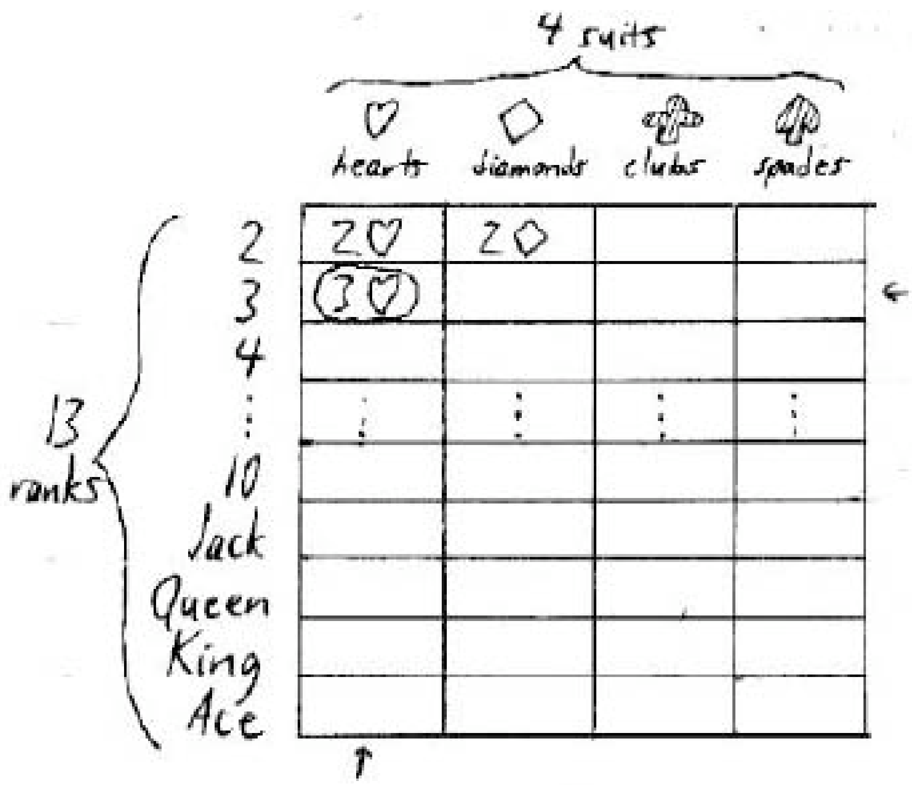
LESSON 11: MULTIPLICATION RULE (SECTION 4-4)

PART A: INDEPENDENT EVENTS

Example 1

Pick (or "draw") a card from a standard deck of 52 cards with no Jokers. (Know this setup!)



$P(3) = \frac{4}{52} = \frac{1}{13}$ (The 13 ranks are equally likely.)

$P(\text{hearts}) = \frac{13}{52} = \frac{1}{4}$ (The 4 suits are equally likely.)

$P(3 \text{ and hearts}) = \frac{1}{52}$ (The 52 cards are equally likely.)

The events "3" and "Hearts" are <u>independent events</u>, because knowing the rank of a card tells us nothing about its suit, and vice-versa. The occurrence of one event does not change our probability assessment for the other event. The rank and the suit of an unknown card are <u>independent random variables</u>, which we will discuss later.

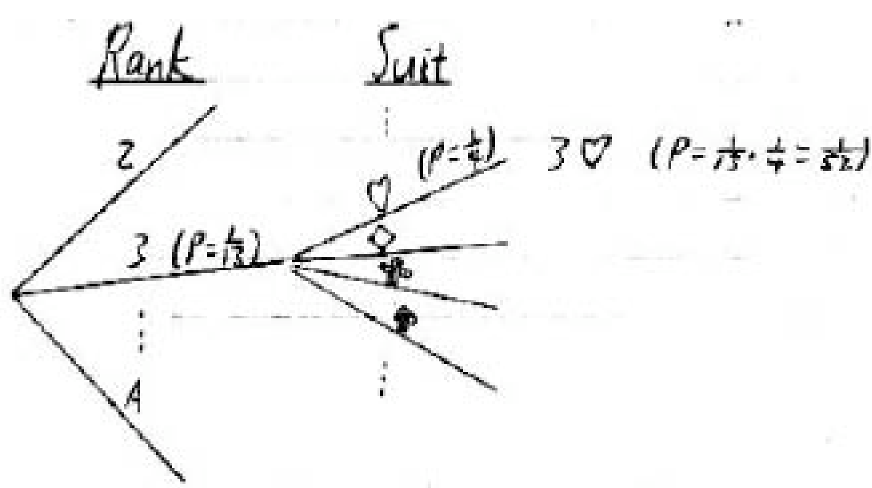
<u>Dependent events</u> are events that are not independent.

<u>Multiplication Rule for Independent Events</u>

If events A, B, C, etc. are independent, then:

$$P\left(A \text{ and } B\right) = P\left(A\right) \cdot P\left(B\right)$$
$$P\left(A \text{ and } B \text{ and } C\right) = P\left(A\right) \cdot P\left(B\right) \cdot P\left(C\right), \text{ etc.}$$

<u>Tree Diagram</u>



<u>Example 2</u>

Draw three cards from a standard deck with replacement.

("<u>With replacement</u>" means that, after we draw a card, we place it back in the deck before we draw the next card.)

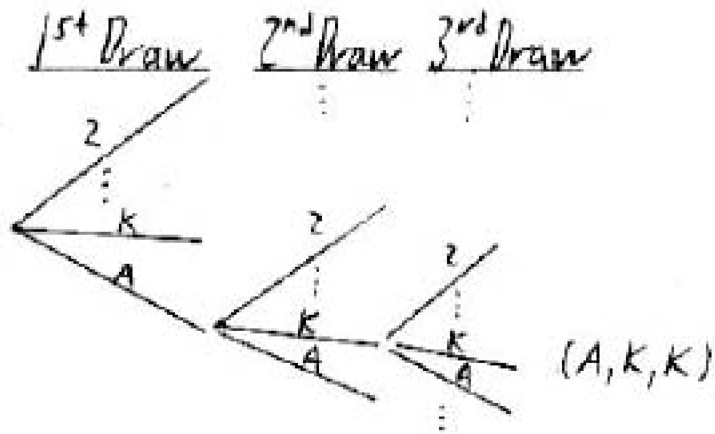
Find the probability that we draw an Ace first, a King second, and a King third. Think: AKK sequence.

Solution to Example 2

Because we are drawing cards with replacement, the draws are independent.

$$P(\text{A-1st and K-2nd and K-3rd}) = P(\text{A-1st}) \cdot P(\text{K-2nd}) \cdot P(\text{K-3rd})$$
$$= \frac{1}{13} \cdot \frac{1}{13} \cdot \frac{1}{13}$$
$$= \frac{1}{2197} \quad (\approx 0.000455) \quad \square$$

Tree



PART B: CONDITIONAL PROBABILITY

$P(B|A)$ = the "updated" probability that B occurs, given that A occurs.

Technical Note: The idea of "updating" probabilities is a popular idea among Bayesian statisticians, though it is controversial.

PART C: DEPENDENT EVENTS

$P(B|A)$ = the "updated" probability that B occurs, given that A occurs.

General Multiplication Rule

For events A, B, C, etc.,

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$
$$P(A \text{ and } B \text{ and } C) = P(A) \cdot P(B|A) \cdot P(C|A \text{ and } B), \text{ etc.}$$

Note: If A and B are independent, then the occurrence of A does not change our probability assessment for B, and $P(B|A) = P(B)$. In that case, the General Multiplication Rule becomes the Multiplication Rule for Independent Events that we discussed earlier: $P(A \text{ and } B) = P(A) \cdot P(B)$.

Example 3

Draw three cards from a standard deck without replacement.

("Without replacement" means that drawn cards are never returned back to the deck.)

Find the probability that we draw an Ace first, a King second, and a King third. Think: AKK sequence.

<u>Solution to Example 3</u>

Because we are drawing cards without replacement, previous draws affect later draws, and the draws are dependent.

$$
\begin{aligned}
&P(\text{A-1st and K-2nd and K-3rd}) \\
&= P(\text{A-1st}) \cdot P(\text{K-2nd} \mid \text{A-1st}) \cdot P(\text{K-3rd} \mid \text{A-1st and K-2nd}) \\
&= \frac{4}{52} \cdot \underbrace{\frac{4}{51}}_{\substack{\text{51 cards,} \\ \text{4 Ks}}} \cdot \underbrace{\frac{3}{50}}_{\substack{\text{50 cards,} \\ \text{3 Ks}}} \\
&\approx 0.000362 \quad \square
\end{aligned}
$$

<u>Think About It</u>: Why is this probability for the AKK sequence lower than the one we found for Example 2, when we were drawing with replacement? Also, why is $P(\text{K-2nd} \mid \text{A-1st})$ in this example higher than $P(\text{K-2nd})$ in Example 2?

PART D: SAMPLING RULE FOR TREATING DEPENDENT EVENTS AS INDEPENDENT

When we conduct polls, we sample without replacement, so that the same person is not contacted twice. Technically, the selections are dependent.

Sometimes, to simplify our calculations, we can treat dependent events as independent, and our results will still be reasonably accurate. We can do this when we take relatively small samples from large populations; then, we can practically assume that we are sampling with replacement, and we can ignore the unlikely possibility of the same item (or person) being selected twice.

Sampling Rule for Treating Dependent Events as Independent

$\boxed{\text{Population (Size } N\text{)}}$	($N \geq 1000$, say)
$\downarrow$	(even if we draw without replacement)
$\boxed{\text{Sample (Size } n\text{)}}$	($n \leq 0.05N$)

If we are drawing a sample of size n from a population of size N without replacement, then, even though the selections are dependent, we can practically treat them as independent if:

• The sample size is no more than 5% of the population size:
$n \leq (5\% \text{ of } N)$, or $n \leq 0.05N$,

and

• The population size, N, is large: say, for our class,
$N \geq 1000$.

Example 4

G.W. Bush won 47.8% of the popular vote in 2000.
(Al Gore won 48.4%, and Ralph Nader won 2.7%.)
Over 105 million voters in the U.S. voted for President in 2000.
Of those, three are randomly selected without replacement.
Find the probability that all three selected voters voted for Bush in 2000.

Solution to Example 4

Observe that we are sampling no more than 5% of a huge population.
By the aforementioned Sampling Rule, we may practically assume that the selections are independent.

Since Bush won 47.8% of the vote, the probability that a randomly selected voter voted for Bush is 47.8%, or 0.478.

Warning: Avoid using percents when doing calculations.

$$
\begin{aligned}
&P\left(\text{all three selected voters voted for Bush}\right) \\
&= P\left(\text{1st-Bush}\right)\cdot P\left(\text{2nd-Bush}\right)\cdot P\left(\text{3rd-Bush}\right) \\
&= \left(0.478\right)\left(0.478\right)\left(0.478\right), \text{ or } \left(0.478\right)^3 \\
&\approx 0.109 \quad \left(\text{or } 10.9\%\right) \quad \square
\end{aligned}
$$

LESSON 12: MORE CONDITIONAL PROBABILITY
(SECTION 4-5)

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

$$\Rightarrow P(A) \cdot P(B|A) = P(A \text{ and } B)$$

$$\Rightarrow P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

$$= \frac{\#(A \text{ and } B)}{\#(A)}$$

# of trials in which A and B occur

A occurs

<u>Ex</u> (Voters in Springfield)

(Assume no other possibs.)

		Voted for... Kerry	Bush	
Voter is a...	Dem.	273	117	390
	Rep.	77	208	285
		350	325	675

Find the prob. that a randomly selected Dem. voter in Springfield voted for Kerry.

$$P(\text{Kerry} \mid \text{Dem.}) = \frac{\#(\text{Dem. and Kerry})}{\#(\text{Dem.})}$$

$$= \frac{273}{390}$$

$$= \boxed{0.7 \text{ (or } 70\%)}$$

(70% of the Dems. voted for Kerry.)

Ex (Test for cancer)

20,000 people live in a town.

Given: $P(\text{tests "+"} \mid \text{has cancer}) = 0.8$
$P(\text{tests "–"} \mid \text{no cancer}) = 0.9$
$P(\text{has cancer}) = 0.01$ prior prob.

Find: $P(\text{has cancer} \mid \text{tests "+"})$.

(If you test "+" for cancer, what is the updated prob. that you actually have cancer?)

$N = 20{,}000$

How many in the town have cancer?

$(0.01)(20{,}000) = 200$

Of these 200...
How many test "+"? $(0.8)(200) = 160$
"–"? $(0.2)(200) = 40$
or $200 - 160$

How many in the town do not have cancer?
$(0.99)(20{,}000)$ or $20{,}000 - 200 = 19{,}800$

Of these...
How many test "–"? $(0.9)(19{,}800) = 17{,}820$
"+"? $(0.1)(19{,}800) = 1{,}980$
or $19{,}800 - 17{,}820$

Table

	Tests "+"	Tests "–"	
Has cancer	160	40	200
No cancer	1,980	17,820	19,800
	2,140	17,860	20,000

"false positives"

$$P(\text{has cancer} \mid \text{tests "+"})$$

$$= \frac{\#(\text{tests "+" and has cancer})}{\#(\text{tests "+"})}$$

$$= \frac{160}{2{,}140}$$

$$\boxed{\approx 0.0748 \quad (7.48\%)}$$

(If you test "+" for cancer, there is only about a 7.48% chance you actually have it!)

Psych study in 1982.
100 physicians were given this problem.
95 thought the answer about 75%.

What's going on? "Bayesian approach"

P(cancer | state of knowledge)

The test is unreliable - can only update prob. so far up!

prior prob. of cancer is tiny!

Conjunctive Fallacy

Someone may write

$P(\text{Joe is a Dem.}) = 0.7$
$P(\text{Joe is a liberal}) = 0.8$
$P(\text{Joe is a liberal Dem.}) = 0.9$

Makes no sense!

Probs. can't rise when you throw in conditions!

Dating service:
Man who is tall and handsome and rich...
Stereotypes
Pentagon scenarios (prob. ≈ 0)

Which words are more common?

_ _ _ _ i n g ← words more "available" but just a special case of

or

_ _ _ _ _ n _

Ex Challenger disaster (1986)

6 independent O-rings. All 6 had to work.
Given the conditions on takeoff, each had a 3% chance of failure.

$$
\begin{aligned}
P(\text{all 6 work}) &= P(1^{st}\text{ works}) \cdot P(2^{nd}\text{ works}) \cdots P(6^{th}\text{ works}) \\
&= (.97) \cdot (.97) \cdots (.97) \\
&= (.97)^6 \\
&\approx .833 \text{ or } 83.3\%
\end{aligned}
$$

$$
\begin{aligned}
P(\text{at least 1 fails}) &\approx 1 - .833 \\
&= .167 \quad (16.7\%)
\end{aligned}
$$

Do you want to shoot up something with a 16.7% chance of failing?