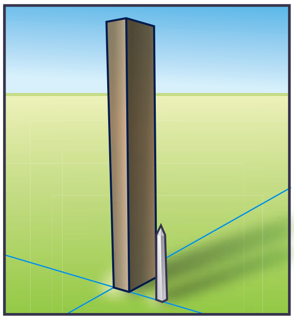
Lesson 10-3 Volumes of Prisms and Cylinders

BIG IDEA From the Box Volume Formula and Cavalieri's Principle, volume formulas for any cylindrical solids can be deduced.

The standard measure of crude oil in the United States is the barrel, which has a capacity equal to 42 gallons. In 2006, total U.S. consumption averaged about 20.7 million barrels every day. To give a dramatic presentation of this fact to the public, many news agencies translate this to how much oil covers a football field. If 20.7 million barrels of oil were poured into a cylindrical tank whose base is the size of a football field, how high would the tank reach? To answer this question, we look at the way that volume is found for any cylindrical surface. The diagram at the right shows such a tank placed next to the Washington Monument, which is 555 feet high.



Mental Math

a. What is the surface area of a cube with sides of length 5?

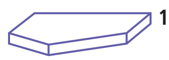
b. One of the faces of a cube with sides of length 5 is removed. What is the surface area of this new shape?

c. How many faces must be removed from a cube with sides of length 5 to get a shape with surface area 75?

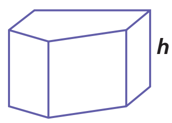
Volumes of Right Prisms and Cylinders

Consider a cylindrical surface whose base is a prism or cylinder having an area of B. Then we think of B unit squares covering the region. We can think this way even if B is not an integer.

area = B square units



volume = B cubic units



volume = Bh cubic units

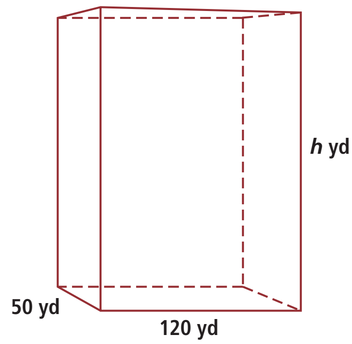
If a prism with this base has height 1 unit, the prism contains B unit cubes, and so the volume of the prism is B cubic units. This is pictured in the middle figure at the right. The bottom figure at the right is a prism with this base and height h. That prism has h times the volume of the middle prism, and so its volume is Bh. This argument shows that, if a right prism or cylinder has height h and a base with area B, then its volume is Bh.

Now you can determine how tall a prism would have to be to contain the amount of oil consumed every day in the United States, if the base of the prism is the size of a football field.

Example 1

An American football field is 120 yards long (with the end zones) and 50 yards wide. If a rectangular prism were built on a football field that would contain the amount of oil consumed in the United States in a day in 2006, how high would the tank be? Make a guess before you go on.

Solution It helps to draw a picture, like the one at the right. The volume V of the tank is found using $V = Bh$, where



$$\begin{aligned} B &= 120 \text{ yd} \cdot 50 \text{ yd} \\ &= 360 \text{ ft} \cdot 150 \text{ ft} \\ &= 54{,}000 \text{ ft}^2. \end{aligned}$$

$$\text{So } V = 54{,}000h \text{ ft}^3.$$

Now convert the usage in the United States from barrels to cubic feet.

$$20.7 \cdot 10^6 \text{ barrels} \cdot \frac{42 \text{ gal}}{1 \text{ barrel}} \cdot \frac{1 \text{ ft}^3}{7.48 \text{ gal}} \approx 116{,}229{,}947 \text{ ft}^3$$

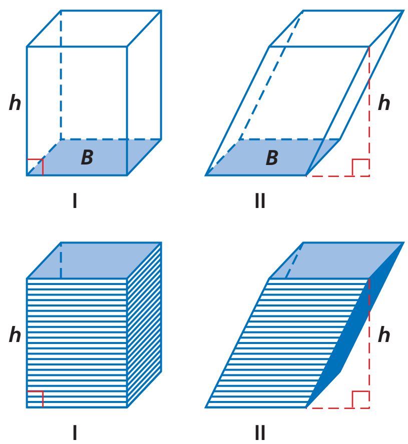
Equate the two expressions for V and solve for h.

$$\begin{aligned} 116{,}229{,}947 &= 54{,}000h \\ h &\approx 2152 \text{ ft} \end{aligned}$$

The picture on page 609 is nearly accurate. The tank would be almost 4 times the height of the Washington Monument.

Volumes of Oblique Prisms and Cylinders

Now suppose you have an *oblique* prism or cylinder. Recall that in these figures, the lateral edges are not perpendicular to the planes of the bases. Pictured at the right are a right prism and an oblique prism with congruent bases and equal heights.



Imagine Prism I to be made up of a stack of thin slices like congruent sheets of paper. Shift the slices of the first stack until it takes the form of Prism II.

Notice that the height, area of the base, and number of slices are the same in Prism I and Prism II. Thus, it makes sense that

$$\text{Volume(Prism II)} = \text{Volume(Prism I)}.$$

Or, because they have equal heights and bases,

$$\text{Volume(Prism II)} = Bh.$$

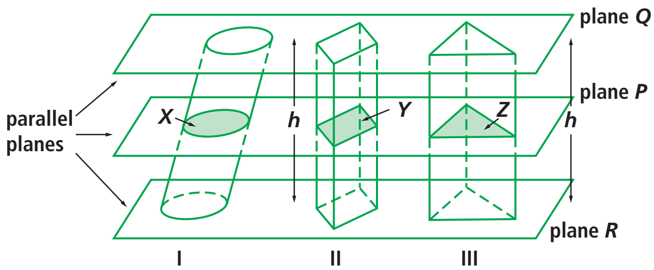
Cavalieri's Principle

The key ideas of this argument are: (1) the prisms have their bases in the same planes; (2) each slice is parallel to the bases; and (3) the slices in each prism have the same area. The conclusion is that these solids have the same volume. The first individuals to use these ideas to obtain volumes seem to have been the Chinese mathematician Zu Chongzhi (429–500) and his son Zu Geng. However, in the West, Bonaventura Cavalieri (1598–1647), an Italian mathematician, first realized the importance of this principle, and in the West it is named after him. *Cavalieri's Principle* is the fifth and last part of the assumed statements about volume.

Volume Postulate

e. *Cavalieri's Principle*
Let I and II be two solids included between parallel planes. If every plane P parallel to the given planes intersects I and II in sections with the same area, then Volume(I) = Volume(II).

At the right, plane P is parallel to the planes Q and R containing the bases, and all three solids have bases with area B. Because plane sections X, Y, and Z are translation images of the bases (this is how prisms and cylinders are defined), they also have area B. Thus, the conditions for Cavalieri's Principle are satisfied. These solids have the same volume. But we know the volume of figure II, the box.



$$\text{Volume(II)} = \ell \cdot w \cdot h$$
$$= B \cdot h.$$

Thus, using Cavalieri's Principle,

$$\text{Volume(I)} = B \cdot h \text{ and}$$
$$\text{Volume(III)} = B \cdot h.$$

This proves the following theorem for *all* cylinders and prisms.

Prism-Cylinder Volume Formula

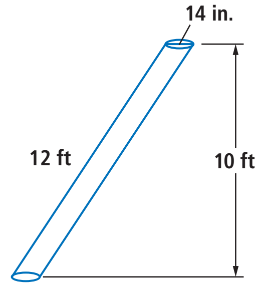
The volume V of any prism or cylinder is the product of its height h and the area B of its base.

$$V = Bh$$

GUIDED

Example 2

A cylindrical duct is used to expel hot air from a basement clothes dryer. The length of the duct is 12 ft. The duct goes down at an angle so that the bottom ring of the duct in the basement is 10 feet lower than the top. The duct diameter is 14 inches. Find the volume of the air that is in the duct.



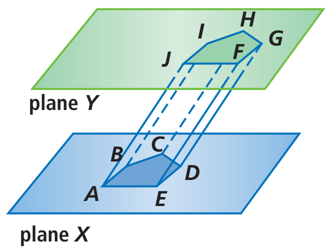
Solution The important idea is that the length of the duct makes no difference in finding the volume of air. Because $V = Bh$, all you need is the area of the base and the height. Change the height to inches.

$$V = Bh$$
$$B = \pi \cdot r^2 = \pi \cdot \underline{\ ?\ }^2 = \underline{\ ?\ } \text{ in}^2$$
$$\text{Thus } V = \underline{\ ?\ } \cdot 120 \approx 18{,}473 \text{ in}^3.$$

There are about 18,473 cubic inches or about 10.7 cubic feet of air in the duct.

Example 3

The pentagonal prism shown at the right was constructed using pentagon *ABCDE* and vector $\overrightarrow{EF}$, where *F* is in plane *Y*. Suppose you move *FGHIJ* in that plane. What will happen to the volume of the prism? Why?



Solution The height of the prism is the distance between the two planes. So, no matter where the point F is placed on plane Y, the height of the prism is always the same. The base area is unaffected by moving FGHIJ. Thus, because the height and the area of the base remain constant, the volume of the prism remains constant as well.

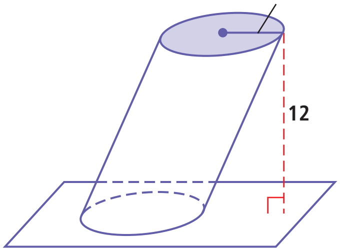
Questions

COVERING THE IDEAS

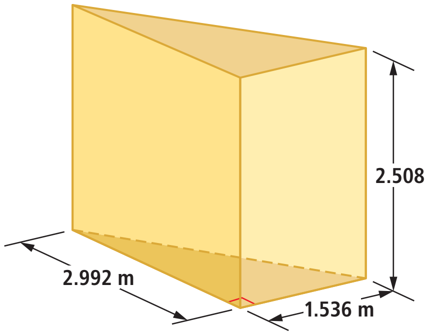
1. A cubic foot of liquid is how many gallons?
2. The United States Strategic Petroleum Reserve was 687.9 million barrels as of July 2006.
   a. If you filled up a football field with this many barrels of oil, how high would the prism be?
   b. Using the average daily consumption of the United States in 2006, calculate how many days worth of oil was in the Reserve.
3. **Multiple Choice** In this lesson, a stack of paper is used to illustrate all but which one of the following?
   A Cavalieri's Principle
   B that an oblique prism and a right prism can have the same volume
   C that the volume of an oblique prism is *Bh*
   D that a cylinder and a prism have the same volume formula

In 4–9, find the volume of each solid.

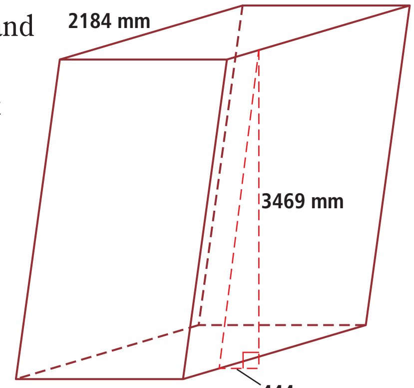
4.



5.

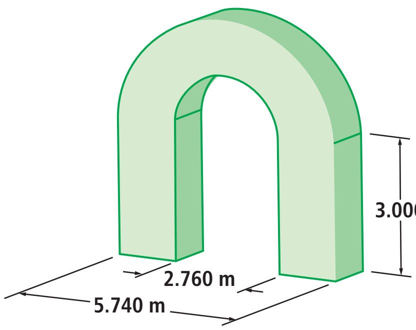


6. a regular hexagonal prism whose base has edge 5 meters, and whose height is 20 meters
7. the oblique prism with rectangular bases drawn at the right
8. a right rectangular prism whose base is 3 feet by 7 feet, and whose height is 10 feet
9. a sewer pipe 100 feet long with a radius of 24 inches.
10. Cavalieri's Principle was discovered by mathematicians of what two nationalities?
11. State Cavalieri's Principle.



APPLYING THE MATHEMATICS

12. Suppose the Roman arch below is made of solid concrete. The bases of the columns are squares and the arch is a semicircle. How much concrete was used?

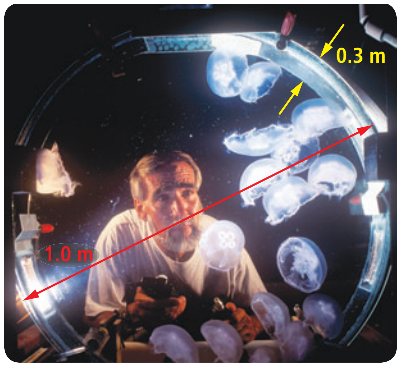


13. In the Georgia Aquarium, the Ocean Voyager display contains 6 million gallons of water. The viewing window is made from acrylic and is 61 ft wide, 24 ft high and 2 ft thick. A recent price for acrylic is about 8 cents per cubic inch. How much would that sheet of acrylic cost?

14. If a cylinder has a height h and base with radius r, find a formula for its volume in terms of h and r.

15. The volume of an oblique prism is 42 cubic meters. Its height is 7 meters. Find the area of its base.

16. A fish tank for jellyfish is called a *Kreisel tank,* from the German word for a spinning top. The spinning of the cylindrical tank forces the jellyfish to stay suspended and not stick to the sides. What is the volume of the Kreisel tank at the right?



17. Suppose you double the height and radius of a cylinder. What is the relationship between the volumes of the smaller and bigger cylinders?

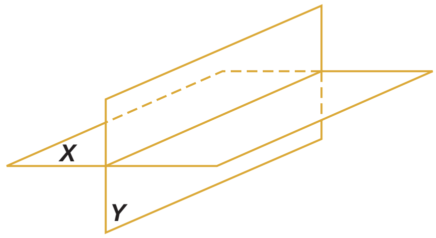
18. A milliliter of water has a mass of 1 gram and occupies 1 cm^3 of space. What mass of water (to the nearest gram) will fill a cylindrical can that is 15 cm high and has radius 3 cm?

REVIEW

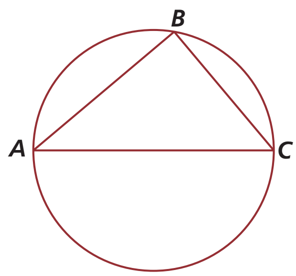
19. Model $(a + 4)(b + c)$ with the area of a rectangle and compute the product. **(Lesson 10-2)**

20. A box with dimensions 1 meter, x meters, and y meters has volume 21 cubic meters and surface area 62 square meters. Find x and y. **(Lessons 10-1, 9-9)**

21. The figure at the right shows two intersecting planes X and Y. Describe where a third plane Z could be placed so its intersection with this figure will look like **(Lesson 9-1)**
   a. two intersecting lines.
   b. two parallel lines.
   c. one line.



22. In the figure at the right, $\overline{AC}$ is a diameter in the circle, $AB = 9$, $AC = 12$. Find the area of the part of the circle that is outside of $\triangle ABC$. **(Lessons 8-9, 8-6, 6-3)**



23. Each of the following is the area formula for a certain type of quadrilateral. Name the quadrilateral that goes with each formula. **(Lesson 8-5)**
   a. $A = s^2$
   b. $A = hb$
   c. $A = \frac{1}{2}h(b_1 + b_2)$
   d. $A = \ell w$

24. Solve $A = \pi r^2 h$ for r where $r > 0$. **(Previous Course)**

EXPLORATION

25. Find and describe one other mathematical contribution made by Zu Chongzhi, and one other mathematical contribution made by Bonaventura Cavalieri.