Legendre Polynomials - Lecture 8

1 Introduction

In spherical coordinates the separation of variables for the function of the polar angle results in Legendre's equation when the solution is independent of the azimuthal angle.

$$(1 - x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + l(l+1)P = 0$$

This equation has $x = cos(\theta)$ with solutions $P_l(x)$. As previously demonstrated, a series solution can be obtained using the form;

$$P(x) = \sum a_n x^{n+s}$$

Taking the derivatives, substituting into the ode, and collecting the coefficient of the same power in x, one obtains the recursion relation for the coefficients.

$$a_{n+2} = \frac{(n+s)(n+s+1) - l(l+1)}{(n+s+2)(n+s+1)}\, a_n$$

The indicial equation must also be satisfied by selection of the initial coefficient and/or starting power of x. Thus

$$a_0\, s(s-1) = 0$$

$$a_1\, (1+s)s = 0$$

We then have the choice of $a_1 = 0$ and $s = 0$ or $a_0 = 0$ and $s = 0$. Other choices are incorporated in these two. Thus if we choose $a_1 = 0$ and $s = 0$, then;

$$a_{n+2} = \frac{n(n+1) - l(l+1)}{(n+2)(n+1)}\, a_n$$

Only even powers of x are included in the series. If a_0 and $s = 0$ then only odd powers are incorporated in the series and the recursion relation remains the same. For the series to be convergent for $|x| = 1$ it must terminate, again as previously pointed out. The series then takes the form;

$$P_l(x) = \sum_{n=0}^{l/2} (-1)^n \frac{(2l - 2n)!}{2^l n!(l-n)!(l-2n)!}\, x^{l-2k}$$

This series results in a set of polynomials which may be obtained from the generating function;

$$g(x,t) \;=\; (1-2xt+t^2)^{-1/2} \;=\; \sum_{l=0}^{\infty} P_l(x)\, t^l$$

Some initial polynomials are;

$$P_0 \;=\; 1$$

$$P_1 \;=\; x$$

$$P_2 \;=\; (1/2)[3x^2-1]$$

2 Orthogonality

Consider the orthogonality integral for the Legendre functions. This follows from the general Sturm-Liouville problem. Put Legendre's equation in self adjoint form;

$$\frac{d}{dx}[(1-x^2)\frac{dP_l(x)}{dx}] + l(l+1)P_l(x) \;=\; 0$$

Then look at the equation for $P_n(x)$ and subtract the equations for P_l and P_n after multiplication of the first by P_n and the later by P_l. Integrate the result between ± 1. This results in

$$[(1-x^2)P_n P_l' - (1-x^2)P_l P_n']_{-1}^{1} + [l(l+1)-n(n+1)]\int_{-1}^{1} dx P_l P_n \;=\; 0$$

The first term vanishes and thus the polynomials are orthogonal but not normalized to 1. The normalization integral takes the form;

$$N^2 \;=\; \int_{-1}^{1} dx\, P_n P_m$$

Use the Rodgiques formula to obtain;

$$P_n(x) \;=\; \frac{1}{2^n n!}(\frac{d}{dx})^n (x^2-1)^n$$

$$N^2 \;=\; \frac{(-1)^{m+n}}{2^{m+n} m! n!}\int_{-1}^{1} dx\, [(\frac{d^m}{dx^m})(1-x^2)^m][(\frac{d^m}{dx^n})(1-x^2)^n]$$

Integrate by parts m times noting that the surface terms vanish at ± 1.

$$N^2 \;=\; \frac{(-1)^{n-m}}{2^{n+m} m! n!}\int_{-1}^{1} dx\, (\frac{d^{n-m}}{dx^{n-m}})(1-x^2)^n$$

This vanishes for $n > m$ (reverse n, m if $n < m$). If $n = m$ then;

$$N^2 = \frac{1}{2^{2n}(n!)^2} \int_{-1}^{1} dx\,(1 - x^2)^n = \frac{2}{2n+1}$$

The orthogonality integral is for the associated Legendre polynomials is expressed as;

$$\int_{-1}^{1} dx\, P_r^m(j) P_k^m(x) = \frac{2}{2j+1} \frac{(j+m)!}{(j-m)!}$$

The normailzation for the Legendre polynomial P_r^m is found for $m = 0$.

3 Recurrence Relations

The recurrence relations between the Legendre polynomials can be obtained from the generating function. The most important recurrence relation is;

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

To generate higher order polynomials, one begins with $P_0(x) = 1$ and $P_1(x) = x$. The generating function also gives the recursion relation for the derivative.

$$P'_{n+1}(x) = (n+1)P_n(x) + xP'_n(x)$$

4 Separation of variables in spherical coordinates

We have previously looked at separation of variables in spherical coordinates, particularly with respect to the radial component. Now consider the component involving the polar angle with $x = cos(\theta)$ when the aximuthal angle id included in the separation.

$$(1 - x^2)\frac{d^2\Theta}{dx^2} - 2x\,\frac{d\Theta}{dx} + [n(n+1) - \frac{m^2}{(1-x^2)}]\Theta = 0$$

The above equation is the associated Legendre equation. For the solution to remain finite at $x = \pm 1$ with n integral. The equation may be obtained from the ordinary Legendre equation by differentiation.

$$\frac{d^m}{dx^m}[(1 - x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x)]$$

Thus we find;

$$P_n^m(x) \;=\; (1-x^2)^{m/2}\frac{d^m P_n(x)}{dx^m}$$

Then;

$$P_n^{-m}(x) \;=\; (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x)$$

The generating function is difficult to use;

$$\frac{(2m)!(1-x^2)^{m/2}}{2^m m!(1-2tx+t^2)^{m+1/2}} \;=\; \sum_{s=0}^{\infty} P_{s+m}^m(x)\, t^s$$

The recursion relation is;

$$(2n+1)xP_n^m(x) \;=\; (n+m)P_{n-1}^m(x) + (n-m+1)P_{n+1}^m(x)$$

Finally;

$$P_n^m(-x) \;=\; (-1)^{n+m} P_n^m(x)$$

$$P_n^m(\pm 1) \;=\; 0$$

5 Spherical harmonics

In spherical coordinates Laplace's equation has the form;

$$\nabla^2 V \;=\; (1/r^2)\frac{\partial}{\partial r}(r^2 \frac{\partial V}{\partial r}) \;+\; \frac{1}{r^2\, sin(\theta)}\frac{\partial}{\partial \theta}(sin(\theta)\, \frac{\partial V}{\partial \theta}) \;+\; \frac{1}{r^2\, sin^2(\theta)}\frac{\partial^2 V}{\partial \phi^2}$$

Now identify an operator on the angles, L, such that;

$$(1/r^2)L^2 V \;=\; \frac{1}{r^2\, sin(\theta)}\frac{\partial}{\partial \theta}(sin(\theta)\, \frac{\partial V}{\partial \theta}) \;+\; \frac{1}{r^2\, sin^2(\theta)}\frac{\partial^2 V}{\partial \phi^2}$$

Then when Laplace's equation is solved by separation of variables;

$$L^2 V \;=\; l(l+1)V$$

Thus identify $V = R_l(r)\, Y_l^m(\theta, \phi)$ where $Y_l^m(\theta, \phi)$ is an angular eigenfunction with eigenvalues, l, and m. We already know that this function has the form;

$$Y_l^m(\theta, \phi) \;\rightarrow\; P_l^m(\theta)\, e^{im\phi}$$

Now we make this function not only orthogonal, which it must be when integrating over θ and ϕ, but also normalized to 1. Therefore integrating over the solid angle, $d\Omega = sin(\theta)\, d\theta\, d\phi$;

$$\delta_{nm}\delta_{lk} = \int_0^{\pi} sin(\theta)\, d\theta \int_0^{2\pi} d\phi\, [Y_l^m\, Y_k^{m*}]$$

Then;

$$Y_l^m(\theta, \phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi\,(l+m)!}}\, P_l^m(\theta, \phi)\, e^{im\phi}$$

The first few functions, Y_l^M, which are the spherical harmonics;

$$Y_0^0 = \sqrt{1/(4\pi)}$$

$$Y_1^1 = -\sqrt{3/(8\pi)}\, sin(\theta)\, e^{i\phi}$$

$$Y_{-1}^1 = \sqrt{3/(8\pi)}\, sin(\theta)\, e^{-i\phi}$$

$$Y_0^1 = \sqrt{3/(4\pi)}\, cos(\theta)$$

Note that $Y_l^{m*} = -Y_l^{-m}$. The functions, $Y_l^m(\theta, \phi)$, are the spherical harmonics, and we will later identify the operator, L, as proportional to the angular momentum operator in Quantum Mechanics.

6 Second solution

We have found a series solutions, $P_l(x)$ with $x = cos(\theta)$, which contain only odd powers and one containing only even powers. But while these are linearly independent, they are not the two linearly independent solutions solutions expected from a 2^{nd} order ode. In fact, we have demonstrated that a second solution must diverge at the singular points. Suppose we choose to allow divergence by not letting the series terminate. Thus write;

$$\zeta = P_l + Q_l$$

In the above, Q_l will be the second solution. For the series solution we look at the form;

$$\zeta = P_l + \sum_{n=1}^{\infty} b_n\, x^n$$

Now l is chosen so that the series in even powers terminates. We then start b_n with $n = 1$ so this series forms even powers of x which cannot terminate because l is already chosen to

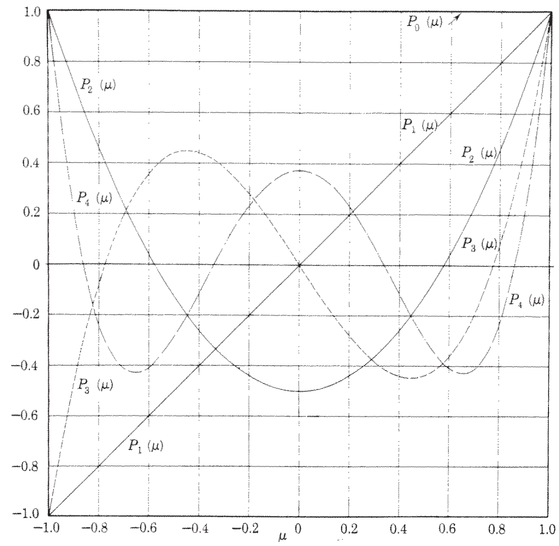

FIG. 5.01 Legendre function of the first kind, $P_n(\mu)$.

Figure 1: A representative example of Legendre functions of the first kind

make the even series terminate. The recursion relation is;

$$b_{n+2} = \frac{n(n+1) - l(l+1)}{(n+2+)(n+1)}\, b_n \quad n \text{ odd}$$

We can also generate a series in even powers in the same way. This series represents the second solution to Legendre's equation and is written, Q_l. Q_l has singular points at $\theta = 0,\ \pi$. Associated functions, as with the function, P_l, are defined and written, $Q_l^m(\theta)$. The first few functional forms are;

$$Q_0 = (1/2)\, ln[\frac{1+x}{1-x}]$$

$$Q_1 = (x/2)\, ln[\frac{1+x}{1-x}] - 1$$

$$Q_2 = (\frac{3x^2-1}{4})\, ln[\frac{1+x}{1-x}] - 3x/2$$

Figure 1 shows the behavior of the first few Legendre functions of the first kind, while Figure 2 shows the behavior of the Legendre functions of the second kind.

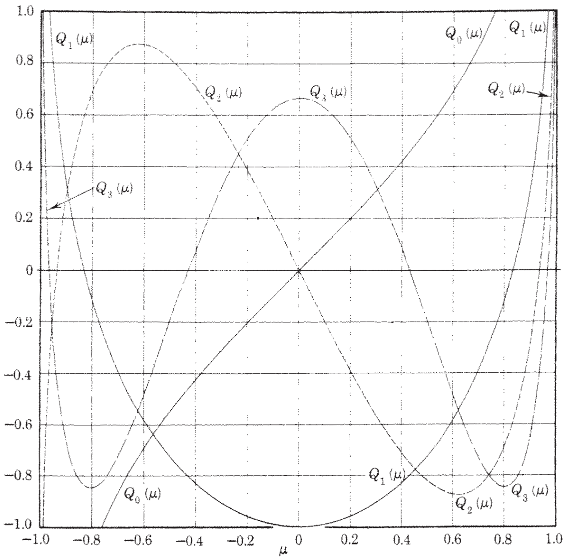

FIG. 5.02 Legendre function of the second kind, $Q_n(\mu)$.

Figure 2: A representative example of Legendre functions of the first kind

7 Green's function in spherical coordinates

Now return to the expression of the Green's function in spherical coordinates. This was developed several times previously but we are now in position to finalize this result. We assume that;

$$\nabla^2 G = -4\pi\delta(\vec{r} - \vec{r}')$$

Now expand the δ function using the complete set of spherical harmonics. Thus;

$$\delta(\theta - \theta')\delta(\phi - \phi') = \sum a_{\nu\mu}\, Y_\nu^\mu(\theta\,\phi)$$

Multiply by $Y_l^{m*}(\theta, \phi)$ and integrate over the solid angle $d\Omega$. Ortho-normality projects out the coefficients, $a_{lm} = Y_l^m(\theta', \phi')$. Then the Green's function takes the form;

$$G = \sum_{lm} g_l\, Y_l^m(\theta, \phi)\, Y_l^m(\theta', \phi')$$

The delta function is;

$$\delta(\vec{r} - \vec{r}') = 4\pi\delta(r - r') \sum_{lm} Y_l^m(\theta, \phi)\, Y_l^m(\theta', \phi')$$

Substitution into Laplaces's equation yields;

$$r\,\frac{d^2}{dr^2}[rg]\;-\;l(l+1)g\;=\;-4\pi\delta(r-r')$$

This is solved by requiring the solution to be continuous and the derivative to match the discontinuity at $r=r'$. The two solutions to the homogeneous equation are r^l and $r^{-(l+1)}$. The result is;

$$g(r,r')\;=\;\frac{4\pi}{2l+1}\left[\begin{array}{ll}\frac{r^l}{r'^{\,(l+1)}} & r<r' \\ \frac{r'^{\,l}}{r^{(l+1)}} & r>r'\end{array}\right]$$

The Green's function is then;

$$G\;=\;\sum_{lm}\frac{4\pi}{2l+1}\,\frac{r_<^l}{r_>^{l+1}}\,Y_l^m(\theta,\phi)Y_l^m(\theta',\phi')$$

Alternatively expand the factor, $\frac{1}{|\vec{r}-\vec{r}'|}$

Choose $r>r'$ and

$$\frac{1}{|\vec{r}-\vec{r}'|}\;=\;(1/r)\frac{1}{1-(r'/r)\,cos(\theta)\;+(r'/r)^2}$$
$$\frac{1}{|\vec{r}-\vec{r}'|}\;=\;\sum_m\frac{r'}{r^{l+1}}\,P_l(cos(\theta))$$

When comparing this to the expression for the Green's function above, it automatically gives the addition theorem.

$$P_l(\cos(\alpha))\;=\;\sum_m\frac{4\pi}{2m+1}\,Y_l^m(\theta,\phi)Y_l^{m*}(\theta',\phi')$$

Return to the expansion of the vector form $\frac{1}{|\vec{r}-\vec{r}'|}$. Figure 3 shows the geometry.

Apply a power series expansion in powers of r'/r when $r>r'$.

$$\frac{1}{R}\;=\;\frac{1}{r[1-(r'/r)cos(\theta)+(r'/r)^2]^{1/2}}$$
$$\frac{1}{R}\;=\;(1/r)+(r'/r)cos(\theta)+(r'/r)^2[1/2(3cos^2(\theta)-1)]+\cdots$$
$$\frac{1}{R}\;=\;\sum_l(r'^{\,l}\,l/r^{l+1})\,P_l(cos(\theta))$$
$$\frac{1}{R}\;=\;\sum_{lm}f_{l,}^m(r,r')\,Y_l^m(\hat{r})Y_l^m(\hat{r'})$$

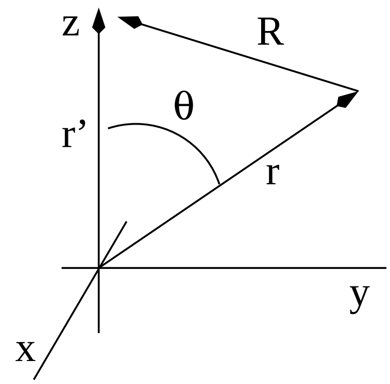

Figure 3: Geometry of the expansion for $\frac{1}{|\vec{r}-\vec{r}'|}$

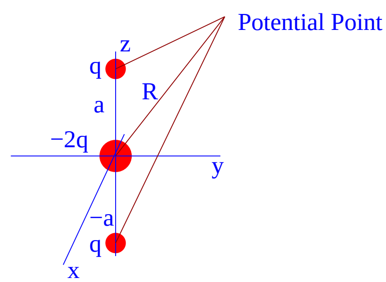

Figure 4: Geometry for the potential problem

8 Example1

From the figure note that the potential can be represented by;

$$V = \frac{1}{4\pi\epsilon}\left[\frac{-2q}{|\vec{R}|} + \frac{q}{|\vec{R}-\vec{a}|} + \frac{q}{|\vec{R}+\vec{a}|}\right]$$

Then by expansion in Legendre polynomials where $R > a$;

$$\frac{1}{|\vec{R}-\vec{a}|} = \sum \frac{a^l}{R^{l+1}} P_l(cos(\theta))$$

Substitute to obtain;

$$V = \frac{1}{4\pi\epsilon}\left[\frac{-2q}{R} + q\sum \frac{a^l}{R^{l+1}} P_l(x) + q\sum \frac{(-1)^l a^l}{R^{l+a}} P_l(x)\right]$$

Here $x = cos(\theta)$

$$V = \frac{2q}{4\pi\epsilon} \sum_{l=2;\, even} \frac{a^l}{R^{l+1}} P_l(x)$$

The lowest non-zero term in the series represents the multipole moment of the charge distribution.

9 Example2

In the development of scattering problems, we will need the expansion of a plane wave in spherical coordinates. Note that the [lane wave is symmetric in azimuthal angle. Thus;

$$e^{ikz} = e^{i\vec{k}\cdot\vec{r}} = \sum_n c_n Y_n^0$$

Use ortho-normality to project out the coefficients

$$c_n = 2\pi \int_0^\pi sin(\theta)\, d\theta\, Y_n^0\, e^{ikr\, cos(\theta)}$$

Apply the integral representation of the spherical Bessel function;

$$j_n(z) = (1/2)(-i)^n \int_0^\pi e^{iz\, cos(\theta)}\, P_n(\theta)\, sin(\theta)\, d\theta$$

This gives;

$$e^{ikr\, cos(\theta)} = \sum_l i^l [4\pi(2l+1)]^{1/2}\, j_l(kr)\, Y_l^0(\theta)$$

By the addition theorem;

$$e^{ikr\, cos(\theta)} = 4\pi \sum_{lm} j_l(kr)\, Y_l^m(\hat{k}) Y_l^{m*}(\hat{r})$$

Figure 5 shows the geometry of the addition theorem.

10 Angular momentum

Neglecting mass, the momentum operator in QM has the form $i\hbar\vec{\nabla}$. The angular momentum operator then has the form;

$$\vec{L} = -\vec{r} \times i\hbar\vec{\nabla} \rightarrow \vec{r} \times \vec{p}$$

The energy operator in spherical coordinates is;

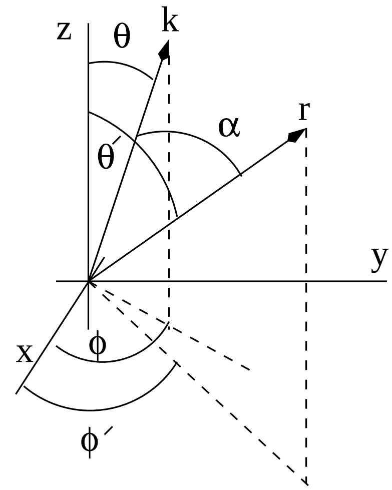

Figure 5: The geometry of the addition theorem

$$T = -\frac{\hbar^2}{2M}\,\nabla^2 = -\frac{\hbar^2}{2M}[\nabla_r^2 + \nabla_\perp^2]$$

where the Laplacian operator is divided into operations on the radial variable, r, and the angular variables, (θ, ϕ). In Cartesian coordinates;

$$L_x = , -i\hbar[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}]$$

$$L_x = , -i\hbar[x\frac{\partial}{\partial z} - x\frac{\partial}{\partial z}]$$

$$L_x = , -i\hbar[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}]$$

The map between Cartesian and spherical coordinates is;

$$x = r\,sin(\theta)\,cos(\phi)$$

$$y = r\,sin(\theta)\,sin(\phi)$$

$$z = r\,cos(\theta)$$

Make a change of variables for the Cartesian angular momentum operators above.

$$L_x = i\hbar[sin(\phi)\,\frac{\partial}{\partial\theta} + cot(\theta)\,cos(\phi)\,\frac{\partial}{\partial\phi}]$$

$$L_y = i\hbar[-cos(\phi)\,\frac{\partial}{\partial\theta} + cot(\theta)\,sin(\phi)\,\frac{\partial}{\partial\phi}]$$

$$L_z = -i\hbar \frac{\partial}{\partial \phi}$$

Then look at $L^2 = \vec{L} \cdot \vec{L}$

$$L^2 = -\hbar^2 [\frac{1}{sin(\theta)} \frac{\partial}{\partial \theta}[sin(\theta) \frac{\partial}{\partial \theta}] + \frac{1}{sin^2(\theta)} \frac{\partial^2}{\partial \phi^2}]$$

This can be obtained by using;

$$\vec{L} = -i\hbar \, \vec{r} \times \vec{\nabla}$$

$$\vec{\nabla} = \hat{r} \frac{\partial}{\partial r} - \frac{\vec{r} \times \vec{r} \times \vec{\nabla}}{r^2}$$

Then we have the eigenvalue equation;

$$L^2 Y_l^m(\theta, \phi) = \hbar^2 \, l(l+1) Y_l^m(\theta, \phi)$$

The commutation rules for the angular momentum operator can be worked out as well using the above description of the operators.

$$[L_i, L_j] = i\epsilon_{ijk} \, \hbar \, L_k$$

$$L^{\pm} = L_x \pm iL_y$$

$$[L_z, L^{\pm}] = \pm \hbar L^{\pm}$$

$$[L^+, L^-] = 2L_z$$

$$L_z \, Y_l^m = \hbar \, m \, Y_l^m$$

$$L_z L^{\pm} Y_l^m = \hbar (m \pm 1) L^{\pm} Y_l^m$$

L^2 commutes with $L^{\pm}$ so from the above, $L^+(L^-)$ is a raising (lowering) operator on the eigenvalue m in the eigenfunction Y_l^m

11 Vector spherical harmonics

When we deal with the vector Laplacian, it is useful to introduce vector spherical harmonics of the form;

$$\vec{\chi}_l^m = \frac{1}{\sqrt{l(l+1)}} \vec{L} Y_l^m$$

The operator L will be the same as the angular momentum operator without the multiplying constant of $\hbar$. Thus $\vec{L}$ is defined by $\vec{L} = -i[\vec{r} \times \vec{\nabla}]$. Then;

$$L^2 = -[\frac{1}{sin(\theta)} \frac{\partial}{\partial\theta}(sin(\theta)\frac{\partial}{\partial\theta}) + \frac{1}{sin^2(\theta)} \frac{\partial^2}{\partial\phi^2}]$$

$$L^2 Y_l^m = l(l+1) Y_l^m$$

All other operations follow in the same way as was the case of the angular momentum terms. The $\vec{\chi}$ functions as defined above are ortho-normal.

$$N^2 = \frac{1}{l(l+1)} \int d\Omega (\vec{L} Y_l^m)^* \cdot \vec{L} Y_l^m$$

Then;

$$\vec{L} Y_l^m = i\vec{r} \times \vec{\nabla}\, Y_l^m$$

Substitute into the integrand to obtain;

$$N^2 = \frac{1}{l(l+1)} \int d\Omega\, r^2\, [\vec{\nabla} Y_l^{m*} \cdot \vec{\nabla} Y_{l'}^{m'}]$$

Integrate by parts to produce a ∇^2 operator operating on Y_l^m. The result is;

$$\int d\Omega\, \vec{\chi_l^m} \cdot \vec{\chi_{l'}^{m'}} = \delta_{ll'} \delta_{m,m'}$$

In the case of non-Cartesian coordinates the vector Laplacian should be used when operating on a vector field. We will then use the expansion of a vector field in a new set of functions called vector spherical harmonics. Define 3 vector operators to span 3-D space;

$$\vec{V}_l^m = \hat{r}[-(\frac{l+1}{2l+1})^{1/2}\, Y_l^m] + \hat{\theta}[\frac{1}{[(l+1)(2l+1)]^{1/2}} \frac{\partial Y_l^m}{\partial\theta}] + \hat{\phi}[\frac{im}{[(l+1)(2l+1)]^{1/2}\, sin(\theta)} Y_l^m]$$

$$\vec{W}_l^m = \hat{r}[(\frac{l}{2l+1})^{1/2}\, Y_l^m] + \hat{\theta}[\frac{1}{[(l)(2l+1)]^{1/2}} \frac{\partial Y_l^m}{\partial\theta}] + \hat{\phi}[\frac{im}{[(l)(2l+1)]^{1/2}\, sin(\theta)} Y_l^m]$$

$$\vec{\chi}_l^m = \hat{\theta}[\frac{-m}{[(l)(2l+1)]^{1/2}\, sin(\theta)} \frac{\partial Y_l^m}{\partial\theta}] + \hat{\phi}[\frac{-i}{[(l)(2l+1)]^{1/2}} \frac{\partial Y_l^m}{\partial\theta}]$$

These functions satisfy orthogonality and are normalized;

$$\int d\Omega\, \vec{A}_l^m \cdot \vec{B}_{l'}^{m'} = \delta_{AB} \delta_{ll'} \delta_{mm'}$$

Where $\vec{A}$, $\vec{B}$ are any of the above functions.

12 Expansion of a vector field

Any vector field can be expanded as;

$$\vec{F} = \vec{\nabla}\Psi_1 + \vec{L}\Psi_2 + curl\vec{L}/i\,\Psi_3$$

To see this note that any vector can be written;

$$F = \vec{\nabla}\Psi_1 + \vec{\nabla} \times \vec{G}$$

Then let $Q = i\vec{r}\Psi_2 - i\vec{L}\Psi_3 - \vec{\nabla}\Psi_4$. Then take $\vec{\nabla} \times \vec{Q}$.

Use the definition $\vec{L} = -i[\vec{r} \times \vec{\nabla}]$ to show that;

$$\vec{\nabla} \times \vec{Q} = \vec{L}\Psi_2 + \frac{\vec{\nabla} \times \vec{L}}{i}\Psi_3$$

In spherical coordinates;

$$Q_r = ir\Psi_2 - \frac{\partial\Psi_4}{\partial r}$$

$$Q_\theta = \frac{1}{sin(\theta)}\frac{\partial\Psi_3}{\partial\phi} - (1/r)\frac{\partial\Psi_4}{\partial\theta}$$

$$Q_\phi = -\frac{\partial\Psi_3}{\partial\theta} - \frac{1}{r\,sin(\theta)}\frac{\partial\Psi_4}{\partial\phi}$$

One can then demonstrate that the above equations may be solved for Ψ_2, Ψ_3, Ψ_4. Then for the vector potential, A, we can write;

$$\vec{A} = \vec{\nabla}\Psi_1 + \vec{L}\Psi_2 + \frac{\vec{\nabla} \times \vec{L}}{i}\Psi_3$$

However, when considering the magnetic field using Coulomb gauge, $\vec{\nabla} \cdot \vec{A} = 0$. Given the definition of the operator, $\vec{L}$, the only zero component is represented by, $\vec{\nabla} \cdot [\vec{L}\Psi_2] = 0$

13 Example3

Apply the vector spherical harmonics to the calculation of the magnetic field of a current loop. From Ampere's law in steady state;

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \mu\vec{J}$$

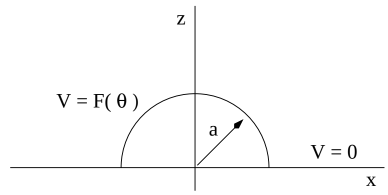

Figure 6: The potential inside a hemisphere over a flat plane

In the above, $\vec{B}$ is the magnetic field, A is the vector potential, and $\vec{J}$ is the current density. We choose the Coulomb gauge, $\vec{\nabla} \cdot \vec{A} = 0$. This reduces the above to;

$$\vec{A} \rightarrow R(r)\vec{\chi}_l^m$$

Then $\nabla^2 \vec{A}$ separates to obtain;

$$\frac{1}{R}[r^2 \frac{d^2 R}{dr^2} + (2r)\frac{dR}{dr}] = \frac{1}{\chi_l^m}[L^2]\chi_l^m = l(l+1)$$

The remainder of the solution is obtained by matching boundary conditions for $\vec{A}$ or by using the Green's function. When $m = 0$ (azimuthal symmetry) and for the solution to remain finite as $r \rightarrow \infty$ the solution will have the form;

$$\vec{F}_l = \sum a_l \, (1/r^{l+1}) \, [\frac{-i}{(l(l+1))^{1/2}} \, \frac{dY_l^0}{d\theta} \, \hat{\phi}]$$

14 Example4

Consider the potential due to a semi-circular screw head on a flat conducting plane. The geometry is illustrated in Figure 6. We look for solutions to Laplace's equation, $\nabla^2 V = 0$, for the potential in a spherical coordinate system. In general the solution must have the form;

$$V = \sum_{lm} A'_{lm} \, r^l \, Y_l^m + \sum_{lm} B'_{lm} \, r^{-(l+1)}$$

Note that while the general solution should include the Legendre functions of the 2^{nd} kind as well as functions of the 1^{st} kind the boundary condition that the solution be finite along the z axis excludes the Q_l functions. The solution is also symmetric in azimuthal angle so $m = 0$. If we wish solutions inside the a hollow screw head, then the solution must be finite at $r = 0$. In this case the solution takes the form;

$$V = \sum_l A_l \, r^l \, P_l$$

For the potential to vanish when $\theta = \pi/2$ choose only values of $l = 1, 3, 5, \cdots$.

$$V = \sum_{n=0} A_n \, r^{2n+1} \, P_{2n+1}(\cos(\theta))$$

Finally, match the boundary condition, $F(x)$ with $x = cos(\theta)$ when $r = a$, by projecting out of the sum the coefficients, A_n

$$A_n = \frac{2n+1}{2a^{(2n+1)}} \int_1^1 dx F(x) \, P_{2n+1}(x)$$

On the other hand suppose we seek a solution for $r > a$. In this case, the solution should have the form;

$$V = \sum B_l \, r^{-(l+1)} \, P_l(x)$$

As above, we wish the solution finite on the z axis, azimuthally symmetric and finite as $r \to \infty$. We will need to change the boundary conditions slightly to find a stable solution in spherical coordinates as the application of Cauchy (Neumann) conditions must be applied on a closed surface. The boundary problem is really due to a mixture of boundary conditions on spherical and cylindrical surfaces. Thus choose to put the potential on the surface at $z = 0$ to a constant value which we can choose as $V = 0$. There will be another potential, $V = V_0$ on a plane which we take as $r \to \infty$. Then the boundary conditions are ;

$$V = 0 \quad \text{for } z = 0, \text{ and } r = a$$

$$V = -E_z \, z \quad \text{as } r \to \infty$$

The boundary condition $V = -E_z \, z$ is really a Neumann condition specifying the electric field, $\vec{E} = \vec{\nabla} V$. Then we must have a solution of the form;

$$V = -E_z \, [r \, cos(\theta)] + \sum_n B_n \, r^{-(2n+2)} \, P_{2n+1}(x)$$

15 Example5

As another example, consider a dielectric sphere of dielectric constant, ϵ, in a uniform field in the $\hat{z}$ direction, $\vec{E} = E_z \hat{z}$. The boundary conditions are;

$$V = -E_z \, z = -E_z \, r \, cos(\theta) \quad \text{as } r \to \infty$$

Tangential $\vec{E}$ is continuous at $r = a$

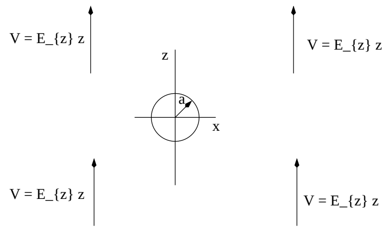

Figure 7: The potential of a dielectric sphere in a uniform electric field

Perpendicular $\vec{D} = \epsilon \vec{E}$ is continuous at $r = a$

The boundary conditions at $r = a$ are derived for a dielectric without surface charge or currents from Gauss' and Ampere's laws.

The solution must have the form after applying azimuthal symmetry and requiring a finite solution on the z axis ($x = cos(\theta)$);

$$V \;=\; \sum_l A_l \, r^l \, P_l(x) \;+\; \sum_l \, B_l \, r^{-(l+1)} \, P_l(x)$$

To match the boundary condition as $r \to \infty$ require the solution to be;

$$V \;=\; -E_z \, r \, cos(\theta) \;+\; \sum_l \, B_l \, r^{-(l+1)} \, P_l(x)$$

Then match the boundary conditions at $r = a$. The boundary condition that tangential $\vec{E}$ is continuous is the same as requiring a continuous potential at $r = a$.

$$\sum A_l \, a^l \, P_l \;=\; -E_z \, r \, P_1 \;+\; \sum_l \, B_l \, a^{-(l+1)} \, P_l$$

For D_r to be continuous;

$$\epsilon \frac{\partial V}{\partial r}|_{in} \;=\; \epsilon \frac{\partial V}{\partial r}|_{out}$$

$$\epsilon_r \sum \, A_l a^{l-1} P_l \;=\; \sum \, B_l \, (l+1) \, a^{-(l+2)} \, P_l \;=\; E_z P_1$$

Solve for the coefficients;

$$A_0 = 0$$

$$B_0 = 0$$

$$A_1 = \frac{3}{\epsilon_r + 2} \, E_z$$

$$B_1 = \frac{(\epsilon_r - 1)a^3}{\epsilon_r + 2}\, E_z$$

All other coefficients vanish. The potential is;

Inside $r < a$

$$V \;=\; \frac{3E_z}{\epsilon_r + 2}\, r\, cos(\theta)$$

Outside $r > a$

$$V \;=\; -E_z\, r\, cos(\theta) \;+\; [\frac{\epsilon_r + 1}{\epsilon_r + 2}]\, E_z\, \frac{a^3}{r^2}\, cos(\theta)$$

16 Multipole EM fields

We look for solutions to Maxwell's equations with the time component suppressed. Let $k = \omega/c$. The Helmholtz equation is;

$$\nabla^2\vec{B} \;+\; k^2\vec{B} \;=\; 0$$

The operator, ∇^2 in the above is the vector Laplacian. We want a set of spherical harmonic solutions (multipoles) for $\vec{B}$ (or for $\vec{E}$. Each component of B satisfies the Helmholtz equation as well as $div\vec{B} = 0$ and we want to find solutions in terms of the spherical Hankel functions as these have the correct boundary conditions as $r \to \infty$. The solution is then expected to have the form;

$$\vec{B} \;\to\; \sum a_{lm} h_l\, \vec{L}\, Y_l^m$$

In the above h_l is the spherical Hankel function giving the radial dependence as $r \to \infty$ and the angular term is the vector spherical harmonic $\vec{\chi}_l^m$ whose divergence vanishes. This solution then satisfies the Helmholtz equation has appropriate boundary conditions as $r \to \infty$, and has $\vec{\nabla} \cdot \vec{B} = 0$. The solution for the electric field is similar, and to get a fully complete set of functions we need to combine the fields, (ie the magnetic field as above and the magnetic field component due to the electric field $\vec{\nabla} \times \vec{E} \;=\; i\omega\vec{B}$, and of course the electric component due to the magnetic field). Therefore the complete solution takes the form;

$$\vec{B} \;=\; \sum_{lm} A(E)_{lm}\, f_l\, \vec{\chi}_l^m \;-\; (i/k)\, A(M)_{lm}\, \vec{\nabla} \;\times\; g_l\, \vec{\chi}_l^m$$

$$\vec{E} \;=\; \sum_{lm} (i/k) A(E)_{lm}\, \vec{\nabla} \;\times\; f_l\, \vec{\chi}_l^m \;+\; A(M)_{lm}\, g_l\, \vec{\chi}_l^m$$

For a wave moving radially outward, f_l, g_l are the Hankel functions of the first kind. The coefficient $A(E)$ $[A(M)]$ represents the amount of electric [magnetic] multipole in the solution. $A(E)$ means the source is due to a charge density and $A(M)$ is due to a magnetic moment density.