# Lesson 11: Solving Mechanical System Dynamics Using Laplace Transforms

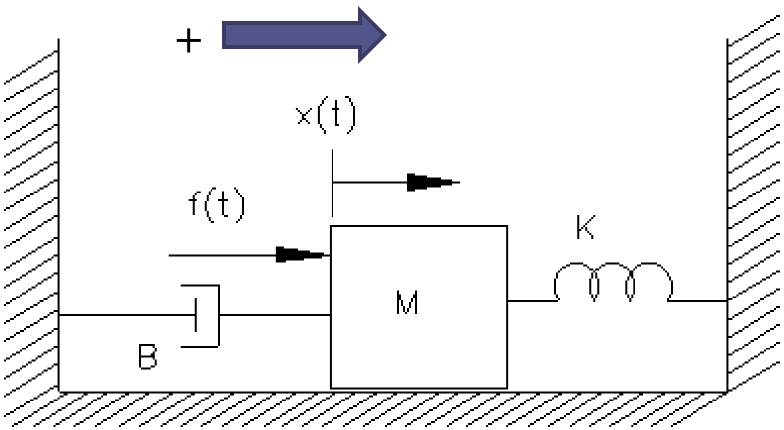
ET 438a Automatic Control Systems Technology

# Learning Objectives

After this presentation you will be able to:

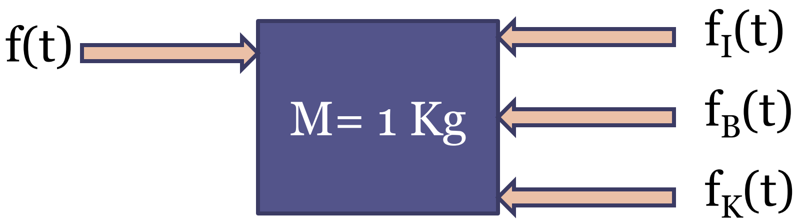
- Draw a free body diagram of a translational mechanical system,
- Write a differential equation that describes the position of a mechanical system as it varies in time.
- Use Laplace transforms to convert differential equations into algebraic equations.
- Take the Inverse Laplace transform and find the time response of a mechanical system.
- Examine the impact of increased and decreased damping on a mechanical system.

# Using Laplace Transforms to Solve Mechanical Systems




**Example 11-1**: Write the differential equation for the system shown with respect to position and solve it using Laplace transform methods. Assume $f(t) = 50 \cdot u_s(t)$ N, M= 1 Kg, K=2.5 N/m and B=0.5 N-s/m. The mass slides on a frictionless surface. x(0)=0.

Draw a free body diagram and label the forces




$$f(t) - f_I(t) - f_B(t) - f_K(t) = 0$$

$$f(t)=f_I(t)+f_B(t)+f_K(t)$$

# Laplace Solution to Mechanical Systems (1)

$$f(t) = 50 \cdot u_s(t) \qquad f_I(t) = M \cdot \frac{d^2x(t)}{dt^2} = 1 \cdot \frac{d^2x(t)}{dt^2} \qquad f_B(t) = B \cdot \frac{dx(t)}{dt} = 0.5 \cdot \frac{dx(t)}{dt}$$

$$f_K(t) = K \cdot x(t) = 2.5 \cdot x(t)$$

$$50u_s(t) = m\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + Kx(t)$$

$$50u_s(t) = 1\frac{d^2x(t)}{dt^2} + 0.5\frac{dx(t)}{dt} + 2.5x(t)$$

$$\frac{50}{s} = s^2X(s) + 0.5sX(s) + 2.5X(s)$$

Solve for X(s)

$$\frac{50}{s} = (s^2 + 0.5s + 2.5)X(s)$$

$$50 = s(s^2 + 0.5s + 2.5)X(s)$$

# Laplace Solution to Mechanical Systems (2)

$$\frac{50}{s(s^2+0.5s+2.5)} = X(s)$$

Solve using inverse Laplace transform and partial fractions expansion

Use Quadratic formula to find factors in denominator

$$S = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad S = \frac{-0.5 \pm \sqrt{0.5^2-4(1)(2.5)}}{2(1)} \qquad S = 0.25 \pm j1.561$$

$$a=1 \quad b=0.5 \quad c=2.5 \qquad S = \frac{-0.5}{2} \pm \frac{\sqrt{9.75}}{2}$$

$$S_1 = 0.25 + j1.561 = 1.581\angle 81^\circ$$

$$S_2 = 0.25 - j1.561 = 1.581\angle -81^\circ$$

# Laplace Solution to Mechanical Systems (3)

Factored form

$$\frac{50}{s(s+1.581\angle 81^\circ)(s+1.581\angle -81^\circ)} = X(s)$$

$s_1$ $s_2$

Partial Fractions Expansion- Find A

$$X(s) = \frac{50}{s(s+1.581\angle 81^\circ)(s+1.581\angle -81^\circ)} = \frac{A}{s} + \frac{B}{s+1.581\angle 81^\circ} + \frac{C}{s+1.581\angle -81^\circ}$$

# Laplace Solution to Mechanical Systems (4)

$$\left.\frac{50s}{s(s+1.581\angle 81^\circ)(s+1.581\angle -81^\circ)}\right|_{s=0} = \left.\frac{A\cancel{s}}{\cancel{s}} + \frac{Bs}{s+1.581\angle 81^\circ} + \frac{Cs}{s+1.581\angle -81^\circ}\right|_{s=0}$$

$$\frac{50}{(0+1.581\angle 81)(0+1.581\angle -81^\circ)} = A + \frac{B0}{0+1.581\angle 81} + \frac{C(0}{0+1.581\angle -81}$$

$$\frac{50}{2.5} = A$$

$$20 = A$$

# Laplace Solution to Mechanical Systems (5)

Find B

$$\left.\frac{50(s+\cancel{1.581\angle 81^\circ})}{s(\cancel{s+1.581\angle 81^\circ})(s+1.581\angle -81^\circ)}\right|_{s=-1.581\angle 81^\circ} = \frac{A(s+1.581\angle 81^\circ)}{s} + \frac{B(\cancel{s+1.581\angle 81^\circ})}{\cancel{s+1.581\angle 81^\circ}} + \left.\frac{C(s+1.581\angle 81^\circ)}{s+1.581\angle -81^\circ}\right|_{s=-1.581\angle 81^\circ}$$

$$\frac{50}{(-1.581\angle 81^\circ)(-1.581\angle 81^\circ+1.581\angle -81^\circ)} = \frac{A(-1.581\angle 81^\circ+1.581\angle 81^\circ)}{s} + B + \frac{C(-1.581\angle 81^\circ+1.581\angle 81^\circ)}{s+1.581\angle -81^\circ}$$

$$\frac{50}{(-1.581\angle 81^\circ)(3.123\angle -90^\circ)} = B \qquad \frac{50}{-4.937\angle -9^\circ} = B$$

$$10.13\angle -171^\circ = B$$

# Laplace Solution to Mechanical Systems (6)

Find C

$$\left.\frac{50(\cancel{s+1.581\angle -81^\circ})}{s(s+1.581\angle 81^\circ)(\cancel{s+1.581\angle -81^\circ})}\right|_{s=-1.581\angle -81^\circ} = \frac{A(s+1.581\angle -81^\circ)}{s} + \frac{B(s+1.581\angle -81^\circ)}{s+1.581\angle 81^\circ} + \left.\frac{C(\cancel{s+1.581\angle -81^\circ})}{\cancel{s+1.581\angle -81^\circ}}\right|_{s=-1.581\angle -81^\circ}$$

$$\frac{50}{(-1.581\angle -81^\circ)(-1.581\angle -81^\circ+1.581\angle -81^\circ)} = \cancelto{0}{\frac{A(-1.581\angle -81^\circ+1.581\angle 81^\circ)}{s}} + \cancelto{0}{\frac{B(-1.581\angle -81^\circ+1.581\angle -81^\circ)}{s+1.581\angle 81^\circ}} + C$$

$$\frac{50}{(-1.581\angle -81^\circ)(3.124\angle 90^\circ)} = C \qquad \frac{50}{-4.939\angle 9^\circ} = C \qquad \frac{50}{4.939\angle 189} = C$$

# Laplace Solution to Mechanical Systems (7)

Reverse angle rotation

$$(10.12\angle{-189})(1\angle{360^\circ}) = C$$

$$10.13\angle{171^\circ} = C$$

Expanded Laplace relationship

$$X(s) = \frac{20}{s} + \frac{10.13\angle{-171^\circ}}{s+1.581\angle{81^\circ}} + \frac{10.12\angle{171^\circ}}{s+1.581\angle{-81^\circ}}$$

$$\mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[\frac{20}{s}\right] + \mathcal{L}^{-1}\left[\frac{10.13\angle{-171^\circ}}{s+1.581\angle{81^\circ}} + \frac{10.12\angle{171^\circ}}{s+1.581\angle{-81^\circ}}\right]$$

Remember

$$1.581\angle{81^\circ} = 0.25 + j1.561$$

$$1.581\angle{-81^\circ} = 0.25 - j1.561$$

# Laplace Solution to Mechanical Systems (8)

$$\mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[\frac{20}{s}\right] + \mathcal{L}^{-1}\left[\frac{10.13\angle -171^\circ}{s+(0.25+j1.561)} + \frac{10.12\angle 171^\circ}{s+(0.25-j1.561)}\right]$$

From Laplace table

$$\mathcal{L}^{-1}\left[\frac{K\angle\theta}{s+(a+jb)} + \frac{K\angle -\theta}{s+(a-jb)}\right] = 2K e^{-at}\cos(bt+\theta)$$

Where $K = 10.12 \quad \theta = 171^\circ$

$b = 1.561 \quad a = 0.25$

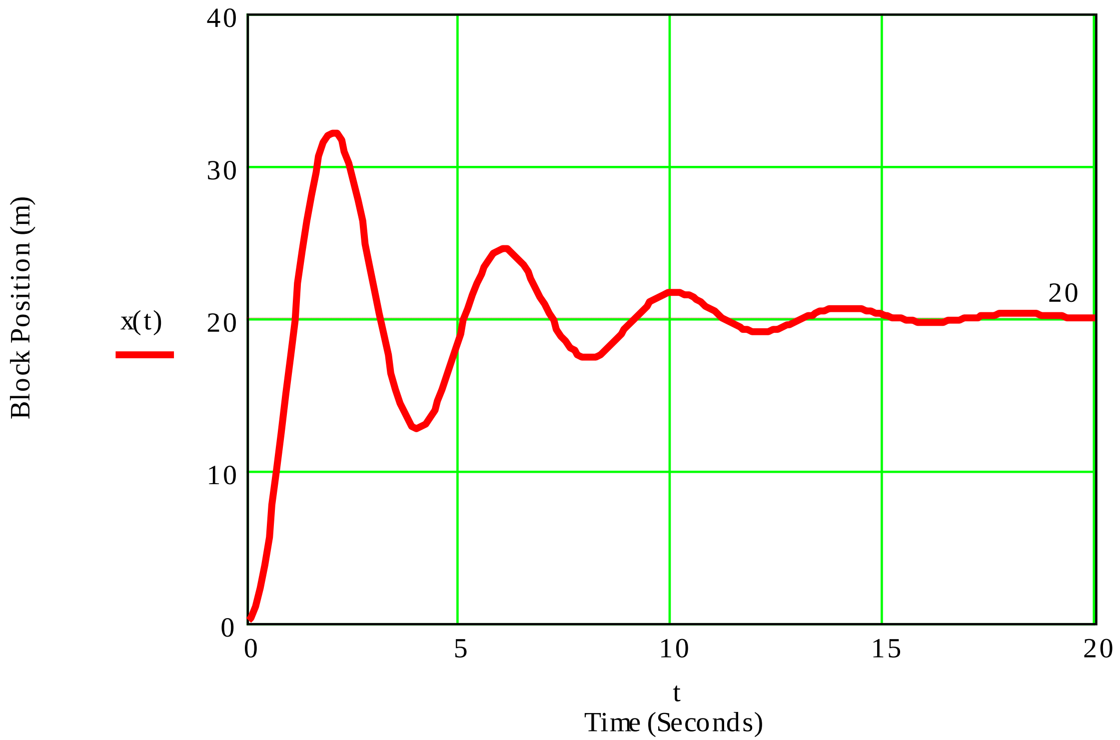
$$\mathcal{L}^{-1}[X(s)] = 2(10.12)e^{-0.25t}\cos(1.561t + 171^\circ) + 20$$

$$\boxed{x(t) = 20 + 20.24e^{-0.25t}\cos(1.561t + 171^\circ)}$$

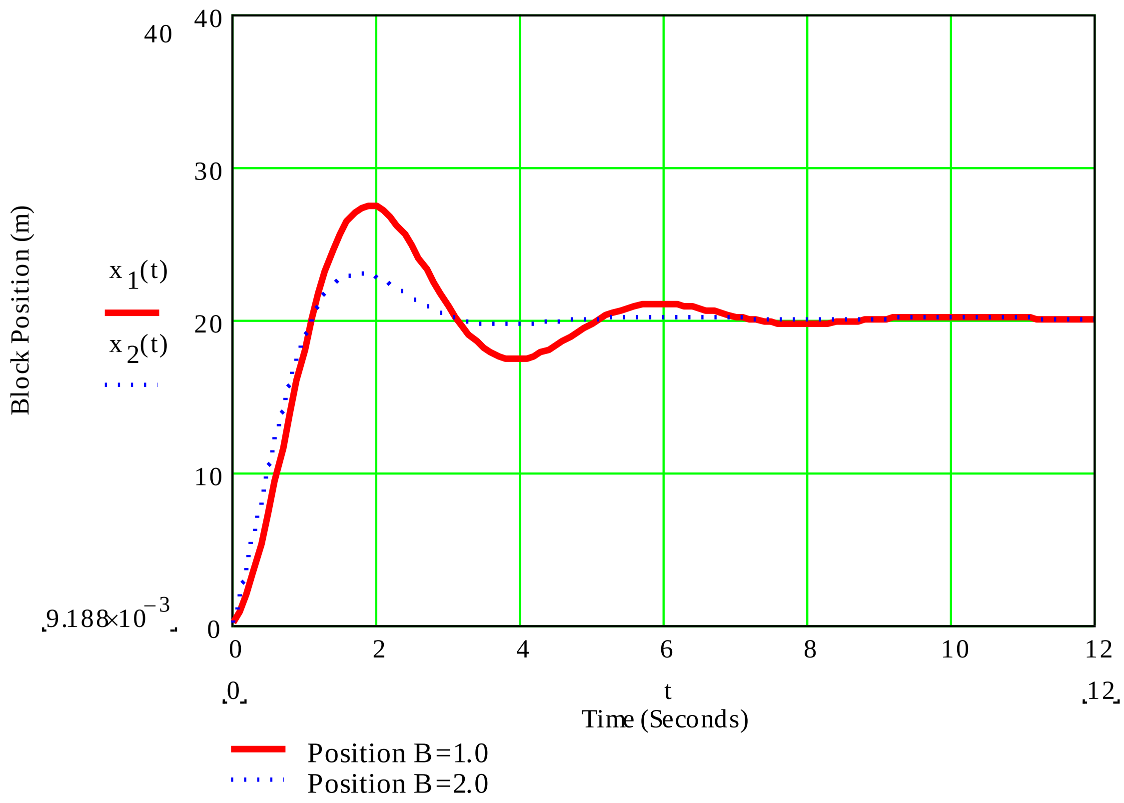
Ans

# Time Plot of System Position

Damping of system is B=0.5



# Effects of Increased Damping




Large damping value reduces oscillations and overshoot but slows response

# End Lesson 11: Solving Mechanical System Dynamics Using Laplace Transforms

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