THE UNIVERSITY OF AKRON
Department of Theoretical and Applied Mathematics

LESSON 1: INTRODUCTION TO ANGLES

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Directory

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1. Introduction

Any study of *trigonometry* requires a basic understanding of the concept and measurement of angles. These are the topics covered in this lesson. The next section entitled The concept of an angle begins with a formal description of an angle and the introduces terminology used to designate their geometrical components. Consequently, the presentation is essentially self-contained and independent of prior training. This section also reviews the notion of degree measurement of angles. The next section section 3 introduces a second method for measuring angles, that of radians. The fourth section entitled The conversion formula examines the relationship between the radian and degree measurements and demonstrates the technique for converting from one to the other. The next section section 5 describes the concept of coterminal angles, angles that have the same geometric appearance but have different measurements. The lesson concludes with a brief discussion of a more general definition of radian measure (section 6).

2. The concept of an angle

Angles are formed whenever two line segments join. The two line segments $\overline{OA}$ and $\overline{OB}$ in Figure 1.1a, which *join* at the point O, determine the $\angle AOB$ (read *angle* AOB). The curved arrow in this figure suggests that the angle is measured from the segment $\overline{OA}$, the **initial side** of the angle, to the **terminal side** $\overline{OB}$. The point O is called the **vertex** of $\angle AOB$. Note the orientation of the curved arrow. This means that $\angle AOB$ is formed by rotating $\overline{OA}$ about the point O to $\overline{OB}$ in a counter-clockwise or **positive** direction. Figure 1.1b illustrates what appears to be the same angle[1] but is actually formed by rotating $\overline{OA}$ about O in a **negative** (clockwise) direction. Any method used to measure an angle must describe this *direction of rotation*. (The two angles in Figure 1.1 demonstrate that the notation $\angle AOB$ can be confusing since it could possibly denote more than one angle. The context in which such symbols are used must prevent any difficulties in their interpretation.)

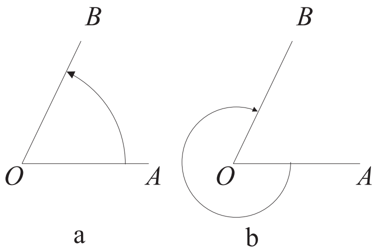
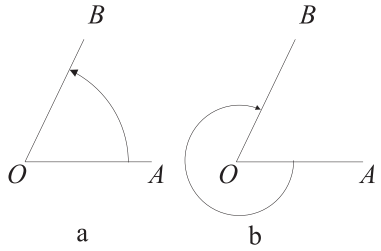


Figure 1.1: Examples of angles in the plane.

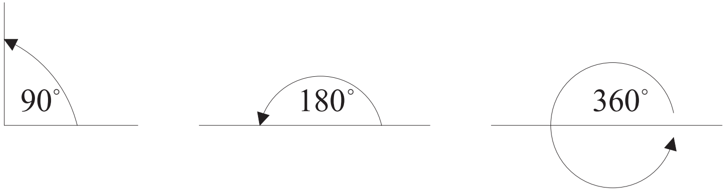
[1]These two angles are coterminal.



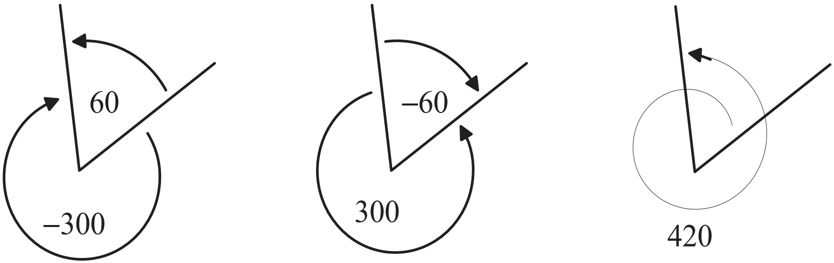
Magnitude, or the size of the opening of an angle, is a second requirement placed on angle measurement. A dimension often used in trigonometry and geometry to measure angles that meets both the demands of magnitude and direction is that of degrees. One degree[2] represents an angle with magnitude 1/360 of a complete revolution of the terminal side of an angle in the positive direction. Consequently, there are 360 one degree angles in one revolution of a terminal side. The angle in Figure 1.1a, which is has been reconstructed at the right for convenience, has approximate measure $65\,^\circ$. The arrow depicting the angle suggests that it is measured in a counter-clockwise or *positive* direction. If the angle is measured from the initial side in a clockwise direction the degree dimension is a *negative* number. For example, the angle in Figure 1.1b has approximate measure $-295\,^\circ$. It is understood that the arc may include one or more complete revolutions or wrappings of the circle. The angle in Figure 1.1a, for example, has the same appearance as an angle with measure $425\,^\circ = 65\,^\circ + 360\,^\circ$.

[2]The symbol $^\circ$ is used to denote degree measurement. For example, $295\,^\circ$ is read 295 degrees.

Example 1 *The figures below graph the* $90\,^\circ, 180\,^\circ$, *and* $360\,^\circ$ *angles.*



Example 2 *The figure below graphically describes several angles, all of which are modifications of a* $60\,^\circ$ *angle. The direction of construction is indicated by the curved arrows, determining the initial and terminal sides of the angles,. Observe that the* $420\,^\circ = 360\,^\circ + 60\,^\circ$ *angle makes one complete revolution of the terminal side before rotating the additional* $60\,^\circ$.



3. Radian dimension

Another frequently encountered dimension used to quantify angles in trigonometric settings is radian measure[3]. Radian measure is based on angles that are determined by points on a circle, usually the circle of radius one centered at the origin O. The equation of this **unit** circle is $x^2 + y^2 = 1$ and its graph appears in Figure 1.2. Notice that the arcAB, the smaller portion of the circle determined by the points A and B in Figure 1.2, is red in color. The angle $\angle AOB$ prescribed by this arc is that angle with terminal side $\overline{OB}$ and initial side $\overline{OA}$[4]. This angle is defined to have **radian measure** t if arcAB is t units long[5]. That is, if the **arclength** of arcAB (the distance determined by the red path) is t units, $\angle AOB$ measures t radians. Since the circumference of the unit circle is 2π, an angle formed by one complete revolution of its terminal side has measure 2π rad . (It is understood that the arc may include one or more complete revolutions or wrappings of the circle.)

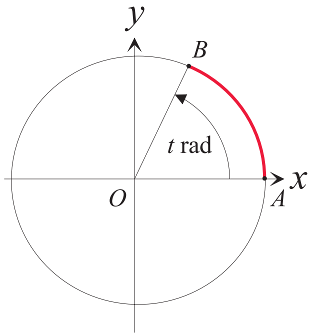
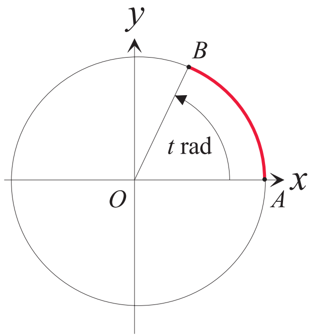


Figure 1.2: The unit circle used to describe an angle.

[3]This dimension is frequently denoted by rad so 2 rad is read two radians.

[4]It can also be said that the $\angle AOB$ **subtents** the arc AB.

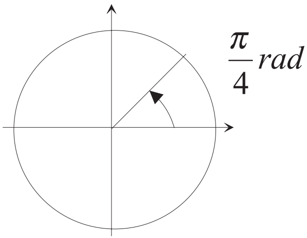
[5]The units used to measure distance (Inches, centimeters, etc.) are not germane to this discussion.



The arrow in Figure 1.2, which is redrawn at the right, suggests that the angle t is measured in a counter-clockwise or positive direction. This is the case for t positive. If t is negative the angle is measured from the x-axis in a clockwise direction. Observe that the initial side $\overline{OA}$ of $\angle AOB$ in Figure 1.2 lies on the x-axis and has its vertex at the origin. Any angle thus formed is said to be in **standard position**.

Example 3 *Draw a figure depicting an angle of radian measure $\pi/4$.*

Solutions: *Since the circumference of the unit circle is 2π, an angle of radian measure $\pi/4$ would subtend an arc of length 1/8th this distance. The angle subtended by this arc is illustrated in the figure below.*



4. The conversion formula

Since the circumference of the unit circle is 2π units, an angle of 2π rad is the same as an angle of 360°, or an angle of π rad is the same as a 180° angle. This suggests the formula

$$\frac{t}{\pi} = \frac{\theta}{180} \tag{1}$$

for converting radian measure t to degree measure θ or vice versa. This relationship and experience with degree measurement provide some intuition in envisioning an angle described in terms of radians. The following examples illustrate the use of Equation 1.

Example 4 *Since*

$$\frac{t\,\mathrm{rad}}{\pi} = \frac{75^\circ}{180} \Longrightarrow t = 1.309\,\mathrm{rad},$$

an angle measuring 75° *is the same as an angle with radian measure* 1.309.

As a quick check we solve

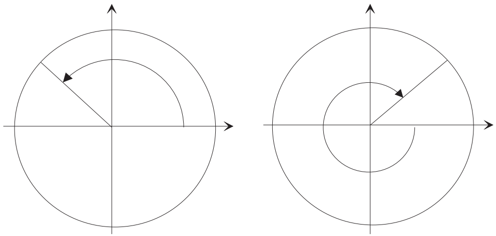
$$\frac{1.309\,\mathrm{rad}}{\pi} = \frac{\theta^\circ}{180}$$

for θ *to obtain* $\theta = 75.0002^\circ$. *(The error of* 0.0002° *is the result of rounding in the calculations.)*

Example 5 *Draw a figure depicting angles of radian measure*
1) $3\pi/4$ *2)* $-7\pi/4$

Solutions:

1. *An angle of radian measure* $3\pi/4$ *would subtend an arc of length 3/8th the circumference of the unit circle and would be the same as an angle of* $135°$. *(See figure (a) below.)*
2. *An angle of radian measure* $-7\pi/4$ *would subtend an arc of length 7/8th the circumference of the unit circle measured in the negative direction and, by Equation 1, would be the same as an angle of* $-315°$. *(See figure (b) below.)*

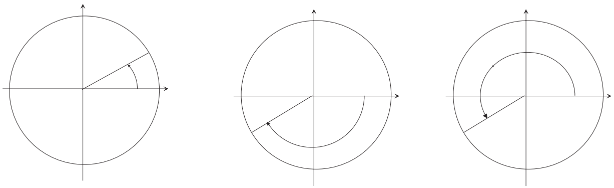


a *b*

Example 6 *Draw a figure depicting angles of radian measure*
1) $\pi/6$ *2)* $-5\pi/6$ *3)* $4\pi/3$.

Solutions:

1. *An angle of radian measure* $\pi/6$ *would subtend an arc of length* $\frac{1}{12}$*th the circumference of the unit circle and would be the same as an angle of* $(360/12)^\circ = 30^\circ$. *(See figure (a) below.)*
2. *An angle of radian measure* $-5\pi/6$ *would subtend an arc of length 5/12th the circumference of the unit circle measured in the negative direction and would be the same as an angle of* -150°. *(See figure (b) below.)*
3. *An angle of radian measure* $4\pi/3$ *would subtend an arc of length 4/6th the circumference of the unit circle and would be the same as an angle of* 240°. *(See figure (c) below.)*



a *b* *c*

Example 7 *Draw a figure depicting an angle with measure* $1.5\,\mathrm{rad}$.

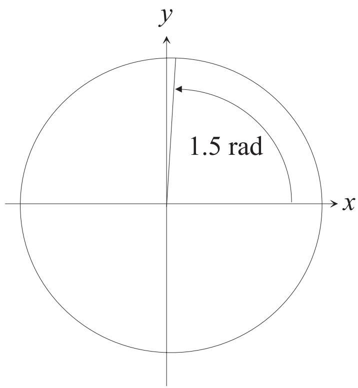
Solution: Since $\pi/2 \cong 1.570\,8$,[6] *an angle of* $1.5\,\mathrm{rad}$ *would subtend an arc on the unit circle somewhat shorter than* $1/4$ *of its circumference. Additional insight as to the size of the angle can be gained by converting it to degree measure using Equation 1. Solving the relation*

$$\frac{1.5}{\pi} = \frac{\theta}{180}$$

for θ *yields*

$$\theta = \frac{(1.5)(180)}{\pi} = 85.944^\circ.$$

That is, the angle is about four degrees smaller than a 90° *angle. The figure to the right portrays an angle of* $1.5\,\mathrm{rad}$.



[6] Recall that $\pi \cong 3.141\,6$.

Example 8 *Determine the radian measure of a 110° angle.*

Solution: *Solving the relation*

$$\frac{t}{\pi} = \frac{110}{180}$$

as suggested by Equation 1 for t yields

$$t = \frac{110\pi}{180}\,\mathrm{rad} = \frac{11}{18}\pi\,\mathrm{rad} = 1.9199\,\mathrm{rad}\,.$$

Example 9 *By the previous example, an angle of measure* $-110\,^\circ$ *has radian measure* -1.9199.

Example 10 *An angle of radian measure* $\pi/8$ *has the same measure as the angle*

$$\theta = \left(\frac{\pi/8}{\pi}180\,^\circ\right) = \frac{180\,^\circ}{8} = \frac{45\,^\circ}{2} = 22.5\,^\circ\,.$$

5. Coterminal angles

Infinitely many angles can have the same initial and terminal sides. For example, angles of radian measure $7\pi/3, 13\pi/3, 19\pi/3...$, as well as the negative values $-5\pi/3, -11\pi/3, -17\pi/3...$ have the same initial and terminal sides as the angle of measure $\pi/3$ rad. More generally, an angle of measure t rad has the same initial and terminal sides as the angles of radian measure $t + 2k\pi$ for any integer k (positive or negative). Conversely, any two or more angles with the same initial and terminal sides must have radian measures that differ by an integer multiple of 2π. In terms of degree measure this means that they must differ by an integer multiple of $360\,°$. Such angles are called **coterminal**. These concepts are illustrated in the following examples.

Example 11 *Show that the angles of radian measure $7\pi/6$ and $-17\pi/6$ are coterminal. That is, show that they have the same initial and terminal sides.*
Solution: *Since the quantity*

$$7\pi/6 - (-17\pi/6) = 24\pi/6 = 2(3)\pi$$

is an integer multiple of 2π, the given angles satisfy the required conditions for coterminality.

Example 12 *Find an angle t that is measured in the clockwise direction that is coterminal with $31\pi/6\,\mathrm{rad}$.*
Solution: *Since t is to be measured in the clockwise direction it must be negative. To satisfy this condition and to be coterminal with $31\pi/6\,\mathrm{rad}$ we must have*

$$t = \frac{31\pi}{6} - 2k\pi < 0$$

for some integer k. Hence, any integer $k \geq 3$ will produce such an angle. In particular, for $k = 3$ we have $t = -\frac{5}{6}\pi$.

Example 13 *Are the angles of measure -1047° and 403° coterminal?*
Solution: *No, since*

$$-1047^\circ - 403^\circ = -1450^\circ$$

is not an integer multiple of 360°. (If $-1450^\circ = k(360^\circ)$, then $k = -1450^\circ/360^\circ = -\frac{145}{36} = -4.0278$ is not an integer.)

Example 14 *Determine the smallest angle of positive measure that is coterminal with the angle of measure* $\frac{30\pi}{7}$ rad.

Solution: *All angles coterminal with the given angle must have radian measure of the form* $(\frac{30\pi}{7} - 2k\pi)$ rad *where* k *is any integer. In view of the problem we can restrict our attention to positive integers. (Why? Try using a few negative values for* k*.) Consider the following table*

$k = 1$	$\frac{30\pi}{7} - 2(1)\pi = 7.1808$
$k = 2$	$\frac{30\pi}{7} - 2(2)\pi = .8976$
$k = 3$	$\frac{30\pi}{7} - 2(3)\pi = -5.3856$

Evidently, the smallest such angle has measure $\left(\frac{30\pi}{7} - 2(2)\pi\right)$ rad $= .8976$ rad *and is obtained by using the value* $k = 2$*. Note that the* largest *angle of negative measure that is coterminal with the given angle is* $\left(\frac{30\pi}{7} - 2(3)\pi\right)$ rad $= -5.3856$ rad.

6. Radian measure on circles of arbitrary radius

The material presented in this section demonstrates that radian measure is a well-defined concept regardless of the radius of the circle used to define angles. It also demonstrates the convenience of using the unit circle when defining radian measure. Recall from geometry that the circumference of a circle of radius r is $2\pi r$. That is the arclength required to form an angle of 2π rad on a circle of radius r is r times the circumference of the unit circle. These geometric considerations suggest that an angle t rad wide will subtend an arc of length tr on a circle of radius r. In Figure 1.3 the length of the red arc is t and the length of the blue arc is tr. Consequently, an arclength of tr is required to produce an angle t rad wide when the angle is formed using points on a circle of radius r.

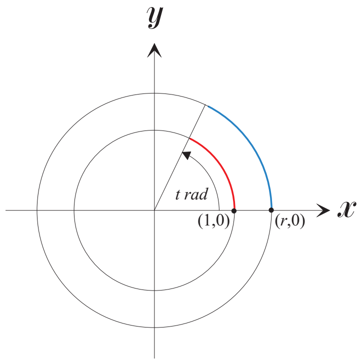


Figure 1.3: Defining angles using a circle of arbitrary radius.

Example 15 *An arclength of* 2 *on a circle of radius* 3 *produces an angle of* $2/3$.rad*.*

7. Exercises

EXERCISE 1. Express each of the following angles in radian measure: (Hint: See Example 8.)

1) $20\,^\circ$ 2) $-35\,^\circ$ 3) $140\,^\circ$ 4) $400\,^\circ$ 5) $-1080\,^\circ$

EXERCISE 2. Express each of the following angles in degree measure:

1) $1\,\mathrm{rad}$ 2) $7\pi/2\,\mathrm{rad}$ 3) $-1.75\,\mathrm{rad}$ 4) $155\pi/200\,\mathrm{rad}$ 5) $-5\pi/8\,\mathrm{rad}$

EXERCISE 3. Which of the following pairs of angles are coterminal?

1) $\pi/4\,\mathrm{rad}$ and $7\pi/4\,\mathrm{rad}$ 2) $3\pi/12\,\mathrm{rad}$ and $27\pi/12\,\mathrm{rad}$ 3) $7\pi/12\,\mathrm{rad}$ and $-19\pi/12\,\mathrm{rad}$ 4) $\pi/12\,\mathrm{rad}$ and $15\,^\circ$ 5) $0\,^\circ$ and $-360\,^\circ$ 6) $-190\,^\circ$ and $90\,^\circ$

EXERCISE 4. Draw a figure depicting angles of measure 1) $-\pi/4\,\mathrm{rad}$ 2) $-11\pi/4\,\mathrm{rad}$

EXERCISE 5. Draw a figure depicting angles of measure 1) $5\pi/3\,\mathrm{rad}$ 2) $-11\pi/6\,\mathrm{rad}$.

EXERCISE 6. Determine the length of the arc on a circle of radius 6 needed to subtend an angle of $-3\pi/4\,\mathrm{rad}$.