Lesson 1 – Introduction to Algebraic Geometry

I. What is Algebraic Geometry?

Algebraic Geometry can be thought of as a (vast) generalization of linear algebra and algebra. Recall that, in linear algebra, you studied the solutions of systems of linear equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0,$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0,$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0,$$

where the coefficients a_{ij} were taken from some field *K*. The set of solutions turned out to be a vector space, whose dimension does not change if we replace *K* by some bigger (or smaller) field.

In algebra (or, more precisely, in Galois Theory), you looked at the solutions of polynomial equations

$$f(x) = a_n x^n + \cdots + a_1 x + a_0 = 0,$$

where the coefficients a_i were take from some field *K*. This time, the set of solutions depended strongly on the choice of the field K. For example, the polynomial $x^2 + 1$ has no solution in$\mathbb{R}$, but two solutions in$\mathbb{C}$.

The natural generalization of these two problems is the following: given a field *K* and set of polynomials$f_1, f_2, \ldots, f_d \in K[x_1, x_2, \ldots, x_n]$, investigate the set of solutions of the system of polynomial equations

$$f_1(x_1, x_2, \ldots, x_n) = 0,$$
$$f_2(x_1, x_2, \ldots, x_n) = 0,$$
$$\vdots$$
$$f_d(x_1, x_2, \ldots, x_n) = 0.$$

On the one hand, this problem belongs to algebra, since the objects we are dealing with are polynomials over some field; on the other hand, we can think of the set of real solutions of $x^2 + y^2 - 1 = 0$ as the unit circle, which is a geometric object. In algebraic geometry, methods from both algebra and geometry are used to gain understanding of the solution set of the system. Such solution sets are known as *algebraic varieties.*

II. Algebraic Varieties – The Main Characters of Algebraic Geometry

Definition Let K be a field, and let $f_1, f_2, \dots, f_d \in K[x_1, x_2, \dots, x_n]$.
Then the **affine variety**, denoted by $\mathbf{V}(f_1, f_2, \dots, f_d)$, is defined by:

$$\mathbf{V}(f_1, f_2, \dots, f_d) = \{(a_1, a_2, \dots, a_n) \in K^n : f_i(a_1, a_2, \dots, a_n) = 0 \text{ for all } 1 \le i \le n\}.$$

Remarks

1. Often times, in introductory books, affine varieties are defined specifically to be over $K = \mathbb{C}$ or $K = \mathbb{R}$, because algebraic geometry is more intuitive and easier there. Indeed, most of the time, our text will be working over such fields. However, it is common in number theory (for example) that the field is either finite or $\mathbb{Q}$.
2. If $f_1, f_2, \dots, f_d \in \mathbb{C}[x_1, x_2, \dots, x_n]$, then it is typical to visualize the variety $\mathbf{V}(f_1, f_2, \dots, f_d)$ by plotting the real points only; i.e., we plot $\mathbf{V}(f_1, f_2, \dots, f_d) \cap \mathbb{R}^n$.

III. Examples and Exercises

Exercise 1 (The Trivial Affine Varieties)

a) What is $\mathbf{V}(1)$, where $1 \in k[x_1, x_2, \dots, x_n]$?

b) What is $\mathbf{V}(0)$?

Definition An **affine plane curve** is the zero set of a single polynomial in $k[x, y]$. Again, k would typically be $\mathbb{R}$ or $\mathbb{C}$.

Exercise 2 How would you use the "**V**" notation to describe the set of points in $\mathbb{R}^2$ on the unit circle?

Definition A **hypersurface** in k^n is the zero set of a single polynomial in $k[x_1, x_2, \dots, x_n]$.

Exercise 3 - True or False:

a) A line is a hypersurface in $\mathbb{R}[x, y, z]$.
b) A point is a hypersurface in $\mathbb{R}[x]$.

Exercise 4 - What about affine varieties defined by more than one polynomial?

How is $\mathbf{V}(x^2+y^2+z^2-1, x^2-x+y^2)$ related to $\mathbf{V}(x^2+y^2+z^2-1)$ and $\mathbf{V}(x^2-x+y^2)$?

Theorem If $V = \mathbf{V}(f_1, f_2, ..., f_s)$ and $W = \mathbf{V}(g_1, g_2, ..., g_t)$ are affine varieties, then

$$V \cap W = \mathbf{V}(f_1, f_2, ..., f_s, g_1, g_2, ..., g_t).$$

Exercise 5. Prove this theorem.

Exercise 6. What about unions? If $V = \mathbf{V}(f_1, f_2, ..., f_s)$ and $W = \mathbf{V}(g_1, g_2, ..., g_t)$ are affine varieties, is the union $V \cup W$ necessarily an affine variety?

Exercise 7 Show that the union of the "Whitney Umbrella" and the "Twisted Cubic":

$$\mathbf{V}(x^2 - y, x^3 - z) \cup \mathbf{V}(x^2 - y^2 z)$$

is an affine variety.

Exercise 8 (Infinite unions)

a) Is the point (1, 2) in $\mathbb{R}^2$ an affine variety?

b) Is the set of two points {(1, 2), (-3, 1)} in $\mathbb{R}^2$ an affine variety?

c) Is the set $\mathbb{Z}^2$ in $\mathbb{R}^2$ an affine variety?

IV. The Zariski Topology (If you have never seen any topology, do not worry about this!)

Affine varieties can be used to define a topology on the set K^n. In particular, open sets in the "Zariski Topology" are defined to be the *complements* of affine varieties in K^n. To be more explicit, define the set of open sets, $\mathcal{Z}$, by $\mathcal{Z} = \{K^n \backslash V: V = \mathbf{V}(f_1, f_2, \ldots, f_2)\}$.

It is not difficult to show that the sets in $\mathcal{Z}$ satisfy the four axioms defining a topology; if V and W are any two affine varieties in K^n, then:

- $\emptyset$ and K^n are both in $\mathcal{Z}$ (i.e., the trivial subsets of K^n are both open sets)
- Since $K^n \backslash V \cup K^n \backslash W = K^n \backslash (V \cap W)$, an arbitrary[1] union of elements in $\mathcal{Z}$ are in $\mathcal{Z}$.
- $K^n \backslash V \cap K^n \backslash W = K^n \backslash (V \cup W)$; so the intersection of finitely many elements in $\mathcal{Z}$ is in $\mathcal{Z}$.

[1] By "arbitrary", we mean that the union could be an infinite union. Since the union of complements is the complement of an intersection, and the infinite intersection of affine varieties is an affine variety, we get that infinite unions of sets in $\mathcal{Z}$ will produce another set in $\mathcal{Z}$. That is, an arbitrary union of open sets is an open set.

A Few Words on Plotting Plane Curves in $\mathbb{R}^2$ and Hypersurfaces in $\mathbb{R}^3$

- To plot the plane curve defined by $f(x, y) = 0$ in Maple use the command:
  >with(plots):
  >implicitplot(*f*(*x*, *y*) = 0, x = -*a*..*a*, y = -*b*..*b*,
  color = black, thickness = 3, gridrefine = 5);

- To plot the same curve on Mathematica use the command:
  In[1]:= **<< Graphics `ImplicitPlot`**
  In[2]:= **ImplicitPlot[*f*(*x*, *y*) == 0, {*x*, -*a*, *a*}, PlotPoints -> 30]**

- To plot the curve with Sage use the command:
  sage: **var('x,y')**
  sage: **implicit_plot(f(x, y) == 0, (x, -*a*, *a*), (y, -*b*, *b*), plot_points=200)**

- To plot the hypersurface defined by $f(x, y, z) = 0$ in Maple use the "implicitplot3d" command:
  >with(plots):
  >implicitplot3d(*f*(*x*, *y*, *z*) = 0, x = -*a*..*a*, y = -*b*..*b*, z = -*c*..*c*);

- To plot the hypersurface defined by $f(x, y, z) = 0$ in Mathematica use
  In[1]:= **<< Graphics `ContourPlot3D`**
  In[2]:= **ContourPlot3D[*f*(*x*, *y*, *z*) , {*x*, -*a*, *a*}, {*y*, -*b*, *b*}, {*z*, -*c*, *c*}]**

- To plot the hypersurface defined by $f(x, y, z) = 0$ in Sage use
  sage: **var('x,y,z')**
  sage: **implicit_plot3d(f(x, y, z) == 0, (x, -*a*, *a*), (y, -*b*, *b*), (z, -c, c))**

- To plot two hypersurfaces, defined by $f(x, y, z) = 0$ and $g(x, y, z) = 0$, together in Maple use the command
  >with(plots):
  >implicitplot3d([*f*(*x*, *y*, *z*) = 0, *g*(*x*, *y*, *z*) = 0] , x = -*a*..*a*, y = -*b*..*b*, z = -*c*..*c*);

- To plot a curve or surface on "SingSurf" go to
  http://www.singsurf.org/index.html