilon_{21} = 1$. Z^{AB} are central charges and $Z^{AB} = -Z^{BA}$. The R-symmetry group is $U(N)$

$$R : Q^A_\alpha \to R^A_B Q^B_\alpha, \quad R \in U(N).$$

The unitary representations of SUSY algebra are similar to Wigner's classification, except that the representation of the little group carries extra odd operators from N copies of minimal spinors.

7.1.1. *N=1 massless supermultiplet.* The little group is the stabilizer of $P_\mu = -i\partial_{x^\mu} = (E, 0, 0, E)$. Then

$$\left[Q_\alpha, \bar{Q}_{\dot\beta}\right] = -2\,\mathrm{i}\,E(\sigma^0_{\alpha\dot\beta} + \sigma^3_{\alpha\dot\beta}) = -4\,\mathrm{i}\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta}.$$

Since $\bar{Q}_{\dot\beta} = Q^\dagger_{\dot\beta} = -\,\mathrm{i}\,Q^*_{\dot\beta}$ where $Q^*_{\dot\beta}$ is the usual Hilbert space adjoint, we find

$$Q_\alpha Q^*_{\dot\beta} + Q^*_{\dot\beta} Q_\alpha = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta}.$$

For unitary representation, $Q_2 Q_2^* + Q_2^* Q_2 = 0$ implies that $Q_2 = 0$.

We consider the helicity case, then the little group is $SO(2) \cong U(1)$

$$U(1) \hookrightarrow \mathrm{Spin}(3,1) = SL(2, \mathbb{C}), \quad e^{\mathrm{i}\,\theta} \to \begin{pmatrix} e^{\mathrm{i}\,\theta} & 0 \\ 0 & e^{-\mathrm{i}\,\theta} \end{pmatrix}.$$

Irreducible representations are spanned by orthonormal basis

$$H_\lambda = \mathbb{C}\,|\lambda\rangle \oplus \mathbb{C}\,|\lambda + 1/2\rangle$$

where λ represents the spin. Then

$$Q_1\,|\lambda\rangle = 0, \quad Q_1^*\,|\lambda\rangle = 2\sqrt{E}\,|\lambda + 1/2\rangle\,, \quad Q_1^*\,|\lambda + 1/2\rangle = 0, \quad Q_1\,|\lambda + 1/2\rangle = 2\sqrt{E}\,|\lambda\rangle\,.$$

(1) $\lambda = 0$. This is called a **chiral multiplet**. It consists of

$$|0\rangle \quad \text{scalar}, \quad |1/2\rangle \quad \text{fermion}$$

(2) $\lambda = 1/2$. This is called a **vector multiplet** or **gauge multiplet**. It consists of

$$|1/2\rangle \quad \text{fermion}, \quad |1\rangle \quad \text{gauge boson}$$

(3) $\lambda = 1$. This is called a **gravity multiplet**. It consists of

$$|3/2\rangle \quad \text{gravitino}, \quad |2\rangle \quad \text{graviton}.$$

7.1.2. *N=1 massive supermultiplet.* The little group is the stabilizer of $P_\mu = -i\partial_{x^\mu} = (m, 0, 0, 0)$. Then

$$\left[Q_\alpha, \bar{Q}_{\dot\beta}\right] = -2\,\mathrm{i}\, m\sigma^0_{\alpha\dot\beta} = -2\,\mathrm{i}\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}_{\alpha\dot\beta}.$$

Equivalently,

$$Q_\alpha Q^*_{\dot\beta} + Q^*_{\dot\beta} Q_\alpha = 2\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}_{\alpha\dot\beta}.$$

The little group is $\mathrm{Spin}(3) \cong SU(2)$, whose irreducible $\mathbb{C}$-representations $\underline{J}$ are indexed by the spin $J \in \frac{1}{2}\mathbb{Z}^{\geq 0}$ with dimension $2J + 1$. Then the irreducible massive representation of N=1 SUSY is represented by

$$\underline{J} \oplus Q^*_\alpha \underline{J} \oplus Q^*_1 Q^*_2 \underline{J} \cong \underline{J} \oplus \left(\underline{\frac{1}{2}} \otimes \underline{J}\right) \oplus \underline{J}.$$

Its dimension is $4(2J + 1)$.

7.1.3. $N > 1$ *massless supermultiplet.* $P_\mu = -i\partial_{x^\mu} = (E, 0, 0, E)$.

$$Q^A_\alpha (Q^B_{\dot\beta}) * + (Q^B_{\dot\beta})^* Q^A_\alpha = \delta^{AB}\begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta}$$

$$Q^A_\alpha Q^B_\beta + Q^B_\beta Q^A_\alpha = -2\,\mathrm{i}\,\epsilon_{\alpha\beta} Z^{AB}.$$

where $1 \leq A, B \leq N$. For unitary representations, $Q^A_2 = 0$. This implies that $Z^{AB} = 0$. The irreducible representation is generated by a state $|\lambda_0\rangle$ of helicity λ_0 via

$$\mathrm{Span}_{\mathbb{C}}\{Q_1^{A_1} \cdots Q_1^{A_k}\,|\lambda_0\rangle\}_{1 \leq A_1 < \cdots < A_k \leq N}.$$

The dimension is 2^N. The helicity λ of $Q_1^{A_1} \cdots Q_1^{A_k}\,|\lambda\rangle$ is $\lambda_0 + \frac{k}{2}$.

(1) $N = 2$ **vector multiplet**. $\lambda_0 = 0$. We have

$$\begin{array}{ccc} & \lambda = 0 & \\ \lambda = \frac{1}{2} & & \lambda = \frac{1}{2} \\ & \lambda = 1 & \end{array}$$

We see that the $N = 2$ vector multiplet consists of one $N = 1$ vector and one $N = 1$ chiral multiplet.

(2) $N=2$ **hyper multiplet**. $\lambda_0=-\frac{1}{2}$. We have

$$\begin{array}{ccc} & \lambda=-\dfrac{1}{2} & \\ \lambda=0 & & \lambda=0 \\ & \lambda=\dfrac{1}{2} & \end{array}$$

The $N=2$ hyper multiplet consists of two $N=1$ chiral multiplets.

(3) $N=4$ **vector multiplet**. $\lambda_0=-1$. We have

$$\begin{aligned} &1\times\{\lambda=-1\}\\ &4\times\{\lambda=-1/2\}\\ &6\times\{\lambda=0\}\\ &4\times\{\lambda=1/2\}\\ &1\times\{\lambda=1\}. \end{aligned}$$

It contains one $N=2$ vector-multiplet and two $N=2$ hyper-multiplet and their conjugates.

7.1.4. $N>1$ *massive supermultiplet.* $P_\mu=-i\partial_{x^\mu}=(m,0,0,0)$. Then

$$Q^A_\alpha(Q^B_{\dot\beta})^*+(Q^B_{\dot\beta})^*Q^A_\alpha=2\delta^{AB}\begin{pmatrix} m & 0\\ 0 & m\end{pmatrix}_{\alpha\dot\beta}$$
$$Q^A_\alpha Q^B_\beta+Q^B_\beta Q^A_\alpha=-2\,\mathrm{i}\,\epsilon_{\alpha\beta}Z^{AB}$$

where $1\leq A,B\leq N$.

(1) $Z^{AB}=0$. Then this is similar to $N=1$ massive case. The irreducible representation is generated from $\underline{J}$ by $(Q^A_\alpha)^*$, whose complex dimemsion is $2^{2N}(2J+1)$.

(2) $Z^{AB}\neq 0$. To see the structure, let us start with $N=2$ and

$$Z=\begin{pmatrix} 0 & z\\ -z & 0\end{pmatrix},\quad Z^{AB}=\epsilon^{AB}z.$$

Consider the combination

$$\Gamma^A_\alpha(\theta)=Q^A_\alpha-e^{\mathrm{i}\,\theta}\epsilon_{\alpha\dot\beta}\epsilon^{AB}(Q^B_\beta)^*.$$

Then

$$\left[\Gamma^A_\alpha(\theta),\Gamma^A_\alpha(\theta)^*\right]=4(m-\mathrm{Im}(e^{-\mathrm{i}\,\theta}z)).$$

Since the left hand side is semi-positive, we have

$$m-\mathrm{Im}(e^{-\mathrm{i}\,\theta}z)\geq 0\quad\forall\theta.$$

Equivalently,

$$\boxed{m\geq|z|}.$$

This is called **BPS bound** (due to Bogomolnyi, Prasad and Sommerfeld). Elements for $m = |z|$ are called **BPS state**, which are characterized by the vanishing of $\Gamma^A_\alpha(\theta)$. In general, for N even and

$$Z^{AB} = \begin{pmatrix} \begin{matrix} 0 & z_1 \\ -z_1 & 0 \end{matrix} & 0 & \cdots & 0 \\ 0 & \begin{matrix} 0 & z_1 \\ -z_1 & 0 \end{matrix} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \begin{matrix} 0 & z_{N/2} \\ -z_{N/2} & 0 \end{matrix} \end{pmatrix},$$

the BPS bound is

$$\boxed{m \geq |z_i|}, \quad 1 \leq i \leq N/2.$$

Let us assume k BPS conditions are saturated: $|z_i| = m, 1 \leq i \leq k$.

(a) $k = 0$. **Long multiplet** with 2^{2N} states, each of dim $2J + 1$.
(b) $0 < k < N/2$. **Short multiplet** with $2^{2(N-k)}$ states, each of dim $2J + 1$.
(c) $k = N/2$. **Ultra-short multiplet** with 2^N states, each of dim $2J + 1$.

7.2. **Superspace.** Let S be a spinor for V and

$$\Gamma : \mathrm{Sym}^2 S \to V$$

generates the SUSY algebra. We obtain a superspace

$$V_S = V \times \Pi S$$

which carries a natural $\mathrm{Spin}(V)$-action with equivariant group law

$$(v_1, s_1) \cdot (v_2, s_2) = (v_1 + v_2 + \frac{1}{2}\Gamma(s_1, s_2), s_1 + s_2).$$

Let $\mathcal{O}(V_S)$ denote smooth functions on V_S, then

$$\mathcal{O}(V_S) = C^\infty(V) \otimes \wedge^*(S^\vee).$$

Let $\mathrm{Der}(V_S)$ be the space of super derivations on $\mathcal{O}(V_S)$, which can be identified with vector fields on V_S. $\mathrm{Der}(V_S)$ is a super Lie algebra

$$\mathrm{Der}(V_S) = \mathrm{Der}_0(V_S) \oplus \mathrm{Der}_1(V_S).$$

For $D_i \in \mathrm{Der}_i(V_S)$, $f, g \in \mathcal{O}(V_S)$,

$$D_i(fg) = (D_i f)g + (-1)^{|i||f|} f D_i g.$$

We choose linear coordinate x^μ on V and θ^α on S,

$$x^\mu x^\nu = x^\nu x^\mu, \quad \theta^\alpha \theta^\beta = -\theta^\beta \theta^\alpha.$$

A general function f on V_S can be expanded by

$$f(x, \theta) = \sum_I f_I(x)\theta^I, \quad \theta^I = \theta^{i_1} \cdots \theta^{i_k} \text{ for } I = \{i_1 < \cdots < i_k\}.$$

We will denote the left derivations with respect to coordinates

$$\partial_{x^\mu} = \frac{\partial}{\partial x^\mu} \in \mathrm{Der}_0(V_S), \quad \partial_{\theta^\alpha} = \frac{\partial}{\partial \theta^\alpha} \in \mathrm{Der}_1(V_S).$$

We treat V_S as a super Lie group with the above multiplication group law. Then we have

$$\begin{cases} \partial_{x^\mu}, \quad D_\alpha = \partial_{\theta^\alpha} - \Gamma^\mu_{\alpha\beta}\theta^\beta\partial^\mu_x, \quad \text{left invariant vector fields} \\ \partial_{x^\mu}, \quad Q_\alpha = \partial_{\theta^\alpha} + \Gamma^\mu_{\alpha\beta}\theta^\beta\partial^\mu_x, \quad \text{right invariant vector fields} \end{cases}$$

Left and right invariant vectors super commute with each other and both satisfy SUSY Lie algebra

$$\left[D_\alpha, D_\beta\right] = -\left[Q_\alpha, Q_\beta\right] = -2\Gamma^\mu_{\alpha\beta}\partial_{x^\mu}, \quad \left[D_\alpha, Q_\beta\right] = 0.$$

This immediately implies that $\mathcal{O}(V_S)$ is naturally a representation of our SUSY algebra.

We recall the definition of integration on superspace. Given an odd variable θ,

$$\int d\theta(a + \theta b) := b.$$

It carries similar properties with ordinary integrations. For example

$$\int d\theta\, \partial_\theta f = 0, \quad \int d\theta^1 d\theta^2 = -\int d\theta^2 d\theta^1.$$

7.3. **N=1 chiral multiplet.** We consider $N = 1$ superspace in $d = 4$. We have Weyl spinor S and its complex conjugate $\bar{S}$. The super coordinates are denoted by $x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}$, where $0 \le \mu \le 3, 1 \le \alpha, \dot{\alpha} \le 2$.

$$Q_\alpha = \partial_{\theta^\alpha} + \sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_{x^\mu} \quad D_\alpha = \partial_{\theta^\alpha} - \sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_{x^\mu}$$
$$\bar{Q}_{\dot{\alpha}} = -\partial_{\bar{\theta}^{\dot{\alpha}}} - \sigma^\mu_{\beta\dot{\alpha}}\theta^\beta\partial_{x^\mu} \quad \bar{D}_{\dot{\alpha}} = -\partial_{\bar{\theta}^{\dot{\alpha}}} + \sigma^\mu_{\beta\dot{\alpha}}\theta^\beta\partial_{x^\mu}$$

The reality conditions are $\theta^\dagger = \bar{\theta}, \partial_\theta^\dagger = -\partial_{\bar{\theta}}$ such that $Q^\dagger = \bar{Q}$. Here $(-)^\dagger$ is the super-adjoint.

Definition 7.1. For convenience, let us introduce the notation

$$\boxed{\partial_{\alpha\dot{\beta}} := \sigma^\mu_{\alpha\dot{\beta}}\partial_\mu}.$$

Then the SUSY operators can be written as

$$Q_\alpha = \partial_{\theta^\alpha} + \bar{\theta}^{\dot{\beta}}\partial_{\alpha\dot{\beta}} \quad D_\alpha = \partial_{\theta^\alpha} - \bar{\theta}^{\dot{\beta}}\partial_{\alpha\dot{\beta}}$$
$$\bar{Q}_{\dot{\alpha}} = -\partial_{\bar{\theta}^{\dot{\alpha}}} - \theta^\beta\partial_{\beta\dot{\alpha}} \quad \bar{D}_{\dot{\alpha}} = -\partial_{\bar{\theta}^{\dot{\alpha}}} + \theta^\beta\partial_{\beta\dot{\alpha}}$$

Definition 7.2. Given two Weyl spinors $\lambda^\alpha, \chi^\alpha, \bar{\lambda}^{\dot{\alpha}}, \bar{\chi}^{\dot{\alpha}}$, we denote

$$\boxed{\lambda_\alpha = \epsilon_{\alpha\beta}\lambda^\beta, \quad \lambda^\alpha = \epsilon^{\alpha\beta}\lambda_\beta, \quad \lambda\chi = \lambda^\alpha\chi_\alpha, \quad \bar{\lambda}\bar{\chi} = \bar{\lambda}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}, \quad \lambda\sigma^\mu\bar{\chi} = \sigma^\mu_{\alpha\dot{\beta}}\lambda^\alpha\bar{\chi}^{\dot{\beta}}, \quad \bar{\lambda}\bar{\sigma}^\mu\chi = \bar{\lambda}^{\dot{\alpha}}\sigma^\mu_{\beta\dot{\alpha}}\chi^\beta}.$$

Here $\epsilon_{12} = -\epsilon_{21} = 1 = \epsilon^{21} = -\epsilon^{12}$.

Note that in the above definition the super adjoint has the property

$$\boxed{\lambda\chi = -(-1)^{|\lambda||\chi|}\chi\lambda, \quad (\lambda\chi)^\dagger = (-1)^{|\lambda||\chi|}\chi^\dagger\lambda^\dagger = -\lambda^\dagger\chi^\dagger, \quad (\lambda\sigma^\mu\bar{\chi})^\dagger = (-1)^{|\lambda||\chi|}\bar{\chi}^\dagger\sigma^\mu\lambda^\dagger = \lambda^\dagger\bar{\sigma}^\mu\bar{\chi}^\dagger}.$$

We denote

$$\theta^2 = \theta^\alpha\theta_\alpha, \quad \bar{\theta}^2 = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}$$

which are nonvanishing since θ's are odd. It is useful to note that

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta^2, \quad \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}^2,$$

which implies the following formula

$$\boxed{\theta\sigma^\mu\bar\theta\theta\lambda = \frac{1}{2}\theta^2\bar\theta\bar\sigma^\mu\lambda, \quad \theta\sigma^\mu\bar\theta\bar\theta\bar\lambda = -\frac{1}{2}\bar\theta^2\theta\sigma^\mu\bar\lambda, \quad \theta\sigma^\mu\bar\theta\theta\sigma^\nu\bar\theta = \frac{1}{2}\theta^2\bar\theta^2\eta^{\mu\nu}}$$

7.3.1. *Super function.* A general $N=1$ super function can be expanded as

$$\Phi = \phi(x) + \theta\psi(x) - \bar\theta\bar\psi(x) + \theta^2 m(x) - \bar\theta^2\bar m(x) + \theta\sigma^\mu\bar\theta v_\mu(x) + \theta^2\bar\theta\bar\lambda(x) + \bar\theta^2\theta\lambda(x) + \theta^2\bar\theta^2 D(x).$$

From the property of super adjoint, the reality condition reads

$$\boxed{\Phi = \Phi^\dagger \Leftrightarrow \phi = \phi^\dagger, \bar\psi = \psi^\dagger, \bar m = m^\dagger, v_\mu^\dagger = -v_\mu, \bar\lambda = \lambda^\dagger, D = D^\dagger}.$$

Proposition 7.3. *Let $\mathcal{O}(V_S)$ be the space of $N=1$ super functions. Then*

- *$\mathcal{O}(V_S)$ is naturally a SUSY representation.*
- *The functional*
$$\int_{V_S} : \mathcal{O}(V_S) \to \mathbb{C}, \quad \Phi \to \int d^4x d\theta^2 d\bar\theta^2 \Phi$$
is invariant under SUSY transformation.

Remark 7.4. Note that
$$\int d^4x d\theta^2 d\bar\theta^2 \Phi = \int d^4x D(x)$$
where $D(x)$ is the top θ-component in the above decomposition.

7.3.2. *Chiral superfield.*

Definition 7.5. A **chiral superfield** is a function $\Phi(x^\mu, \theta^\alpha, \bar\theta^{\dot\alpha})$ on $N=1$ superspace such that
$$\bar D_{\dot\alpha}\Phi = 0.$$

Since D commutes Q, chiral superfields form a representation of $N=1$ SUSY. Let us denote
$$y^\mu = x^\mu - \sigma^\mu_{\alpha\dot\beta}\theta^\alpha\bar\theta^{\dot\beta}.$$
In the new coordinates $\{y^\mu, \theta^\alpha, \bar\theta^{\dot\alpha}\}$,
$$Q_\alpha = \partial_{\theta^\alpha} \quad D_\alpha = \partial_{\theta^\alpha} - 2\bar\theta^{\dot\beta}\partial_{\alpha\dot\beta}$$
$$\bar Q_{\dot\alpha} = -\partial_{\bar\theta^{\dot\alpha}} - 2\theta^\beta\partial_{\beta\dot\alpha} \quad \bar D_{\dot\alpha} = -\partial_{\bar\theta^{\dot\alpha}}$$

If Φ is a chiral superfield, then Φ is a function of $\{\theta^\alpha, y^\mu\}$ and can be expanded in components
$$\begin{aligned}\Phi = \Phi(\theta^\alpha, y^\mu) &= \phi(y) + \theta^\alpha\psi_\alpha(y) + \theta^2 F(y)\\ &= \phi(x) + \theta\psi(x) - \theta\sigma^\mu\bar\theta\partial_\mu\phi + \theta^2 F(x) - \frac{1}{2}\theta^2\bar\theta\bar\sigma^\mu\partial_\mu\psi + \frac{1}{4}\theta^2\bar\theta^2\partial^\mu\partial_\mu\phi(x).\end{aligned}$$
Equivalently,
$$\phi = \Phi_{\theta=\bar\theta=0}, \quad \psi_\alpha = D_\alpha\Phi|_{\theta=\bar\theta=0}, \quad F = -\frac{1}{4}D^\alpha D_\alpha\Phi|_{\theta=\bar\theta=0}.$$
The SUSY transformation
$$\delta_\epsilon = \epsilon^\alpha Q_\alpha + \bar\epsilon^\alpha\bar Q_\alpha$$

on chiral superfields can be read off easily

$$\begin{cases} \delta_\epsilon \phi = \epsilon^\alpha \psi_\alpha \\ \delta_\epsilon \psi_\alpha = 2\epsilon_\alpha F - 2\bar{\epsilon}^{\dot{\beta}} \partial_{\alpha\dot{\beta}} \phi \\ \delta_\epsilon F = -\bar{\epsilon}^{\dot{\beta}} \partial_{\alpha\dot{\beta}} \psi^\alpha. \end{cases}$$

Observe that F component transforms as a total derivative under SUSY, therefore

$$\int d^4 x F(x) = \int d^4 x d\theta^2 \Phi|_{\bar{\theta}=0}$$

defines a SUSY invariant functional on chiral superfields.

7.3.3. *D-term and Kahler potential.* Let

$$\mathcal{O}^{ch}(V_S) = \{\text{chiral superfields}\} \subset \mathcal{O}(V_S).$$

We consider n copies of chiral superfields $\{\Phi^i\}_{1 \leq i \leq n}$. Given a smooth function $K(z^i, \bar{z}^i)$ on $\mathbb{C}^n$, we can define an functional

$$S_K(\Phi^i) = \int d^4 x d\theta^2 d\bar{\theta}^2 K(\Phi^i, (\Phi^i)^\dagger).$$

Since S_K comes from the D-term of a superfield, S_K is automatically SUSY invariant.

Geometrically, let us think about $\mathcal{O}^{ch}(V_S)$ as a function ring of a superspace V_S^{ch}. Then

$$\Phi^i : V_S^{ch} \to \mathbb{C}^n$$

can be viewed as a σ-model whose target is $\mathbb{C}^n$ with Kahler potential K. It is easy to see that S_K is invariant under the transformation:

$$K \to K(z, \bar{z}) + f(z) + \bar{f}(\bar{z}).$$

This implies that the above construction can be glued to a general Kahler manifold (X, ω). The space of fields describes the σ-model

$$\Phi : V_S^{ch} \to X$$

where the action functional S_K is the same above with the interpretation that

$$\Phi^i = \Phi^*(z^i)$$

where z^i's are local coordinates on X, and $K(z, \bar{z})$ is a local choice of Kahler potential. The SUSY transformation acts on $Map(V_S^{ch}, X)$ via its domain geometry, and S_K is SUSY invariant.

7.3.4. *F-term and superpotential.* Let us again consider n chiral superfields

$$\Phi : V_S^{ch} \to \mathbb{C}^n.$$

Let $W(z^i)$ be a holomorphic function on $\mathbb{C}^n$. We define the following action functional

$$S_F(\Phi) = \int d^4 x d\theta^2 W(\Phi^i)|_{\bar{\theta}=0}.$$

SInce this picks up the F-term of the chiral superfield $W(\Phi^i)$, this action functional is automatically SUSY invariant. This is sometimes call the F-term construction.

In general, given a Kahler manifold (X, ω) with a holomorphic function

$$W : X \to \mathbb{C},$$

we can define both D-term and F-term SUSY invariant action functionals on the mapping space

$$\Phi : V_S^{ch} \to X.$$

For a nontrivial W, X is necessarily non-compact.

7.4. **N=1 vector multiplet.** Geometrically, we consider a principal G-bundle P over the $N=1$ superspace V_S. We can assume P is trivial, and $\mathfrak{g}$ be the Lie algebra of G. Let ∇ denote the covariant derivative and

$$\nabla_\mu = \nabla_{\partial_\mu} = \partial_\mu + A_\mu, \quad \nabla_\alpha = \nabla_{D_\alpha} = D_\alpha + A_\alpha, \quad \nabla_{\dot\alpha} = \nabla_{D_{\dot\alpha}} = D_{\dot\alpha} + A_{\dot\alpha}$$

be the covariant derivatives along $\partial_\mu, D_\alpha, D_{\dot\alpha}$ respectively. Here $A_\mu, A_\alpha, A_{\dot\alpha}$ are $\mathfrak{g}$-valued superfunctions.

Let $F_{\mu\nu}, F_{\mu\alpha}, F_{\alpha\beta}$ be the corresponding curvatures (along the vectors $\partial_\mu, D_\alpha, D_{\dot\alpha}$).

Definition 7.6. ∇ is called a $N=1$ **connection** if the following components vanish

$$\boxed{F_{\alpha\beta} = F_{\alpha\dot\beta} = F_{\dot\alpha\dot\beta} = 0}.$$

We let $\mathcal{A}^{N=1} = \{N=1 \text{ connections}\}$, and $Gauge^{N=1} = \{\text{gauge transformations}\}$

$$\nabla \to e^S \nabla e^{-S}$$

where S is a $\mathfrak{g}$-valued superfield.

Using $[D_\alpha, D_{\dot\beta}] = 2\partial_{\alpha\dot\beta}$, this is equivalent to

$$\boxed{[\nabla_\alpha, \nabla_\beta] = 0, \quad [\nabla_{\dot\alpha}, \nabla_{\dot\beta}] = 0, \quad [\nabla_\alpha, \nabla_{\dot\beta}] = 2\nabla_{\alpha\dot\beta}}.$$

The Bianchi identity

$$[\nabla_\mu, [\nabla_\alpha, \nabla_{\dot\beta}]] = [[\nabla_\mu, \nabla_\alpha], \nabla_{\dot\beta}] + [\nabla_\alpha, [\nabla_\mu, \nabla_{\dot\beta}]]$$

implies that

$$\boxed{2\sigma^\nu_{\alpha\dot\beta} F_{\mu\nu} = [F_{\mu\alpha}, \nabla_{\dot\beta}] + [\nabla_\alpha, F_{\mu\dot\beta}]}$$

which expresses $F_{\mu\nu}$ in terms of $F_{\mu\alpha}, F_{\mu\dot\alpha}$. The Bianchi identity

$$[\nabla_\alpha, [\nabla_{\dot\beta}, \nabla_{\dot\gamma}]] = [[\nabla_\alpha, \nabla_{\dot\beta}], \nabla_{\dot\gamma}] - [[\nabla_{\dot\beta}, [\nabla_\alpha, \nabla_{\dot\gamma}]].$$

implies that

$$\boxed{\sigma^\mu_{\alpha\dot\beta} F_{\mu\dot\gamma} = \epsilon_{\dot\beta\dot\gamma}\Sigma_\alpha}, \quad \text{and similarly} \quad \boxed{\sigma^\mu_{\alpha\dot\beta} F_{\mu\gamma} = \epsilon_{\alpha\gamma}\Sigma_{\dot\beta}}.$$

Σ can be expressed in terms of curvatures by

$$\boxed{\Sigma_\alpha = -2\sigma^\mu_{\alpha\dot\beta} F^{\dot\beta}_\mu}, \quad \boxed{\Sigma_{\dot\beta} = -2\sigma^\mu_{\alpha\dot\beta} F^\alpha_\mu}.$$

The Bianchi identity

$$[\nabla_\mu, [\nabla_{\dot\alpha}, \nabla_{\dot\beta}]] = [[\nabla_\mu, \nabla_{\dot\alpha}], \nabla_{\dot\beta}] + [[\nabla_{\dot\alpha}, [\nabla_\mu, \nabla_{\dot\beta}]]$$

further implies that

$$\boxed{\nabla_{\dot\alpha}\Sigma_\beta = 0} \quad \text{and similarly} \quad \boxed{\nabla_\alpha \Sigma_{\dot\beta} = 0}.$$

Definition 7.7. Let ∇ be a $N=1$ connection. ∇ is said to be

(1) **chiral** if $A_{\dot\alpha} = 0$, i.e., $\mathcal{D}_{\dot\alpha} = D_{\dot\alpha}$. We write $\nabla = \nabla^c$ and define a chiral gauge transformation to be

$$\nabla^c \to e^\Lambda \nabla^c e^{-\Lambda}.$$

Here Λ is a $\mathfrak{g}$-valued chiral superfield. We let

$$\mathcal{A}^c = \{\text{chiral connections}\}, \quad Gauge^c = \{\text{chiral gauge transformations}\}.$$

(2) **anti-chiral** if $A_\alpha = 0$, i.e., $\mathcal{D}_\alpha = D_\alpha$. We write $\nabla = \nabla^{ac}$ and define an anti-chiral gauge transformation to be

$$\nabla^{ac} \to e^{\Lambda^\dagger} \nabla^{ac} e^{-\Lambda^\dagger}.$$

Here $\Lambda^\dagger$ is a $\mathfrak{g}$-valued anti-chiral sueprfield. We let

$$\mathcal{A}^{ac} = \{\text{anti-chiral connections}\}, \quad Gauge^{ac} = \{\text{anti-chiral gauge transformations}\}.$$

Proposition 7.8. *There are natural identifications of quotients*

$$\mathcal{A}^{N=1}/Gauge^{N=1} \cong \mathcal{A}^c/Gauge^c \cong \mathcal{A}^{ac}/Gauge^{ac}.$$

Definition 7.9. Given a (real) N=1 connection ∇, let ∇^c and ∇^{ac} be a choice of chiral and anti-chiral gauge. We define the N=1 vector superfield representing the gauge transformation

$$\nabla^{ac} = e^V \nabla^c e^{-V}.$$

Here V is a $\mathfrak{g}$-valued real superfield: $V = V^\dagger$. V is defined up to the chiral gauge transformation

$$\boxed{e^V \to e^{-\Lambda^\dagger} e^V e^{-\Lambda}}$$

where Λ is a $\mathfrak{g}$-valued chiral superfield. The reality condition is compatible with $\nabla^{ac} = (\nabla^c)^\dagger$.

Let us work with the chiral gauge ∇^c. In the D_α component $\nabla^c_\alpha = D_\alpha + A^c_\alpha$, $\nabla^{ac}_\alpha = D_\alpha$, we find

$$D_\alpha = e^V \circ (D_\alpha + A^c_\alpha) \circ e^{-V} \Longrightarrow A^c_\alpha = e^{-V} D_\alpha e^V.$$

The curvature relation

$$[\nabla_\alpha, \nabla_{\dot\beta}] = 2\sigma^\mu_{\alpha\dot\beta} \nabla_\mu$$

implies that

$$D_{\dot\beta} A^c_\alpha = 2\sigma^\mu_{\alpha\dot\beta} A_\mu.$$

In the chiral gauge, the curvature

$$\Sigma_\alpha = -2\sigma^\mu_{\alpha\dot\beta} F^{\dot\beta}_\mu = -2\sigma^\mu_{\alpha\dot\beta}[\nabla_\mu, \nabla^{\dot\beta}] = 2\sigma^\mu_{\alpha\dot\beta} D^{\dot\beta} A^c_\mu = D^{\dot\beta} D_{\dot\beta} A^c_\alpha,$$

i.e.,

$$\boxed{\Sigma_\alpha = -\bar{D}^2(e^{-V} D_\alpha e^V)},$$

which is a chiral superfield. Under the gauge transformation $e^V \to e^{-\Lambda^\dagger} e^V e^{-\Lambda}$, we find

$$\boxed{\Sigma_\alpha \to e^\Lambda \Sigma_\alpha e^{-\Lambda}}.$$

Definition 7.10. The N=1 Super Yang-Mills functional

$$S^{N=1}_{YM} : \{\text{N=1 vector superfield}\}/\text{Gauge} \to \mathbb{C}$$

is defined by

$$\boxed{S^{N=1}_{YM}(V) = \frac{1}{16} \int d^4x d^2\theta \, \mathrm{Tr}\, \Sigma^\alpha \Sigma_\alpha|_{\bar\theta=0} + c.c.}$$

where $\Sigma_\alpha = -\bar{D}^2(e^{-V} D_\alpha e^V)$.

Under the gauge transformation, the vector superfield V can be put into the form

$$V_{WZ} = \theta\sigma^\mu\bar{\theta}v_\mu + \theta^2\bar{\theta}\bar{\lambda} + \bar{\theta}^2\theta\lambda + \theta^2\bar{\theta}^2 D, \quad v_\mu^\dagger = -v_\mu, \quad \lambda^\dagger = \bar{\lambda}, \quad D^\dagger = D.$$

This is called **Wess-Zumino gauge**. Under this gauge, $S_{YM}^{N=1}$ becomes the N=1 SYM that we discussed before, except with an extra auxiliary field D.

More general, let us introduce a complex coupling constant

$$\tau = \frac{\Theta}{2\pi} + \frac{4\pi\,\mathrm{i}}{g}.$$

Then in the Wess-Zumino gauge,

$$\begin{aligned} S_{YM}^{N=1}(V) &= \frac{1}{32}\,\mathrm{Im}\int d^4x d^2\theta\tau\,\mathrm{Tr}\,\Sigma^\alpha\Sigma_\alpha|_{\bar{\theta}=0} \\ &= \frac{1}{g^2}\int d^4x\,\mathrm{Tr}\left(-\frac{1}{4}\langle F, F\rangle - \frac{1}{2}\langle\lambda, \not{D}\lambda\rangle + \frac{1}{2}D^2\right) + \frac{\Theta}{32\pi^2}\int \mathrm{Tr}\,F\wedge F. \end{aligned}$$

7.5. **N=1 gauge theory with matter.** We consider $N=1$ vector superfield V coupled with chiral superfields Φ in a representation R of the gauge group G. The gauge transformation reads

$$\Phi \to e^\Lambda\Phi, \quad e^V \to e^{-\Lambda^\dagger}e^V e^{-\Lambda}.$$

The full $N=1$ action for gauge theory coupled with matter is

$$S = \int d^4x\mathcal{L}_{gauge} + \mathcal{L}_{matter} + \mathcal{L}_{FI}$$

where

$$\mathcal{L}_{gauge} = \frac{1}{32}\,\mathrm{Im}\int \tau d^2\theta\,\mathrm{Tr}\,\Sigma^\alpha\Sigma_\alpha|, \quad \tau = \frac{\Theta}{2\pi} + \frac{4\pi\,\mathrm{i}}{g},$$

$$\mathcal{L}_{matter} = \int d^2\theta d^2\bar{\theta}\Phi^\dagger e^V\Phi + \int d^2\theta W(\Phi) + \int d^2\bar{\theta}W(\Phi)^\dagger.$$

Here the superpotential $W(\Phi)$ is a G-invariant holomorphic function.

The term $\mathcal{L}_{FI}$ is called the Fayet-Iliopoulos term. It is present if there is a $U(1)$ factor of the gauge group. Let V^A be the component of the vector multiplet along the abelian $U(1)$ factor. Then

$$\mathcal{L}_{FI} = \sum_{A\in\text{abelian factors}}\int d^2\theta d^2\bar{\theta}V^A$$

which is invariant under the gauge transformation $V^A \to V^A - \Lambda - \Lambda^\dagger$ for a chiral superfield Λ.

7.6. **N=2 vector multiplet.** An $N=2$ vector multiplet consists of an $N=1$ vector multiplet and an $N=1$ chiral multiplet in the adjoint representation of $\mathfrak{g}$:

$\lambda_\alpha, \quad A_\mu$	N=1 vector multiplet
$\phi, \quad \psi_\alpha,$	N=1 chiral multiplet

The easiest way to construct $N=2$ SUSY is to start with $N=1$ SUSY and require the $SU(2)_R$ symmetry rotating λ and ψ. Then $N=1$ SUSY and $SU(2)_R$ will generate $N=2$ SUSY.

The simplest $N=2$ gauge action is

$$\frac{1}{4\pi}\,\mathrm{Im}\int d^4x\left(\tau\int d^2\theta d^2\bar{\theta}\Phi^\dagger e^V\Phi + \frac{\tau}{2}\int d^2\theta\,\mathrm{Tr}\,\Sigma^\alpha\Sigma_\alpha\right).$$

More generally, $N = 2$ SUSY action for $N = 2$ vector multiplet is expressed by a single gauge-invariant holomorphic function $\mathcal{F}(\Phi)$, called the **prepotential**. In the abelian case of our main interest here, the action takes the form

$$\frac{1}{4\pi}\operatorname{Im}\int d^4x\left(\int d^2\theta d^2\bar{\theta}\Phi^\dagger e^V\frac{\partial\mathcal{F}(\Phi)}{\partial\Phi}+\frac{1}{2}\int d^2\theta\frac{\partial\mathcal{F}(\Phi)}{\partial\Phi^2}\Sigma^\alpha\Sigma_\alpha\right).$$

In other words, $N = 2$ SUSY relates the Kahler potential with the superpotential. When $\mathcal{F} = \frac{\tau}{2}\Phi^2$, this reduces to the above form.

7.7. **N=2 hyper multiplet.** An $N = 2$ hyper multiplet consists of two $N = 1$ chiral multiplets $H, \tilde{H}$.

$$\begin{array}{ll} q, \quad \psi_\alpha & H : \text{N=1 chiral} \\ \tilde{\psi}^\dagger_{\dot\alpha}, \quad \tilde{q}^\dagger, & \tilde{H}^\dagger : \text{N=1 anti-chiral} \end{array}$$

$N = 2$ SUSY is achieved by requiring $SU(2)_R$ symmetry rotating q and $\tilde{q}^\dagger$. The $N = 2$ action is

$$\int d^4xd^2\theta d^2\bar{\theta}^2(H^\dagger H+\tilde{H}^\dagger\tilde{H})+m\int d^4xd^2\theta(\tilde{H}H)+m\int d^4d^2\bar{\theta}\tilde{H}^\dagger H^\dagger.$$

7.8. **N=2 gauge theory with matter.** We consider $N = 2$ vector multiplet coupled with $N = 2$ hyper multiplet. We have $N = 2$ $\mathfrak{g}$-valued vector multiplet V, Φ, and $N = 2$ hyper multiplet $H, \tilde{H}$ living in conjugate representations of $\mathfrak{g}$. The massless $N = 2$ action reads

$$\frac{1}{4\pi}\operatorname{Im}\int d^4x\left(\tau\int d^2\theta d^2\bar{\theta}\left(\Phi^\dagger e^V\Phi+H^\dagger e^VH+\tilde{H}^\dagger e^{-V}\tilde{H}\right)+\frac{\tau}{2}\int d^2\theta\operatorname{Tr}\Sigma^\alpha\Sigma_\alpha+\tilde{H}\Phi H\right).$$

8. SEIBERG-WITTEN THEORY

Seiberg-Witten theory deals with non-perturbative dynamics of $N = 2$ super-Yang-Mills theory in the low energy limit.

8.1. Electro-Magnetic duality.

8.1.1. *Dirac quantization condition.* Let us consider Maxwell's abelian gauge theory. The field strength F is 2-form on spacetime, decomposed as

$$F = dt\wedge\mathbf{E}+\star(dt\wedge\mathbf{B})$$

where $\mathbf{E} = \sum_{i=1}^3\mathbf{E}_idx^i$ is the electric field and $\mathbf{B} = \sum_{i=1}^3\mathbf{B}_idx^i$ is the magnetic field. $\star$ is the Hodge star in Minkowski space and $\star^2 = -1$. The field strength satisfies Maxwell equation

$$dF = J_m$$
$$d\star F = J_e$$

Here J_e is the electric current for the electric source, and J_m is the magnetic current. J_e, J_m's are closed 3-forms, usually representing the Poincare dual of curves in spacetime. We write

$$J_e = q_e\delta(\gamma_e),\quad J_m = q_m\delta(\gamma_m).$$

q_e is called the electric charge and q_m is called the magnetic charge. For example, a static electron or monopole produces a current supported in the curve with x^i's fixed and moving along the time t. Even though the monopole has not been observed in the lab, the above Maxwell equation has a manifest duality

$$F\leftrightarrow\star F,\quad J_e\leftrightarrow J_m$$

which switches the role of electric field and magnetic field.

Geometrically, in the electric picture and away from the support of J_m, Maxwell theory describes a $U(1)$-bundle with connection 1-form A such that $F = dA$ plays the role of curvature form. For simplicity, let us assume $J_m = 0$. Then Maxwell equation comes from the action functional

$$-\frac{1}{4}\int F \wedge \star F + 2\pi q_e \int_{\gamma_e} A.$$

When γ_e is a loop, $2\pi q_e \int_{\gamma_e} A$ represents the holonomy. There is a similar description in the magnetic picture.

Let us assume the existence of a monopole sitting at the origin o of the space $\mathbb{R}^3$, in other words,

$$\gamma_m = \mathbb{R}_t \times \{o\} \subset \mathbb{R}_t \times \mathbb{R}^3.$$

Let us consider electric current moving along γ_e away from the origin o with electric charge q_e. Then $2\pi q_e A$ defines a connection for a $U(1)$-bundle on $\mathbb{R}^4 - \{\gamma_m\}$, whose curvature is $2\pi q_e dA = 2\pi q_e F$. Note

$$\mathbb{R}^4 - \{\gamma_m\} = \mathbb{R}_t \times (\mathbb{R}^3 - o)$$

which is topologically S^2. Let S^2 be the unit sphere in the space $\mathbb{R}^3$, then

$$\int_{S^2} q_e F \in \mathbb{Z}$$

is integer valued representing the 1st Chern class. On the other hand, Maxwell equation says that

$$\int_{S^2} F = \int_B dF = \int_B q_m \delta(\gamma_m) = q_m$$

where B is the unit ball. It follows that

$$\boxed{q_e q_m \in \mathbb{Z}}$$

which is the celebrated **Dirac quantization condition**. In particular, the assumption of existence of monopole would imply that electric charges are integer multiples of a single unit.

Example 8.1 (Dirac monopole)**.** The field strength of Dirac monopole is

$$F = \frac{k}{2|x|^3}(x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2), \quad |x| = \sqrt{|x^1|^2 + |x^2|^2 + |x^3|^2}, \quad k \in \mathbb{Z}.$$

Then $\mathbf{E} = 0$ and $\mathbf{B} = -\frac{1}{2|r|^3}\sum_{i=1}^3 x^i dx^i$. Topologically, it represents the complex line bundle $O(k)$ on $\mathbb{P}^1 = S^2$.

In general, we can consider dyons, which are particles with both electric and magnetic charges. Consider the configuration with two dyons moving with charges (q_e, q_m) and (q_e', q_m'), then a similar argument gives the Dirac quantization condition

$$q_e q_m' - q_m q_e' \in \mathbb{Z}.$$

In other words, the charges (q_e, q_m) form a symplectic lattice.

8.1.2. *$SL(2,\mathbb{Z})$ duality.* Let us consider Maxwell theory with a topological Θ-angle

$$S = -\frac{1}{4g^2}\int F \wedge \star F + \frac{\Theta}{32\pi^2}\int F \wedge F.$$

If we introduce the complex coupling constant

$$\tau = \frac{\Theta}{2\pi} + \frac{4\pi\,\mathrm{i}}{g^2},$$

then the above action can be written as

$$-\frac{1}{32\pi}\,\mathrm{Im}\int \tau(F + \mathrm{i}\,\tilde{F}) \wedge \star(F + \mathrm{i}\,\tilde{F}), \quad \text{where} \quad \tilde{F} = \star F.$$

The physics is described by a path integral

$$\int e^{\frac{iS}{2\pi}}.$$

By the topological property, $\frac{1}{32\pi^2}\int F\wedge F$ represents second Chern-class an is always integer. It follows that the physics is period in Θ by the T-transformation

$$\boxed{T:\tau\to\tau+1}.$$

To implement the constraint $dF=0$, we introduce a 1-form lagrangian multiplier A_D

$$\begin{aligned}&-\frac{1}{32\pi}\operatorname{Im}\int\tau(F+\mathrm{i}\tilde{F})\wedge\star(F+\mathrm{i}\tilde{F})+\frac{1}{8\pi}\int F_D\wedge F,\quad (F_D=dA_D)\\ =&-\frac{1}{32\pi}\operatorname{Im}\int\tau(F+\mathrm{i}\tilde{F})\wedge\star(F+\mathrm{i}\tilde{F})+\frac{1}{16\pi}\operatorname{Im}\int i(F_D+\mathrm{i}\tilde{F}_D)\wedge(F+\mathrm{i}\tilde{F}),\quad(\tilde{F}_D=\star F_D).\end{aligned}$$

By completing the square and integrating out F, we find the dual description

$$\boxed{F_D=\frac{4\pi}{g^2}\star F-\frac{\Theta}{2\pi}F}=\operatorname{Im}\tau\star(F+\mathrm{i}\tilde{F}).$$

and the dual action

$$S_D=-\frac{1}{32\pi}\operatorname{Im}\int\frac{-1}{\tau}(F_D+\mathrm{i}\tilde{F}_D)\wedge\star(F_D+\mathrm{i}\tilde{F}_D),\quad \text{where}\quad \tilde{F}_D=\star F_D.$$

We find $\int e^{iS}=\int e^{iS_D}$ as a ∞-dim version of Fourier transform. In particular, S and S_D describe equivalent physics, and we find the following duality S-transformation

$$\boxed{S:\tau\to-\frac{1}{\tau}}.$$

When $\Theta=0$, this gives $g\to\frac{4\pi}{g}$, which turns a strongly coupled system to weakly coupled system.

All together, T and S generate $PSL(2,\mathbb{Z})$ transformations

$$\boxed{\tau\to\frac{a\tau+b}{c\tau+d}},\quad a,b,c,d\in\mathbb{Z},ad-bc=1.$$

The electric magnetic charge (q_e,q_m) will transform under $SL(2,\mathbb{Z})$ via

$$\begin{pmatrix}q_e & q_m\end{pmatrix}\to\begin{pmatrix}q_e & q_m\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1}$$

We take a closer look at the duality transformation. Let us denote

$$G=-8\pi\frac{\delta S}{\delta F}=\frac{4\pi}{g^2}\star F-\frac{\Theta}{2\pi}F.$$

The action functional takes the form

$$-\frac{1}{16\pi}\int F\wedge G.$$

The S-transformation gives

$$S:\begin{pmatrix}F & G\end{pmatrix}\to\begin{pmatrix}F_D & G_D\end{pmatrix}=\begin{pmatrix}F & G\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

Similarly, the T-transformation gives

$$T:\begin{pmatrix}F & G\end{pmatrix}\to\begin{pmatrix}F & G\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$$

We find the general $SL(2,\mathbb{Z})$-transformation

$$\boxed{\begin{pmatrix} F & G \end{pmatrix} \to \begin{pmatrix} F & G \end{pmatrix}\gamma^{-1}}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

Such transformation rotates Bianchi identities and equations of motion.

Geometrically, let Ω^2 be 2-forms on spacetime, and we consider

$$(F,G) \in \Omega^2 \otimes \mathbb{R}^2.$$

Ω^2 carries a natural symmetric pairing via integration, and $\mathbb{R}^2$ carries a natural symplectic paring that is invariant under $SL(2,\mathbb{Z})$-transformations. Therefore $\Omega^2 \otimes \mathbb{R}^2$ has a $SL(2,\mathbb{Z})$-invariant symplectic structure. The expression

$$\mathcal{L}_\tau = \{(F,G) | G = \frac{4\pi}{g^2} \star F - \frac{\Theta}{2\pi} F\} \subset \Omega^2 \otimes \mathbb{R}^2$$

defines a linear Lagrangian subspace parametrized by τ, whose generating function is $-\frac{1}{4g^2}\int F \wedge \star F + \frac{\Theta}{32\pi^2}\int F \wedge F$. Under a $SL(2,\mathbb{Z})$ transformation γ, it changes the lagrangian to

$$\mathcal{L}_\tau \to \mathcal{L}_{\gamma\tau}.$$

8.2. **Seiberg-Witten's exact solution.** We consider $N = 2$ super Yang-Mills theory with gauge group $SU(2)$. It contains a $SU(2)$ $N = 1$ vector multiplet V and a chiral multiplet Φ valued in the adjoint representation of $SU(2)$. The $N = 2$ action takes the form

$$S = \frac{1}{4\pi} \operatorname{Im} \int d^4x \left(\tau \int d^2\theta d^2\bar{\theta} \operatorname{Tr} \Phi^\dagger e^V \Phi + \frac{\tau}{2} \int d^2\theta \operatorname{Tr} \Sigma^\alpha \Sigma_\alpha \right).$$

If we expand in terms of components for

$$\begin{array}{ll} \lambda_\alpha, \quad A_\mu & \text{N=1 vector multiplet} \\ \phi, \quad \psi_\alpha, & \text{N=1 chiral multiplet} \end{array}$$

we find the lagrangian density

$$\mathcal{L} = \frac{1}{g^2} \operatorname{Tr} \left(-\frac{1}{4}\langle F, F\rangle - \frac{1}{2}\langle \lambda, \not{D}\lambda\rangle - \frac{1}{2}\langle \psi, \not{D}\psi\rangle + \frac{1}{2}(D_\mu\phi)^\dagger D^\mu\phi + \phi^\dagger[\lambda,\psi] + [\bar{\psi},\bar{\lambda}]\phi + \frac{1}{2}[\phi^\dagger,\phi]^2 \right) + \frac{\Theta}{32\pi^2} \operatorname{Tr} F \wedge F$$

To find the low energy physics, we need to figure out the vacua (ground state) of the theory. At the semi-classical level, a solution of vanishing energy will require

$$F_{\mu\nu} = 0, \quad D_\mu\phi = 0, \quad \operatorname{Tr}[\phi^\dagger,\phi]^2 = 0.$$

The first equation says that the gauge field A_μ is pure gauge and can be chosen to be zero under gauge transformation. Then the second equation says that ϕ is a constant. The third equation says that

$$[\phi^\dagger, \phi] = 0.$$

Then up to a $SU(2)$ gauge transformation

$$\phi = \frac{1}{2}\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

where a is a complex number labelling the vacua. Using the identification

$$g/G = h/W,$$

we see that a nonvanishing a breaks the $SU(2)$ gauge symmetry into $U(1)$ with Weyl group $\mathbb{Z}_2 : a \to -a$. The gauge invariant quantity that parametrizes the space of vacus is

$$u = \frac{1}{2}a^2 = \operatorname{Tr}\phi^2.$$

Therefore we find the low energy effective theory by a family of abelian $N = 2$ super Yang-Mills theory parametrized by the vacua moduli u.

By $N = 2$ SUSY, the low energy effective lagrangian with at most two derivatives is completely determined by a prepotential $\mathcal{F}$

$$\frac{1}{4\pi}\operatorname{Im}\int d^4x\left(\int d^2\theta d^2\bar{\theta}\Phi^\dagger e^V\frac{\partial\mathcal{F}(\Phi)}{\partial\Phi} + \frac{1}{2}\int d^2\theta\frac{\partial\mathcal{F}(\Phi)}{\partial\Phi^2}\Sigma^\alpha\Sigma_\alpha\right).$$

Seiberg and Witten figured out an exact solution of $\mathcal{F}$ via symmetry considerations that we now describe.

First of all, renormalization of $N = 2$ SUSY implies that $\mathcal{F}$ would take the general form

$$\mathcal{F}(\Phi) = \mathcal{F}_{one-loop} + \mathcal{F}_{inst} = \frac{i}{2\pi}\Phi^2\ln\frac{\Phi^2}{\Lambda^2} + \sum_{k=1}^{\infty}\mathcal{F}_k\left(\frac{\Lambda}{\Phi}\right)^{4k}\Phi^2.$$

Here the first term is for the exact one-loop perturbative renormalization, and the second term comes from instanton corrections. Λ is a parameter for the renormalization scale.

Geometrically, let $\mathcal{M}$ be the complex manifold of vacua moduli parametrized by u. The D-term has the interpretation of Kahler metric on $\mathcal{M}$, whose Kahler potential is

$$K = \operatorname{Im}(\bar{a}\mathcal{F}'(a)).$$

The Kahler metric is

$$ds^2 = \operatorname{Im}\tau(a)da d\bar{a}, \quad \tau(a) = \mathcal{F}''(a).$$

The positivity of the metric requires $\operatorname{Im}\tau(a) > 0$.

There is an analogue of the electro-magnetic duality transformation for $N = 2$ super Yang-Mills. It is again by a version of Fourier transformation

$$\phi \to \phi_D, \mathcal{F} \to \mathcal{F}_D$$

such that

$$\boxed{\phi_D = \mathcal{F}'(\phi), \quad \mathcal{F}_D'(\phi_D) = -\phi}.$$

It follows that

$$\tau_D(a_D) = \frac{d\mathcal{F}_D'(a_D)}{da_D} = -\frac{da}{da_D} = -\frac{1}{\mathcal{F}''(a)} = -\frac{1}{\tau(a)}.$$

In other words, τ transforms

$$\tau \to -\frac{1}{\tau}$$

as what we expect for electro-magnetic duality.

To give a geometric interpretation, let us introduce a complex manifold $X = \mathbb{C}^2$ parametrized by (a, a_D) with a holomorphic 2-form

$$\Omega = da \wedge da_D$$

and a sympletic 2-form

$$\omega = \frac{1}{2}(da \wedge d\bar{a}_D - da_D \wedge d\bar{a}).$$

The function $a(u), a_D(u)$ can be described by a map

$$f : \mathcal{M} \to X, \quad f^*\Omega = 0.$$

This says that $\mathcal{M}$ is mapped to a holomorphic lagragian submanifold of X, whose generating function gives rise to the prepotential $\mathcal{F}(a)$. Then the Kahler metric on $\mathcal{M}$ is simply

$$ds^2 = f^*\omega.$$

Note that this requires a special positivity condition: $f^*\omega$ is a Kähler form even though ω is far from being Kähler. There is a $SL(2,\mathbb{Z})$-action on X which preserves Ω, ω. It changes the role of a, a_D as well as the lagrangian submanifold of X, hence the prepotential.

It turns out that the pair (a, a_D) has a physical meaning: for a dyon with eletric charge n_e and magnetic charge n_m, its central charge from N=2 SUSY is given by

$$Z = an_e + a_D n_m.$$

Therefore this gives also the mass formula for the corresponding BPS particle. Now we analyze the possible monodromy property when we move around the moduli.

(1) Singularity at ∞: when $u \to \infty$ ($a \to \infty$), $\mathcal{F}_{one-loop}$ dominates the large a behaviour. Then

$$a_D = \frac{\partial \mathcal{F}}{\partial a} \sim \frac{2i}{\pi} a \ln \frac{a}{\Lambda} + \frac{ia}{\pi}.$$

Under the monodromy

$$u \to e^{2\pi i} u,$$

we find

$$a \to -a, \quad a_D \to -a_D + 2a.$$

Therefore there is an nontrivial monodromy at ∞ in the u-plane

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$$

such that

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to M_\infty \begin{pmatrix} a_D \\ a \end{pmatrix}$$

(2) Singularity at small u: this is the region for strong coupling. Singularities will appear when certain BPS particle becomes massless at certain point of the moduli. It turns out that there are two singularities

$$u = \pm u_0.$$

We may assume that $u_0 = 1$. At u_0, a magnetic monopole becomes massless, implying that

$$a_D(u_0) = 0.$$

At this point the coupling constant will become divergent, and it is better to go to the dual magnetic picture to describe the weak coupling theory. The monopole in the dual theory will be described by $N = 2$ hypermultiplet. Another analysis of one-loop renormalization implies that near $a_D = 0$,

$$\tau_D \approx -\frac{i}{\pi} \ln a_D.$$

On the other hand, since a_D is a good local coordinate

$$a_D \approx c_0(u - u_0).$$

Then

$$a = -\mathcal{F}_D'(a_D) = -\int da_D \tau_D = \frac{i}{\pi} a_D \ln a_D + a_0 \approx a_0 + \frac{i}{\pi} c_0(u - u_0) \ln(u - u_0).$$

Then the monodromy around $(u-u_0) \to e^{2\pi i}(u-u_0)$ reads

$$a_D \to a_D, \quad a \to a - 2a_D.$$

The monodromy matrix is

$$M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

The third monodromy M_{-1} around the other singularity $-u_0 = -1$ has to satisfy the relation

$$M_1 M_{-1} = M_\infty,$$

which is solved by

$$M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}.$$

It turns out that the above three monodromies for (a, a_D) are enough to solve the model exactly. We can think about (a, a_D) as defining a holomorphic section of a flat $SL(2,\mathbb{Z})$ bundle V over

$$V \to \mathcal{M}^* := \mathcal{M} - \{\pm 1\}$$

whose monodromy around $\pm 1, \infty$ are described above. Consider the following family of elliptic curves

$$E_u : y^2 = (x-1)(x+1)(x-u).$$

It becomes singular precisely when $u = \pm 1, \infty$. We find an elliptic fibration

$$\pi : \mathcal{E} \to \mathcal{M}^*.$$

Now the bundle V is given by

$$V = R^1 \pi_* \mathcal{E},$$

whose fiber over u is

$$V_u = H^1(E_u, \mathbb{C}).$$

We consider the following Seiberg-Witten differential

$$\lambda = \frac{\sqrt{2}}{2\pi} \frac{y dx}{x^2 - 1}.$$

We check that λ has no pole of order 1. It follows that

$$\oint \lambda : H_1(E_u, \mathbb{Z}) \to \mathbb{C}, \quad \gamma \to \oint_\gamma \lambda$$

is well-define. Now locally if we choose A-cycle and B-cycle on E_u such that

$$A \cdot B = 1.$$

Then the coupling constants are given by

$$a_D = \oint_A \lambda, \quad a = \oint_B \lambda.$$

To get precisely our model, A is the cycle loop around the branch points at 1 and u, and B is the cycle loop around the branch points at ± 1. Observe that

$$\tau = \frac{da_D/du}{da/du} = \frac{\oint_A \frac{dx}{y}}{\oint_B \frac{dx}{y}}$$

where $\frac{dx}{y}$ is a holomorphic 1-form on E_u. It follows that τ is an element of the upper half plane which parametrizes the complex structure of E_u. In particular,

$$\operatorname{Im}\tau > 0$$

is achieved for a well-defined Kahler metric on the moduli space.

9. N=4 Super Yang-Mills

9.1. **Reduction from N=1 d=10.** The easiest way to obtain $N = 4$ super Yang-Mills theory in $d = 4$ is to start with $N = 1$ super Yang-Mills in $d = 10$ and do dimensional reduction to $d = 4$.

The lagrangian density of $N = 1$ $d = 10$ super Yang-Mills is

$$\mathcal{L}[A, \psi] = \frac{1}{4}\langle F_A, F_A\rangle + \frac{1}{2}\langle \psi, \not{D}_A \psi\rangle .$$

Here $A \in \Omega^1(V, \mathfrak{g})$ is even and represents a connection 1-form, while $\psi \in \Omega^0(V, S_+ \otimes \mathfrak{g})$ is odd and represents a chiral fermion. S_+ is a 16-dim chiral spinor. If we identify $\mathrm{Spin}(9, 1) \cong SL(2, \mathbb{O})$, then $S_+ = \mathbb{O}^2$ is given by two copies of octonions.

We consider its dimension reduction to $d = 4$. If we write

$$\mathbb{R}^{9,1} = \mathbb{R}^{3,1} \times \mathbb{R}^6,$$

then this is amount to declare that our fields A, λ only vary along $\mathbb{R}^{3,1}$ and are constant in the extra $\mathbb{R}^6$.

The connection 1-form in $d = 10$

$$\begin{cases} A_\mu \to A_\mu & 0 \le \mu \le 3 \\ A_{3+i} \to \phi_i & 1 \le i \le 6 \end{cases}$$

decomposes into a connection 1-form A_μ in $d = 4$ and six scalars ϕ_i in the adjoint representation.

The spin group is reduced to

$$\mathrm{Spin}(9, 1) \to \mathrm{Spin}(3, 1) \times \mathrm{Spin}(6).$$

$\mathrm{Spin}(3, 1)$ plays the role of spin group in $d = 4$ while $\mathrm{Spin}(6) = SU(4)_R$ plays the role of R-symmetry group. Under this reduction, the chiral spinor S_+ is decomposed into

$$S_+ \otimes_{\mathbb{R}} \mathbb{C} = (S \otimes \bar{\underline{\mathbf{4}}}) \oplus (\bar{S} \otimes \underline{\mathbf{4}})$$

Here S is the Weyl spinor in $d = 4$, $\bar{S}$ its complex conjugate, $\underline{\mathbf{4}}$ is the fundamental reppresentation of $SU(4)$, and $\bar{\underline{\mathbf{4}}}$ its complex conjugate. Therefore the chiral fermion is reduced to four Weyl fermions

$$\psi \to \lambda^a_\alpha, \quad \alpha = 1, 2 \quad a = 1, 2, 3, 4.$$

Under dimensional reduction, the Yang-Mills term becomes

$$\int d^4x \operatorname{Tr}\left(\frac{1}{4}\langle F, F\rangle + \frac{1}{2}\sum_{i=1}^{6} D_\mu\phi_i D^\mu\phi_i + \frac{1}{4}\sum_{i,j=1}^{6}[\phi_i, \phi_j]^2\right).$$

The fermionic part becomes

$$\int d^4x \frac{1}{2}\langle \bar{\lambda}^a, \not{D}_A \lambda^a\rangle + \sum_{a=1}^{4}\langle \lambda^a, [\phi, \lambda^a]\rangle + \sum_{a=1}^{4}\langle \bar{\lambda}^a, [\phi, \bar{\lambda}^a]\rangle .$$

Here $[\phi, \lambda_a]$ involves a bracket in $\mathfrak{g}$ as well as a Clifford multiplication in the $SU(4)_R$ factor.

The SUSY relations are now reduced to

$$\left[Q^a_\alpha, \bar{Q}^b_{\dot{\beta}}\right] = -2\delta^{ab}\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu, \quad [Q,Q] = [\bar{Q},\bar{Q}] = 0.$$

The last condition can be modified to have central charges due to certain boundary conditions.

From the point of view of $N=1$ superspace, the theory contains one $N=1$ vector multiplet V and three $N=1$ chiral multiplets Φ^A

$$V = (A_\mu, \lambda^4), \quad \Phi^A = (\varphi^A, \lambda^A), \quad A = 1,2,3.$$

$\{\phi_i\}_{i=1}^6$ is related to $\{\mathrm{Re}\,\varphi^A, \mathrm{Im}\,\varphi^A\}_{A=1}^3$.

9.2. **Geometric Langlands twist.** The twist in the context of supersymmetric gauge theories was introduced by Witten. The basic idea is to twist Poincare symmetry by global symmetry to formulate the theory on a curved manifold while preserving certain global supercharge Q. Then we can obtain a cohomological field theory which makes Q-operation homologous. The twisting process will in general change the spins of the original fields. The procedure is usually taken in the following steps

(1) Consider a theory on V with symmetry $\mathrm{Spin}(V) \times G_R$, where G_R is the R-symmetry group. Choose a homomorphism

$$\rho : \mathrm{Spin}(V) \to G_R.$$

(2) Find an odd operator Q (this is one of the reason we need SUSY) such that $[Q,Q]=0$ and invariant under twisted action of $\mathrm{Spin}(V)$ by $id \times \rho$.

In this case, we can use $id \times \rho$ to declare as part of the "new" Poincare symmetry (this amounts to shift the spins of the fields) and consider observables which are Q-closed. Moreover, Q-exact observables will drop out naturally in this sector. Since Q is a scalar in the new Poincare symmetry, it is often possible to formulate the theory on a nontrivial manifold V while Q survives as a globally defined fermionic symmetry.

(3) If furthermore we find the energy momentum tensor $T_{\mu\nu}$ to be Q-exact, then the twisted theory is independent of the metric tensor. Precisely, varying metric is Q-homologous. This often leads to theory which can be computed exactly at the semiclassical limit.

A typical example is that twisting N=2 super Yang-Mills theory produces Donaldson's theory. Another example is that twisting $N=(2,2)$ σ-model in two dimensions produces topological A-model and B-model that play essential roles in mirror symmetry.

For geometric applications, we consider $N=4$ in $d=4$ Euclidean space with global symmetry

$$\mathrm{Spin}(4) \times \mathrm{Spin}(6) \cong SU(2)_l \times SU(2)_r \times SU(4)_R.$$

We will indicate a complex representation of $SU(n)$ and its conjugate by its dimension $\underline{\mathbf{d}}$ and $\underline{\bar{\mathbf{d}}}$. The supercharges $Q^a_\alpha, \bar{Q}_{\dot{\alpha},a}$ lie in

$$(\underline{\mathbf{2}}, \underline{\mathbf{1}}, \underline{\bar{\mathbf{4}}}) \oplus (\underline{\mathbf{1}}, \underline{\mathbf{2}}, \underline{\mathbf{4}}).$$

Specifying a topological twist amounts to understand how the $\underline{\mathbf{4}}$ of $SU(4)_R$ is decomposed into a representation of $SU(2)_r \times SU(2)_l$. There are essentially three inequivalent twists

(1) $\underline{\mathbf{4}} \to (\underline{\mathbf{2}}, \underline{\mathbf{1}}) \oplus (\underline{\mathbf{2}}, \underline{\mathbf{1}})$. This is the Vafa-Witten twist with two scalar supercharges. It leads to explicit test of S-duality on four-manifolds.
(2) $\underline{\mathbf{4}} \to (\underline{\mathbf{2}}, \underline{\mathbf{1}}) \oplus (\underline{\mathbf{1}}, \underline{\mathbf{1}}) \oplus (\underline{\mathbf{1}}, \underline{\mathbf{1}})$. This is "half-twisted theory" with only one scalar supercharge. It is reminiscent of twisted $N=2$ supersymmetric gauge theory.

(3) $\underline{\mathbf{4}} \to (\underline{\mathbf{2}}, \underline{\mathbf{1}}) \oplus (\underline{\mathbf{1}}, \underline{\mathbf{2}})$. This is the GL twist with two scalar supercharges. It is shown by Kapustin-Witten that this twist is related to the geometric Langlands program.

We will be interested here in the GL twist, which amounts to

$$\rho : SU(2)_l \times SU(2)_r \to SU(4)_R, \quad \text{via} \quad \begin{pmatrix} SU(2)_l & 0 \\ 0 & SU(2)_r \end{pmatrix}.$$

This embedding is compatible with an additional $U(1)$ action

$$K : U(1) \to SU(4)_R, \quad \text{via} \begin{pmatrix} \begin{pmatrix} e^{i\theta} & \\ & e^{i\theta} \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} e^{-i\theta} & \\ & e^{-i\theta} \end{pmatrix} \end{pmatrix}.$$

We will write our decomposition as

$$\underline{\mathbf{4}} \to (\underline{\mathbf{2}}, \underline{\mathbf{1}})^1 \oplus (\underline{\mathbf{1}}, \underline{\mathbf{2}})^{-1}$$

where the superscript represents the $U(1)$-charge. Let us work out the field contents in the twist:

(1) The supercharges $\bar{Q} \in (\underline{\mathbf{2}}, \underline{\mathbf{1}}, \underline{\bar{\mathbf{4}}})$ transform in the new $\mathrm{Spin}(4) \times U(1)$ as

$$(\underline{\mathbf{2}}, \underline{\mathbf{1}})^0 \otimes \left((\underline{\mathbf{2}}, \underline{\mathbf{1}})^{-1} \oplus (\underline{\mathbf{1}}, \underline{\mathbf{2}})^1 \right) = (\underline{\mathbf{1}}, \underline{\mathbf{1}})^{-1} \oplus (\underline{\mathbf{3}}, \underline{\mathbf{1}})^{-1} \oplus (\underline{\mathbf{2}}, \underline{\mathbf{2}})^1.$$

We find a scalar supercharge denoted by

$$Q_l \in (\underline{\mathbf{1}}, \underline{\mathbf{1}})^{-1}.$$

The supercharges $Q \in (\underline{\mathbf{1}}, \underline{\mathbf{2}}, \underline{\mathbf{4}})$ transform in the new $\mathrm{Spin}(4) \times U(1)$ as

$$(\underline{\mathbf{1}}, \underline{\mathbf{2}})^0 \otimes \left((\underline{\mathbf{2}}, \underline{\mathbf{1}})^1 \oplus (\underline{\mathbf{1}}, \underline{\mathbf{2}})^{-1} \right) = (\underline{\mathbf{1}}, \underline{\mathbf{1}})^{-1} \oplus (\underline{\mathbf{1}}, \underline{\mathbf{3}})^{-1} \oplus (\underline{\mathbf{2}}, \underline{\mathbf{2}})^1.$$

We find a scalar supercharge denoted by

$$Q_r \in (\underline{\mathbf{1}}, \underline{\mathbf{1}})^{-1}.$$

(2) A_μ is central for $SU(4)_R$, hence remains as connection 1-form.

(3) ϕ_i lies in $\underline{\mathbf{6}}$ of $SU(4)_R$. Since

$$\underline{\mathbf{6}} = \wedge^2 \underline{\mathbf{4}} = \wedge^2 \left((\underline{\mathbf{2}}, \underline{\mathbf{1}})^1 \oplus (\underline{\mathbf{1}}, \underline{\mathbf{2}})^{-1} \right) = (\underline{\mathbf{2}}, \underline{\mathbf{2}})^0 \oplus (\underline{\mathbf{1}}, \underline{\mathbf{1}})^2 \oplus (\underline{\mathbf{1}}, \underline{\mathbf{1}})^{-2},$$

we see that four components of ϕ_i becomes a 1-form ϕ_μ while two other scalers combine into a complex scalar σ with $U(1)$ charge 2 and its complex conjugate $\bar{\sigma}$.

(4) λ behaves the same as supercharges. It decomposes into

$$\lambda \to (\psi_\mu, \tilde{\psi}_\mu, \chi_{\mu\nu}, \eta, \tilde{\eta})$$

where the fields

$$\psi, \tilde{\psi} \in (\underline{\mathbf{2}}, \underline{\mathbf{2}})^1$$

are 1-forms. The fields

$$\chi^+ \in (\underline{\mathbf{3}}, \underline{\mathbf{1}})^{-1}, \quad \chi^- \in (\underline{\mathbf{1}}, \underline{\mathbf{3}})^{-1}$$

are selfdual and anti-selfdual parts of a two-form χ. The fields

$$\eta \in (\underline{\mathbf{1}}, \underline{\mathbf{1}})^{-1}, \quad \tilde{\eta} \in (\underline{\mathbf{1}}, \underline{\mathbf{1}})^{-1}$$

are two 0-forms.

We have a $\mathbf{P}^1$-family of topological twisted theory parametrized by

$$Q = uQ_l + vQ_r, \quad [u,v] \in \mathbf{P}^1.$$

The SUSY variation can be read off by

$$\begin{aligned} Q\phi_\mu &= iv\psi_\mu - iu\tilde{\psi}_\nu \\ Q\sigma &= 0 \\ Q\bar{\sigma} &= iu\eta + iv\tilde{\eta}. \end{aligned}$$

$$\begin{aligned} Q\chi^+ &= u(F - \phi \wedge \phi)^+ + v(D\phi)^+ \\ Q\chi^- &= v(F - \phi \wedge \phi)^- - u(D\phi)^- \\ Q\eta &= vD^*\phi + u[\bar{\sigma}, \sigma] \\ Q\tilde{\eta} &= -uD^*\phi + v[\bar{\sigma}, \sigma]. \end{aligned}$$

$$\begin{aligned} Q\psi &= uD\sigma + v[\phi, \sigma] \\ Q\tilde{\psi} &= vD\sigma - u[\phi, \sigma]. \end{aligned}$$

After twisting, the theory can be defined on an arbitrary four manifold M. The topological action is

$$\begin{aligned} S =& \frac{2}{e^2} \int_M dvol \operatorname{Tr} \left(-\frac{1}{2}(D^*\phi)^2 + \frac{1}{2}[\bar{\sigma}, \sigma]^2 - D_\mu \bar{\sigma} D^\mu \sigma - [\phi_\mu, \sigma][\phi^\mu, \bar{\sigma}] + i\tilde{\eta} D_\mu \tilde{\psi}^\mu + i\eta D_\mu \psi^\mu \right. \\ & \left. -i\tilde{\eta}[\psi_\mu, \phi^\mu] + i\eta[\tilde{\phi}_\mu, \phi^\mu] - \frac{i}{2}[\sigma, \tilde{\eta}]\tilde{\eta} - \frac{i}{2}[\sigma, \eta]\eta + i[\bar{\sigma}, \psi_\mu]\psi^\mu + i[\bar{\sigma}, \tilde{\psi}_\mu]\tilde{\psi}^\mu \right) \\ & - \frac{1}{e^2} \int_M dvol \operatorname{Tr} \left(\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi_\nu D^\mu \phi^\nu + R_{\mu\nu} \phi^\mu \phi^\nu + \frac{1}{2}[\phi_\mu, \phi_\nu]^2 - (D^*\phi)^2 \right. \\ & \left. -2i\chi_{\mu\nu}(D\psi + i[\tilde{\psi}, \phi])^{\mu\nu} - 2i\chi^-_{\mu\nu}(D\tilde{\psi} - [\psi, \phi])^{\mu\nu} - i\chi^+_{\mu\nu}[\sigma, \chi^{+\mu\nu}] + i\chi^-_{\mu\nu}[\sigma, \chi^{-\mu\nu}] \right) \\ & + i\frac{\theta}{8\pi^2} \int_M \operatorname{Tr} F \wedge F. \end{aligned}$$

Here we again introduce the complex parameter

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi}{e^2}.$$

S itself is independent of t. However, in terms of the Q_t-twisting, it can be written in the form

$$S = Q_t V + \frac{i\Psi}{4\pi} \int_M \operatorname{Tr} F \wedge F$$

for some local action V. Here $\boxed{t = v/u}$ and

$$\boxed{Q_t = Q_l + tQ_r}, \quad \boxed{\Psi = \frac{\tau + \bar{\tau}}{2} + \frac{\tau - \bar{\tau}}{2} \left(\frac{t - t^{-1}}{t + t^{-1}} \right)}.$$

This expression shows that S is Q_t-closed, and the variation of metric is Q_t-exact. Moreover, the topological theory only depends on the coefficient Ψ, which is called the **canonical parameter**. The $SL(2, \mathbb{Z})$ duality transformation acts on Ψ as

$$\Psi \to \frac{a\Psi + b}{c\Psi + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

In the Q_t-twisted theory, the path integral will usually be localized to subspace where the Q_t-variation of fermions vanishes. This becomes the equations ($t = v/u$)

$$\begin{aligned}(F - \phi \wedge \phi + tD\phi)^+ &= 0\\ (F - \phi \wedge \phi - t^{-1}D\phi)^- &= 0\\ D^*\phi + t^{-1}[\bar{\sigma}, \sigma] &= 0\\ D^*\phi - t[\bar{\sigma}, \sigma] &= 0\\ D\sigma + t[\phi, \sigma] &= 0\\ D\sigma - t^{-1}[\phi, \sigma] &= 0.\end{aligned}$$

We will call this **topological equations** for the twist t. We just mention that for real t it is analogous to a family of two-dimensional A-model, while for $t = \pm i$ it is analogous to two-dimensional B-model.

9.3. **Higgs bundle and Hitchin moduli.** Let C be a compact Riemann surface, G be a compact Lie group and $P \to C$ be a principal G-bundle. We consider pairs (A, ϕ) where A is a unitary connection on P and

$$\phi \in \Omega^{1,0}(C, \mathfrak{g}_{\mathbb{C}}).$$

Here we have used $\mathfrak{g}$ to denote the adjoint bundle ad P when it is clear from the context, and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. We are interested in pairs satisfying **Hitchin's equation**

$$F_A + [\phi, \phi^*] = 0, \quad \bar{\partial}_A \phi = 0.$$

Here F_A is the curvature of A, and $\bar{\partial}_A$ is the (0,1)-part of the covariant derivative with respect to A. Hitchin's equation is the dimensional reduction of four dimensional self-dual Yang-Mills equation. We will denote $\mathcal{M}_H(C, G)$ the moduli space of solutions of Hitchin's equations modulo gauge transformations. This moduli space carries a hyperkähler structure that we now describe.

9.3.1. *Hyperkähler manifold.* Let us first recall the construction of **symplectic reduction**. Let (M, ω) be a symplectic manifold. Let G be a connected Lie group with Hamiltonian action on M. Infinitesimally, this is characterized by the moment map

$$\mu : M \to \mathfrak{g}^*$$

where $\mathfrak{g}$ is the Lie algebra of G. Given an element $v \in \mathfrak{g}$, the natural pairing $\langle \mu, v \rangle$ is the Hamiltonian function for the vector field on M generated by v. The classical Marsden-Weinstein Theorem says that

$$M /\!/ G := \mu^{-1}(0)/G$$

is again a symplectic manifold, whose symplectic form is induced by restricting ω to $\mu^{-1}(0)$.

A complex structure on M is described by a bundle map

$$J : TM \to TM, \quad J^2 = -1$$

satisfying certain integrability condition. We can use J to decompose

$$TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$$

where $T^{1,0}M$ and $T^{0,1}M$ are eigenvectors of J with eigenvalues i and $-$i. The integrability condition says that $T^{1,0}M$ is closed under commutators of vector fields.

Definition 9.1. M is a **Kähler manifold** if it is endowed with a triple (g, J, ω) where g is a metric, ω is a symplectic form, J is an integrable complex structure such that the following compatibility condition holds

$$\omega(-, -) = g(J(-), -).$$

Kähler structure can be incorporated with symplectic reduction as follows. Let (M, g, J, ω) be a Kähler manifold with a Hamiltonian isometric action by a connected Lie group G. Then the symplectic reduction $M /\!/ G$ is a Kähler manifold with the naturally induced Kähler structure.

Definition 9.2. A Riemannian manifold (M, g) is **hyperkähler** if it is equipped with three complex structures I, J, K, each of which defines a Kähler structure with g, and which satisfy the quaternion relations

$$I^2 = J^2 = K^2 = -1, \quad IJ = -JI = K, JK = -KJ = I, KI = -IK = J.$$

On hyperkähler manifold, we have three symplectic forms

$$\omega_I(-,-) = g(I(-),-) \quad \omega_J(-,-) = g(J(-),-), \quad \omega_K(-,-) = g(K(-),-).$$

Let M carry a G isometric action which is hamiltonian with respect to all the above three symplectic structures. We can organize three moment maps into a single one

$$\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3, \quad \mu = (\mu_I, \mu_J, \mu_K).$$

Then the quotient

$$\mu^{-1}(0)/G$$

is naturally a hyperkähler manifold. This construction is called **hyperkähler quotient**.

9.3.2. *Hitchin moduli as hyperkähler quotient.* Let us consider an infinite dimensional space

$$\mathcal{W} = \mathscr{A} \times \Omega$$

where $\mathscr{A}$ denotes the space of unitary connections on P, and $\Omega = \Omega^1(C, \mathfrak{g})$.

$$T\mathcal{W} = T\mathscr{A} \oplus T\Omega = \Omega^1(C, \mathfrak{g}) \oplus \Omega^1(C, \mathfrak{g}).$$

We define three complex structures on $T\mathcal{W}$ by

$$\begin{aligned} I(\delta A, \delta\phi) &= (\star\delta A, - \star \delta\phi) \\ J(\delta A, \delta\phi) &= (-\delta\phi, \delta A) \\ K(\delta A, \delta\phi) &= (- \star \delta\phi, - \star \delta A). \end{aligned}$$

Here $\delta A, \delta\phi$ represent infinitesimal variation on $\mathcal{W}$, and $\star$ is the Hodge star on 1-forms. They define hyperkähler structure with respect to the metric

$$ds^2 = \int_C \mathrm{Tr}\, \delta A \wedge \star\delta A + \delta\phi \wedge \star\delta\phi.$$

The three corresponding symplectic forms read

$$\begin{aligned} \omega_I &= \frac{1}{2} \int_C \mathrm{Tr}\,(-\delta A \wedge \delta A + \delta\phi \wedge \delta\phi) \\ \omega_J &= \int_C \mathrm{Tr}\,(-\delta A \wedge \star\delta\phi) \\ \omega_K &= \int_C \mathrm{Tr}\, \delta A \wedge \delta\phi. \end{aligned}$$

We consider the gauge transformation $\mathscr{G}$

$$(A, \phi) \to (gAg^{-1} + gdg^{-1}, g\phi g^{-1})$$

whose infinitesimal form is

$$\delta(A, \phi) = (D_A u, [\phi, u]), \quad u \in \Omega^0(C, \mathfrak{g}).$$

It can be checked that the gauge transformation is hamiltonian with respect to the three symplectic forms. The moment maps are given by

$$\begin{aligned}\mu_I &= F_A - \frac{1}{2}[\phi, \phi]\\ \mu_J &= D_A \star \phi\\ \mu_K &= -D_A \phi.\end{aligned}$$

The vanishing locus of the moment maps are

$$F_A - \frac{1}{2}[\phi, \phi] = 0, \quad D_A \phi = D_A^* \phi = 0.$$

We have found that Hitchin's equation arises naturally as a hyerkähler quotient of $\mathcal{W}$. It follows that the Hitchin moduli $\mathcal{M}_H(C, G)$ inherits a hyperkähler structure. We can describe this moduli in terms of holomorphic data.

(1) We choose the complex structure I and consider the holomorphic symplectic form $\Omega_I = \omega_j + \mathrm{i}\,\omega_k$. The holomorphic moment map is

$$\mu_J + \mathrm{i}\,\mu_K = \bar{\partial}_A \varphi.$$

Here $\bar{\partial}_A$ is the $(0,1)$-component of the connection, defining a holomorphic G-bundle. $\varphi \in \Omega^{1,0}(C, \mathfrak{g}_{\mathbb{C}})$ such that $\phi = \varphi + \varphi^{\dagger}$. The vanishing of the holomorphic moment map says that φ defines a holomorphic section $H^0(C, K_C \otimes \mathfrak{g}_{\mathbb{C}})$, which is called a **Higgs field**. The pair $(\bar{\partial}_A, \varphi)$ of a holomorphic bundle with a Higgs field is also called a **Higgs bundle**. The other equation

$$F_A - [\varphi, \varphi^{\dagger}] = 0$$

is the stability condition. From this perspective, the Hitchin moduli describes stable Higgs bundles.

(2) We choose the complex structure J and consider the holomorphic symplectic form $\Omega_I = \omega_K + \mathrm{i}\,\omega_I$. The holomorphic moment map is

$$\mu_K + \mathrm{i}\,\mu_I = \mathrm{i}\,\mathcal{F}$$

where $\mathcal{F}$ is the curvature of the complex connection $A + \mathrm{i}\,\phi$. The vanishing of the holomorphic moment map says that $A + \mathrm{i}\,\phi$ defines a flat connection, or a group homomorphism

$$\pi_1(C) \to G_{\mathbb{C}}.$$

The vanishing of μ_J is again about stability condition.

(3) For a general complex structure

$$I_\xi = \frac{1 - \bar{\xi}\xi}{1 + \bar{\xi}\xi} I + \frac{\mathrm{i}(\xi - \bar{\xi})}{1 + \bar{\xi}\xi} J + \frac{\xi + \bar{\xi}}{1 + \bar{\xi}\xi} K,$$

for $\xi \neq 0, \infty$, an analog holomorphic moment map gives the vanishing

$$\left[\bar{\partial}_A + \xi\varphi^{\dagger}, \partial_A - \xi^{-1}\varphi\right] = 0.$$

They are equivalent to the case when $I_\xi = J$.

9.3.3. *Hitchin fibration.* For simplicity, we assume $G = SU(N)$. The discussion is similarly for other compact Lie groups. We define the Hitchin base

$$\mathcal{B} = \bigoplus_{n=2}^{N} H^0(C, K_C^n).$$

There exists the **Hitchin fibration**

$$\pi : \mathcal{M}_H(C,G) \to \mathcal{B}$$

by sending a Higgs bundle (A,φ)

$$\pi(A,\varphi) \to \bigoplus_{n=2}^{N} \operatorname{Tr} \varphi^n.$$

The fibration π gives a complete integrable system for the holomorphic symplectic form $\omega_J + i\omega_K$ on $\mathcal{M}_H$.

9.4. **Low energy effective theory.** We formulate the twisted theory on the product of two Riemann surfaces

$$M = \Sigma \times C$$

where the size of Σ is much larger than the size of C.

We look at the low energy effective theory of the twisted theory around configurations that minimize the topological action. Apart from the topological θ-term, the bosonic part of the action is minimized by

$$\mathcal{F} = 0, \quad D^*\phi = 0, \quad D\sigma = [\phi,\sigma] = [\sigma,\bar{\sigma}] = 0.$$

Here $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is the curvature of the complex connection $\mathcal{A} = A + i\phi$. This is equivalent to topological equations for all t.

With certain appropriate assumption on the singularity, solutions of these equations is obtained by taking $\mathcal{A}$ to be pullback from the curve C with $\sigma = 0$. The equations on C is

$$F - \phi \wedge \phi = 0, \quad D\phi = D^*\phi = 0$$

which is precisely Hitchin's equation. Let us denote by $\mathcal{M}_H$ for the moduli space of solutions of Hitchin's equations on C up to gauge transformations. Therefore the zero energy configurations are given by constant maps $\Sigma \to \mathcal{M}_H$. Assume that the size of Σ is much larger than C, the almost zero energy energy effective theory on $M = \Sigma \times C$ is given by a supersymmetric σ-model

$$\Sigma \to \mathcal{M}_H.$$

By choosing $\Sigma = \mathbb{R}^{1,1}$, we find that the σ-model has induced $N = (4,4)$ supersymmetry. This implies that the target $\mathcal{M}_H$ should be a hyper-Kahler manifold which is indeed the case. The topological twist Q_t in four dimension becomes a topologically twisted theory in two dimension:

(1) At $t = \pm\mathrm{i}$. We get B-model for the complex structure J on $\mathcal{M}_H$.
(2) At $t = \bar{t}$ real. We get A-model for the complex structure I_t on $\mathcal{M}_H$.

Kapustin-Witten shows that S-duality maps G to its Langlands dual group ${}^L G$. In particular, it identifies

$$\boxed{\mathcal{M}_H(C,G) \text{ B-model in complex structure J}} \overset{\text{S-duality}}{\longleftrightarrow} \boxed{\mathcal{M}_H(C,{}^L G) \text{ A-model in complex structure K}}$$

as a mirror pair. The second one is also equivalent to A-model in complex structure J. S-duality acts on the Hitchin fibration by T-duality on the fibers.

More generally, S-duality changes

$$I_B \longleftrightarrow I_B$$

$$I_A \longleftrightarrow I_A$$
$$J_B \longleftrightarrow K_A$$
$$J_A \longleftrightarrow K_B$$

As we will see, boundary conditions of 2d topological theory form a category, called the category of branes. Branes in the topological A-model are described by flat bundles on lagrangian submanifolds, and branes in the topological B-modelare described by holomorphic bundles on holomorphic submanifolds. We can talk about branes with respect to the three complex structures I, J, K. For example, a point p is holomorphic with respect to any complex structure, defining a brane of type (B, B, B). Under S-duality, it will be transformed into a brane of type (B, A, A).

9.5. **Boundary conditions and line operators.** We have seen that GL-twist of N=4 SYM gives rise to a P^1-family of topological field theory in four dimension. Formulated on $\Sigma \times C$, its low energy effective theory becomes topological σ-model $\Sigma \to \mathcal{M}_H(C, G)$ in two dimension. Let us write $Y_G(C)$ for the moduli stack of flat G-bundles on C, and $Z_G(C)$ for the moduli stack of holomorphic G-bundles. The very rough idea of geometric Langlands correspondence asserts that the category of coherent sheaves in $Y_{{}^LG}(C)$ is equivalent to the category of D-modules on $Z_G(C)$. The topological twist of N=4 SYM will be related to Geometric Langlands when boundary conditions and line operators are considered under S-transformation.

9.5.1. *TQFT.* Let us first discuss some generalities about topological quantum field theory (TQFT). In terms of the Atiyah-Segal formalism, path integrals of TQFT in n-dimension gives rise to a functor

$$\mathbf{F} : (\mathrm{Bord}_n, \sqcup) \to (\mathrm{Vect}, \otimes).$$

Here Bord_n is the category of n-dim bordisms, whose objects are oriented $(n-1)$-manifolds without boundary and morphisms are oriented bordisms. $\mathbf{F}$ is required to be monoidal sending the disjoint union

$$\mathbf{F}(N_1 \sqcup N_2) = \mathbf{F}(N_1) \otimes \mathbf{F}(N_2)$$

and sending the empty manifold

$$\mathbf{F}(\emptyset) = \mathbb{C}.$$

The functor $\mathbf{F}$ is related to the path integral as follows:

(1) Let $N = \partial M$. Let ϕ represent fields on M and ϕ_∂ represents fields on the boundary ∂M. Typically,

$$\mathbf{F}(N) = \{\phi_\partial | |\phi_\partial|_{L^2} < \infty\}$$

is given by certain L^2-Hilbert space of boundary fields. $\mathbf{F}(M)$ gives rise to a vector $|M\rangle$, which is determined by its inner product with an arbitrary element $\phi_\partial \in \mathbf{F}(N)$ by the path integral over ϕ with fixed boundary condition ϕ_∂

$$\langle \phi_\partial | M \rangle = \int_{\phi|_{\partial M} = \phi_\partial} D\phi e^{iS[\phi]/\hbar}.$$

(2) Let $\overline{N}$ be the orientation reversion of N, then

$$\mathbf{F}(\overline{N}) = \mathbf{F}(N)^\vee.$$

(3) When $\partial M = N_1 \sqcup \overline{N_2}$, similar path integrals with boundary conditions give

$$\mathbf{F}(M) : \mathbf{F}(N_1) \to \mathbf{F}(N_2).$$

(4) When $\partial M = \emptyset$, then
$$\mathbf{F}(M) \in \mathbb{C} = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$$
is a complex number, called the **partition function** on M.

This definition is sufficient for $n = 1$, i.e., topological quantum mechanics. When $n > 1$, there exists finer structures, and the above definition of TQFT will be generalized to the notion of **Extended TQFT**. To motivate the definition, consider a bordism M between two manifolds with boundaries $(N_1, \partial N_1), (N_2, \partial N_2)$. An extended TQFT $\mathbf{F}$ assigns a $\mathbb{C}$-linear category $\mathbf{F}(\partial N_i)$ to the $(n-2)$-manifolds ∂N_i and assigns
$$\mathbf{F}(N_i) \in \mathrm{Obj}(\mathbf{F}(N_i))$$
an object of this category. Then $\mathbf{F}$ assigns a morphism to the bordism M
$$\mathbf{F}(M) \in \mathrm{Mor}(\mathbf{F}(N_1), \mathbf{F}(N_2)).$$

Similar discussions apply to higher codimensional objects in n-dim TQFT. $\mathbf{F}$ assigns a $\mathbb{C}$-linear category to an $(n-2)$-manifold, $\mathbb{C}$-linear 2-category to an $(n-3)$-manifold, and in general a k-category to an $(n-k-1)$-manifold. It is a subtle question on how to formulate a correction definition of n-categories that apply appropriately to a quantum field theory. Let us live with the above naive picture so far. Another way of saying this is that

$$\boxed{\text{boundary conditions of a } n\text{-dim TQFT form a } (n-1)\text{-category.}}$$

Example 9.3. Boundary conditions in 2d extended TQFT form a $\mathbb{C}$-linear category. They are known as the category of **D-branes**. In topological A-model, D-branes are described by langrangian submanifolds.In topological B-model, D-branes are described by coherent sheaves.

Besides boundary conditions, we can also talk about **defects**. Consider a TQFT on M and let $L \subset M$ be a submanifold of dim k. Roughly speaking, a defect on L is a local modification of states supported on L. 0-dim defects are local operators, 1-dim defects are line operators, 2-dim defects are surface operators and etc. The point is that

$$\boxed{\text{k-dim defects of a TQFT form a k-category.}}$$

The idea is that we can view a k-dim defect on L as a boundary conditions for an effective $(k+1)$-TQFT with time direction pointing to the normal direction of L inside M. For example, a local operator creates a state of the Hilbert space. As an illustration, we consider a path integral
$$\int D\phi e^{iS[\phi]} \mathcal{O}_L(\phi)$$
where $\mathcal{O}_L$ is an operator supported at L. Let N be a tuber neighborhood of L. The path integral over $M - N$ creates a vector in the Hilbert space $\mathbf{F}(\partial N)$ associated to ∂N. On the other hand, ∂N is a S^{n-k-1}-bundle over L, and $N - L$ can be viewed as giving an effective theory on $L \times I$, whose boundary conditions are given by the k-category $\mathbf{F}(S^{n-k-1})$. The local operator $\mathcal{O}_L$ creates a vector in the effective Hilbert space associated to L, or a vector in $\mathbf{F}(\partial N)$. Then the above path integral is simply the inner product of these two vectors.

The above idea of effective theory is related to the notion of **Kaluza-Klein reduction**. In general, suppose we have a TQFT on the product $M^k \times N^{n-k}$, we can get an effective TQFT on M^k. The boundary conditions of the effective theory is precisely given by the $(k-1)$-category $\mathbf{F}(N^{n-k})$. However, the fact that the theory comes from higher dimension is reflected by the fact that the boundary conditions carry additional structures from the internal N. As an illustration, consider the case when $N = [0,1]$ is an interval. Then objects of the boundary condition $\mathbf{F}(I)$ on M^k are 1-morphisms, which carries a natural monoidal structure.

Turning this around, if the boundary conditions of the theory on M carries a monoidal structure, it is very likely that it comes from a higher dimensional theory.

9.5.2. *Wilson and 't Hooft operators.* Geometric Langlands is a statement above two categories living on two dimensional space C. With a physics interpretation, it is natural from a duality on four dimensional TQFT.

This is precisely what Kapustin-Witten formulated: Geometric Langlands is related to the S-duality of the GL-twist of 4d N=4 SYM on $\Sigma \times C$. In terms of effective 2d topological σ-model, coherent sheaves come from the D-brane category of topological B-model, while flat bundles/D-modules come from the D-brane category of topological A-model.

In our discussion of Electro-Magnetic duality for Maxwell's equations

$$
\begin{aligned}
dF &= J_m \\
d \star F &= J_e
\end{aligned}
$$

The currents J_m, J_e are supported on one-dimensional objects, which can be viewed as line defects. J_e is coupled to the gauge connection via the **Wilson line operator**

$$\int_{J_e} A.$$

On the other hand, J_m creates certain singularity for A, which is called the **'t Hooft line operator**. Under S-duality transformation, Wilson line operators and 't Hooft line operators transform to each other.

In the non-abelian case, we pick a representation R of the gauge group G to get the Wilson loop operator as the holonomy around a loop S

$$W(R, S) = \mathrm{Tr}_R \, P \exp\left(\oint_S A \right).$$

Similarly, for a loop S, we can associate the 't Hooft operator $T(\rho, S)$ by specifying the singularity of the connection along a $U(1)$-component of the gauge group, i.e. , a homomorphism $\rho : U(1) \to G$. Such ρ picks essentially a representation ${}^L R$ of the dual group ${}^L G$. So we also denote the 't Hooft operator by $T({}^L R, S)$. Again, S-duality transforms G to ${}^L G$ and turn Wilson and 't Hooft operators into each other.

At $i = \pm 1$ ($\Psi = \infty$), we get an effective topological model for the complex structure J on $\mathcal{M}_H$. The Wilson loop operator can be completed into a topological operator by replacing $\mathcal{A} \to \mathcal{A} \pm \mathrm{i}\,\phi$. Similarly, at $t = 1, \tau =$ imaginary ($\Psi = 0$), we can define topological t' Hooft operator by picking up a compatible singular behaviour of ϕ along S.

In the effective 2d topological model, we put topological line operator $\mathscr{X}_p$ of the form $L \times p \subset \Sigma \times C$. Here L is a line or loop on Σ, and p is a point on C. When L is approaching the boundary $\partial\Sigma$, $\mathscr{X}_p$ acts on boundary conditions in the 2d topological theory. Because of the extra freedom to move around C, operators $\mathscr{X}_p$ will all commute with each other. In summary, we find

$$\boxed{\mathscr{X}_p\text{'s form commuting functors on the category of branes in 2d topological theory}}$$

It makes sense to talk about a joint eigenbrane for relevant line operators. A joint eigenbrane of the Wilson line operators will be called an **electric eigenbrane**, and a joint eigenbrane of the 't Hooft line operator will be called an **magnetic eigvenbrane**.

Kapustin-Witten argued that S-duality transforms the (B, B, B)-brane of a single point of $\mathcal{M}_H(C, G)$ to the (B, A, A)-brane of a lagrangian fiber of the Hitchin fibration of $\mathcal{M}_H(C, {}^L G)$. It enjoys features of SYZ mirror transformation. The above (B, B, B)-brane will be an electric eigenbrane, and the dual (B, A, A)-brane will be a magnetic eigenbrane. The 't Hooft line operators at $t = 1, \tau =$ imaginary will correspond to the Hecke operators of the geometric Langlands program.